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GREEN'S FUNCTIONAL FOR SECOND-ORDER LINEAR DIFFERENTIAL EQUATION WITH NONLOCAL CONDITIONS

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Abstract. In this work, we present a new constructive technique which is based on Green's functional concept. According to this technique, a linear completely nonhomogeneous nonlocal problem for a second-order ordinary differential equation is reduced to one and only one integral equation in order to identify the Green's solution. The coefficients of the equation are assumed to be generally variable nonsmooth functions satisfying some general properties such as p-integrability and boundedness. A system of three integro-algebraic equations called the special adjoint system is obtained for this problem. A solution of this special adjoint system is Green's functional which enables us to determine the Green's function and the Green's solution for the problem. Some illustrative applications and comparisons are provided with some known results.

1. Introduction

Green functions of linear boundary-value problems for ordinary differential equations with smooth coefficients have been investigated in detail in several studies [10, 13, 16, 17, 18]. In this work, a linear nonlocal problem is studied for a second-order differential equation. The coefficients of the equation are assumed to be generally nonsmooth functions satisfying some general properties such as p-integrability and boundedness. The operator of this equation, in general, does not have a formal adjoint operator, or any extension of the traditional type for this operator exists only on a space of distributions [7, 16]. In addition, the considered problem does not have a meaningful traditional type adjoint problem, even for simple cases of a differential equation and nonlocal conditions. Due to these facts, some serious difficulties arise in the application of the classical methods for such a problem. As can be seen from [10], similar difficulties arise even for classical type boundary value problems if the coefficients of the differential equation are, for example, continuous nonsmooth functions. For this reason, a Green's functional approach is introduced for the investigation of the considered problem. This approach is based on [1, 2, 3, 4, 14] and on some methods of functional analysis. The main idea of this approach is related to the usage of a new concept of the adjoint problem named adjoint system. Such an adjoint system includes three integro-algebraic equations

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with an unknown element $(f_2(\xi), f_1, f_0)$ in which $f_2(\xi)$ is a function, and f_j for $j = 0, 1$ are real numbers. One of these equations is an integral equation with respect to $f_2(\xi)$ and generally includes f_j as parameters. The other two equation can be considered as a system of algebraic equations with respect to f_0 and f_1, and they may include some integral functionals defined on $f_2(\xi)$. The form of the adjoint system depends on the operators of the equation and the conditions. The role of the adjoint system is similar to that of the adjoint operator equation in the general theory of the linear operator equations in Banach spaces [5, 8, 10]. The integral representation of the solution is obtained by a concept of the Green functional which is introduced as a special solution $f(x) = (f_2(\xi, x), f_1(x), f_0(x))$ of the corresponding adjoint system having a special free term depending on x as a parameter. The first component $f_2(\xi, x)$ of Green functional $f(x)$ is corresponded to Green's function for the problem. The other two components $f_j(x)$ for $j = 0, 1$ correspond to the unit effects of the conditions. If the homogeneous problem has a nontrivial solution, then the Green functional does not exist. In summary, this approach is principally different from the classical methods used for constructing Green functions[17].

2. Statement of the problem

Let $\mathbb{R}$ be the set of real numbers. Let $G = (x_0, x_1)$ be a bounded open interval in $\mathbb{R}$. Let $L_p(G)$ with $1 \leq p < \infty$ be the space of p-integrable functions on G. Let $L_\infty(G)$ be the space of measurable and essentially bounded functions on G, and let $W_p^{(2)}(G)$ with $1 \leq p \leq \infty$ be the space of all functions $u = u(x) \in L_p(G)$ having derivatives $d^k u/dx^k \in L_p(G)$, where $k = 1, 2$. The norm on the space $W_p^{(2)}(G)$ is defined as

$$\|u\|_{W_p^{(2)}(G)} = \sum_{k=0}^{2} \|\frac{d^k u}{dx^k}\|_{L_p(G)}\,.$$

We consider the second-order boundary value problem

$$(V_2 u)(x) \equiv u''(x) + A_1(x)u'(x) + A_0(x)u(x) = z_2(x), \quad x \in G, \tag{2.1}$$

subject to the nonlocal boundary conditions

$$\begin{aligned} V_1 u &\equiv a_1 u(x_0) + b_1 u'(x_0) + \int_{x_0}^{x_1} g_1(\xi) u''(\xi) d\xi = z_1, \\ V_0 u &\equiv a_0 u(x_0) + b_0 u'(x_0) + \int_{x_0}^{x_1} g_0(\xi) u''(\xi) d\xi = z_0, \end{aligned} \tag{2.2}$$

which are more general conditions than the ones in [4]. We investigate a solution to the problem in the space $W_p = W_p^{(2)}(G)$. Furthermore, we assume that the following conditions are satisfied: $A_i \in L_p(G)$ and $g_i \in L_q(G)$ for $i = 0, 1$ are given functions; a_i, b_i for $i = 0, 1$ are given real numbers; $z_2 \in L_p(G)$ is a given function and z_i for $i = 0, 1$ are given real numbers.

Problem (2.1)-(2.2) is a linear completely nonhomogeneous problem which can be considered as an operator equation:

$$Vu = z, \tag{2.3}$$

with the linear operator $V = (V_2, V_1, V_0)$ and $z = (z_2(x), z_1, z_0)$.

The assumptions considered above guarantee that V is bounded from W_p to the Banach space $E_p \equiv L_p(G) \times \mathbb{R} \times \mathbb{R}$ consisting of element $z = (z_2(x), z_1, z_0)$ with

$$\|z\|_{E_p} = \|z_2\|_{L_p(G)} + |z_1| + |z_0|, \quad 1 \le p \le \infty.$$

If, for a given $z \in E_p$, the problem (2.1)-(2.2) has a unique solution $u \in W_p$ with $\|u\|_{W_p} \le c_0 \|z\|_{E_p}$, then this problem is called a well-posed problem, where c_0 is a constant independent of z. Problem (2.1)-(2.2) is well-posed if and only if $V : W_p \to E_p$ is a (linear) homeomorphism.

3. Adjoint space of the solution space

Problem (2.1)-(2.2) is investigated by means of a new concept of the adjoint problem. This concept is introduced in the papers [2, 3] by the adjoint operator V^* of V. Some isomorphic decompositions of the space W_p of solutions and its adjoint space W_p^* are employed. Any function $u \in W_p$ can be represented as

$$u(x) = u(\alpha) + u'(\alpha)(x - \alpha) + \int_\alpha^x (x - \xi) u''(\xi) d\xi \tag{3.1}$$

where α is a given point in $\overline{G}$ which is the set of closure points for G. Furthermore, the trace or value operators $D_0 u = u(\gamma), D_1 u = u'(\gamma)$ are bounded and surjective from W_p onto $\mathbb{R}$ for a given point γ of $\overline{G}$. In addition, the values $u(\alpha), u'(\alpha)$ and the derivative $u''(x)$ are unrelated elements of the function $u \in W_p$ such that for any real numbers ν_0, ν_1 and any function $\nu_2 \in L_p(G)$, there exists one and only one $u \in W_p$ such that $u(\alpha) = \nu_0, u'(\alpha) = \nu_1$ and $u''(x) = \nu_2(x)$. Therefore, there exists a linear homeomorphism between W_p and E_p. In other words, the space W_p has the isomorphic decomposition $W_p = L_p(G) \times \mathbb{R} \times \mathbb{R}$.

Theorem 3.1 ([4]). *If $1 \le p < \infty$, then any linear bounded functional $F \in W_p^*$ can be represented as*

$$F(u) = \int_{x_0}^{x_1} u''(x) \varphi_2(x) dx + u'(x_0)\varphi_1 + u(x_0)\varphi_0 \tag{3.2}$$

with a unique element $\varphi = (\varphi_2(x), \varphi_1, \varphi_0) \in E_q$ where $p + q = pq$. Any linear bounded functional $F \in W_\infty^*$ can be represented as

$$F(u) = \int_{x_0}^{x_1} u''(x) d\varphi_2 + u'(x_0)\varphi_1 + u(x_0)\varphi_0 \tag{3.3}$$

with a unique element $\varphi = (\varphi_2(e), \varphi_1, \varphi_0) \in \widehat{E_1} = (BA(\sum, \mu)) \times \mathbb{R} \times \mathbb{R}$ where μ is the Lebesgue measure on $\mathbb{R}$, $\sum$ is σ-algebra of the μ-measurable subsets $e \subset G$ and $BA(\sum, \mu)$ is the space of all bounded additive functions $\varphi_2(e)$ defined on $\sum$ with $\varphi_2(e) = 0$ when $\mu(e) = 0$ [8]. The inverse is also valid; that is, if $\varphi \in E_q$, then (3.2) *is bounded on W_p for $1 \le p < \infty$ and $p + q = pq$. If $\varphi \in \widehat{E_1}$, then* (3.3) *is bounded on W_∞.*

Proof. [4] The operator $Nu \equiv (u''(x), u'(x_0), u(x_0)) : W_p \to E_p$ is bounded and has a bounded inverse N^{-1} represented by

$$\begin{gathered} u(x) = (N^{-1}h)(x) \equiv \int_{x_0}^x (x - \xi) h_2(\xi) d\xi + h_1(x - x_0) + h_0, \\ h = (h_2(x), h_1, h_0) \in E_p. \end{gathered} \tag{3.4}$$

The kernel $\ker N$ of N is trivial and the image $\operatorname{Im} N$ of N is equal to E_p. Hence, there exists a bounded adjoint operator $N^* : E_p^* \to W_p^*$ with $\ker N^* = \{0\}$ and $\operatorname{Im} N^* = W_p^*$. In other words, for a given $F \in W_p^*$ there exists a unique $\psi \in E_p^*$ such that

$$F = N^*\psi \quad \text{or} \quad F(u) = \psi(Nu), \quad u \in W_p. \tag{3.5}$$

If $1 \le p < \infty$, then $E_p^* = E_q$ in the sense of an isomorphism [8]. Therefore, the functional ψ can be represented by

$$\psi(h) = \int_{x_0}^{x_1} \varphi_2(x)h_2(x)dx + \varphi_1 h_1 + \varphi_0 h_0, \quad h \in E_p, \tag{3.6}$$

with a unique element $\varphi = (\varphi_2(x), \varphi_1, \varphi_0) \in E_q$. By expressions (3.5) and (3.6), any $F \in W_p^*$ can uniquely be represented by (3.2). For a given $\varphi \in E_q$, the functional F represented by (3.2) is bounded on W_p. Hence, (3.2) is a general form for the functional $F \in W_p^*$.

The proof is complete due to that the case $p = \infty$ can also be shown [4]. $\square$

Theorem 3.1 guarantees that $W_p^* = E_q$ for all $1 \le p < \infty$, and $W_\infty^* = E_\infty^* = \widehat{E_1}$. The space E_1 can also be considered as a subspace of the space $\widehat{E_1}$ (see [3, 4]).

4. Adjoint operator and adjoint system of the integro-algebraic equations

Investigating an explicit form for the adjoint operator V^* of V is taken into consideration in this section. To this end, any $f = (f_2(x), f_1, f_0) \in E_q$ is taken as a linear bounded functional on E_p and also

$$f(Vu) \equiv \int_{x_0}^{x_1} f_2(x)(V_2u)(x)dx + f_1(V_1u) + f_0(V_0u), \quad u \in W_p, \tag{4.1}$$

can be assumed. By substituting expressions (2.1) and (2.2), and expression (3.1) (for $\alpha = x_0$) of $u \in W_p$ into (4.1), we have

$$\begin{aligned} f(Vu) \equiv \int_{x_0}^{x_1} & f_2(x)[u''(x) + A_1(x)\{u'(x_0) + \int_{x_0}^{x} u''(\xi)d\xi\} \\ & + A_0(x)\{u(x_0) + u'(x_0)(x-x_0) + \int_{x_0}^{x}(x-\xi)u''(\xi)d\xi\}]dx \\ & + f_1\{a_1u(x_0) + b_1u'(x_0) + \int_{x_0}^{x_1} g_1(\xi)u''(\xi)d\xi\} \\ & + f_0\{a_0u(x_0) + b_0u'(x_0) + \int_{x_0}^{x_1} g_0(\xi)u''(\xi)d\xi\}. \end{aligned} \tag{4.2}$$

After some calculations, we can obtain

$$\begin{aligned} f(Vu) &\equiv \int_{x_0}^{x_1} f_2(x)(V_2u)(x)dx + \sum_{i=0}^{1} f_i(V_iu) \\ &= \int_{x_0}^{x_1} (w_2f)(\xi)u''(\xi)d\xi + (w_1f)u'(x_0) + (w_0f)u(x_0) \\ &\equiv (wf)(u), \quad \forall f \in E_q, \quad \forall u \in W_p, \quad 1 \le p \le \infty \end{aligned} \tag{4.3}$$

where

$$
\begin{aligned}
(w_2 f)(\xi) &= f_2(\xi) + f_1 g_1(\xi) + f_0 g_0(\xi) + \int_{\xi}^{x_1} f_2(s)\{A_1(s) + A_0(s)(s-\xi)\}ds, \\
w_1 f &= b_1 f_1 + b_0 f_0 + \int_{x_0}^{x_1} f_2(s)\{A_1(s) + A_0(s)(s-x_0)\}ds \\
w_0 f &= a_1 f_1 + a_0 f_0 + \int_{x_0}^{x_1} f_2(s) A_0(s) ds.
\end{aligned}
\tag{4.4}
$$

The operators w_2, w_1, w_0 are linear and bounded from the space E_q of the triples $f = (f_2(x), f_1, f_0)$ into the spaces $L_q(G), \mathbb{R}, \mathbb{R}$ respectively. Therefore, the operator $w = (w_2, w_1, w_0) : E_q \to E_q$ represented by $wf = (w_2 f, w_1 f, w_0 f)$ is linear and bounded. By (4.3) and Theorem 3.1, the operator w is an adjoint operator for the operator V when $1 \le p < \infty$, in other words, $V^* = w$. When $p = \infty$, $w : E_1 \to E_1$ is bounded; in this case, the operator w is the restriction of the adjoint operator $V^* : E_\infty^* \to W_\infty^*$ of V onto $E_1 \subset E_\infty^*$.

Equation (2.3) can be transformed into the following equivalent equation

$$
VSh = z, \tag{4.5}
$$

with an unknown $h = (h_2, h_1, h_0) \in E_p$ by the transformation $u = Sh$ where $S = N^{-1}$. If $u = Sh$, then $u''(x) = h_2(x)$, $u'(x_0) = h_1$, $u(x_0) = h_0$. Hence, equation (4.3) can be rewritten as

$$
\begin{aligned}
f(VSh) &\equiv \int_{x_0}^{x_1} f_2(x)(V_2 Sh)(x)dx + \sum_{i=0}^{1} f_i(V_i Sh) \\
&= \int_{x_0}^{x_1} (w_2 f)(\xi) h_2(\xi) d\xi + (w_1 f)h_1 + (w_0 f)h_0 \\
&\equiv (wf)(h), \quad \forall f \in E_q, \quad \forall h \in E_p, \quad 1 \le p \le \infty.
\end{aligned}
\tag{4.6}
$$

Therefore, one of the operators VS and w becomes an adjoint operator for the other one. Consequently, the equation

$$
wf = \varphi, \tag{4.7}
$$

with an unknown function $f = (f_2(x), f_1, f_0) \in E_q$ and a given function $\varphi = (\varphi_2(x), \varphi_1, \varphi_0) \in E_q$ can be considered as an adjoint equation of (4.5)(or of (2.3)) for all $1 \le p \le \infty$. Equation (4.7) can be written in explicit form as the system of equations

$$
\begin{aligned}
(w_2 f)(\xi) &= \varphi_2(\xi), \quad \xi \in G, \\
w_1 f &= \varphi_1, \\
w_0 f &= \varphi_0.
\end{aligned}
\tag{4.8}
$$

By the expressions (4.4), the first equation in (4.8) is an integral equation for $f_2(\xi)$ and includes f_1 and f_0 as parameters; on the other hand, the other equations in (4.8) constitute a system of two algebraic equations for the unknowns f_1 and f_0 and they include some integral functionals defined on $f_2(\xi)$. In other words, (4.8) is a system of three integro-algebraic equations. This system called the adjoint system for (4.5)(or (2.3)) is constructed by using (4.3) which is actually a formula of integration by parts in a nonclassical form. The traditional type of an adjoint problem is defined by the classical Green's formula of integration by parts [17], therefore, has a sense only for some restricted class of problems.

5. Solvability conditions for completely nonhomogeneous problem

The operator $Q = w - I_q$ is considered where I_q is the identity operator on E_q; i.e., $I_q f = f$ for all $f \in E_q$. This operator can also be defined as $Q = (Q_2, Q_1, Q_0)$ with

$$\begin{gathered}(Q_2 f)(\xi) = (w_2 f)(\xi) - f_2(\xi), \quad \xi \in G, \\ Q_i f = w_i f - f_i, \quad i = 0, 1.\end{gathered} \tag{5.1}$$

By the expressions (4.4) and the conditions imposed on A_i and g_i for $i = 0, 1$, $Q_m : E_q \to L_q(G)$ is a compact operator, and also $Q_i : E_q \to \mathbb{R}$ for $i = 0, 1$ are compact operators where $1 < p < \infty$. That is, $Q : E_q \to E_q$ is a compact operator, and therefore has a compact adjoint operator $Q^* : E_p \to E_p$. Since $w = Q + I_q$ and $VS = Q^* + I_p$, where $I_p = I_q^*$, (4.5) and (4.7) are canonical Fredholm type equations, and S is a right regularizer of (2.3) [10]. Consequently, we have the following theorem.

Theorem 5.1 ([4]). *If $1 < p < \infty$, then $Vu = 0$ has either only the trivial solution or a finite number of linearly independent solutions in W_p:*
***(1)** If $Vu = 0$ has only the trivial solution in W_p, then also $wf = 0$ has only the trivial solution in E_q. Then, the operators $V : W_p \to E_p$ and $w : E_q \to E_q$ become linear homeomorphisms.*
***(2)** If $Vu = 0$ has m linearly independent solutions $u_1, u_2, \dots, u_m$ in W_p, then $wf = 0$ also has m linearly independent solutions*

$$f^{\star 1\star} = (f_2^{\star 1\star}(x), f_1^{\star 1\star}, f_0^{\star 1\star}), \dots, f^{\star m\star} = (f_2^{\star m\star}(x), f_1^{\star m\star}, f_0^{\star m\star})$$

in E_q. In this case, (2.3) and (4.7) have solutions $u \in W_p$ and $f \in E_q$ for given $z \in E_p$ and $\varphi \in E_q$ if and only if the conditions

$$\int_{x_0}^{x_1} f_2^{\star i\star}(\xi) z_2(\xi) d\xi + f_1^{\star i\star} z_1 + f_0^{\star i\star} z_0 = 0, \quad i = 1, 2, \dots, m, \tag{5.2}$$

and

$$\int_{x_0}^{x_1} \varphi_2(\xi) u_i''(\xi) d\xi + \varphi_1 u_i'(x_0) + \varphi_0 u_i(x_0) = 0, \quad i = 1, 2, \dots, m, \tag{5.3}$$

are satisfied, respectively.

6. Green's Functional

Consider the following equation given in the form of a functional identity

$$(wf)(u) = u(x), \quad \forall u \in W_p, \tag{6.1}$$

where $f = (f_2(\xi), f_1, f_0) \in E_q$ is an unknown triple and $x \in \overline{G}$ is a parameter.

Definition. [4] Suppose that $f(x) = (f_2(\xi, x), f_1(x), f_0(x)) \in E_q$ is a triple with parameter $x \in \overline{G}$. If $f = f(x)$ is a solution of (6.1) for a given $x \in \overline{G}$, then $f(x)$ is called a Green's functional of V (or of (2.3)).

Due to that the operator $I_{W_p,C}$ of the imbedding of W_p into the space $C(\overline{G})$ of continuous functions on $\overline{G}$ is bounded, the linear functional $\theta(x)$ defined by $\theta(x)(u) = u(x)$ is bounded on W_p for a given $x \in \overline{G}$. On the other hand, $(wf)(u) = (V^* f)(u)$. Thus, (6.1) can also be written as [2, 3]

$$(V^* f) = \theta(x).$$

In other words, (6.1) can be considered as a special case of the adjoint equation $V^*f = \psi$ for some $\psi = \theta(x)$.

By substituting $\alpha = x_0$ into (3.1) and using (4.3), we can rewrite (6.1) as

$$\begin{aligned} &\int_{x_0}^{x_1} (w_2 f)(\xi) u''(\xi) d\xi + (w_1 f) u'(x_0) + (w_0 f) u(x_0) \\ &= \int_{x_0}^{x} (x-\xi) u''(\xi) d\xi + u'(x_0)(x - x_0) + u(x_0), \quad \forall f \in E_q, \quad \forall u \in W_p. \end{aligned} \tag{6.2}$$

The elements $u''(\xi) \in L_p(G), u'(x_0) \in \mathbb{R}$ and $u(x_0) \in \mathbb{R}$ of the function $u \in W_p$ are unrelated. Then, we can construct the system

$$\begin{gathered} (w_2 f)(\xi) = (x-\xi) H(x-\xi), \quad \xi \in G, \\ (w_1 f) = (x - x_0), \\ (w_0 f) = 1, \end{gathered} \tag{6.3}$$

where $H(x-\xi)$ is a Heaviside function on $\mathbb{R}$.

Equation (6.1) is equivalent to the system (6.3) which is a special case for the adjoint system (4.8) when $\varphi_2(\xi) = (x-\xi)H(x-\xi)$, $\varphi_1 = x - x_0$ and $\varphi_0 = 1$. Therefore, $f(x)$ is a Green's functional if and only if $f(x)$ is a solution of system (6.3) for an arbitrary $x \in \overline{G}$. For a solution $u \in W_p$ of (2.3) and a Green's functional $f(x)$, we can rewrite (4.3) as

$$\begin{aligned} &\int_{x_0}^{x_1} f_2(\xi, x) z_2(\xi) d\xi + f_1(x) z_1 + f_0(x) z_0 \\ &= \int_{x_0}^{x_1} (x-\xi) H(x-\xi) u''(\xi) d\xi + u'(x_0)(x - x_0) + u(x_0). \end{aligned} \tag{6.4}$$

The right hand side of (6.4) is equal to $u(x)$. Therefore, we can state the following theorem.

Theorem 6.1 ([4]). *If (2.3) has at least one Green's functional $f(x)$, then any solution $u \in W_p$ of (2.3) can be represented by*

$$u(x) = \int_{x_0}^{x_1} f_2(\xi, x) z_2(\xi) d\xi + f_1(x) z_1 + f_0(x) z_0. \tag{6.5}$$

Additionally, $Vu = 0$ has only the trivial solution.

Since one of the operators $V : W_p \to E_p$ and $w : E_q \to E_q$ is a homeomorphism, so is the other, and, there exists a unique Green's functional, where $1 \le p \le \infty$. Necessary and sufficient conditions for the existence of a Green's functional can be given in the following theorem for $1 < p < \infty$.

Theorem 6.2 ([4]). *If there exists a Green's functional, then it is unique. Additionally, a Green's functional exists if and only if $Vu = 0$ has only the trivial solution.*

From Theorems 5.1, 6.1, 6.2 can be easily shown.

Remark. If $Vu = 0$ has a nontrivial solution, then a Green's functional corresponding to $Vu = z$ does not exist due to Theorem 6.1. In this case, $Vu = z$ usually has no solution unless z is of a specific type. Therefore, a representation of the existing solution of $Vu = z$ are constructed by a concept of the generalized Green's functional [3, 4].

It must be noted that the proposed Green's functional approach can also be employed some classes of nonlinear equations involving linear nonlocal conditions to transform into the corresponding integral equations and then solve them. The corresponding integral equations will naturally become of nonlinear type. These nonlinear integral equations can be solved approximately even if they can not be solved exactly.

7. Some applications

In this section, some applications to such problems involving nonlocal boundary conditions are implemented in order to emphasize the preferability of the presented approach.

Example 7.1. First, we seek for the Green's solution to the following problem, which has been considered in [15]:

$$u''(x) = -f(x), \quad x \in G = (0, 1), \tag{7.1}$$

$$u(0) = \gamma_0 u'(\xi_0), \quad u(1) = \gamma_1 u'(\xi_1), \tag{7.2}$$

where $f(x) \in L_p(G)$, $\xi_0, \xi_1 \in \overline{G}$ and $\gamma_0, \gamma_1 \in \mathbb{R}$. We can rewrite this problem as

$$\begin{gathered}
(V_2 u)(x) \equiv u''(x) = -f(x) = z_2(x), \quad x \in G = (0, 1),\\
V_1 u \equiv u(1) - \gamma_1 u'(\xi_1) = 0 = z_1,\\
V_0 u \equiv u(0) - \gamma_0 u'(\xi_0) = 0 = z_0.
\end{gathered}$$

Thus, we have

$$\begin{gathered}
a_1 = 1, \quad b_1 = 1 - \gamma_1, \quad g_1(\xi) = 1 - \xi - \gamma_1 H(\xi_1 - \xi),\\
a_0 = 1, \quad b_0 = -\gamma_0, \quad g_0(\xi) = -\gamma_0 H(\xi_0 - \xi),
\end{gathered}$$

and $A_i(x) = z_i = 0$ for $i = 0, 1$, where $H(\xi_1 - \xi)$ and $H(\xi_0 - \xi)$ are Heaviside functions on $\mathbb{R}$.

Consequently, the special adjoint system (6.3) corresponding to this problem can be constructed in the form

$$f_2(\xi) + f_1\{1 - \xi - \gamma_1 H(\xi_1 - \xi)\} - f_0\gamma_0 H(\xi_0 - \xi) = (x - \xi)H(x - \xi), \tag{7.3}$$

$$f_1(1 - \gamma_1) - f_0\gamma_0 = x, \tag{7.4}$$

$$f_1 + f_0 = 1, \tag{7.5}$$

where $\xi \in (0, 1)$. We firstly determine f_1 and f_0 with using only (7.4) and (7.5) under the condition $\Delta_1 = 1 - \gamma_1 + \gamma_0 \neq 0$ in order to solve (7.3)-(7.5). Thus, we have

$$f_1 = \frac{x + \gamma_0}{\Delta_1}, \quad f_0 = \frac{1 - \gamma_1 - x}{\Delta_1}.$$

After substituting f_1 and f_0 into equation (7.3), $f_2(\xi)$ becomes

$$\begin{aligned}
f_2(\xi) = (x - \xi)H(x - \xi) + \frac{(1 - \gamma_1 - x)}{\Delta_1}\gamma_0 H(\xi_0 - \xi)\\
- \frac{(x + \gamma_0)}{\Delta_1}\{1 - \xi - \gamma_1 H(\xi_1 - \xi)\}.
\end{aligned}$$

Thus, the Green's functional $f(x) = (f_2(\xi, x), f_1(x), f_0(x))$ for the problem has been determined. The first component $f_2(\xi, x) = f_2(\xi)$ is the Green's function for the problem. After substituting $\xi = s$ for notational compatibility, $f_2(\xi, x)$ is equal to the Green's function constructed in [15] for the problem. By (6.5) in Theorem 6.1, the representation of the existing solution for the problem can be given as

$$\begin{aligned} u(x) = \int_0^1 & -[(x-\xi)H(x-\xi) + \frac{(1-\gamma_1-x)}{\Delta_1}\gamma_0 H(\xi_0-\xi) \\ & - \frac{(x+\gamma_0)}{\Delta_1}\{1-\xi-\gamma_1 H(\xi_1-\xi)\}]f(\xi)d\xi. \end{aligned}$$

Example 7.2. Next, we construct the Green's solution to the following problem, which has been considered in [15]:

$$u''(x) = -f(x), \quad x \in G = (0,1), \tag{7.6}$$

$$u(0) = \gamma_0 \int_0^1 (1+t)u(t)dt, \quad u(1) = \gamma_1 \int_0^1 u(t)dt, \tag{7.7}$$

where $f(x) \in L_p(G)$ and $\gamma_0, \gamma_1 \in \mathbb{R}$. We can rewrite this problem as

$$\begin{gathered} (V_2 u)(x) \equiv u''(x) = -f(x) = z_2(x), \quad x \in G = (0,1), \\ V_1 u \equiv u(1) - \gamma_1 \int_0^1 u(t)dt = 0 = z_1, \\ V_0 u \equiv u(0) - \gamma_0 \int_0^1 (1+t)u(t)dt = 0 = z_0. \end{gathered}$$

Then, we have

$$\begin{gathered} a_1 = 1-\gamma_1, \quad b_1 = 1 - \frac{\gamma_1}{2}, \quad g_1(\xi) = 1-\xi-\gamma_1(\frac{1}{2}-\xi+\frac{\xi^2}{2}), \\ a_0 = 1-\frac{3}{2}\gamma_0, \quad b_0 = -\frac{5}{6}\gamma_0, \quad g_0(\xi) = -\gamma_0(\frac{\xi^3}{6}+\frac{\xi^2}{2}-\frac{3}{2}\xi+\frac{5}{6}), \end{gathered}$$

and $A_i(x) = z_i = 0$ for $i = 0,1$.

Consequently, the special adjoint system (6.3) corresponding to this problem is of the form

$$\begin{aligned} & f_2(\xi) + f_1\{1-\xi-\gamma_1(\frac{1}{2}-\xi+\frac{\xi^2}{2})\} \\ & - f_0\gamma_0(\frac{\xi^3}{6}+\frac{\xi^2}{2}-\frac{3}{2}\xi+\frac{5}{6}) = (x-\xi)H(x-\xi), \end{aligned} \tag{7.8}$$

$$(1-\frac{\gamma_1}{2})f_1 - \frac{5}{6}\gamma_0 f_0 = x, \tag{7.9}$$

$$(1-\gamma_1)f_1 + (1-\frac{3}{2}\gamma_0)f_0 = 1, \tag{7.10}$$

where $\xi \in (0,1)$ and, $H(x-\xi)$ is Heaviside function on $\mathbb{R}$. We firstly determine f_1 and f_0 with using only (7.9) and (7.10) under the condition $\Delta_2 = (1-\frac{\gamma_1}{2})(1-\frac{3}{2}\gamma_0) + \frac{5}{6}\gamma_0(1-\gamma_1) \neq 0$ in order to solve (7.8)-(7.10). Hence, we have

$$f_1 = \frac{(1-\frac{3}{2}\gamma_0)x + \frac{5}{6}\gamma_0}{\Delta_2}, \quad f_0 = \frac{1-\frac{\gamma_1}{2} - x(1-\gamma_1)}{\Delta_2}.$$

After substituting f_1 and f_0 into equation (7.8), $f_2(\xi)$ becomes

$$f_2(\xi) = (x-\xi)H(x-\xi) + \frac{[1-\frac{\gamma_1}{2} - x(1-\gamma_1)]}{\Delta_2}\gamma_0(\frac{\xi^3}{6}+\frac{\xi^2}{2}-\frac{3}{2}\xi+\frac{5}{6})$$

$$-\frac{[(1-\frac{3}{2}\gamma_0)x+\frac{5}{6}\gamma_0]}{\Delta_2}\{1-\xi-\gamma_1(\frac{1}{2}-\xi+\frac{\xi^2}{2})\}.$$

Thus, the Green's functional $f(x) = (f_2(\xi, x), f_1(x), f_0(x))$ for the problem has been determined. The first component $f_2(\xi, x) = f_2(\xi)$ is the Green's function for the problem. After substituting $\xi = s$ for notational compatibility, $f_2(\xi, x)$ is equal to the Green's function constructed in [15] for the problem. By (6.5) in Theorem 6.1, the representation of the existing solution for the problem can be given as

$$\begin{aligned} u(x) = \int_0^1 &-[(x-\xi)H(x-\xi)+\frac{[1-\frac{\gamma_1}{2}-x(1-\gamma_1)]}{\Delta_2}\gamma_0(\frac{\xi^3}{6}+\frac{\xi^2}{2}-\frac{3}{2}\xi+\frac{5}{6}) \\ &-\frac{[(1-\frac{3}{2}\gamma_0)x+\frac{5}{6}\gamma_0]}{\Delta_2}\{1-\xi-\gamma_1(\frac{1}{2}-\xi+\frac{\xi^2}{2})\}]f(\xi)d\xi. \end{aligned}$$

Example 7.3. Finally, we consider the following problem to reduce an integral equation by using the Green's functional concept:

$$u''(x) + \lambda u(x) = f(x), \quad x \in G = (0, 1), \tag{7.11}$$

$$u(0) = \alpha_{00}u(\beta_{00}) + \alpha_{01}u'(\beta_{01}) + \gamma_0 \int_0^1 u(t)dt, \tag{7.12}$$

$$u(1) = \alpha_{10}u(\beta_{10}) + \alpha_{11}u'(\beta_{11}) + \gamma_1 \int_0^1 u(t)dt, \tag{7.13}$$

where $f(x) \in L_p(G)$, $\alpha_{00}, \alpha_{01}, \alpha_{10}, \alpha_{11}, \gamma_0, \gamma_1 \in \mathbb{R}$, $\beta_{00}, \beta_{01}, \beta_{10}, \beta_{11} \in \overline{G}$ and, λ is a constant. We can rewrite this problem as

$$\begin{gathered} (V_2u)(x) \equiv u''(x) + \lambda u(x) = f(x) = z_2(x), \quad x \in G = (0, 1), \\ V_1u \equiv u(1) - \alpha_{10}u(\beta_{10}) - \alpha_{11}u'(\beta_{11}) - \gamma_1 \int_0^1 u(t)dt = 0 = z_1, \\ V_0u \equiv u(0) - \alpha_{00}u(\beta_{00}) - \alpha_{01}u'(\beta_{01}) - \gamma_0 \int_0^1 u(t)dt = 0 = z_0. \end{gathered}$$

Then, we have

$$\begin{gathered} a_1 = 1-\alpha_{10}-\gamma_1, \quad b_1 = 1-\alpha_{10}\beta_{10}-\alpha_{11}-\frac{\gamma_1}{2}, \\ a_0 = 1-\alpha_{00}-\gamma_0, \quad b_0 = -\beta_{00}-\alpha_{01}-\frac{\gamma_0}{2}, \\ g_1(\xi) = 1-\xi-\alpha_{10}(\beta_{10}-\xi)H(\beta_{10}-\xi)-\alpha_{11}H(\beta_{11}-\xi)-\gamma_1\frac{(1-\xi)^2}{2}, \\ g_0(\xi) = -\alpha_{00}(\beta_{00}-\xi)H(\beta_{00}-\xi)-\alpha_{01}H(\beta_{01}-\xi)-\gamma_0\frac{(1-\xi)^2}{2}, \end{gathered}$$

and $A_0(x) = \lambda, A_1(x) = z_1 = z_0 = 0$ and, $H(\cdot)$ is Heaviside function on $\mathbb{R}$.

Consequently, the special adjoint system (6.3) corresponding to this problem is of the form

$$
\begin{aligned}
&f_2(\xi) + \int_\xi^1 f_2(s)\lambda(s-\xi)ds \\
&+ f_1\{1-\xi-\alpha_{10}(\beta_{10}-\xi)H(\beta_{10}-\xi)-\alpha_{11}H(\beta_{11}-\xi)-\gamma_1\frac{(1-\xi)^2}{2}\} \\
&+ f_0\{-\alpha_{00}(\beta_{00}-\xi)H(\beta_{00}-\xi)-\alpha_{01}H(\beta_{01}-\xi)-\gamma_0\frac{(1-\xi)^2}{2}\} \\
&= (x-\xi)H(x-\xi),
\end{aligned}
\tag{7.14}
$$

$$
(1-\alpha_{10}\beta_{10}-\alpha_{11}-\frac{\gamma_1}{2})f_1 + (-\beta_{00}-\alpha_{01}-\frac{\gamma_0}{2})f_0 + \int_0^1 f_2(s)\lambda s ds = x, \tag{7.15}
$$

$$
(1-\alpha_{10}-\gamma_1)f_1 + (1-\alpha_{00}-\gamma_0)f_0 + \int_0^1 f_2(s)\lambda ds = 1, \tag{7.16}
$$

where $\xi \in (0,1)$. We denote $\int_0^1 f_2(s)\lambda s ds$ and $\int_0^1 f_2(s)\lambda ds$ by E and F respectively, and then determine f_1 and f_0 with using only (7.15) and (7.16) under the condition $\Delta_3 = b_1a_0 - b_0a_1 = (1-\alpha_{10}\beta_{10}-\alpha_{11}-\frac{\gamma_1}{2})(1-\alpha_{00}-\gamma_0) - (-\beta_{00}-\alpha_{01}-\frac{\gamma_0}{2})(1-\alpha_{10}-\gamma_1) \neq 0$ in order to solve (7.14)-(7.16). As a result, we have

$$
f_1 = \frac{a_0(x-E)-b_0(1-F)}{\Delta_3}, \quad f_0 = \frac{b_1(1-F)-a_1(x-E)}{\Delta_3}. \tag{7.17}
$$

After substituting f_1 and f_0 into (7.14), we have

$$
\begin{aligned}
&f_2(\xi) + \int_\xi^1 f_2(s)\lambda(s-\xi)ds + \{\frac{a_0(x-E)-b_0(1-F)}{\Delta_3}\} \\
&\times \{1-\xi-\alpha_{10}(\beta_{10}-\xi)H(\beta_{10}-\xi)-\alpha_{11}H(\beta_{11}-\xi)-\gamma_1\frac{(1-\xi)^2}{2}\} \\
&+ \{\frac{b_1(1-F)-a_1(x-E)}{\Delta_3}\} \\
&\times \{-\alpha_{00}(\beta_{00}-\xi)H(\beta_{00}-\xi)-\alpha_{01}H(\beta_{01}-\xi)-\gamma_0\frac{(1-\xi)^2}{2}\} \\
&= (x-\xi)H(x-\xi).
\end{aligned}
\tag{7.18}
$$

As can be seen from the denotations E and F, (7.18) is an integral equation for $f_2(\xi)$. After $f_2(\xi)$ is determined by solving this integral equation, and then $f_1(x)$ and $f_0(x)$ by (7.17); the Green's functional $f(x) = (f_2(\xi,x), f_1(x), f_0(x))$ for the problem will have been constructed. The first component $f_2(\xi,x) = f_2(\xi)$ will become the Green's function for the problem. Consequently, it must be noticed that the Green's function is constructed under the condition $\Delta_3 \neq 0$ in addition to the solvability conditions for the problem.

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