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STURM-LIOUVILLE EIGENVALUE CHARACTERIZATIONS

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ABSTRACT. We study the relationship between the eigenvalues of separated self-adjoint boundary conditions and coupled self-adjoint conditions. Given an arbitrary real coupled boundary condition determined by a coupling matrix K we construct a one parameter family of separated conditions and show that all the eigenvalues for K and $-K$ are extrema of the eigencurves of this family. This characterization makes it possible to use the well known Prüfer transformation which has been used very successfully, both theoretically and numerically, for separated conditions, also in the coupled case. In particular, this characterization makes it possible to compute the eigenvalues for any real coupled self-adjoint boundary condition using any code which works for separated conditions.

1. INTRODUCTION

We study regular self-adjoint Sturm-Liouville problems:

$$-(py')' + qy = \lambda wy \quad \text{on } J = (a, b),\ -\infty < a < b < \infty, \tag{1.1}$$

$$AY(a) + BY(b) = 0, \quad A, B \in M_2(\mathbb{C}), \quad Y = \begin{bmatrix} y \\ py' \end{bmatrix}. \tag{1.2}$$

Here the coefficients satisfy the regularity conditions

$$\frac{1}{p}, q, w \in L(J, \mathbb{R}), \quad p > 0, w > 0 \text{ a.e. on } J, \quad \lambda \in \mathbb{C}; \tag{1.3}$$

and the boundary conditions satisfy the well known self-adjointness condition

$$\det(A : B) = 2 \text{ and } AEA^* = BEB^*, \quad E = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad A, B \in M_2(\mathbb{C}), \tag{1.4}$$

where $M_2(\mathbb{C})$ denotes the 2×2 matrices over the complex numbers $\mathbb{C}$.

It is well known that, given (1.1), the conditions (1.2) are well defined and they are categorized into two disjoint mutually exclusive classes: separated and coupled. The separated conditions have the well known canonical representation:

$$\begin{aligned} \cos(\alpha)y(a) - \sin(\alpha)(py')(a) = 0, \quad \alpha \in [0, \pi), \\ \cos(\beta)y(b) - \sin(\beta)(py')(b) = 0, \quad \beta \in (0, \pi]. \end{aligned} \tag{1.5}$$

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Also known, but not as well known [5], is the canonical form of the coupled self-adjoint conditions

$$Y(b) = e^{i\gamma} K Y(a), \quad -\pi < \gamma \le \pi,\ K \in M_2(\mathbb{R}),\ \det(K) = 1, \tag{1.6}$$

where $M_2(\mathbb{R})$ denotes the 2×2 matrices over the reals $\mathbb{R}$.

The matrices $K \in M_2(\mathbb{R})$ satisfying $\det(K) = 1$ are known as the special linear group of order two over the reals and are denoted by $SL_2(\mathbb{R})$. Note that $K \in SL_2(\mathbb{R})$ if and only if $-K \in SL_2(\mathbb{R})$. When $\gamma = 0$ the self-adjoint condition (1.6) is real; its spectrum is denoted by $\sigma(K) = \{\lambda_n(K) : n \in \mathbb{N}_0 = \{0, 1, 2, 3, \dots\}\}$ and we refer to the $\lambda_n(K)$ as the eigenvalues for K. When $\gamma \in (0, \pi) \cup (-\pi, 0)$ then (1.6) is a complex self-adjoint coupled boundary condition; its spectrum is denoted by $\sigma(K, \gamma) = \{\lambda_n(K, \gamma) : n \in \mathbb{N}_0 = \{0, 1, 2, 3, \dots\}\}$ and the eigenvalues $\lambda_n(K, \gamma)$ are referred to as the eigenvalues for (K, γ). The spectrum of the separated conditions (1.5) is denoted by $\sigma(\alpha, \beta) = \{\lambda_n(\alpha, \beta) : n \in \mathbb{N}_0\}$. Let $\mathbb{N} = \{1, 2, 3, \dots\}$.

For separated conditions there is a well known and powerful tool based on the Prüfer transformation for studying properties of eigenvalues and eigenfunctions; for example to prove that the n-th eigenfunction has exactly n zeros in the open domain interval (a, b). In 1966 Bailey [1] showed that this transformation can be used to compute the eigenvalues of separated conditions very effectively and efficiently. The *nth* eigenvalue can be computed without any prior knowledge of the previous or subsequent eigenvalues.

To the best of our knowledge, the only general purpose code available for the computation of the eigenvalues for coupled conditions (1.2) is the Bailey, Everitt, Zettl code SLEIGN2 [2]. The algorithm used in [2] is based on inequalities found by Eastham, Kong, Wu and Zettl [5]. These inequalities locate the coupled eigenvalues uniquely between two separated ones. The Prüfer transformation is then used to compute the separated eigenvalues followed by a search mechanism to compute the coupled eigenvalue within the bounds given by the separated ones.

In this paper, given any $K \in M_2(\mathbb{R})$, we construct a one parameter family of separated conditions and prove that the extrema of this family are eigenvalues for K or $-K$ and all eigenvalues for K and $-K$ can be obtained in this way. Given the index, n, for any eigenvalue of K we determine the appropriate separated boundary condition and determine which eigenvalue of this separated condition is equal to the coupled one with this index n. Thus we show that the information obtained for separated conditions - which have been studied much more extensively than coupled ones (in the direct theory and even more so in the inverse theory where little is known for coupled conditions) and have a history and voluminous literature dating back more than 170 years - can be applied to any real coupled condition. If λ_n is a simple eigenvalue for K, the number of its zeros in the open domain interval is determined exactly by this characterization. Furthermore, our characterization can be used to compute the eigenvalues for any $K \in SL_2(\mathbb{R})$ using any code which works for separated conditions.

In stark contrast to the above mentioned results about the close relationships between the eigenvalues $\lambda_n(K)$ and $\lambda_n(\alpha, \beta)$ we also show that no eigenvalue of the constructed family of separated conditions determined by K or $-K$ is an eigenvalue $\lambda_n(K, \gamma)$ when $\gamma \neq 0$.

There is a voluminous literature on applications of Sturm-Liouville problems in Applied Mathematics, Science, Engineering and other fields. A description of this literature is well outside the scope of this paper. See the book [9] for an extensive list of references and as a general reference for known results, also see [8], [6] and [3]. An example of a more recent reference is the paper of Ren and Chang [7] where knowledge of the dependence of eigenvalues on the boundary conditions is applied to the study of crystals.

The organization of this paper is as follows. For the convenience of the reader we review some basic results used below in Section 2. The Algorithm is presented in Section 3. Section 4 contains theorems used in the proof of the Algorithm given in Section 5. We believe these theorems are of independent interest.

2. Basic Review

In (1.2) if $\det(A) = 0$ then $\det(B) = 0$ by (1.4), the boundary conditions are separated and can be taken to have the form

$$A_1 y(a) + A_2 (py')(a) = 0, \quad A_1, A_2 \in \mathbb{R},\ (A_1, A_2) \neq (0,0), \tag{2.1}$$

$$B_1 y(b) + B_2 (py')(b) = 0, \quad B_1, B_2 \in \mathbb{R},\ (B_1, B_2) \neq (0,0), \tag{2.2}$$

where $\mathbb{R}$ denotes the real numbers.

Remark 2.1. The different parameterizations for α and β in (1.5) are for convenience in stating results and in using the Prüfer transformation. These play an important role below. As mentioned above, this transformation is a well known and powerful tool, in Pure and Applied Mathematics, for proving the existence of eigenvalues, studying their properties, and for their numerical computation, see [8, 6, 3, 2]. There is no comparable tool for coupled conditions. Until the publication of [4] the standard method of proving the existence of eigenvalues for coupled problems was to construct the Green's function, use it as a kernel to define an integral operator, then show that this operator is self-adjoint in an appropriate Hilbert space and use operator theory [3, 6]. In contrast, the proof using the Prüfer transformation is elementary [3, 9]. An elementary proof for the existence of the eigenvalues for coupled conditions was first established in [5, 4]; this forms the basis for one of the algorithms used in the SLEIGN2 code [2] to compute the eigenvalues for all coupled (real and complex) conditions (as well as for singular conditions, separated or coupled, real or complex). Another consequence of the study of the relationships between the eigenvalues for different boundary conditions is that the continuous and discontinuous dependence of the n-th eigenvalue on the boundary conditions is now completely understood. All discontinuities are finite jumps when $n \in \mathbb{N}$ and are infinite jumps when $n = 0$. The 'jump sets' - the sets of boundary conditions where the eigenvalues have jump discontinuities - are now completely known; see [5, 9].

The next theorem summarizes known results, including some relatively recent ones, on the dependence of the eigenvalues on the boundary conditions for a fixed equation.

Theorem 2.2 (Eigenvalue Properties)**.** *Let* (1.1) *to* (1.6) *hold. Let* $K = (k_{ij}) \in SL_2(\mathbb{R})$, $\alpha \in [0, \pi)$, $\beta \in (0, \pi]$. *Then*

(a) *(1) The spectrum* $\sigma(\alpha, \beta)$ *of the boundary value problem determined by the separated conditions* (1.5) *is real, simple, countably infinite with no finite accumulation point, bounded below and not bounded above. Let* $\sigma(\alpha, \beta) = \{\lambda_n(\alpha, \beta) : n \in \mathbb{N}_0 = \{0, 1, 2, 3, \dots\}\}$. *Then the eigenvalues* $\lambda_n(\alpha, \beta)$ *can be ordered to satisfy*

$$-\infty < \lambda_0(\alpha, \beta) < \lambda_1(\alpha, \beta) < \lambda_2(\alpha, \beta) < \lambda_3(\alpha, \beta) < \dots \tag{2.3}$$

For each $n \in \mathbb{N}_0$ the eigenvalue function $\lambda_n(\alpha, \beta)$ is jointly continuous in α, β for $\alpha \in [0, \pi)$, $\beta \in (0, \pi]$. For fixed $\beta \in (0, \pi]$, $\lambda_n(\alpha, \beta)$ is a differentiable function of $\alpha \in [0, \pi)$, and for fixed $\alpha \in [0, \pi)$, $\lambda_n(\alpha, \beta)$ is a differentiable function of $\beta \in (0, \pi]$. The eigenfunctions are unique up to constant multiples and $\lambda_n(\alpha, \beta)$ has exactly n zeros in the open domain interval.

(2) Assume that $\gamma = 0$. The spectrum $\sigma(K)$ of the boundary value problem determined by the coupled conditions (1.6) is real, countably infinite with no finite accumulation point, bounded below and not bounded above. Let $\sigma(K) = \{\lambda_n(K) : n \in \mathbb{N}_0\}$. Each eigenvalue $\lambda_n(K)$ may be simple or double and the eigenvalues can be ordered to satisfy

$$-\infty < \lambda_0(K) \le \lambda_1(K) \le \lambda_2(K) \le \lambda_3(K) \le \dots \tag{2.4}$$

where equality cannot occur in two consecutive terms.

(3) If $\gamma \neq 0$, then the spectrum $\sigma(K, \gamma)$ of (1.6), denoted by $\sigma(K, \gamma) = \{\lambda_n(K, \gamma) : n \in \mathbb{N}_0\}$, is real, simple, countably infinite with no finite accumulation point, bounded below and not bounded above and can be ordered to satisfy

$$-\infty < \lambda_0(K, \gamma) < \lambda_1(K, \gamma) < \lambda_2(K, \gamma) < \lambda_3(K, \gamma) < \dots \tag{2.5}$$

Furthermore, for each $n \in N_0$, $\lambda_n(K, \gamma) = \lambda_n(K, -\gamma)$, and if y_n is an eigenfunction of $\lambda_n(K, \gamma)$ then its conjugate $\overline{y_n}$ is an eigenfunction of $\lambda_n(K, -\gamma)$. (Note that every eigenvalue $\lambda_n(K, \gamma)$ is simple, $n = 0, 1, 2, 3, \dots$).

***(b)** Let υ_n, ν_n, $n \in \mathbb{N}_0$ denote the eigenvalues of the special separated boundary conditions*

$$y(a) = 0, \quad k_{22} y(b) - k_{12}(py')(b) = 0; \tag{2.6}$$

$$(py')(a) = 0, \quad k_{21} y(b) - k_{11}(py')(b) = 0; \tag{2.7}$$

respectively. Note that $(k_{22}, k_{12}) \neq (0, 0) \neq (k_{21}, k_{11})$ since $\det(K) = 1$ so each of these is a self-adjoint boundary condition.

- *Suppose $k_{12} < 0$ and $k_{11} \le 0$. Then $\lambda_0(K)$ is simple, $\lambda_0(K) < \lambda_0(-K)$ and the following inequalities hold for $-\pi < \gamma < 0$ and $0 < \gamma < \pi$*

$$\begin{aligned} \lambda_0(K) &< \lambda_0(K, \gamma) < \lambda_0(-K) < \{\upsilon_0, \nu_0\} \\ &\le \lambda_1(-K) < \lambda_1(K, \gamma) < \lambda_1(K) < \{\upsilon_1, \nu_1\} \\ &\le \lambda_2(K) < \lambda_2(K, \gamma) < \lambda_2(-K) < \{\upsilon_2, \nu_2\} \\ &\le \lambda_3(-K) < \lambda_3(K, \gamma) < \lambda_3(K) < \{\upsilon_3, \nu_3\} \le \dots \end{aligned} \tag{2.8}$$

- *Suppose $k_{12} \le 0$ and $k_{11} > 0$. Then $\lambda_0(K)$ is simple, $\lambda_0(K) < \lambda_0(-K)$ and the following inequalities hold for $-\pi < \gamma < 0$ and $0 < \gamma < \pi$*

$$\begin{aligned} \nu_0 \le \lambda_0(K) &< \lambda_0(K, \gamma) < \lambda_0(-K) < \{\upsilon_0, \nu_1\} \\ &\le \lambda_1(-K) < \lambda_1(K, \gamma) < \lambda_1(K) < \{\upsilon_1, \nu_2\} \\ &\le \lambda_2(K) < \lambda_2(K, \gamma) < \lambda_2(-K) < \{\upsilon_2, \nu_3\} \\ &\le \lambda_3(-K) < \lambda_3(K, \gamma) < \lambda_3(K) < \{\upsilon_3, \nu_4\} \le \dots \end{aligned} \tag{2.9}$$

If neither of the above two cases holds for K then one of them must hold for $-K$. Here the notation $\{\upsilon_n, \nu_m\}$ is used to indicate either υ_n or ν_m but no comparison is made between υ_n and ν_m.

- *Furthermore, for* $0 < \gamma < \delta < \pi$ *we have*

$$\begin{aligned} \lambda_0(K,\delta) &< \lambda_0(K,\gamma) < \lambda_1(K,\gamma) < \lambda_1(K,\delta) < \lambda_2(K,\delta) \\ &< \lambda_2(K,\gamma) < \lambda_3(K,\gamma) < \lambda_3(K,\delta) < \dots \end{aligned} \tag{2.10}$$

For a proof, or references to proofs, of the above theorem, see the book [9].

Remark 2.3. We comment on Theorem 2.2 (a)(1) and Remark 2.1. There is no contradiction between Theorem 2.2 (a)(1) and Remark 2.1 because the jump discontinuities occur when α or β are outside the normalization intervals $[0,\pi)$ and $(0,\pi]$, respectively.

Remark 2.4. We comment on the inequalities (2.8) and (2.9). By (2.9),

$$\nu_n \le \lambda_n(K) < \nu_{n+1}, n \in \mathbb{N}_0,$$

when $k_{12} \le 0$ and $k_{11} > 0$ or $k_{12} \ge 0$ and $k_{11} < 0$. By (2.8)

$$\nu_n \le \lambda_{n+1}(K) < \nu_{n+1}, n \in \mathbb{N}_0,$$

when $k_{12} < 0$ and $k_{11} \le 0$ or $k_{12} > 0$ and $k_{11} \ge 0$. Thus for each $K \in SL_2(\mathbb{R})$, $\lambda_n(K)$ is uniquely located between two consecutive separated eigenvalues ν_j for any $n \in \mathbb{N}$. In addition $\nu_0 \le \lambda_0(K) < \nu_1$ when $k_{12} \le 0$ and $k_{11} > 0$ or $k_{12} \ge 0$ and $k_{11} < 0$. Also $\lambda_0(K) < \nu_0$ when $k_{12} < 0$ and $k_{11} \le 0$ or $k_{12} > 0$ and $k_{11} \ge 0$ but, in this case, $\lambda_0(K)$ has no lower bound in terms of the ν_j. In Section 4 below we will establish a lower bound for this case in terms of a family of separated boundary conditions. In all cases where the eigenvalue $\lambda_n(K)$ has been uniquely located in an interval $[\nu_n, \nu_{n+1}]$, the Prüfer transformation can be used to compute the lower and upper bounds ν_n and ν_{n+1}. Then a zero finder algorithm can be used to find $\lambda_n(K)$ as the unique zero of the characteristic function $D(\lambda)$ in the interval $[\nu_n, \nu_{n+1}]$. Such a method is used by the code SLEIGN2 [2]. Our algorithm, given in the next section, uses the constructed separated family rather than the characteristic function $D(\lambda)$.

Remark 2.5. In this remark we comment on what happens when the normalization conditions $\alpha \in [0,\pi)$ and $\beta \in (0,\pi]$ are violated. For fixed $\beta \in (0,\pi]$ as $\alpha \to 0^-$, $\lambda_n(\alpha,\beta)$ has an infinite jump discontinuity when $n = 0$ and a finite jump discontinuity when $n \in \mathbb{N}$. Similarly, for fixed $\alpha \in [0,\pi)$, as $\beta \to \pi^+$, $\lambda_n(\alpha,\beta)$ has an infinite jump discontinuity when $n = 0$ and a finite jump discontinuity when $n \in \mathbb{N}$. In each case where $\lambda_n(\alpha,\beta)$ has a jump discontinuity the eigenvalue can be embedded in a 'continuous eigenvalue branch' which is defined by two indices n and $n+1$; in other words the eigencurves from the left and right of the point where the jump occurs 'match up' continuously when one of the indices n is changed to $n+1$. Furthermore, the resulting matched eigencurve determined by two consecutive indices is not only continuous but also differentiable everywhere including at the matched point.

Remark 2.6. If $k_{12} \ne 0$, then each eigenvalue $\lambda_n(K)$ is a continuous function of $K = (k_{ij}) \in SL_2(\mathbb{R})$. If $k_{12} = 0$, then $\lambda_n(K)$ may have a jump discontinuity; this jump is infinite when $n = 0$ and finite when $n \in \mathbb{N}$. Thus the 'jump sets' of boundary conditions consist of the separated conditions when either $y(a) = 0$ or $y(b) = 0$ and the (real or complex) coupled conditions when $k_{12} = 0$.

3. The New Algorithm

As mentioned above, for each $K \in SL_2(\mathbb{R})$ all eigenvalues for K and $-K$ can be found from the eigenvalues of a related family of separated conditions constructed from K. In this section we define this separated family and present the Algorithm.

Definition 3.1 (α-family of K)**.** Let $K = (k_{ij}) \in SL_2(\mathbb{R})$. For each $\alpha \in [0, \pi)$ consider the separated boundary condition

$$\begin{gathered} y(a)\cos\alpha - (py')(a)\sin\alpha = 0, \\ y(b)(k_{21}\sin\alpha + k_{22}\cos\alpha) - (py')(b)(k_{11}\sin\alpha + k_{12}\cos\alpha) = 0. \end{gathered} \tag{3.1}$$

Define $\alpha^* \in [0, \pi)$ by

$$\alpha^* = \begin{cases} \arctan(-k_{12}/k_{11}) & \text{if } k_{11} \neq 0, \\ \pi/2 & \text{if } k_{11} = 0. \end{cases} \tag{3.2}$$

Note that $\alpha^* = 0$ when $k_{12} = 0$ since $k_{11} \neq 0$ in this case.

Remark 3.2. Note that condition (3.1) for $-K$ is equivalent to (3.1) for K. Since $K \in SL_2(\mathbb{R})$, $(k_{i1}, k_{i2}) \neq (0, 0) \neq (k_{1i}, k_{2i})$, $i = 1, 2$ and (3.1) is a self-adjoint boundary condition for each $\alpha \in [0, \pi)$. When $\alpha = 0$ (3.1)) reduces to $y(a) = 0 = y(b)k_{22} - (py')(b)k_{12}$; when $\alpha = \pi/2$ (3.1) is equivalent with $(py')(a) = 0 = y(b)k_{21} - (py')(b)k_{11}$. When $\alpha = \pi/2$ and $k_{11} = 0$ (3.1) becomes $(py')(a) = 0 = y(b)$.

Definition 3.3. Let $K = (k_{ij}) \in SL_2(\mathbb{R})$. For each $\alpha \in [0, \pi)$ let

$$\{\mu_n(\alpha) : n \in \mathbb{N}_0\} \tag{3.3}$$

denote the eigenvalues of (3.1). (See Figure 1 below.)

It is these eigenvalues $\{\mu_n(\alpha) : n \in \mathbb{N}_0\}$ which determine all eigenvalues for K and for $-K$ for each $K \in SL_2(\mathbb{R})$. The next theorem defines the continuous eigenvalue curves whose maxima and minima are the eigenvalues for K and $-K$.

Theorem 3.4. *Let $K = (k_{ij}) \in SL_2(\mathbb{R})$.*

(1) *Suppose $k_{12} \neq 0$ and α^* is defined by (3.2). Then $\alpha^* \in (0, \pi)$. For each $n \in \mathbb{N}_0$ define the eigencurves R_n and L_n as follows:*

$$R_n(\alpha) = \mu_n(\alpha), \quad \alpha^* \leq \alpha < \pi; \tag{3.4}$$

$$L_n(\alpha) = \mu_n(\alpha), \quad 0 \leq \alpha < \alpha^*. \tag{3.5}$$

Then $R_n(\alpha)$ is continuous on $[\alpha^*, \pi)$ and $L_n(\alpha)$ is continuous on $[0, \alpha^*)$.

(2) *Suppose $k_{12} = 0$. Define R_n by*

$$R_n(\alpha) = \mu_n(\alpha), \ 0 \leq \alpha < \pi. \tag{3.6}$$

Then $R_n(\alpha)$ is continuous on $[0, \pi)$. (There is no L_n in this case.)

The proof of the above theorem follows from Theorem 2.2 part (a)(1). The selection process for the eigenvalues for K and $-K$ is given by the following algorithm:

Algorithm 3.5. Let $K \in SL_2(\mathbb{R})$.

- If, for some $n \in \mathbb{N}$, $\lambda_n(K) = \lambda_{n+1}(K)$, then $\mu_n(\alpha) = \lambda_n(K)$ for all $\alpha \in [0, \pi)$.

(1) Assume $k_{12} = 0$ and $k_{11} > 0$. Then $\lambda_0(K)$ is simple and

(a)

$$\lambda_0(K) = \max R_0(\alpha), \quad 0 \leq \alpha < \pi. \tag{3.7}$$

(b) If n is even, then

$$\lambda_n(K) = \max R_n(\alpha), \quad 0 \le \alpha < \pi. \tag{3.8}$$

(c) If n is odd, then

$$\lambda_n(K) = \min R_{n+1}(\alpha), \quad 0 \le \alpha < \pi. \tag{3.9}$$

(2) Assume $k_{12} = 0$ and $k_{11} < 0$. (Note that if $k_{12} = 0$ then $k_{11} \neq 0$ since $\det(K) = 1$.) Then

(a)

$$\lambda_0(K) = \min R_1(\alpha), \quad 0 \le \alpha < \pi. \tag{3.10}$$

(b) If n is even, then

$$\lambda_n(K) = \min R_{n+1}(\alpha), \quad 0 \le \alpha < \pi. \tag{3.11}$$

(c) If n is odd, then

$$\lambda_n(K) = \max R_n(\alpha), \quad 0 \le \alpha < \pi. \tag{3.12}$$

(3) Assume $k_{12} < 0$. Then

(a)

$$\lambda_0(K) = \max R_0(\alpha),\ \alpha^* \le \alpha < \pi. \tag{3.13}$$

(b) If n is even, then

$$\lambda_n(K) = \max R_n(\alpha),\ \alpha^* \le \alpha < \pi. \tag{3.14}$$

(c) If n is odd, then

$$\lambda_n(K) = \min L_n(\alpha),\ 0 \le \alpha < \alpha^*. \tag{3.15}$$

(4) Assume $k_{12} > 0$. Then

(a)

$$\lambda_0(K) = \min L_0(\alpha),\ 0 \le \alpha < \alpha^*. \tag{3.16}$$

(b) If n is even, then

$$\lambda_n(K) = \min L_n(\alpha), \quad 0 \le \alpha < \alpha^*. \tag{3.17}$$

(c) If n is odd, then

$$\lambda_n(K) = \max R_n(\alpha), \quad \alpha^* \le \alpha < \pi. \tag{3.18}$$

For the proof of the above statements, see Section 5 below. Except for the case when $k_{12} < 0$ and $n = 0$, the inequalities given by Theorem (2.2) locate each eigenvalue $\lambda_n(K)$, $n \in \mathbb{N}_0$ uniquely between two consecutive eigenvalues of the separated conditions (2.7). The next Corollary fills this gap.

Corllary 3.6. *Assume that $k_{12} > 0$. Then for any $\alpha \in [\alpha^*, \pi)$ we have that*

$$\lambda_0(K) \le \mu_0(\alpha). \tag{3.19}$$

In particular, for any $\varepsilon > 0$, $\mu_0(\alpha) - \varepsilon$ is a lower bound of $\lambda_0(K)$.

The proof follows from part (3)(a) of Algorithm 3.5.

4. Another family of separated boundary conditions

In this section we study another family of separated boundary conditions generated by a coupling matrix K. This family and its relationship to the α-family (3.1) is used in the proof of the Algorithm and, we believe, is of independent interest. But first we recall the characterization of the eigenvalues by means of the characteristic function.

For any $\lambda \in \mathbb{C}$ define two linearly independent solutions, $\varphi(x,\lambda)$, $\psi(x,\lambda)$, of the differential equation (1.1) by the initial conditions

$$\begin{aligned} \varphi(a,\lambda) = 0, \quad (p\varphi')(a,\lambda) = 1, \\ \psi(a,\lambda) = 1, \quad (p\psi')(a,\lambda) = 0. \end{aligned} \tag{4.1}$$

Then any solution $y(x,\lambda)$ of (1.1) can be expressed in the form

$$\begin{aligned} y(x,\lambda) &= y(a,\lambda)\psi(x,\lambda) + (py')(a,\lambda)\varphi(x,\lambda), \\ (py')(x,\lambda) &= y(a,\lambda)(p\psi')(x,\lambda) + (py')(a,\lambda)(p\varphi')(x,\lambda), \end{aligned} \tag{4.2}$$

for all $x \in [a,b]$ and all $\lambda \in \mathbb{C}$. In particular we have

$$\begin{aligned} y(b,\lambda) &= y(a,\lambda)\psi(b,\lambda) + (py')(a,\lambda)\varphi(b,\lambda), \\ (py')(b,\lambda) &= y(a,\lambda)(p\psi')(b,\lambda) + (py')(a,\lambda)(p\varphi')(b,\lambda), \end{aligned} \tag{4.3}$$

and two basic results follow:

Theorem 4.1. *Let $\lambda \in \mathbb{C}$. The differential equation* (1.1) *has a non-trivial solution satisfying the separated boundary conditions* (2.1), (2.2) *if, and only if,*

$$\delta(\lambda) := -A_1B_2(p\varphi')(b,\lambda) + A_2B_1\psi(b,\lambda) + A_2B_2(p\psi')(b,\lambda) - A_1B_1\varphi(b,\lambda) = 0. \tag{4.4}$$

Proof. Substitute (4.2) in (2.1), (2.2) getting two equations in $y(a,\lambda)$ and $(py')(a,\lambda)$, which must be consistent. This condition is (4.4). □

Theorem 4.2. *Let $K \in SL_2(\mathbb{R})$, $\lambda \in \mathbb{C}$, $-\pi < \gamma \le \pi$. The differential equation* (1.1) *has a non-trivial solution satisfying the coupled boundary conditions* (1.6) *if, and only if,*

$$D(\lambda) = k_{11}(p\varphi')(b,\lambda) - k_{21}\varphi(b,\lambda) + k_{22}\psi(b,\lambda) - k_{12}(p\psi')(b,\lambda) = 2\cos\gamma. \tag{4.5}$$

Proof. Proceed as in the proof of Theorem 4.1 using the coupled boundary conditions (1.6), see [9] for details. (See Figure 2 below.) □

Now we define a family of separated boundary conditions in terms of $r \in \mathbb{R} \cup \{\pm\infty\}$ as follows:

Given $K \in SL_2(\mathbb{R})$, for each $r \in \mathbb{R}$ consider the boundary conditions

$$\begin{aligned} y(a) - r(py')(a) = 0, \\ (k_{21}r + k_{22})y(b) - (k_{11}r + k_{12})(py')(b) = 0; \end{aligned} \tag{4.6}$$

and the conditions

$$(py')(a) = 0 = k_{21}y(b) - k_{11}(py')(b) = 0. \tag{4.7}$$

Condition (4.7) corresponds to $r = \pm\infty$; its eigenvalues are denoted by $\nu_n = \lambda_n(\pm\infty), n \in \mathbb{N}_0$. Here it is important to keep in mind that conditions (4.6) for all $r \in \mathbb{R}$ and condition (4.7) together form one family of separated conditions generated by K, we refer to this family as the r-family of K. Next we study this family.

Notation. Let $\sigma(r) = \{\lambda_n(r),\, n \in \mathbb{N}_0\}$ denote the eigenvalues of the r-family with $\nu_n = \lambda_n(\pm\infty)$ corresponding to $r \pm \infty$.

The next lemma discusses the continuity properties of the eigenvalues of the r-family.

Lemma 4.3. *For a fixed $n \in \mathbb{N}_0$ the eigenvalue function $\lambda_n(r)$ is a continuous function of $r \in \mathbb{R}$ except in the following three cases:*

(1) *as $r \to 0^-$.*
(2) *when $k_{12} = 0$ and $r = 0$ (note that $r_{22} \neq 0$ in this case).*
(3) *when $k_{11} \neq 0$ and $r = -k_{12}/k_{11}$.*

The above lemma follows from Theorem (2.2). Next for each $K = (k_{ij}) \in SL_2(\mathbb{R})$ we construct a family of separated boundary conditions associated with K. Let $r \in \mathbb{R} \cup \{\pm\infty\}$ and let

$$R = \frac{rk_{11} + k_{12}}{rk_{21} + k_{22}}. \tag{4.8}$$

Note that:

(1) Not both of k_{11}, k_{12} or k_{21}, k_{22} can be 0 since $\det(K) = 1$.
(2) If $k_{12} = 0$ then $k_{11} \neq 0$ and $R = 0$ if and only if $r = 0$.
(3) If $k_{21} \neq 0$ and $r = -k_{22}/k_{21}$, then R is undefined.
(4) If $k_{21} = 0$ then $k_{22} \neq 0$ and R is well defined for all $r \in \mathbb{R}$.

The next theorem relates the eigenvalues of K and $-K$ with those of the r-family of K.

Theorem 4.4. *Let $K \in SL_2(\mathbb{R})$ and let $D(\lambda)$ be defined by (4.5). If λ is an eigenvalue of any member of the r-family of K, then $D^2(\lambda) - 4 \geq 0$; i.e., $D(\lambda) \geq 2$ or $D(\lambda) \leq -2$.*

Proof. We first prove the case when $r \in \mathbb{R}$. If λ is such an eigenvalue, then its boundary condition is of the form (2.1), (2.2) with

$$A_1 = 1, \quad A_2 = -r, \quad B_1 = 1, \quad B_2 = -R. \tag{4.9}$$

Substituting into (4.4) gives a quadratic in r,

$$Ar^2 + Br + C = 0, \tag{4.10}$$

where

$$\begin{gathered} A = k_{21}\psi(b,\lambda) - k_{11}(p\psi')(b,\lambda), \\ B = k_{21}\varphi(b,\lambda) + k_{22}\psi(b,\lambda) - k_{11}(p\psi')(b,\lambda) - k_{12}(p\varphi')(b,\lambda), \\ C = k_{22}\varphi(b,\lambda) - k_{12}(p\varphi')(b,\lambda). \end{gathered} \tag{4.11}$$

Since λ is an eigenvalue for some fixed number r, the left hand side of (4.10) must vanish. Hence

$$4A^2\big\{(r + \frac{B}{2A})^2 - \frac{B^2 - 4AC}{4A^2}\big\} = 0, \tag{4.12}$$

or

$$4A^2(r + \frac{B}{2A})^2 = B^2 - 4AC. \tag{4.13}$$

A direct computation shows that $B^2 - 4AC = D^2(\lambda) - 4$. Therefore,

$$4A^2(r + \frac{B}{2A})^2 = D^2(\lambda) - 4. \tag{4.14}$$

Since r, A, B, C are real it follows that

$$D^2(\lambda) \geq 4 \quad \text{and} \quad D(\lambda) \geq 2 \quad \text{or} \quad D(\lambda) \leq -2. \tag{4.15}$$

This concludes the proof for $r \in \mathbb{R}$.

For $r = \pm\infty$ the member of the r-family is (4.7) whose eigenvalues are ν_n, $n \in \mathbb{N}_0$. The conclusion (4.15) for ν_n, $n \in \mathbb{N}_0$ was established in [3], see also [5, pp. 80-84]. This concludes the proof. □

Although the next result is a Corollary of Theorem 4.4 and other known results, we state it here as a theorem because we think it is surprising and provides a stark contrast with the Algorithm.

Theorem 4.5. *Let $K \in SL_2(\mathbb{R})$. Let $\sigma(r)$ for $r \in \mathbb{R} \cup \{\pm\infty\}$ and $\sigma(K, \gamma)$ for $\gamma \in (-\pi, 0) \cup (0, \pi)$ be defined as above. Then no eigenvalue of any member of any r-family is an eigenvalue in $\sigma(K, \gamma)$ for any $\gamma \in (-\pi, 0) \cup (0, \pi)$. More explicitly*

$$\sigma(r) \cap \sigma(K, \gamma) = \emptyset, \tag{4.16}$$

for all $r \in \mathbb{R} \cup \{\pm\infty\}$ and all $\gamma \in (-\pi, 0) \cup (0, \pi)$.

Proof. If λ is an eigenvalue in $\sigma(K, \gamma)$ for any $\gamma \in (-\pi, 0) \cup (0, \pi)$ then $|D(\lambda)| < 2$ by (4.5) and the conclusion follows from Theorem 4.4. □

Theorem 4.6. *Let $K \in SL_2(\mathbb{R})$. If $\lambda_n'(r_0) = 0$ for some $n \in \mathbb{N}_0$ and some $r_0 \in \mathbb{R} \cup \{\pm\infty\}$, then $\lambda_n(r_0)$ is an eigenvalue of either K or $-K$.*

Proof. Suppose $\lambda = \lambda_n(r)$ satisfies (4.10). Each term of this equation is a function of r, so we differentiate the left-hand side of the equation with respect to r and obtain

$$2Ar + B + \big\{r^2 \frac{\partial A}{\partial \lambda} + r \frac{\partial B}{\partial \lambda} + C \frac{\partial C}{\partial \lambda}\big\} \frac{d\lambda}{dr} = 0 \quad \text{at } \lambda = \lambda_n(r). \tag{4.17}$$

By assumption $\lambda_n'(r_0) = 0$. Hence (4.17) reduces to

$$2Ar_0 + B = 0 \quad \text{at } \lambda_n(r_0). \tag{4.18}$$

This equation together with (4.13) yields $B^2 - 4AC = 0$. This implies that $D^2(\lambda) - 4 = 0$, which means that λ is an eigenvalue for either K or $-K$. □

Remark 4.7. In other words, Theorem 4.6 says that the eigenvalues for K and $-K$ are the extrema of the continuous eigencurves $L_n(\alpha)$ and $R_n(\alpha)$ defined above. The Algorithm states explicitly, for any K and any n, which extremum is equal to $\lambda_n(K)$.

Next we study the relationship of the α and r families with each other. Let $r \in \mathbb{R} \cup \{\pm\infty\}$ be determined by

$$\tan(\alpha) = r, \quad \alpha \in [0, \pi), \tag{4.19}$$

here $r = \pm\infty$ when $\alpha = \pi/2$.

Let R be given by (4.8). Now define $\beta = \beta(\alpha) = \beta(\alpha(r))$ by

$$\tan(\beta) = R, \quad \beta \in (0, \pi], \tag{4.20}$$

and observe that

(1) $\beta = \pi/2$ when $k_{21} \neq 0$ and $r = -k_{22}/k_{12}$,
(2) $\beta = \pi$ if and only if $k_{12} = 0$ and $r = 0$.
(3) By (4.20) $\beta = \pi/2$ corresponds to $R = \pm\infty$.

By (4.19) $\alpha = \pi/2$ corresponds to $r = \pm\infty$.

5. Proof of the Algorithm

Basically the proof is obtained by combining Theorems 4.5 and 4.4 with the known inequalities given by Theorem 2.2.

Proof of Algorithm 3.5. First consider the special case: $k_{12} = 0$. Then $\alpha^* = 0$. If $k_{11} > 0$, from Theorem 2.2 the interval $[\lambda_{n-1}(K), \lambda_n(K)]$ for even n contains $\nu_n = \upsilon_n(\pi/2)$ and the function $\upsilon_n(\alpha)$ is continuous at $\alpha = \pi/2$. By Theorem 4.4, $\upsilon_n(\alpha)$ cannot move outside the interval $[\lambda_{n-1}(K), \lambda_n(K)]$ as α varies continuously away from $\alpha = \pi/2$ since $\upsilon_n(\alpha) > \lambda_n(K)$ or $\upsilon_n(\alpha) < \lambda_{n-1}(K)$ would contradict $D^2(\upsilon_n(\alpha)) - 4 \geq 0$. Therefore $\lambda_{n-1}(K) \leq \upsilon_n(\alpha) \leq \lambda_n(K)$ for all $\alpha \in [0, \pi)$. If $\lambda_{n-1}(K) = \lambda_n(K)$ then $\upsilon_n(\alpha) = \lambda_n(K)$ for all $\alpha \in [0, \pi)$.

If $\lambda_{n-1}(K) < \lambda_n(K)$ then the continuous function $\upsilon_n(\alpha)$ has a maximum and a minimum in the compact interval $[\lambda_{n-1}(K), \lambda_n(K)]$ as α varies in $[0, \pi)$. If the maximum is not $\lambda_n(K)$ then it occurs in the interior of this interval and by Theorem 4.5 $\lambda_n(K) = \max\{\upsilon_n(\alpha) : \alpha \in [0, \pi)\}$. Similarly $\lambda_{n-1}(K) = \min\{\upsilon_n(\alpha) : \alpha \in [0, \pi)\}$. The proof for $k_{12} = 0$ and $k_{11} < 0$ is similar.

When $k_{12} \neq 0$ then $0 < \alpha^*$. In this case the proof is also similar to the above proof but with one important difference: The interval $[0, \pi)$ in the above argument is replaced by the two intervals $[0, \alpha^*)$ and $[\alpha^*, \pi)$. This is due to the fact that the function $\upsilon_n(\alpha)$ has a jump discontinuity at α^* - see part (1) of Theorem 2.2 for a discussion of jump discontinuities of eigencurves for separated boundary conditions. This discontinuity is due to the fact that $\tan(\beta) = 0$ when $\alpha = \alpha^*$ and hence (recall the normaliztion for β in (1.5)) $\beta = \pi$ when $\alpha = \alpha^*$. See part (1) of Theorem 2.2 for a discussion of jump discontinuities. Although this discussion is for the case when β is fixed as α varies it extends readily to our situation here where β is a continuous function of α. In fact the results mentioned in part (1) of Theorem 2.2 have far reaching extensions, see [9, Sections 3.4, 3.5, 3.6]. □

Lemma 5.1. *In each of these three cases, there is an infinite jump discontinuity when $n = 0$ and a finite jump discontinuity when $n > 0$.*

In Figure 1 are plotted some of the eigenvalues of an α-family when $n = 0, 1, 2, 3$ and the element k_{12} of the matrix K is not zero. As appears in the figure, the eigenvalues corresponding to each index n lie along two distinct continuous curves; one on $[0, \alpha^*)$ and the other on $[\alpha^*, \pi)$.

Figure 2 depicts a typical characteristic function $D(\lambda)$. When the parameter $\gamma = 0$ (or π), then $D(\lambda) = 2$ if, and only if, λ is an eigenvalue for the problem with coupled boundary conditions defined by a matrix K; $D(\lambda) = -2$ when λ is an eigenvalue of the problems defined by $-K$. Thus to compute eigenvalues of such coupled boundary condition problems one simply computes the values of functions $\varphi(x, \lambda)$, $\psi(x, \lambda)$ at $x = b$, evaluates $D(\lambda)$, and searches for values of λ for which $D(\lambda) = 2$.

The coupled boundary condition problem to which both Figure 1 and Figure 2 refer consists of the differential equation having $p(x) = 1$, $q(x) = (5/16)/x^2$,

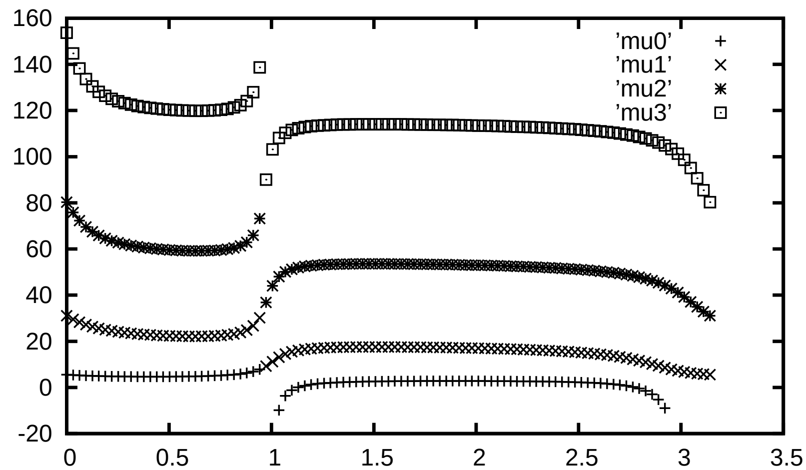

FIGURE 1. Eigenvalues of an α-family

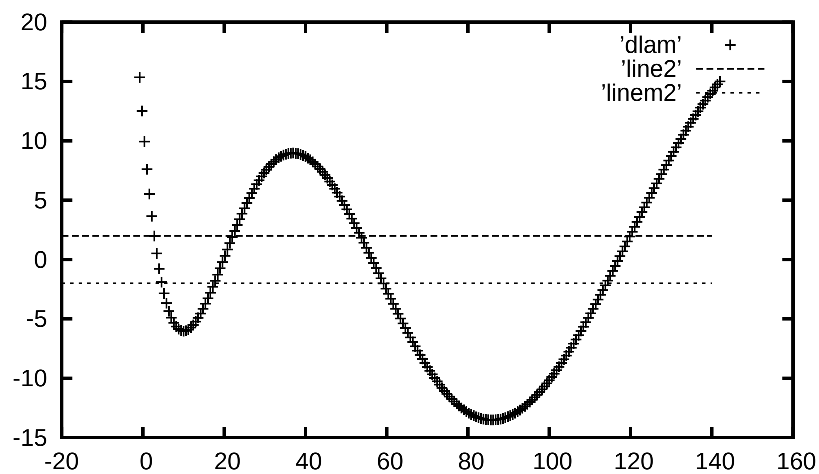

FIGURE 2. Characteristic function $D(\lambda)$

$w(x) = 1$ on the interval $(0.1, 1.0)$, subject to boundary condition (1.6). The first few eigenvalues corresponding to the matrix $K = (k_{ij})$,

$$k_{11} = 1.0576, \quad k_{12} = -1.5125, \quad k_{21} = -0.2270, \quad k_{22} = 1.2701,$$

have been computed to be

$$\lambda_0 = 2.7933, \quad \lambda_1 = 22.1285, \quad \lambda_2 = 53.5645, \quad \lambda_3 = 119.9029;$$

while the eigenvalues for matrix $-K$ are

$$\lambda_0 = 4.6599, \quad \lambda_1 = 17.5508, \quad \lambda_2 = 59.1740, \quad \lambda_3 = 114.0604.$$

Remark 5.2. The essential point of this paper is that the intersections of the function $D(\lambda)$ with the horizontal lines at $+2$ and -2, as depicted in Figure 2, which are the eigenvalues for K and $-K$, correspond precisely with the local extrema of the continuous eigencurves $R_n(\alpha)$ and $L_n(\alpha)$ of the related α-family in an appropriate α interval, as in Figure 1.

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