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BIFURCATION ALONG CURVES FOR THE p-LAPLACIAN WITH RADIAL SYMMETRY

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Abstract. We study the global structure of the set of radial solutions of a nonlinear Dirichlet eigenvalue problem involving the p-Laplacian with $p > 2$, in the unit ball of $\mathbb{R}^N$, $N \geqslant 1$. We show that all non-trivial radial solutions lie on smooth curves of respectively positive and negative solutions, bifurcating from the first eigenvalue of a weighted p-linear problem. Our approach involves a local bifurcation result of Crandall-Rabinowitz type, and global continuation arguments relying on monotonicity properties of the equation. An important part of the analysis is dedicated to the delicate issue of differentiability of the inverse p-Laplacian, and holds for all $p > 1$.

1. Introduction

In this article we consider the nonlinear Dirichlet problem

$$\begin{gathered} -\Delta_p(u) = \lambda f(|x|, u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $\Delta_p(u) := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian, $p > 1$, $\lambda \geqslant 0$, and Ω is the unit ball in $\mathbb{R}^N$, $N \geqslant 1$. The function f is continuous, such that $f(r, 0) = 0$ for all $r \in [0, 1]$, and will be subject to various additional assumptions.

We look for C^1 radial solutions by studying the problem

$$\begin{gathered} -(r^{N-1}\phi_p(u'))' = \lambda r^{N-1} f(r, u), \quad 0 < r < 1, \\ u'(0) = u(1) = 0, \end{gathered} \tag{1.2}$$

where $\phi_p(\xi) := |\xi|^{p-2}\xi,\ \xi \in \mathbb{R}$, and $'$ denotes differentiation with respect to r. By a solution of (1.2) will be meant a couple (λ, u), with $\lambda \in \mathbb{R}$ and $u \in C^1[0, 1]$, such that $\phi_p(u') \in C^1[0, 1]$, that satisfies (1.2). Note that, since $f(r, 0) = 0$ for all $r \in [0, 1]$, $(\lambda, 0)$ is a solution for all $\lambda \in \mathbb{R}$. Such solutions will be called trivial. We are interested in existence and bifurcation of non-trivial solutions of (1.2).

Bifurcation results for quasilinear equations in bounded domains have been considered for instance in [7, 5, 8] — further references can be found in these papers. Del Pino and Manásevich [5] prove global bifurcation from the first eigenvalue of the

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p-Laplacian in a general bounded domain, and global bifurcation from every eigenvalue in the radial case. They also obtain nodal properties of solutions along the bifurcating continua. These results generalize the well-known results of Rabinowitz [9] to the quasilinear setting, using degree theoretic arguments.

More recently, Girg and Takáč [8] obtained results in the spirit of Dancer [4], about bifurcation from the first eigenvalue of a homogeneous quasilinear problem, in the cones of positive and negative solutions. They consider a large class of quasilinear problems in a general bounded domain Ω and they allow the asymptotic problems as $|u| \to 0/\infty$ to depend on $x \in \Omega$. They also prove their results using topological arguments, combined with a technical asymptotic analysis.

The last contribution we want to mention here, which is probably the most closely related to our work, is the paper [7] by García-Melián and Sabina de Lis. The famous Crandall-Rabinowitz theorem [3] is extended in [7] to p-Laplacian equations, in the radial setting, see [7, Theorem 1]. This result yields a continuous local branch of solutions bifurcating from every eigenvalue of the p-Laplacian, and uniqueness of the branch in a neighbourhood of the bifurcation point. García-Melián and Sabina de Lis then use this local result to obtain further information about the global structure of the continua of solutions obtained by the topological method in [5]. In particular, they show that there exist two unbounded continua $\mathcal{C}^{\pm}$ of respectively positive and negative solutions, which only meet at the bifurcation point $(\lambda_0, 0) \in \mathbb{R} \times C^1[0,1]$, where λ_0 is the first eigenvalue of the homogeneous problem

$$\begin{gathered} -(r^{N-1}\phi_p(u'))' = \lambda r^{N-1}\phi_p(u), \quad 0 < r < 1, \\ u'(0) = u(1) = 0. \end{gathered} \tag{1.3}$$

Since [7, Theorem 1] is only a local result, it is not known from [7] whether the global continua $\mathcal{C}^{\pm}$ are actually continuous curves or only connected sets. In fact, the picture obtained from [7] is somewhat hybrid, due to a mixture of analytical arguments (essentially the implicit function theorem) used to get local bifurcation, and the topological method yielding the global continua $\mathcal{C}^{\pm}$ in [5].

Our main purpose in this paper is to show that, under additional assumptions on the function f in (1.2) — in particular monotonicity assumptions —, it is possible to describe the global structure of solutions bifurcating from the first eigenvalue using purely analytical arguments. We obtain smooth curves of respectively positive and negative solutions, parametrized by the bifurcation parameter λ.

Besides, we consider a more general homogeneous problem than (1.3) in the limit $|u| \to 0$. In fact, we allow both asymptotics as $|u| \to 0/\infty$ to depend on $r \in [0,1]$, in the same spirit as [8]. The asymptotic problems as $|u| \to 0/\infty$ — see equations (2.3) below — are weighted homogeneous problems respectively associated with the asymptotes

$$f_0(r) := \lim_{\xi \to 0} \frac{f(r,\xi)}{\phi_p(\xi)} > 0 \quad \text{and} \quad f_\infty(r) := \lim_{|\xi| \to \infty} \frac{f(r,\xi)}{\phi_p(\xi)} > 0, \quad r \in [0,1].$$

The properties of (2.3) with f_0 enable us to obtain a local bifurcation theorem as in [7], while the asymptotic problem (2.3) with f_∞ governs the behaviour as $|u| \to \infty$.

We will consider two different situations. In the first case, we will assume that $f(r,\xi) > 0$ for all $(r,\xi) \in [0,1] \times \mathbb{R}^*$ and $f(r,0) = 0$ for all $r \in [0,1]$. It follows that the set of non-trivial solutions of (1.2) is a smooth curve of positive solutions — see Theorem 2.3. If we rather assume $f(r,\xi)\xi > 0$ for $(r,\xi) \in [0,1] \times \mathbb{R}^*$ and $f(r,0) \equiv 0$, then we get two smooth curves of respectively positive and negative solutions, containing all non-trivial solutions of (1.2) — see Theorem 2.4. Furthermore, if $N = 1$, we are also able to deal with the case where f is 'sublinear' at infinity, that is, $f(r, \xi)/\phi_p(\xi) \to 0$ as $|\xi| \to \infty$, uniformly for $r \in [0, 1]$.

A one-dimensional problem similar to (1.2) was studied by Rynne in [10], from which the present work is substantially inspired. In particular, the explicit form we get for the inverse p-Laplacian in the radial setting allows us to study the differentiability of this operator following arguments of [1], where the one-dimensional p-Laplacian was considered. This differentiability issue is probably the most delicate part of the analysis. It should be noted that the results regarding the inverse p-Laplacian in Section 3 hold for any $p > 1$, while we had to restrict ourselves to $p > 2$ in the bifurcation analysis for other reasons — see Remark 5.3.

We conclude this section with a brief description of the contents of the paper. In Section 2, we give some information about the functional setting, our precise hypotheses, and we state our main results, Theorems 2.3 and 2.4. Then, in Section 3, we study an integral operator corresponding to the inverse of the p-Laplacian in (1.2). The main results about this operator are Theorems 3.3 and 3.5. It should be noted that [7] already dealt with differentiability results similar to those of Theorem 3.5. However, we believe that the discussion in [7] is incomplete and so Theorem 3.5 is of importance in its own right. In Section 4, we establish some *a priori* properties of solutions of (1.2), notably positivity/negativity, as well as the asymptotic behaviour of solutions (λ, u) as $|u| \to 0/\infty$. Section 5 is devoted to the local bifurcation analysis, where we establish, in particular, a Crandall-Rabinowitz-type result, Lemma 5.1. Finally, the proofs of Theorems 2.3 and 2.4 are completed in Section 6, where we show that the local branches of solutions obtained in Section 5 can be extended globally.

2. Setting and main results

We will work in various function spaces. We will denote by $L^1(0,1)$ the Banach space of real Lebesgue integrable functions over $(0,1)$ and by $W^{1,1}(0,1)$ the Sobolev space of functions $u \in L^1(0,1)$ having a weak derivative $u' \in L^1(0,1)$. For $n = 0, 1$, $C^n[0,1]$ will denote the space of n times continuously differentiable functions, with the usual sup-type norm $|\cdot|_n$.

In our operator formulation of (1.2), it will be convenient to use the shorthand notation

$$X_p := \{u \in C^1[0,1] : \phi_p(u') \in C^1[0,1] \text{ and } u'(0) = u(1) = 0\}, \quad Y := C^0[0,1].$$

An important part of our discussion in the next section will concern the differentiability of an integral operator, that will depend on the value of $p > 1$. This analysis will rely on results in [1], and we borrow the following notation from there:

$$B_p := \begin{cases} C^1[0,1], & 1 < p \leqslant 2, \\ W^{1,1}(0,1), & p > 2. \end{cases} \tag{2.1}$$

However, our main results require $p \geqslant 2$ and, apart from Section 3, we will suppose $p > 2$ throughout the paper — the results are well known for $p = 2$.

Denoting by $\partial_2 f$ the partial derivative of f with respect to $\xi \in \mathbb{R}$, we make the following hypotheses on the continuous function $f : [0,\infty) \times \mathbb{R} \to \mathbb{R}$:

(H1) $f(r,\cdot) \in C^1(\mathbb{R})$ for all $r \in [0,1]$ and $\partial_2 f \in C^0([0,1] \times \mathbb{R})$;

(H2) $f(r,\xi) > 0$ for $(r,\xi) \in [0,1] \times \mathbb{R}^*$ and $f(r,0) \equiv 0$;
(H3) $(p-1)f(r,\xi) \geqslant \partial_2 f(r,\xi)\xi$ for $(r,\xi) \in [0,1] \times [0,\infty)$, and there exist $\delta, \epsilon > 0$ such that $(p-1)f(r,\xi) > \partial_2 f(r,\xi)\xi$ for all $(r,\xi) \in (1-\delta,1] \times (0,\epsilon)$.

It follows from (H3) that, for any fixed $r \in [0,1]$, the mapping $\xi \mapsto f(r,\xi)/\phi_p(\xi)$ is decreasing on $(0,\infty)$. Therefore, for each $r \in [0,1]$ there exist $f_0(r)$ and $f_\infty(r)$ such that

$$f(r,\xi)/\phi_p(\xi) \to f_{0/\infty}(r) \quad \text{as } \xi \to 0^+/+\infty,$$

with

$$0 \leqslant f_\infty(r) \leqslant f(r,\xi)/\phi_p(\xi) \leqslant f_0(r), \quad \text{for all } (r,\xi) \in [0,1] \times (0,\infty). \tag{2.2}$$

We will further suppose that $f_0(r)$ is finite for all $r \in [0,1]$, that $f_0, f_\infty \in C^0[0,1]$, and (for $p > 2$):

(H4) $\lim_{\xi\to 0^+} |f(\cdot,\xi)/\phi_p(\xi) - f_0|_0 = \lim_{\xi\to 0^+} |\partial_2 f(\cdot,\xi)/\xi^{p-2} - (p-1)f_0|_0 = 0$;
(H5) $\lim_{\xi\to\infty} |f(\cdot,\xi)/\phi_p(\xi) - f_\infty|_0 = 0$.

Remark 2.1.
(i) Note that (H2) and (2.2) imply $f_0 > 0$ on $[0,1]$.
(ii) Also, (H3) implies that, for $r \in (1-\delta,1]$, the function $\xi \mapsto f(r,\xi)/\phi_p(\xi)$ is not constant on $(0,\infty)$. In particular, $f_\infty \not\equiv f_0$ on $[0,1]$.

To state our main results, we need to relate problem (1.2) to the homogeneous eigenvalue problems corresponding to the asymptotes $f_0, f_\infty \in C^0[0,1]$:

$$\begin{gathered} -(r^{N-1}\phi_p(v'))' = \lambda r^{N-1} f_{0/\infty}(r)\phi_p(v), \quad 0 < r < 1, \\ v'(0) = v(1) = 0. \end{gathered} \tag{2.3}$$

The following result follows from [11, Sec. 5].

Lemma 2.2. *If $f_{0/\infty} > 0$ on $[0,1]$ then problem (2.3) has a simple eigenvalue $\lambda_{0/\infty} > 0$ with a corresponding eigenfunction $v_{0/\infty} > 0$ in $[0,1)$, and no other eigenvalue having a positive eigenfunction. Furthermore, $f_\infty \lneqq f_0$ implies $\lambda_0 < \lambda_\infty$.*

We know from Remark 2.1 that $f_0 > 0$ and $0 \leqslant f_\infty \lneqq f_0$. For λ_∞ to be well-defined, we will still make the following assumption.

(H6) Either
(a) $N \geqslant 1$ is arbitrary and $f_\infty > 0$ on $[0,1]$, or
(b) $N = 1$ and $f_\infty \equiv 0$ on $[0,1]$.
If (a) holds, $\lambda_\infty > 0$ is defined in Lemma 2.2; if (b) holds, we set $\lambda_\infty = \infty$.

We are now in a position to state our first result about the solutions of (1.2). From now on, we will refer to the collection of hypotheses (H1) to (H6) as (H).

Theorem 2.3. *Suppose that $p > 2$. If (H) holds, there exists $u \in C^1((\lambda_0,\lambda_\infty),Y)$ such that $u(\lambda) \in X_p$, $u(\lambda) > 0$ on $[0,1)$ and, for any $\lambda \in (\lambda_0,\lambda_\infty)$, $(\lambda,u(\lambda))$ is the unique non-trivial solution of (1.2). Furthermore,*

$$\lim_{\lambda\to\lambda_0} |u(\lambda)|_0 = 0 \quad \textit{and} \quad \lim_{\lambda\to\lambda_\infty} |u(\lambda)|_0 = \infty. \tag{2.4}$$

We will see in the proof of Theorem 2.3 that the condition (H2) forces the solutions of (1.2) to be positive. If, instead of (H2) to (H5), we suppose:

(H2') $f(r,\xi)\xi > 0$ for $(r,\xi) \in [0,1] \times \mathbb{R}^*$ and $f(r,0) \equiv 0$;
(H3') in addition to (H3), $(p-1)f(r,\xi) \leqslant \partial_2 f(r,\xi)\xi$ for $(r,\xi) \in [0,1] \times (-\infty,0]$ and $(p-1)f(r,\xi) < \partial_2 f(r,\xi)\xi$ for all $(r,\xi) \in (1-\delta,1] \times (-\epsilon,0)$;

(H4') $\lim_{\xi\to 0} |f(\cdot,\xi)/\phi_p(\xi) - f_0|_0 = \lim_{\xi\to 0} |\partial_2 f(\cdot,\xi)/|\xi|^{p-2} - (p-1)f_0|_0 = 0$;
(H5') $\lim_{|\xi|\to\infty} |f(\cdot,\xi)/\phi_p(\xi) - f_\infty|_0 = 0$,

then the solutions need not be positive any more and we have the following result. We refer to the collection of hypotheses (H1), (H2') to (H5') and (H6) as (H').

Theorem 2.4. *Suppose that $p > 2$. If* (H') *holds, there exist $u_\pm \in C^1((\lambda_0,\lambda_\infty), Y)$ such that $u_\pm(\lambda) \in X_p$, $\pm u_\pm(\lambda) > 0$ on $[0,1)$ and, for any $\lambda \in (\lambda_0,\lambda_\infty)$, $(\lambda, u_\pm(\lambda))$ are the only non-trivial solutions of* (1.2). *Furthermore, both u_- and u_+ satisfy the limits in* (2.4).

We will prove Theorems 2.3 and 2.4 by giving the detailed arguments for the case where (H) holds, and explaining what needs to be modified to account for (H').

Remark 2.5. Theorems 2.3-2.4 yield a complete description of the set of solutions of (1.2), hence the set of radial solutions of (1.1). Under the additional assumption that $f(r,\xi)$ is non-increasing in $r \in [0,1]$, it follows from [2, Theorem 1, p. 51] that positive solutions $u \in W_0^{1,p}(\Omega) \cap C^1(\overline{\Omega})$ of (1.1) are radial, and hence satisfy (1.2). Thus in this case, Theorems 2.3-2.4 characterize all positive solutions of (1.1).

As immediate corollaries of Theorems 2.3 and 2.4, we have the following results of existence and uniqueness/multiplicity for the problem

$$\begin{gathered} -(r^{N-1}\phi_p(u'))' = r^{N-1} f(r,u), \quad 0 < r < 1, \\ u'(0) = u(1) = 0. \end{gathered} \tag{2.5}$$

Corollary 2.6. *Let $p > 2$, suppose that* (H) *holds, and that $1 \in (\lambda_0,\lambda_\infty)$. Then problem* (2.5) *has a unique non-trivial solution $u \in X_p$, such that $u > 0$ on $[0,1)$.*

Corollary 2.7. *Let $p > 2$, suppose that* (H') *holds, and that $1 \in (\lambda_0,\lambda_\infty)$. Then problem* (2.5) *has exactly two non-trivial solutions $u_\pm \in X_p$, satisfying $\pm u_\pm > 0$ on $[0,1)$.*

Notation. For $h \in C^0([0,1]\times\mathbb{R})$, we define the Nemitskii mapping $h : Y \to Y$ by $h(u)(r) := h(r,u(r))$. Then $h : Y \to Y$ is bounded and continuous. We will always use the same symbol for a function and for the induced Nemitskii mapping.

When computing estimates, the symbol C will denote positive constants which may change from line to line. Their exact values are not essential to the analysis.

3. The inverse p-Laplacian

In this section we study an integral operator corresponding to the inverse of the p-Laplacian in the radial setting. Although our main goal in this paper is the proof of Theorems 2.3 and 2.4, which requires $p \geqslant 2$, the results in this section will be stated in greater generality, for $p > 1$. Let us start by introducing some useful notation. For $p > 1$, we let

$$p^* := \tfrac{1}{p-1} \quad \text{and} \quad p' := \tfrac{p}{p-1} = p^* + 1.$$

Then for $\xi, \eta \in \mathbb{R}$, the continuous function $\phi_p : \mathbb{R} \to \mathbb{R}$ satisfies

$$\phi_p(\xi) = \eta \iff \xi = \phi_{p'}(\eta) = \phi_{p^*+1}(\eta). \tag{3.1}$$

For any $h \in C^0[0,1]$, the problem

$$\begin{gathered} -(r^{N-1}\phi_p(u'))' = r^{N-1} h(r), \quad 0 < r < 1, \\ u'(0) = u(1) = 0, \end{gathered} \tag{3.2}$$

has a unique solution $u(h) \in X_p$, given by

$$u(h)(r) = \int_r^1 \phi_{p'}\Big(\int_0^s \Big(\frac{t}{s}\Big)^{N-1} h(t)\,\mathrm{d}t\Big)\,\mathrm{d}s. \tag{3.3}$$

The formula (3.3) defines a mapping

$$S_p : C^0[0,1] \to C^1[0,1], \quad h \mapsto S_p(h) = u(h),$$

that we shall now study. It will be convenient to rewrite S_p as

$$S_p = T_p \circ J = \mathcal{I} \circ \Phi_{p'} \circ J, \tag{3.4}$$

where we define the following operators:

$$\begin{gathered} J : C^0[0,1] \to C^1[0,1], \quad J(h)(s) := \int_0^s \Big(\frac{t}{s}\Big)^{N-1} h(t)\,\mathrm{d}t; \\ \Phi_q : C^0[0,1] \to C^0[0,1], \quad \Phi_q(g)(s) := \phi_q(g(s)), \quad \text{for any } q > 1; \\ \mathcal{I} : C^0[0,1] \to C^1[0,1], \quad \mathcal{I}(k)(r) := \int_r^1 k(s)\,\mathrm{d}s; \\ T_p : C^0[0,1] \to C^1[0,1], \quad T_p := \mathcal{I} \circ \Phi_{p'} \quad \text{for any } p > 1. \end{gathered}$$

It is clear that $\Phi_q, \mathcal{I}$ and T_p are continuous and bounded, for any $q, p > 1$, and that $\mathcal{I}$ is linear. Furthermore, S_p is p^*-homogeneous.

Remark 3.1. For $1 < q < 2$, Φ_q does not map $C^1[0,1]$ into itself, which causes trouble in differentiating S_p for $p > 2$ (i.e. $p' < 2$). Nevertheless, if $g \in C^1[0,1]$ has only simple zeros, then Φ_q maps a neighbourhood of g in $C^1[0,1]$ continuously into $L^1(0,1)$ (see Lemma 2.1 in [1]).

The following lemma gives important properties of J.

Lemma 3.2.
- (i) *$J : C^0[0,1] \to C^1[0,1]$ is well-defined.*
- (ii) *$J : C^0[0,1] \to C^1[0,1]$ is a bounded linear operator, with norm $\|J\| \leqslant 2$.*
- (iii) *J is Fréchet differentiable on $C^0[0,1]$ with $DJ = J$.*
- (iv) *$J : C^0[0,1] \to C^0[0,1]$ is compact.*

Proof. (i) Using de l'Hospital's rule, we get

$$\lim_{s\to 0} J(s) = \tfrac{1}{N-1} \lim_{s\to 0} s h(s) = 0,$$

and it follows that $J(h) \in C^0[0,1]$ for all $h \in C^0[0,1]$. Furthermore,

$$\lim_{s\to 0} \frac{J(s)}{s} = \lim_{s\to 0} \frac{s^{N-1}h(s)}{N s^{N-1}} = \frac{h(0)}{N},$$

and so $J(h) \in C^1[0,1]$, with $J'(0) = h(0)/N$.

(ii) We first have

$$|J(h)|_0 \leqslant \sup_{0\leqslant s\leqslant 1} \Big| \int_0^s \Big(\frac{t}{s}\Big)^{N-1} \mathrm{d}t \Big| |h|_0 \leqslant \frac{|h|_0}{N}.$$

Then, since

$$\begin{aligned} J(h)'(s) &= (1-N)s^{-N} \int_0^s t^{N-1} h(t)\,\mathrm{d}t + h(s) \\ &= (1-N)s^{-1} J(h)(s) + h(s) \quad \text{for all } s \in (0,1], \end{aligned} \tag{3.5}$$

we have

$$|J(h)'|_0 \leqslant \sup_{0\leqslant s\leqslant 1} \Big|(1-N)s^{-N}\int_0^s t^{N-1}\,\mathrm{d}t\Big||h|_0 + |h|_0 \leqslant \frac{2N-1}{N}|h|_0.$$

It follows that J is bounded, with norm

$$\|J\| := \sup_{|h|_0=1} |J(h)|_1 \leqslant \frac{1}{N} + \frac{2N-1}{N} = 2.$$

(iii) follows from (ii).

(iv) follows from (ii) and the compact embedding $C^1[0,1] \hookrightarrow C^0[0,1]$. □

We can now state important properties of S_p, following from the above results.

Theorem 3.3. *The mapping $S_p : C^0[0,1] \to C^1[0,1]$ defined by* (3.4) *is continuous, bounded and compact.*

The following result is a simple adaptation of [1, Theorem 3.2] to the present context. This is the first step towards the differentiability of S_p.

Proposition 3.4.

(i) *Suppose $1 < p < 2$. Then $T_p : C^0[0,1] \to B_p$ is C^1, and for all $g, \bar g$ in $C^0[0,1]$,*

$$DT_p(g)\bar g = p^*\mathcal{I}(|g|^{p^*-1}\bar g). \tag{3.6}$$

(ii) *Suppose $p > 2$ and let $g_0 \in C^1[0,1]$ have only simple zeros (i.e. $g_0(s) = 0 \Rightarrow g_0'(s) \neq 0$). Then $T_p : C^1[0,1] \to B_p$ is C^1 on a neighbourhood U_0 of g_0 in $C^1[0,1]$, and* (3.6) *holds for all $g \in U_0$, $\bar g \in C^1[0,1]$.*

We can now prove the main result of this section, about the differentiability of S_p. Its statement and proof are very similar to those of Theorem 3.4 in [1].

Theorem 3.5.

(i) *Suppose $1 < p < 2$. Then $S_p : C^0[0,1] \to B_p$ is C^1, and for all $h, \bar h$ in $C^0[0,1]$,*

$$DS_p(h)\bar h = p^*\mathcal{I}(|u(h)'|^{2-p}J(\bar h)), \tag{3.7}$$

where $u(h) = S_p(h)$. Furthermore,

$$v = DS_p(h)\bar h \Longrightarrow v \in B_p \text{ and } \begin{cases} -(r^{N-1}|u(h)'(r)|^{p-2}v'(r))' = p^* r^{N-1}\bar h(r), \\ v'(0) = v(1) = 0, \end{cases} \tag{3.8}$$

(ii) *Suppose $p > 2$ and let $h_0 \in C^0[0,1]$ be such that $u(h_0)'(r) = 0 \Rightarrow h_0(r) \neq 0$. Then there exists a neighbourhood V_0 of h_0 in $C^0[0,1]$ such that the mapping $h \mapsto |u(h)'|^{2-p} : V_0 \to L^1(0,1)$ is continuous, $S_p : V_0 \to B_p$ is C^1, and DS_p satisfies* (3.7) *and* (3.8)*, for all $h \in V_0$, $\bar h \in C^0[0,1]$.*

Proof. (i) In view of (3.4), the differentiability of $S_p : C^0[0,1] \to B_p$ follows from Lemma 3.2(iii) and Proposition 3.4(i). Then, for $h, \bar h \in C^0[0,1]$,

$$DS_p(h)\bar h = DT_p(J(h))J(\bar h) = p^*\mathcal{I}(|J(h)|^{p^*-1}J(\bar h)).$$

Now letting $u(h) = S_p(h)$ and differentiating (3.3) yields

$$u(h)' = \phi_{p'}(J(h)) = \phi_{p^*+1}(J(h)) \Longrightarrow |u(h)'|^{2-p} = |J(h)|^{p^*-1},$$

proving (3.7), from which the continuity of DS_p follows. We will prove below that (3.8) holds in both cases (i) and (ii).

(ii) The case $p > 2$ is more delicate and uses Proposition 3.4(ii). We define $g_0 := J(h_0) \in C^1[0,1]$ and $u_0 := u(h_0)$. Then

$$-\phi_p(u_0'(r)) = g_0(r) \quad \text{and} \quad -\left(r^{N-1}\phi_p(u_0'(r))\right)' = r^{N-1}h_0(r), \quad 0 \leqslant r \leqslant 1.$$

We will show that g_0 has only simple zeros. First remark that

$$g_0(r) = 0 \Longrightarrow \phi_p(u_0'(r)) = 0 \Longrightarrow u_0'(r) = 0 \Longrightarrow h_0(r) \neq 0,$$

by our hypothesis. But now by (3.5),

$$g_0'(r) = (1-N)r^{-1}g_0(r) + h_0(r), \quad 0 \leqslant r \leqslant 1. \tag{3.9}$$

Therefore, if $g_0(r) = 0$ with $r > 0$, then $g_0'(r) = h_0(r) \neq 0$. On the other hand, if $g_0(0) = 0$, it follows from (3.9) that $Ng_0'(0) = h_0(0) \neq 0$. Hence, g_0 has only simple zeros. Apart from our statement (3.8), which is slightly more precise than its analogue in [1], the proof then follows that of [1, Theorem 3.4], using Proposition 3.4(ii) and the analogue of [1, Lemma 2.1] for the present setting.

To prove statement (3.8), let $v = DS_p(h)\bar{h}$, $h, \bar{h} \in C^0[0,1]$. Then, from (3.7),

$$v(r) = p^* \int_r^1 |u(h)'(s)|^{2-p} \int_0^s \left(\frac{t}{s}\right)^{N-1} \bar{h}(t)\,\mathrm{d}t\,\mathrm{d}s, \quad r \in [0,1].$$

Since $|u(h)'|^{2-p} \in L^1(0,1)$ in both cases (i) and (ii), it follows that $v(1) = 0$. Furthermore,

$$v'(r) = -p^*|u(h)'(r)|^{2-p} \int_0^r \left(\frac{t}{r}\right)^{N-1} \bar{h}(t)\,\mathrm{d}t, \quad r \in [0,1], \tag{3.10}$$

from which the equation in (3.8) easily follows. But (3.10) also implies

$$|v'(r)| \leqslant C_1|u(h)'(r)|^{2-p}r, \quad r \in [0,1].$$

Now since $u(h)'(r) = -\phi_{p'}(\int_0^r (t/r)^{N-1}h(t)\,\mathrm{d}t)$, it follows that $|u(h)'(r)| \leqslant C_2 r^{p'-1}$, so that

$$|v'(r)| \leqslant Cr^{(p'-1)(2-p)+1} = Cr^{p^*}, \quad r \in [0,1],$$

showing that $v'(0) = 0$ and finishing the proof. □

Remark 3.6. Note that Theorem 3.5 reduces to well-known results for $p = 2$.

4. Properties of solutions

In this section we discuss some *a priori* properties of solutions. We first study the sign of solutions and then we determine their behaviour as $|u|_0 \to 0/\infty$.

By the results of Section 3, $(\lambda, u) \in [0,\infty) \times X_p$ is a solution of (1.2) if and only if

$$F(\lambda, u) := u - \lambda^{p^*} S_p(f(u)) = 0, \quad (\lambda, u) \in [0,\infty) \times Y. \tag{4.1}$$

Note that $F : [0,\infty) \times Y \to Y$ is continuous. Furthermore, $F(0,u) = 0 \implies u = 0$, so we will only consider solutions in

$$\mathcal{S} := \{(\lambda, u) \in (0,\infty) \times Y : (\lambda, u) \text{ is a solution of (4.1) with } u \not\equiv 0\}.$$

4.1. **The case where (H) holds.** We start with the positivity of solutions.

Proposition 4.1. *Let $(\lambda, u) \in \mathcal{S}$. Then $u > 0$ on $[0,1)$, u is decreasing and satisfies $u'(1) < 0$.*

Proof. Equation (4.1) yields

$$u(r) = \lambda^{p^*} \int_r^1 \phi_{p'}\Big(\int_0^s \Big(\frac{t}{s}\Big)^{N-1} f(t, u(t))\, \mathrm{d}t \Big)\, \mathrm{d}s.$$

Since $u \not\equiv 0$ is continuous, it follows from (H2) that $u(0) > 0$. Furthermore,

$$\phi_p(u'(r)) = -\lambda \int_0^r \Big(\frac{t}{r}\Big)^{N-1} f(t, u(t))\, \mathrm{d}t \leqslant 0, \quad r \in [0,1],$$

showing that $u'(r) \leqslant 0$ for all $r \in [0,1]$, so u is decreasing on $[0,1]$. Finally,

$$\phi_p(u'(1)) = -\lambda \int_0^1 t^{N-1} f(t, u(t))\, \mathrm{d}t < 0.$$

This implies $u'(1) < 0$, from which $u > 0$ on $[0,1)$ now follows. □

For the following results, we will use the function $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ defined by

$$g(r, \xi) := \begin{cases} f(r,\xi)/\phi_p(\xi), & \xi \neq 0, \\ f_0(r), & \xi = 0. \end{cases} \tag{4.2}$$

It follows from our assumptions that $g \in C^0([0,1] \times \mathbb{R})$ and g satisfies

$$0 \leqslant f_\infty(r) \leqslant g(r,\xi) \leqslant f_0(r), \quad (r,\xi) \in [0,1] \times \mathbb{R}. \tag{4.3}$$

Lemma 4.2. *Consider a sequence $\{(\lambda_n, u_n)\} \subset \mathcal{S}$. Suppose that $|u_n|_0 \to 0/\infty$ as $n \to \infty$. Then $\lambda_n \to \lambda_{0/\infty}$.*

Proof. Setting $v_n := u_n/|u_n|_0$, we have

$$v_n = S_p(\lambda_n g(u_n) \phi_p(v_n)) \tag{4.4}$$

or, equivalently,

$$\begin{gathered} -(r^{N-1}\phi_p(v_n'))' = \lambda_n r^{N-1} g(u_n) \phi_p(v_n), \quad 0 < r < 1, \\ v_n'(0) = v_n(1) = 0, \end{gathered} \tag{4.5}$$

where $u \mapsto g(u)$ denotes the Nemitskii mapping induced by g. Since $v_n > 0$ in $[0,1)$ for all n, it follows from (4.3), (4.5), and the Sturmian-type comparison theorem in [11, Sec. 4] that

$$0 < \lambda_0 \leqslant \lambda_n \leqslant \lambda_\infty \leqslant \infty. \tag{4.6}$$

Let us first suppose that hypothesis (H6)(a) holds. Then $\lambda_\infty < \infty$ and we can suppose that $\lambda_n \to \bar\lambda \in [\lambda_0, \lambda_\infty]$ as $n \to \infty$. Now $|v_n|_0 = 1$ for all n, so $\{\lambda_n g(u_n)\phi_p(v_n)\}$ is bounded in $C^0[0,1]$ and $\{S_p(\lambda_n g(u_n)\phi_p(v_n))\}$ is bounded in $C^1[0,1]$. Therefore, by (4.4), we can suppose that $|v_n - \bar v|_0 \to 0$ as $n \to \infty$, for some $\bar v \in C^0[0,1]$. It then follows by fairly standard arguments (see e.g. the proof of [6, Lemma 5.4]) that

$$g(u_n)\phi_p(v_n) \to f_{0/\infty}\phi_p(\bar v) \quad \text{provided } |u_n|_0 \to 0/\infty.$$

Hence $\bar v$ satisfies $\bar v = S_p(\bar\lambda f_{0/\infty}\phi_p(\bar v))$ if $|u_n|_0 \to 0/\infty$; that is,

$$\begin{gathered} -(r^{N-1}\phi_p(\bar v'))' = \bar\lambda r^{N-1} f_{0/\infty}\phi_p(\bar v), \quad 0 < r < 1, \\ \bar v'(0) = \bar v(1) = 0. \end{gathered} \tag{4.7}$$

Now the proof of Proposition 4.1 shows that $\bar{v} > 0$ in $[0, 1)$ and it follows from the properties of the eigenvalue problem (4.7) (see [11, Sec. 5]) that $\bar{\lambda} = \lambda_{0/\infty}$.

We next suppose that hypothesis (H6)(b) holds; i.e., $N = 1$ and $f_\infty \equiv 0$. We first prove that $\lambda_n \to \infty$ if $|u_n|_0 \to \infty$. Indeed, if we suppose instead that $\{\lambda_n\}$ is bounded, then the above argument yields a $\bar{v} \in C^0[0, 1]$ such that $v_n \to \bar{v}$ in $C^0[0, 1]$ (up to a subsequence), and it follows that $g(u_n)\phi_p(v_n) \to 0$ in $C^0[0, 1]$. Then (4.4) implies $\bar{v} = 0$, contradicting $|\bar{v}|_0 = 1$.

Regarding the behaviour as $|u_n|_0 \to 0$, the argument for the case $f_\infty > 0$ will hold in exactly the same way for $f_\infty \equiv 0$ if we can show that $\{\lambda_n\}$ is bounded. It follows from (4.4) that

$$1 = |v_n|_0 = v_n(0) = \lambda_n^{p^*} \int_0^1 \phi_{p'}\Big(\int_0^s g(u_n)\phi_p(v_n)\,\mathrm{d}t\Big)\,\mathrm{d}s.$$

Since u_n, v_n are decreasing on $[0, 1]$ and, for any fixed $t \in [0, 1]$, the mapping $\xi \mapsto f(t, \xi)/\xi^{p-1}$ is decreasing on $(0, \infty)$, we have

$$\begin{aligned}
\lambda_n^{-p^*} &= \int_0^1 \phi_{p'}\Big(\int_0^s g(u_n)\phi_p(v_n)\,\mathrm{d}t\Big)\,\mathrm{d}s \\
&\geqslant \int_0^{1/2} \phi_{p'}\Big(\int_0^s \phi_p(v_n(1/2))\frac{f(t, u_n(0))}{u_n(0)^{p-1}}\,\mathrm{d}t\Big)\,\mathrm{d}s \\
&\geqslant C v_n(1/2) \min_{0\leqslant t\leqslant \frac{1}{2}} \Big(\frac{f(t, u_n(0))}{u_n(0)^{p-1}}\Big)^{p^*}.
\end{aligned}$$

Since $N = 1$, it follows from (1.2) that u is concave for all $(\lambda, u) \in \mathcal{S}$. Hence, there is a constant $M > 0$ (independent of n) such that $v_n(1/2) \geqslant M|v_n|_0 = M$ for all n. Furthermore, $f(t, u_n(0))/u_n(0)^{p-1} \to f_0(t) > 0$ uniformly for $t \in [0, \frac{1}{2}]$ and so there exists $\delta > 0$ such that $\lambda_n^{-p^*} \geqslant \delta$ for n large enough. Therefore, $\{\lambda_n\}$ is bounded.

Note that the arguments above only show that $\lambda_{n_k} \to \lambda_{0/\infty}$ for a subsequence $\{\lambda_{n_k}\}$. Since they can be applied to any subsequence of $\{\lambda_n\}$, it follows that the whole sequence must converge. This concludes the proof. □

Remark 4.3. The proof of (4.6) shows that $\lambda_0 \leqslant \lambda \leqslant \lambda_\infty$ for all $(\lambda, u) \in \mathcal{S}$.

4.2. **The case where (H') holds.** In this case we consider solutions in the sets

$$\mathcal{S}^\pm := \{(\lambda, u) \in (0, \infty) \times Y : (\lambda, u) \text{ is a solution of (4.1) with } u \not\equiv 0, \text{ and } \pm u \geqslant 0\}.$$

The following result can be proved as Proposition 4.1, using (H2') instead of (H2).

Proposition 4.4. *Let $(\lambda, u) \in \mathcal{S}^\pm$. Then $\pm u > 0$ on $[0, 1)$, $\pm u$ is decreasing and satisfies $\pm u'(1) < 0$.*

Regarding the asymptotic behaviour, we have

Lemma 4.5. *Consider a sequence $\{(\lambda_n, u_n)\} \subset \mathcal{S}^\pm$. Suppose that $|u_n|_0 \to 0/\infty$ as $n \to \infty$. Then $\lambda_n \to \lambda_{0/\infty}$.*

Proof. The proof is the same as for Lemma 4.2 in the case where $\{(\lambda_n, u_n)\} \subset \mathcal{S}^+$. In case $u_n \leqslant 0$, as similar proof can be carried out, setting $v_n := -u_n/|u_n|_0 \geqslant 0$ and remarking that, with this new definition, v_n still satisfies (4.4). □

Remark 4.6. We also have $\lambda_0 \leqslant \lambda \leqslant \lambda_\infty$ for all $(\lambda, u) \in \mathcal{S}^\pm$.

5. Local bifurcation

To prove Theorems 2.3 and 2.4, we begin with a local bifurcation result in the spirit of Crandall and Rabinowitz [3]. This will allow us to start off the bifurcating branch from the line of trivial solutions at the point $(\lambda_0, 0)$ in $\mathbb{R} \times Y$. Crandall and Rabinowitz' original result pertained to semilinear equations, i.e. $p = 2$. A first generalization to $p > 2$ was given in [7] for a problem very similar to (1.2). The main difference in our setting is that we allow the asymptote f_0 to depend on r, so that we get the weighted eigenvalue problem (2.3) with f_0 instead of problem (1.3).

In the following, we assume the principal eigenfunction v_0 given in Lemma 2.2 is normalized so that

$$\int_0^1 r^{N-1} f_0(r)|v_0(r)|^p \, \mathrm{d}r = 1.$$

We define the subspace

$$Z := \{z \in Y : \int_0^1 r^{N-1} f_0 |v_0|^{p-2} v_0 z \, \mathrm{d}r = 0\}$$

and we remark that

$$Y = \operatorname{span}\{v_0\} \oplus W. \tag{5.1}$$

To be able to discuss later cases (H) and (H'), it will be convenient to state our local bifurcation result more generally, in terms of the function $G : \mathbb{R}^2 \times Z \to Y$ defined by

$$G(s, \lambda, z) := \begin{cases} v_0 + z - S_p(\lambda f(sv_0 + sz)/\phi_p(s)), & s \neq 0, \\ v_0 + z - S_p(\lambda f_0 \phi_p(v_0 + z)), & s = 0. \end{cases}$$

Note that $G(s, \lambda, z) = F(\lambda, s(v_0 + z))/s$ for all $s \neq 0$, where $F : \mathbb{R} \times Y \to Y$ was defined in (4.1). It follows from the definitions of λ_0 and v_0 that $G(0, \lambda_0, 0) = 0$.

Lemma 5.1. *Let $p > 2$ and suppose that (H1) and (H4) hold. There exist $\varepsilon > 0$, a neighbourhood U of $(\lambda_0, 0)$ in $\mathbb{R} \times Z$ and a continuous mapping $s \mapsto (\lambda(s), z(s)) : (-\varepsilon, \varepsilon) \to U$ such that $(\lambda(0), z(0)) = (\lambda_0, 0)$ and*

$$\{(s, \lambda, z) \in (-\varepsilon, \varepsilon) \times U : G(s, \lambda, z) = 0\} = \{(s, \lambda(s), z(s)) : s \in (-\varepsilon, \varepsilon)\}.$$

Proof. Our proof follows that of [7, Theorem 1] but we give it here for completeness. Under hypotheses (H1) and (H4), it is easily seen that G is continuous. It follows from Theorem 3.5(ii) that S_p is C^1 in a neighbourhood of $\lambda_0 f_0 \phi_p(v_0)$ in Y. A routine verification then shows that there is a neighbourhood U of $(0, \lambda_0, 0)$ in $\mathbb{R}^2 \times Z$ such that the mapping $(\lambda, z) \mapsto G(s, \lambda, z)$ is differentiable in

$$A_s := \{(\lambda, z) \in \mathbb{R} \times Z : (s, \lambda, z) \in U\},$$

for any $s \in \mathbb{R}$ such that $A_s \neq \emptyset$. Furthermore, the Fréchet derivative $D_{(\lambda,z)}G$ is continuous on U. Since

$$v_0 = S_p(\lambda_0 f_0 \phi_p(v_0)), \tag{5.2}$$

it follows from (3.7) that

$$\begin{aligned} &D_{(\lambda,z)}G(0, \lambda_0, 0)(\bar{\lambda}, \bar{z}) \\ &= \bar{z} - \lambda_0 (p^*)^{-1} DS_p(\lambda_0 f_0 \phi_p(v_0)) f_0 |v_0|^{p-2} \bar{z} - p^*(\bar{\lambda}/\lambda_0) v_0, \quad (\bar{\lambda}, \bar{z}) \in \mathbb{R} \times Z. \end{aligned} \tag{5.3}$$

To conclude the proof using the implicit function theorem as stated in Appendix A of [3], we need only check that $D_{(\lambda,z)}G(0, \lambda_0, 0) : \mathbb{R} \times Z \to Y$ is an isomorphism.

Let us first show that the mapping

$$L\bar{z} := \lambda_0 (p^*)^{-1} DS_p(\lambda_0 f_0 \phi_p(v_0)) f_0 |v_0|^{p-2} \bar{z}$$

leaves the subspace Z invariant. Suppose $\bar{z} \in Z$ and let $z = L\bar{z}$. By (3.8) and (5.2), we have

$$\begin{gathered} -(r^{N-1}|v_0'|^{p-2} z')' = \lambda_0 r^{N-1} f_0 |v_0|^{p-2} \bar{z}, \quad 0 < r < 1, \\ z'(0) = z(1) = 0. \end{gathered} \tag{5.4}$$

Multiplying both sides of the equation by v_0 and integrating by parts twice yields

$$\int_0^1 r^{N-1} f_0 |v_0|^{p-2} v_0 z \,\mathrm{d}r = \int_0^1 r^{N-1} f_0 |v_0|^{p-2} v_0 \bar{z} \,\mathrm{d}r = 0,$$

showing that $z \in Z$. In view of the decomposition (5.1),

$$D_{(\lambda,z)} G(0, \lambda_0, 0)(\bar{\lambda}, \bar{z}) = 0 \implies \bar{\lambda} = 0 \text{ and } \bar{z} = L\bar{z}.$$

Then $\bar{z}$ is a solution of (5.4) and an argument similar to the proof of [7, Theorem 7] shows that there exists $c \in \mathbb{R}$ such that $\bar{z} = cv_0$. Since $\bar{z} \in Z$, it follows that $c = 0$, showing that the null space $N(D_{(\lambda,z)} G(0, \lambda_0, 0)) = \{0\}$.

Finally, from (5.1) and the invariance of Z under L, $D_{(\lambda,z)} G(0, \lambda_0, 0)$ is isomorphically equivalent to the operator $T : \mathbb{R} \times Z \to \mathbb{R} \times Z$ defined by

$$T(\bar{\lambda}, \bar{z}) = (\bar{\lambda}, \bar{z}) - ((1 + p^*/\lambda_0)\bar{\lambda}, L\bar{z}).$$

It follows from Theorem 3.3 that T is a compact perturbation of the identity on $\mathbb{R} \times Z$. Therefore, the triviality of $N(D_{(\lambda,z)} G(0, \lambda_0, 0))$ implies that $D_{(\lambda,z)} G(0, \lambda_0, 0)$ is an isomorphism, finishing the proof. □

Remark 5.2. As noted earlier, the above result was presented in [7] in the case where $f_0 \equiv 1$. However, the proof relies heavily on the differentiability properties of the integral operator S_p, given by Theorem 3.5, and the arguments establishing these properties in [7] seem incomplete. Hence, in addition to the slightly more general context considered here, our work completes the proof of [7, Theorem 1].

Remark 5.3. Since the differentiability results in Theorem 3.5 cover the whole range $p > 1$, we first had some hope to obtain bifurcation for all $p > 1$. It turns out that the integration by parts arguments involved in the proof of Lemma 5.1 require at least $p \geqslant 1 + 1/N$ (for the boundary terms to vanish). Moreover, the differentiability of the function G in the present functional setting requires $p \geqslant 2$, and we have not been able to find another suitable setting allowing for $p < 2$.

We can now state the local bifurcation results for equation (4.1).

Theorem 5.4. *Let $p > 2$ and suppose that* (H) *holds. There exist $\varepsilon_0 \in (0, \varepsilon)$ and a neighbourhood U_0 of $(\lambda_0, 0)$ in $\mathbb{R} \times Y$ such that*

$$\{(\lambda, u) \in U_0 : F(\lambda, u) = 0\} = \{(\lambda(s), s(v_0 + z(s))) : s \in [0, \varepsilon_0)\}. \tag{5.5}$$

Proof. It follows from Lemma 5.1 that $(\lambda(s), s(v_0 + z(s)))$ is a solution of (4.1) for all $s \in [0, \varepsilon)$. To prove the reverse inclusion in (5.5), let us first remark that, by Proposition 4.1, $s \in (-\varepsilon, 0)$ yields no solutions of (4.1). Furthermore, a compactness argument similar to that in [7, p. 39] shows that any solution in a small enough neighbourhood of $(\lambda_0, 0)$ in $\mathbb{R} \times Y$ must have the form $(\lambda(s), s(v_0 + z(s)))$ for some $s \in [0, \varepsilon)$. This completes the proof. □

Theorem 5.5. *Let $p > 2$ and suppose that* (H') *holds. There exist $\varepsilon_1 \in (0, \varepsilon)$ and a neighbourhood U_1 of $(\lambda_0, 0)$ in $\mathbb{R} \times Y$ such that*

$$\{(\lambda, u) \in U_1 : F(\lambda, u) = 0\} = \{(\lambda(s), s(v_0 + z(s))) : s \in (-\varepsilon_1, \varepsilon_1)\}, \tag{5.6}$$

with

$$\{(\lambda(s), s(v_0 + z(s))) : s \in (-\varepsilon_1, 0)\} \subset \mathcal{S}^- \tag{5.7}$$

and

$$\{(\lambda(s), s(v_0 + z(s))) : s \in (0, \varepsilon_1)\} \subset \mathcal{S}^+. \tag{5.8}$$

Proof. The local characterization of solutions in (5.6) follows similarly to (5.5) in Theorem 5.4. For $\varepsilon_1 > 0$ small enough, statements (5.7) and (5.8) follow from the construction of the solutions $(\lambda(s), s(v_0 + z(s)))$. □

6. Global continuation

Our goal in this final section is to complete the proofs of Theorems 2.3 and 2.4. Namely, we will first show that the local curves of solutions obtained in Section 5 can be parametrized by λ and then we will prove that they can be extended globally.

6.1. **Proof of Theorem 2.3.** We begin with a non-degeneracy result implying that, in fact, through any non-trivial solution of (4.1), there passes a (local) continuous curve of solutions, parametrized by λ.

Lemma 6.1. *The function $F \in C([0, \infty) \times Y, Y)$ defined in* (4.1) *is continuously differentiable in a neighbourhood of any point $(\lambda, u) \in \mathcal{S}$, with*

$$D_u F(\lambda, u)v = v - \lambda D S_p(\lambda f(u)) \partial_2 f(u) v, \quad v \in Y.$$

Furthermore, for any $(\lambda, u) \in \mathcal{S}$, $D_u F(\lambda, u) : Y \to Y$ is an isomorphism.

Proof. The statement about the differentiability of F follows from Theorem 3.5(ii) and Proposition 4.1. Furthermore, we see that $D_u F(\lambda, u) : Y \to Y$ is a compact perturbation of the identity. Therefore, to show that it is an isomorphism, we only need to prove that $N(D_u F(\lambda, u)) = \{0\}$. Let $v \in N(D_u F(\lambda, u))$. By (3.8), we have

$$\begin{gathered} -(r^{N-1}|u'|^{p-2}v')' = p^* \lambda r^{N-1} \partial_2 f(u) v, \quad 0 < r < 1, \\ v'(0) = v(1) = 0. \end{gathered} \tag{6.1}$$

Multiplying the equation in (6.1) by u, that in (1.2) by v, subtracting and integrating by parts yield

$$r^{N-1}|u'|^{p-2}(uv' - u'v)(r) = \lambda \int_0^r s^{N-1}[f(u) - p^* \partial_2 f(u) u] v \, \mathrm{d}s, \quad r \in [0, 1]. \tag{6.2}$$

Suppose that $v \not\equiv 0$, and let $r_1 > 0$ be the smallest positive zero of v. Without loss of generality, we can suppose $v > 0$ on $(0, r_1)$. If $r_1 < 1$, we have $u(r_1)v'(r_1) < 0$. However by (H3),

$$r_1^{N-1}|u'(r_1)|^{p-2}u(r_1)v'(r_1) = \lambda \int_0^{r_1} s^{N-1}[f(u) - p^* \partial_2 f(u) u] v \, \mathrm{d}s \geqslant 0,$$

a contradiction. If $r_1 = 1$, (H3) and Proposition 4.1 imply

$$0 = |u'(1)|^{p-2}u(1)v'(1) = \lambda \int_0^1 s^{N-1}[f(u) - p^* \partial_2 f(u) u] v \, \mathrm{d}s > 0,$$

again a contradiction. Hence, $v \not\equiv 0$ is impossible and so $N(D_u F(\lambda, u)) = \{0\}$. □

By Remark 4.3, Theorem 5.4 and Lemma 6.1, the implicit function theorem yields a maximal open interval $(\lambda_0, \widetilde{\lambda})$ with $\lambda_0 < \widetilde{\lambda} \leqslant \lambda_\infty$ and a mapping $u \in C^1((\lambda_0, \widetilde{\lambda}), Y)$ such that $(\lambda, u(\lambda)) \in \mathcal{S}$ for all $\lambda \in (\lambda_0, \widetilde{\lambda})$, and $\lim_{\lambda\to\lambda_0} u(\lambda) = 0$. Let us show that $\widetilde{\lambda} = \lambda_\infty$. Suppose by contradiction that $\lambda_0 < \widetilde{\lambda} < \lambda_\infty \leqslant \infty$, and consider a sequence $\lambda_n \to \widetilde{\lambda}$. If $|u(\lambda_n)|_0$ is unbounded, it follows by Lemma 4.2 that $\widetilde{\lambda} = \lambda_\infty$ and we are done. On the other hand, if $|u(\lambda_n)|_0$ is bounded, a compactness argument similar to that yielding the convergence of $\{v_n\}$ in the proof of Lemma 4.2 shows that there exists $\widetilde{u} \in Y$ such that $u(\lambda_n) \to \widetilde{u}$ (up to a subsequence), and

$$\widetilde{u} = S_p(\widetilde{\lambda} f(\widetilde{u})).$$

Note that, by Lemma 4.2, we cannot have $u \equiv 0$. Hence, $(\widetilde{\lambda}, \widetilde{u}) \in \mathcal{S}$, $F(\widetilde{\lambda}, \widetilde{u}) = 0$, and so by Lemma 6.1 and the implicit function theorem, we can extend the curve $u(\lambda)$ through the point $(\widetilde{\lambda}, \widetilde{u})$, contradicting the maximality of $\widetilde{\lambda}$. Therefore, $\widetilde{\lambda} = \lambda_\infty$, and we have a solution curve

$$\mathcal{S}_0 := \{(\lambda, u(\lambda)) : \lambda \in (\lambda_0, \lambda_\infty)\} \subset \mathcal{S}.$$

We next prove that $\lim_{\lambda\to\lambda_\infty} |u(\lambda)|_0 = \infty$. In the case where (H6)(a) holds, this readily follows by the above argument for if $|u(\lambda)|_0$ were bounded as $\lambda \to \lambda_\infty < \infty$, we could continue the solution curve beyond $\lambda = \lambda_\infty$. In case (H6)(b) holds, the result follows from

Lemma 6.2. *Suppose that* (H6)(b) *holds, and consider* $(\lambda_n, u_n) \in \mathcal{S}$ *with* $\lambda_n \to \infty$. *Then* $|u_n|_0 \to \infty$.

Proof. By contradiction, suppose there exists a constant $R > 0$ and a subsequence, still denoted by (λ_n, u_n), such that $|u_n|_0 \leqslant R$ for all n. Then, by (H2) and (H3),

$$\begin{aligned} |u_n|_0 = u_n(0) &= \int_0^1 \phi_{p'}\Big(\int_0^s \lambda_n f(t, u_n)\,\mathrm{d}t\Big)\,\mathrm{d}s \\ &\geqslant \lambda_n^{p^*} \int_0^{1/2} \phi_{p'}\Big(\int_0^s g(t, u_n) u_n(t)^{p-1}\,\mathrm{d}t\Big)\,\mathrm{d}s \\ &\geqslant \lambda_n^{p^*} u_n(1/2) \int_0^{1/2} \phi_{p'}\Big(\int_0^s g(t, R)\,\mathrm{d}t\Big)\,\mathrm{d}s \\ &\geqslant \lambda_n^{p^*} C |u_n|_0, \end{aligned}$$

where the last inequality follows from the concavity of the solutions u_n on $[0, 1]$. Hence $\lambda_n^{p^*} \leqslant C^{-1} < \infty$, a contradiction. □

We still need to prove the uniqueness statement of Theorem 2.3, that is, $\mathcal{S}_0 = \mathcal{S}$. Suppose instead that there exists $(\bar{\lambda}, \bar{u}) \in \mathcal{S} \setminus \mathcal{S}_0$, and let $\mathcal{S}_1$ be the connected subset of $\mathcal{S}$ such that $(\bar{\lambda}, \bar{u}) \in \mathcal{S}_1$. It follows by Lemma 6.1 that $\mathcal{S}_1$ is a smooth curve, parametrized by λ in a maximal interval I_1. In fact, the previous arguments imply that $I_1 = (\lambda_0, \lambda_\infty)$. Let us denote by $u_1 : (\lambda_0, \lambda_\infty) \to Y$ the parametrization of $\mathcal{S}_1$ and consider a sequence $\lambda_n \to \lambda_0$. Since $|u_1(\lambda_n)|_0$ is bounded by Lemma 4.2, it follows that there exists $u_0 \in Y$ such that $u(\lambda_n) \to u_0$ in Y as $n \to \infty$. Then by continuity, we have

$$u_0 = S_p(\lambda_0 f(u_0)).$$

Since $u_1(\lambda_n) \geqslant 0$ for all n, it follows that $u_0 \geqslant 0$. We will show that, in fact, $u_0 \equiv 0$. Hence we will have $(\lambda_n, u_1(\lambda_n)) \to (\lambda_0, 0)$ in Y and, by the characterization (5.5) in

Theorem 5.4, $\mathcal{S}_1 = \mathcal{S}_0$. If $u_0 \not\equiv 0$, we set $w_0 = u_0/|u_0|_0$. Then $w_0 \geqslant 0$ and satisfies

$$w_0 = S_p(\lambda_0 g(u_0)\phi_p(w_0)). \tag{6.3}$$

Having in mind (H3) and (4.3), it follows from the comparison theorem of [11, Sec. 4] applied to (6.3) and (2.3) with f_0 that we must have $w_0 \equiv 0$. This contradiction finishes the proof of Theorem 2.3. □

6.2. **Proof of Theorem 2.4.** We start with the analogue of Lemma 6.1 under hypothesis (H').

Lemma 6.3. *The function $F \in C([0,\infty) \times Y, Y)$ defined in* (4.1) *is continuously differentiable in a neighbourhood of any point $(\lambda, u) \in \mathcal{S}^{\pm}$, with*

$$D_uF(\lambda,u)v = v - \lambda DS_p(\lambda f(u))\partial_2 f(u)v, \quad v \in Y.$$

Furthermore, for any $(\lambda, u) \in \mathcal{S}^{\pm}$, $D_uF(\lambda,u) : Y \to Y$ is an isomorphism.

Proof. The proof is almost identical to that of Lemma 6.1, so we only indicate the minor modifications. The differentiability part follows as in Lemma 6.1, using Theorem 3.5(ii), and Proposition 4.4 instead of Proposition 4.1. The non-singularity of $D_uF(\lambda,u) : Y \to Y$ follows in the same way if $(\lambda, u) \in \mathcal{S}^+$. For $(\lambda,u) \in \mathcal{S}^-$, we proceed in a similar manner, considering $v \in N(D_uF(\lambda,u))$. Then the identity (6.2) still holds, and we suppose by contradiction that $v > 0$ on a maximal interval $(0, r_2)$, with $v(r_2) = 0$ and $v'(r_2) < 0$. If $r_2 < 1$, we have $u(r_2)v'(r_2) > 0$ while (H3') implies

$$r_2^{N-1}|u'(r_2)|^{p-2}u(r_2)v'(r_2) = \lambda \int_0^{r_2} s^{N-1}[f(u) - p^*\partial_2 f(u)u]v \,\mathrm{d}s \leqslant 0,$$

a contradiction. On the other hand, if $r_2 = 1$, it follows from (H3') and Proposition 4.4 that

$$0 = |u'(1)|^{p-2}u(1)v'(1) = \lambda \int_0^1 s^{N-1}[f(u) - p^*\partial_2 f(u)u]v \,\mathrm{d}s < 0,$$

another contradiction. Hence $v \equiv 0$ and $N(D_uF(\lambda,u)) = \{0\}$. □

Now, similarly to the proof of Theorem 2.3, using Remark 4.6, Theorem 5.5 and Lemma 6.3, the implicit function theorem yields maximal open intervals $(\lambda_0, \widetilde{\lambda}_{\pm})$ with $\lambda_0 < \widetilde{\lambda}_{\pm} \leqslant \lambda_\infty$ and two solution curves $u_{\pm} \in C^1((\lambda_0, \widetilde{\lambda}_{\pm}), Y)$. It follows as in the proof of Theorem 2.3 that $\widetilde{\lambda}_{\pm} = \lambda_\infty$, so we get two global solution curves

$$\mathcal{S}_0^{\pm} = \{(\lambda, u_{\pm}(\lambda)) : \lambda \in (\lambda_0, \lambda_\infty)\} \subset \mathcal{S}.$$

Furthermore, $\lim_{\lambda\to\lambda_\infty} |u(\lambda)|_0 = \infty$ follows as in the proof of Theorem 2.3, using the version of Lemma 6.2 holding under hypothesis (H'):

Lemma 6.4. *Suppose that* (H6)(b) *holds, and consider $(\lambda_n, u_n) \in \mathcal{S}^{\pm}$ such that $\lambda_n \to \infty$. Then $|u_n|_0 \to \infty$.*

Proof. The proof is the same as that of Lemma 6.2 when $(\lambda_n, u_n) \in \mathcal{S}^+$.

For $(\lambda_n, u_n) \in \mathcal{S}^-$, if $|u_n|_0 \leqslant R$, it follows by (H2') and (H3') that

$$\begin{aligned}|u_n|_0 = -u_n(0) &= -\int_0^1 \phi_{p'}\left(\int_0^s \lambda_n f(t,u_n)\,\mathrm{d}t\right)\mathrm{d}s\\ &\geqslant \lambda_n^{p^*}\int_0^{1/2} \phi_{p'}\Big(\int_0^s g(t,u_n)|u_n(t)|^{p-1}\,\mathrm{d}t\Big)\,\mathrm{d}s\end{aligned}$$

$$\geqslant \lambda_n^{p^*} |u_n(1/2)| \int_0^{1/2} \phi_{p'}\Big(\int_0^s g(t,R)\,\mathrm{d}t \Big)\,\mathrm{d}s$$
$$\geqslant \lambda_n^{p^*} C |u_n|_0,$$

showing that the sequence $\{\lambda_n\}$ must be bounded, a contradiction. □

Using the characterization (5.6) in Theorem 5.5, it follows similarly to the last part of the proof of Theorem 2.3 that $\mathcal{S} = \mathcal{S}_0^- \cup \mathcal{S}_0^+$. Hence, to finish the proof of Theorem 2.4, we only need to prove that $\mathcal{S}_0^- \subset \mathcal{S}^-$ and $\mathcal{S}_0^+ \subset \mathcal{S}^+$.

Let $\mathcal{C}_0^\pm := \mathcal{S}_0^\pm \cap \mathcal{S}^\pm$. First, we know from Theorem 5.5 that, for λ close to λ_0, $(\lambda, u_\pm(\lambda)) \in \mathcal{S}^\pm$, so that $\mathcal{C}_0^\pm \neq \emptyset$. Thus, we need only show that $\mathcal{C}_0^\pm$ is both open and closed in $\mathcal{S}_0^\pm$, for the product topology inherited from $\mathbb{R} \times Y$. We will only consider $\mathcal{C}_0^+$, the proof for $\mathcal{C}_0^-$ is similar.

For $(\lambda, u) \in \mathcal{C}_0^+$, it follows from Proposition 4.4 that $u > 0$ on $[0, 1)$ (with $u(1) = 0$). If $(\mu, v) \in \mathcal{S}_0^+$ with $|\mu - \lambda| + |v - u|_0$ small enough, we will have $\lambda \in (\lambda_0, \lambda_\infty)$ and $0 \not\equiv v \geqslant 0$ on $[0, 1]$, hence $(\mu, v) \in \mathcal{C}_0^+$. This proves that $\mathcal{C}_0^+$ is open in $\mathcal{S}_0^+$.

Now consider a sequence $\{(\lambda_n, u_n)\} \subset \mathcal{C}_0^+$ and suppose there exists $(\lambda, u) \in \mathcal{S}_0^+$ such that $(\lambda_n, u_n) \to (\lambda, u)$. By continuity, $F(\lambda, u) = 0$, and we have $u \geqslant 0$. Since $\lambda > \lambda_0$, it follows from Lemma 4.5 that $u \not\equiv 0$. Hence, $(\lambda, u) \in \mathcal{S}_0^+$ and $\mathcal{C}_0^+$ is closed in $\mathcal{S}_0^+$. Thus, $\mathcal{C}_0^+ = \mathcal{S}_0^+$, and it follows in a similar way that $\mathcal{C}_0^- = \mathcal{S}_0^-$, showing that $\mathcal{S}_0^\pm \subset \mathcal{S}^\pm$, and so actually $\mathcal{S}_0^\pm = \mathcal{S}^\pm$. This completes the proof of Theorem 2.4. □

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References

[1] P. A. Binding, B. P. Rynne; The spectrum of the periodic p-Laplacian, *J. Differential Equations* **235** (2007), no. 1, 199–218.

[2] F. Brock; Radial symmetry of nonnegative solutions of semilinear elliptic equations involving the p-Laplacian, in H. Amann, C. Bandle, M. Chipot, F. Conrad, I. Shafrir (Eds.), Progress in Partial Differential Equations, Pont-à-Mousson (1), Pitman Res. Notes Math. Ser., vol. 383, 1997.

[3] M. G. Crandall, P. H. Rabinowitz; Bifurcation from simple eigenvalues, *J. Functional Analysis* **8** (1971), 321–340.

[4] E. N. Dancer; On the structure of solutions of non-linear eigenvalue problems, *Indiana Univ. Math. J.* **23** (1973/74), 1069–1076.

[5] M. del Pino, R. F. Manàsevich; Global bifurcation from the eigenvalues of the p-Laplacian, *J. Differential Equations* **92** (1991), no. 2, 226–251.

[6] N. Dodds, B. P. Rynne; Spectral properties and nodal solutions for second-order, m-point, p-Laplacian boundary value problems, *Topol. Methods Nonlinear Anal.* **32** (2008) 21–40.

[7] J. García-Melián, J. Sabina de Lis; A local bifurcation theorem for degenerate elliptic equations with radial symmetry, *J. Differential Equations* **179** (2002), no. 1, 27–43.

[8] P. Girg, P. Takáč; Bifurcations of positive and negative continua in quasilinear elliptic eigenvalue problems, *Ann. Henri Poincaré* **9** (2008), no. 2, 275–327.

[9] P. H. Rabinowitz; Some global results for nonlinear eigenvalue problems, *J. Functional Analysis* **7** (1971), 487–513.

[10] B. P. Rynne; A global curve of stable, positive solutions for a p-Laplacian problem, *Electron. J. Differential Equations* **2010**, no. 58.

[11] W. Walter; Sturm-Liouville theory for the radial Δ_p-operator, *Math. Z.* **227** (1998), no. 1, 175–185.

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