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EXISTENCE OF SOLUTIONS FOR ELLIPTIC SYSTEMS IN $\mathbb{R}^N$ INVOLVING THE $p(x)$-LAPLACIAN

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Abstract. This article presents sufficient conditions for the existence of nontrivial solutions for a nonlinear elliptic system. To establish this result, we use a classical existence theorem in reflexive Banach spaces, under some growth conditions on the non-linearities.

1. Introduction

In this article we establish the existence of nontrivial weak solution for nonlinear elliptic system

$$\begin{aligned} -\Delta_{p(x)}u &= \frac{\partial F}{\partial u}(x,u,v) \quad \text{in } \mathbb{R}^N \\ -\Delta_{q(x)}v &= \frac{\partial F}{\partial v}(x,u,v) \quad \text{in } \mathbb{R}^N \end{aligned} \tag{1.1}$$

Here $p(x)$ and $q(x)$ are continuous real-valued functions such that $1 < p(x), q(x) < N$ $(N \geq 2)$ for all $x \in \mathbb{R}^N$. The real-valued function F belongs to $C^1(\mathbb{R}^N \times \mathbb{R}^2)$, and $\Delta_{p(x)}$ is the so-called $p(x)$-Laplacian operator; i.e., $\Delta_{p(x)}u = \operatorname{div}(|\nabla u|^{p(x)}\nabla u)$.

This decade bears witness to a considerable sum of results on non standard growth conditions problems. This abundance is due to the recent research developments in elasticity problems, electrorheological fluids, image processing, flow in porous media, etc.; see for example [2, 12].

In a natural way, the introduction of the generalized Lebesgue-Sobolev spaces turned out to be crucial [3, 5, 8]. In this way, many authors could successfully deal with $p(x)$-Laplacian problems [7, 8]. Many additional works concern elliptic systems in relationship to standard and nonstandard growth conditions. We refer the readers to [1, 10, 15] and the references therein. In [4, 14], the authors show the existence of nontrivial solutions for the (p, q)-Laplacian system

$$\begin{aligned} -\Delta_p u &= \frac{\partial F}{\partial u}(x,u,v) \quad \text{in } \mathbb{R}^N \\ -\Delta_q v &= \frac{\partial F}{\partial v}(x,u,v) \quad \text{in } \mathbb{R}^N \end{aligned}$$

where the potential function F satisfies mixed and subcritical growth conditions and, in addition, to be intimately connected with the first eigenvalue of p-Laplacian

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operator. They apply the Mountain Pass theorem to obtain the nontrivial solutions of the system.

In [6], the authors obtained the existence and multiplicity of solutions for the vector valued elliptic system

$$
\begin{gathered}
-\Delta_{p(x)} u=\frac{\partial F}{\partial u}(x, u, v) \quad \text{in } \Omega \\
-\Delta_{p(x)} v=\frac{\partial F}{\partial v}(x, u, v) \quad \text{in } \Omega \\
u=v=0 \quad \text{on } \partial \Omega,
\end{gathered}
$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with a smooth boundary $\partial\Omega$, $N \geq 2$, $(p,q) \in [C(\overline{\Omega})]^2$, $F \in C(\mathbb{R}^N \times \mathbb{R}^2, \mathbb{R})$. Existence and multiplicity results are subjected to some natural growth conditions which guarantee the Mountain Pass geometry and Palais-Smale condition.

In [16], the authors studied the system

$$
\begin{gathered}
-\Delta_{p(x)} u+|u|^{p(x)-2} u=\frac{\partial F}{\partial u}(x, u, v) \quad \text{in } \mathbb{R}^N \\
-\Delta_{p(x)} v+|v|^{q(x)-2} v=\frac{\partial F}{\partial v}(x, u, v) \quad \text{in } \mathbb{R}^N
\end{gathered}
$$

The potential function F needs to satisfy Caratheodory conditions. Using critical point theory, they establish existence results in sub-linear and super-linear cases.

In [12], by the Mountain Pass theorem, the authors show the existence of nontrivial solutions for the following $(p(x), q(x))$-Laplacian system

$$
\begin{gathered}
-\Delta_{p(x)} u=\frac{\partial F}{\partial u}(x, u, v) \quad \text{in } \mathbb{R}^N \\
-\Delta_{q(x)} v=\frac{\partial F}{\partial v}(x, u, v) \quad \text{in } \mathbb{R}^N
\end{gathered}
$$

where $F \in C^1(\mathbb{R}^N \times \mathbb{R}^2, \mathbb{R})$ verifies some mixed growth conditions.

With regard to existence results, we use critical point theory. Our main goal is to establish that the energy functional of the system is lower semi- continuous and coercive in reflexive Banach space.

2. Notation and hypotheses

To discuss system (1.1), we recall some results on generalized Lebesgue-Sobolev spaces.

Let $E(\Omega)$ be a space of functions defined on Ω. We set

$$
E_+(\Omega)=\{h \in E(\Omega): \inf_{x \in \Omega} h(x)>1\}.
$$

So, for all $h \in C_+(\mathbb{R}^N)$, we set

$$
h^- := \inf_{x \in \mathbb{R}^N} h(x), \quad h^+ := \sup_{x \in \mathbb{R}^N} h(x).
$$

Let $M(\mathbb{R}^N)$ be the set of all measurable real-valued functions defined on $\mathbb{R}^N$. For $p \in C_+(\mathbb{R}^N)$, we designates the variable exponent Lebesgue space by

$$
L^{p(x)}(\mathbb{R}^N)=\{u \in M(\mathbb{R}^N): \int_{\mathbb{R}^N}|u(x)|^{p(x)} dx<\infty\},
$$

equipped with the so called Luxemburg norm

$$|u|_{p(x)} := |u|_{L^{p(x)}(\mathbb{R}^N)} = \inf\{\lambda > 0 : \int_{\mathbb{R}^N} |\frac{u(x)}{\lambda}|^{p(x)} dx \le 1\}.$$

This is a Banach space. Define the Lebesgue-Sobolev space $W^{1,p(x)}(\mathbb{R}^N)$ by

$$W^{1,p(x)}(\mathbb{R}^N) = \{u \in L^{p(x)}(\mathbb{R}^N) : |\nabla u| \in L^{p(x)}(\mathbb{R}^N)\},$$

equipped with the norm

$$\|u\|_{1,p(x)} = \|u\|_{W^{1,p(x)}(\mathbb{R}^N)} = |u|_{p(x)} + |\nabla u|_{p(x)}.$$

The space $W_0^{1,p(x)}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$ with respect to the norm $\|u\|_{1,p(x)}$. For $u \in W_0^{1,p(x)}(\Omega)$, we can define an equivalent norm $\|u\| = |\nabla u|_{p(x)}$; since the well known Poincaré inequality holds.

Next, we recall some previous results. This way, we want to make the proofs of the main results as transparent as possible.

Proposition 2.1 ([5, 9]). *If $p \in C_+(\mathbb{R}^N)$, then the spaces $L^{p(x)}(\mathbb{R}^N)$, $W^{1,p(x)}(\mathbb{R}^N)$ and $W_0^{1,p(x)}(\mathbb{R}^N)$ are separable and reflexive Banach spaces.*

Proposition 2.2 ([5, 9]). *The topological dual space of $L^{p(x)}(\mathbb{R}^N)$ is $L^{p'(x)}(\mathbb{R}^N)$, where*

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1.$$

Moreover for any $(u, v) \in L^{p(x)}(\mathbb{R}^N) \times L^{p'(x)}(\mathbb{R}^N)$, we have

$$|\int_{\mathbb{R}^N} uv dx| \le (\frac{1}{p^-} + \frac{1}{(p')^-})|u|_{p(x)}|v|_{p'(x)} \le 2|u|_{p(x)}|v|_{p'(x)}.$$

Set $\rho(u) = \int_{\mathbb{R}^N} |u|^{p(x)} dx$.

Proposition 2.3 ([5, 9]). *For all $u \in L^{p(x)}(\mathbb{R}^N)$, we have*

$$\min\{|u|_{p(x)}^{p^-}, |u|_{p(x)}^{p^+}\} \le \rho(u) \le \max\{|u|_{p(x)}^{p^-}, |u|_{p(x)}^{p^+}\}.$$

In addition, we have

(i) $|u|_{p(x)} < 1$ *(resp.* $= 1, > 1$*)* $\Leftrightarrow \rho(u) < 1$ *(resp.* $= 1, > 1$*);*
(ii) $|u|_{p(x)} > 1 \Rightarrow |u|_{p(x)}^{p^-} \le \rho(u) \le |u|_{p(x)}^{p^+}$*;*
(iii) $|u|_{p(x)} > 1 \Rightarrow |u|_{p(x)}^{p^+} \le \rho(u) \le |u|_{p(x)}^{p^-}$*;*
(iv) $\rho(\frac{u}{|u|_{p(x)}}) = 1$.

Proposition 2.4 ([5]). *Let $p(x)$ and $q(x)$ be measurable functions such that $p(x) \in L^\infty(\mathbb{R}^N)$ and $1 \le p(x)q(x) \le \infty$ almost every where in $\mathbb{R}^N$. If $u \in L^{q(x)}(\mathbb{R}^N)$, $u \ne 0$. Then*

$$|u|_{p(x)q(x)} \le 1 \Rightarrow |u|_{p(x)q(x)}^{p^-} \le \big||u|^{p(x)}\big|_{q(x)} \le |u|_{p(x)q(x)}^{p^+},$$

$$|u|_{p(x)q(x)} \ge 1 \Rightarrow |u|_{p(x)q(x)}^{p^+} \le \big||u|^{p(x)}\big|_{q(x)} \le |u|_{p(x)q(x)}^{p^-}.$$

In particular, if $p(x) = p$ is a constant, then $||u|^p|_{q(x)} = |u|_{pq(x)}^p$.

Proposition 2.5 ([9]). *If $u, u_n \in L^{p(x)}(\mathbb{R}^N)$, $n = 1, 2, \dots$, then the following statements are mutually equivalent:*

(1) $\lim_{n\to\infty} |u_n - u|_{p(x)} = 0$,

(2) $\lim_{n\to\infty} \rho(u_n - u) = 0$,
(3) $u_n \to u$ *in measure in* $\mathbb{R}^N$ *and* $\lim_{n\to\infty} \rho(u_n) = \rho(u)$.

Let $p^*(x)$ be the critical Sobolev exponent of $p(x)$ defined by

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{for } p(x) < N \\ +\infty & \text{for } p(x) \geq N \end{cases}$$

and let $C^{0,1}(\mathbb{R}^N)$ be the Lipschitz-continuous functions space.

Proposition 2.6 ([3, 5])**.** *If $p(x) \in C_+^{0,1}(\mathbb{R}^N)$, then there exists a positive constant c such that*

$$|u|_{p^*(x)} \leq c\|u\|_{p(x)}, \quad \forall u \in W_0^{1,p(x)}(\mathbb{R}^N).$$

Let $p \in L_+^\infty(\mathbb{R}^N)$ be an uniformly continuous function such that $p^+ < N$ and let $\Omega \subset \mathbb{R}^N$ be a bounded domain.

Proposition 2.7 ([3, 5])**.** *(1) If $q \in L_+^\infty(\mathbb{R}^N)$ and $p(x) \leq q(x) \ll p^*(x)$, for all $x \in \mathbb{R}^N$, then the embedding $W^{1,p(x)}(\mathbb{R}^N) \hookrightarrow L^{q(x)}(\mathbb{R}^N)$ is continuous but not compact.*

(2) If p is continuous on $\overline{\Omega}$ and q is a measurable function on Ω, with $p(x) \ll q(x) \ll p^*(x)$, for all $x \in \Omega$, then the embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is compact.

Observe that the solution of (1.1) will belong to the product space

$$W_{p(x),q(x)} := W_0^{1,p(x)}(\mathbb{R}^N) \times W_0^{1,q(x)}(\mathbb{R}^N),$$

equipped with the norm

$$\|(u,v)\|_{p(x)} = |\nabla u|_{p(x)} + |\nabla v|_{p(x)}.$$

The space $W'_{p(x),q(x)}$ is the topological dual of $W_{p(x),q(x)}$ equipped with the usual dual norm. For (u,v) in $W_{p(x),q(x)}$, let us define the functionals I, J, K

$$F(u,v) = \int_{\mathbb{R}^N} F(x, u(x), v(x))dx,$$
$$J(u,v) = \int_{\mathbb{R}^N} \frac{1}{p(x)}|\nabla u|^{p(x)}dx + \int_{\mathbb{R}^N} \frac{1}{q(x)}|\nabla v|^{q(x)}dx,$$
$$I(u,v) = J(u,v) - F(u,v).$$

Hypotheses. We assume some growth conditions:

(H1) $F \in C^1(\mathbb{R}^N \times \mathbb{R}^2, \mathbb{R})$ and $F(x,0,0) = 0$.
(H2) There exist positive functions a_i, b_i such that

$$|\frac{\partial F}{\partial u}(x,u,v)| \leq a_1(x)|u|^{p_1^- -1} + a_2(x)|v|^{p_1^+ -1},$$
$$|\frac{\partial F}{\partial v}(x,u,v)| \leq b_1(x)|u|^{q_1^- -1} + b_2(x)|v|^{q_1^+ -1},$$

where $1 < p_1(x), q_1(x) < \inf(p(x), q(x))$, and $p(x)$, $q(x) > \frac{N}{2}$, for all $x \in \mathbb{R}^N$. The weight-functions a_i and b_i, $i = 1, 2$, belong respectively to the generalized Lebesgue spaces $L^{\alpha_i}(\mathbb{R}^N)$ and $L^\beta(\mathbb{R}^N)$, where

$$\alpha_1(x) = \frac{p(x)}{p(x)-1}, \beta(x) = \frac{p^*(x)q^*(x)}{p^*(x)q^*(x) - p^*(x) - q^*(x)}, \quad \alpha_2(x) = \frac{q(x)}{q(x)-1}.$$

(H3) There exist constants $R > 0, \theta > 0$, and a positive function $H : \mathbb{R}^N \times \mathbb{R}^2 \to \mathbb{R}$ such that for $x \in \mathbb{R}^N$, $|u|, |v| \le R$ and $t > 0$ sufficiently small, we have
$$F(x, t^{1/p(x)}u, t^{1/q(x)}v) \ge t^\theta H(x, u, v).$$

Assumption (H3) implies that the potential function F is sufficiently positive in a neighborhood of zero.

Lemma 2.8. *Under assumptions* (H1)–(H2)*, the functional F is well defined and Frechet differentiable. Its derivative is*
$$F'(u, v)(\omega, z) = \int_{\mathbb{R}^N} \frac{\partial F}{\partial u}(x, u, v)\omega + \frac{\partial F}{\partial v}(x, u, v)z dx, \forall (u, v), (\omega, z) \in W_{p(x),q(x)}.$$

Proof. The functional F is well defined on $W_{p(x),q(x)}$. Indeed, for all pair of real-valued functions $(u, v) \in W_{p(x),q(x)}$, we have in virtue of (H1) and (H2),
$$\begin{aligned} F(x, u, v) &= \int_0^u \frac{\partial F}{\partial s}(x, s, v)ds + F(x, 0, v) \\ &= \int_0^u \frac{\partial F}{\partial s}(x, s, v)ds + \int_0^v \frac{\partial F}{\partial s}(x, 0, s)ds + F(x, 0, 0). \end{aligned}$$
Then
$$F(x, u, v) \le c_1[a_1(x)|u|^{p_1^-} + a_2(x)|v|^{p_1^+-1}|u| + b_2(x)|v|^{q_1^+}] \tag{2.1}$$
Since $W^{1,p(x)}(\mathbb{R}^N) \hookrightarrow L^{s(x)p(x)}(\mathbb{R}^N)$ for $s(x) > 1$, we have
$$\big||u|^{p_1^-}\big|_{p(x)} = |u|^{p_1^-}_{p_1^- p(x)} \le c\|u\|^{p_1^-}_{p(x)}.$$
So, taking into account Hölder inequality, Propositions 2.2, 2.4, 2.6, 2.7 and (H2), we obtain
$$\begin{aligned} F(u, v) &= \int_{\mathbb{R}^N} F(x, u, v)\, dx \\ &\le c_2\Big(|a_1|_{\alpha_1(x)}|u|^{p_1^-}_{p_1^- p(x)} + |a_2|_{\beta(x)}|v|^{p_1^+-1}_{(p_1^+-1)q^*(x)}|u|_{p^*(x)} + |b_2|_{\alpha_2(x)}|v|^{q_1^+}_{q_1^+ q(x)}\Big) \\ &\le c_3(|a_1|_{\alpha_1(x)}\|u\|^{p_1^-}_{p(x)} + |a_2|_{\beta(x)}\|v\|^{p_1^+-1}_{q(x)}\|u\|_{p(x)} + |b_2|_{\alpha_2(x)}\|v\|^{q_1^+}_{q(x)}) < \infty \end{aligned}$$
The proof is complete. □

Similarly, we show that F' is also well defined. Indeed, for all $(u, v), (\omega, z) \in W_{p(x),q(x)}$, we can write
$$\begin{aligned} F'(u, v)(\omega, z) &= \int_{\mathbb{R}^N} \frac{\partial F}{\partial u}(x, u, v)\omega dx + \int_{\mathbb{R}^N} \frac{\partial F}{\partial v}(x, u, v)z\, dx \\ &\le \int_{\mathbb{R}^N} (a_1(x)|u|^{p_1^- -1} + a_2(x)|v|^{p_1^+-1})\omega\, dx \\ &\quad + \int_{\mathbb{R}^N} (b_1(x)|u|^{q_1^- -1} + b_2(x)|v|^{q_1^+-1})z\, dx \end{aligned}$$
Following Hölder inequality, we obtain
$$\begin{aligned} F'(u, v)(\omega, z) \le c_4(&|a_1|_{\alpha_1(x)}||u|^{p_1^- -1}|_{p^*(x)}|\omega|_{p(x)} + |a_2|_{\beta(x)}||v|^{p_1^+-1}|_{q^*(x)}|\omega|_{p^*(x)} \\ &+ |b_1|_{\beta(x)}||u|^{q_1^- -1}|_{p^*(x)}|z|_{q^*(x)} + |b_2|_{\alpha_2(x)}||v|^{q_1^+-1}|_{q^*(x)}|z|_{q(x)}) \end{aligned}$$

The above propositions yield

$$\begin{aligned} F'(u,v)(\omega,z) \le c_5(&|a_1|_{\alpha_1(x)}\|u\|_{p(x)}^{p_1^- -1}\|\omega\|_{p(x)} + |a_2|_{\beta(x)}\|v\|_{q(x)}^{p_1^+ -1}\|\omega\|_{p(x)} \\ &+ |b_1|_{\beta(x)}\|u\|_{p(x)}^{q_1^- -1}\|z\|_{q(x)} + |b_2|_{\alpha_2(x)}\|v\|_{q(x)}^{q_1^+ -1}\|z\|_{q(x)}) < \infty. \end{aligned}$$

Moreover F is Frechet differentiable; namely, for any fixed point $(u,v) \in W_{p(x),q(x)}$, and for any $\varepsilon > 0$, there exist $\delta = \delta_{\varepsilon,u,v} > 0$ such that for all $(\omega,z) \in W_{p(x),q(x)}$, satisfying $\|(\omega,z)\|_{p(x),q(x)} < \delta$ we have

$$|F(u+\omega, v+z) - F(u,v) - F'(u,v)(\omega,z)| \le \varepsilon\|(\omega,z)\|_{p(x),q(x)}.$$

First, let B_R be the ball in $\mathbb{R}^N$ centered at the origin and of radius R. Set $B'_R = \mathbb{R}^N - B_R$.

It is well-known that the functional F_R defined on $W_0^{1,p(x)}(B_R) \times W_0^{1,q(x)}(B_R)$ by

$$F_R(u,v) = \int_{B_R} F(x,u,v)dx$$

belongs to $C^1(W_0^{1,p(x)}(B_R) \times W_0^{1,q(x)}(B_R))$, by in virtue of (H1) and (H2). In addition, the operator F'_R defined from $W_0^{1,p(x)}(B_R) \times W_0^{1,q(x)}(B_R)$ to $(W_0^{1,p(x)}(B_R) \times W_0^{1,p(x)}(B_R))'$ by

$$F'_R(u,v)(\omega,z) = \int_{B_R} \frac{\partial F}{\partial u}(x,u,v)\omega + \frac{\partial F}{\partial v}(x,u,v)z dx,$$

is compact (see [9]). Clearly, for all $(u,v), (\omega,z) \in W_{p(x),q(x)}$, we can write

$$\begin{aligned} &|F(u+\omega, v+z) - F(u,v) - F'(u,v)(\omega,z)| \\ &\le |F_R(u+\omega, v+z) - F_R(u,v) - F'_R(u,v)(\omega,z)| \\ &\quad + |\int_{B'_R} (F(x,u+\omega,v+z) - F(x,u,v)) - \frac{\partial F}{\partial u}(x,u,v)\omega - \frac{\partial F}{\partial v}(x,u,v)z dx| \end{aligned}$$

According to a classical theorem, there exist $\zeta_1, \zeta_2 \in]0,1[$, such that

$$\begin{aligned} &|\int_{B'_R} (F(x,u+\omega,v+z) - F(x,u,v)) - \frac{\partial F}{\partial u}(x,u,v)\omega - \frac{\partial F}{\partial v}(x,u,v)z dx| \\ &= |\int_{B'_R} \frac{\partial F}{\partial u}(x,u+\zeta_1\omega,v)\omega + \frac{\partial F}{\partial v}(x,u,v+\zeta_2 z)z \\ &\quad - \frac{\partial F}{\partial u}(x,u,v)\omega - \frac{\partial F}{\partial v}(x,u,v)z\, dx|. \end{aligned}$$

Consequently, by growth conditions (H2), we obtain

$$\begin{aligned} &|\int_{B'_R} (F(x,u+\omega,v+z) - F(x,u,v)) - \frac{\partial F}{\partial u}(x,u,v)\omega - \frac{\partial F}{\partial v}(x,u,v)z\, dx| \\ &\le \int_{B'_R} a_1(x)(|u+\zeta_1\omega|^{p_1^- -1} + |u|^{p_1^- -1})\omega dx \\ &\quad + \int_{B'_R} a_2(x)(|v+\zeta_2 z|^{p_1^+ -1} + |v|^{p_1^+ -1})\omega\, dx \\ &\quad + \int_{B'_R} b_1(x)(|u+\zeta_1\omega|^{q_1^- -1} + |u|^{q_1^- -1})z dx \end{aligned}$$

$$+\int_{B'_R} b_2(x)(|v+\zeta_2 z|^{q_1^+-1}+|v|^{q_1^+-1})z\,dx.$$

By an elementary inequality, Propositions 2.4, 2.6 and the fact that

$$\begin{gathered} |a_i|_{L^{p'(x)}(B'_R)} \to 0, \quad |a_i|_{L^{\beta(x)}(B'_R)} \to 0 \\ |b_i|_{L^{q'(x)}(B'_R)} \to 0, \quad |b_i|_{L^{\beta(x)}(B'_R)} \to 0, \end{gathered} \tag{2.2}$$

for R sufficiently large and $i=1,2$, we obtain the estimate

$$\begin{aligned} &|\int_{B'_R}(F(x,u+\omega,v+z)-F(x,u,v)-\frac{\partial F}{\partial u}(x,u,v)\omega-\frac{\partial F}{\partial v}(x,u,v)z)\,dx| \\ &\le \varepsilon(\|\omega\|_{p(x)}+\|z\|_{q(x)}). \end{aligned}$$

We prove now that F' is continuous on $W_{p(x),q(x)}$. To this end, we let $(u_n,v_n)\to(u,v)$ in $W_{p(x),q(x)}$ as $n\to\infty$. Then for any $(\omega,z)\in W_{p(x),q(x)}$, we have

$$\begin{aligned} &|F'(u_n,v_n)(\omega,z)-F'(u,v)(\omega,z)| \\ &\le |F'_R(u_n,v_n)(\omega,z)-F'_R(u,v)(\omega,z)|+|\int_{B'_R}\big(\frac{\partial F}{\partial u}(x,u_n,v_n)-\frac{\partial F}{\partial u}(x,u,v)\big)\omega dx| \\ &\quad+|\int_{B'_R}\big(\frac{\partial F}{\partial v}(x,u_n,v_n)-\frac{\partial F}{\partial v}(x,u,v)\big)z\,dx| \end{aligned}$$

Note that

$$|F'_R(u_n,v_n)(\omega,z)-F'_R(u,v)(\omega,z)|\to 0 \quad \text{as } n\to\infty,$$

since F'_R is continuous on $W_0^{1,p(x)}(B_R)\times W_0^{1,q(x)}(B_R)$ (see [9]). Using (H2) once again and (2.1), the other terms on the wrigth-hand side of the above inequality tend to zero.

Lemma 2.9. *Under assumptions* (H1)–(H2)*, F is lower weakly semicontinuous in $W_{p(x),q(x)}$.*

Proof. Let (u_n,v_n) be a weakly convergent sequence to (u,v) in $W_{p(x),q(x)}$. In the same way, we write

$$|F(u_n,v_n)-F(u,v)|\le|F_R(u_n,v_n)-F_R(u,v)|+|\int_{B'_R}(F(x,u_n,v_n)-F(x,u,v))dx|$$

Since the restriction operator is continuous, the sequence (u_n,v_n) is weakly convergent to (u,v) in $W_0^{1,p(x)}(B_R)\times W_0^{1,q(x)}(B_R)$. However F_R is weakly lower semicontinuous. This result comes from growth conditions (H1) and (H2), and Sobolev compact inclusion

$$W_0^{1,p(x)}(B_R)\times W_0^{1,q(x)}(B_R)\hookrightarrow L^{s(x)}(B_R)\times L^{t(x)}(B_R),$$

for all $(s,t)\in[p(x),p^*(x)[\times[q(x),q^*(x)[$. Using (2.1) and (2.2), both the terms on the right-hand side of the last inequality tend to zero. □

We remark that the C^1-functional J is weakly lower semi-continuous, and its derivative is given by

$$J'(u,v)(\omega,z)=\int_{\mathbb{R}^N}|\nabla u|^{p(x)-2}\nabla u\nabla\omega\,dx+\int_{\mathbb{R}^N}|\nabla v|^{q(x)-2}\nabla v\nabla z\,dx$$

The Euler-Lagrange functional associated to the system (1.1) takes the form

$$I(u,v)=\int_{\mathbb{R}^N}\frac{1}{p(x)}|\nabla u|^{p(x)}+\frac{1}{q(x)}|\nabla v|^{q(x)}dx-\int_{\mathbb{R}^N}F(x,u,v)\,dx.$$

In other words $I(u,v)=J(u,v)-F(u,v)$. Observe that the weak solutions of the system (1.1) are precisely the critical points of the functional I.

Lemma 2.10. *Under assumptions* (H1)–(H2)*, the functional I is coercive.*

Proof. We have

$$\begin{aligned}
I(u,v)&=\int_{\mathbb{R}^N}\frac{1}{p(x)}|\nabla u|^{p(x)}+\frac{1}{q(x)}|\nabla v|^{q(x)}-F(x,u,v)\,dx\\
&\ge\int_{\mathbb{R}^N}\frac{1}{p^+}|\nabla u|^{p(x)}+\frac{1}{q^+}|\nabla v|^{q(x)}\,dx\\
&\quad-\int_{\mathbb{R}^N}(a_1(x)|u|^{p_1^-}+a_2(x)|v|^{p_1^+-1}|u|+b_2(x)|v|^{q_1^+})dx\\
&\ge\frac{1}{p^+}\rho(\nabla u)+\frac{1}{q^+}\rho(\nabla v)\\
&\quad-(|a_1|_{\alpha_1(x)}||u|^{p_1^-}|_{p(x)}+|a_2|_{\beta(x)}||v|^{p_1^+-1}|_{q^*(x)}|u|_{p^*(x)}+|b_2|_{\alpha_2(x)}||v|^{q_1^+}|_{q(x)}).
\end{aligned}$$

By Propositions 2.3, 2.4, 2.6 and the Young inequality, we obtain

$$\begin{aligned}
I(u,v)&\ge\frac{1}{p^+}\|u\|_{p(x)}^{p^-}+\frac{1}{q^+}\|v\|_{q(x)}^{q^-}-\Big(|a_1|_{\alpha_1(x)}\|u\|_{p(x)}^{p_1^-}\\
&\quad+|a_2|_{\beta(x)}\big(\frac{p_1^+-1}{p_1^+}\|v\|_{q(x)}^{p_1^+}+\frac{1}{p_1^+}\|u\|_{p(x)}^{p_1^+}\big)+|b_2|_{\alpha_2(x)}\|v\|_{q(x)}^{q_1^+}\Big)\\
&\ge\frac{1}{p^+}\|u\|_{p(x)}^{p^-}+\frac{1}{q^+}\|v\|_{q(x)}^{q^-}-c_6\Big(|a_1|_{\alpha_1(x)}\|u\|_{p(x)}^{p_1^-}\\
&\quad+|a_2|_{\beta(x)}\|v\|_{q(x)}^{p_1^+-1}+|a_2|_{\beta(x)}\|u\|_{p(x)}+|b_2|_{\alpha_2(x)}\|v\|_{q(x)}^{q_1^+}\Big)
\end{aligned}$$

Clearly, $I(u,v)$ tends to infinity as $\|(u,v)\|_{p(x),q(x)}\to\infty$, since $1<p_1(x),q_1(x)<\inf(p(x),q(x))$. $\square$

Theorem 2.11. *Under assumptions* (H1)–(H3)*, the system* (1.1) *has a non-trivial weak solution.*

Proof. By lemmas 2.8, 2.9 and 2.10, the functional I is weakly lower semi-continuous and coercive in $W_{p(x),q(x)}$. Consequently, the functional I has a global minimum (see [13, Theorem 12]. On the other hand I is C^1. Hence this minimum is necessarily characterized by a critical point of I, which is a weak solution of (1.1). This solution is nontrivial. Indeed, as $I(0,0)=0$, it is sufficient to show that there exists $(u_1,v_1)\in W_{p(x),q(x)}$ such that $I(u_1,v_1)<0$. Let $R>0$, $\theta<1$ and $(0,0)\neq(\varphi,\psi)\in C_0^\infty(\mathbb{R}^N)\times C_0^\infty(\mathbb{R}^N)$ with $|\varphi|,|\psi|\le R$. According to (H3), one has

$$\begin{aligned}
&I(t^{1/p(x)}\varphi,t^{1/q(x)}\psi)\\
&=J(t^{1/p(x)}\varphi,t^{1/q(x)}\psi)-F(t^{1/p(x)}\varphi,t^{1/q(x)}\psi)\\
&\le t\int_{\mathbb{R}^N}[\frac{1}{p^-}|\nabla\varphi|^{p(x)}+\frac{1}{q^-}|\nabla\psi|^{q(x)}]dx-\int_{\mathbb{R}^N}F(x,t^{1/p(x)}\varphi,t^{1/q(x)}\psi)dx
\end{aligned}$$

$$
\begin{aligned}
&\le t[\frac{1}{p^-}\rho(\nabla\varphi)+\frac{1}{q^-}\rho(\nabla\psi)]-t^\theta\int_{\mathbb{R}^N}H(x,\varphi,\psi)dx\\
&\le t[\frac{1}{p^-}\max\{|\nabla\varphi|_{p(x)}^{p^-},|\nabla\varphi|_{p(x)}^{p^+}\}+\frac{1}{q^-}\max\{|\nabla\psi|_{q(x)}^{q^-},|\nabla\psi|_{q(x)}^{q^+}\}]\\
&\quad -t^\theta\int_{\mathbb{R}^N}H(x,\varphi,\psi)dx\\
&\le t[\frac{1}{p^-}\max\{\|\nabla\varphi\|_{p(x)}^{p^-},\|\nabla\varphi\|_{p(x)}^{p^+}\}+\frac{1}{q^-}\max\{\|\nabla\psi\|_{q(x)}^{q^-},\|\nabla\psi\|_{q(x)}^{q^+}\}]\\
&\quad -t^\theta\int_{\mathbb{R}^N}H(x,\varphi,\psi)\,dx<0,
\end{aligned}
$$

for $t > 0$ sufficiently small. □

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