1. Introduction

Let \( z_j = x_j + iy_j \) denote a complex variable, \( 1 \leq j \leq n \). Let \( z = (z_1, \ldots, z_n) \), \( z^{2k} = z_1^{2k_1}, \ldots, z_n^{2k_n} \) where \( k \) is the vector \((k_1, \ldots, k_n)\) with \( k_j \) a nonnegative integer \((j = 1, \ldots, n)\) and let \( \|k\| = k_1 + \cdots + k_n \). Let \( \phi(z) \) be an entire function of \( z_1^2, \ldots, z_n^2 \) in a domain \( D \) that includes the origin and let \( \Delta_j = D^2_{z_j} + \frac{a_j}{z_j} D_z, \alpha_j \geq 0, j = 1, \ldots, n \). Also, let \( a > -1 \) and \( \varepsilon_j = 1 \) if \( j = 1, \ldots, m \) and \( \varepsilon_j = -1 \) if \( j = m + 1, \ldots, n \). Now consider the representations of an entire function solutions of the problem

\[
(D_t^2 + \frac{a}{t} D_t) u(z, t) = \sum_{j=1}^{n} \varepsilon_j \Delta_j u(z, t)
\]

with initial data

\[
u(z, 0) = \phi(z), \quad u_t(z, 0) = 0
\]

in terms of a set of associated multinomials \( \{R_k(z, t)\} \) throughout \((z, t)\) space, \( t \) real. These multinomials are solutions of \([1.1]\) corresponding to the choice of \( \phi(z) = z^{2k} \) in \([1.1]\).

Let \( G \) be a region in \( \mathbb{R}^n \) (positive hyper octant) and let \( G_R \subset \mathbb{C}^n \) denote the region obtained from \( G \) by a similarity transformation about the origin, with ratio of similitude \( R \).

**Definition 1.1.** Let \( \phi(z) = \sum_{\|k\| = 0}^{\infty} a_k z_1^{2k_1} \) be an entire function of several complex variables. Then \( \phi(z) \) is of growth \((\rho, T)\) if

\[
T = \limsup_{\|k\| \to \infty} \frac{\|2k\|^{\rho/\|2k\|}}{e^\rho \|a_k d_k(G)\|^{\rho/\|2k\|}}, \quad (0 < \rho < \infty)
\]

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where
\[ d_k(G) = \max_{k \in G} \{R^{2k}\}, \quad R^{2k} = R^{2k_1}, \ldots, R^{2k_n}. \]

This implies the existence of a positive constant \( M \) such that
\[ |\phi(z)| \leq Me^{T|z|^\rho} \quad \forall z \in \mathbb{C}^n. \]

Using (1.2), for each \( \varepsilon > 0 \) there exists a positive integer \( k_0 \) such that if \( k \geq k_0 \), then
\[ |a_kd_k(G)| \leq \left[ \frac{e^\rho(T + \varepsilon)}{\|k\|/\rho} \right] \|2k\|/\rho. \quad (1.3) \]

We can easily estimate, from (1.4), that
\[ |R_k(z,t)| \leq \left( \frac{2k}{\rho T} \right)^{\|2k\|/\rho} M(\rho, T)e^{-\|2k\|/\rho e^{K|t|+\sum_{j=1}^n T_j|z_j|^\rho}}, \quad (1.4) \]

where
\[ M(\rho, T) = \int_0^\infty e^{-\rho|t|^\rho + \rho t^2} d\rho, \]

and \( K \) is the sum of the absolute values of the coefficients of multinomial and \( M(\rho, T) \) is a generic constant depending only on the \( \rho_j \)s and \( T_j \)s.

Now we can establish a theorem.

**Theorem 1.2.** Let \( \phi(z) = \sum_{k=0}^\infty a_kz_1^{2k_1}, \ldots, z_n^{2k_n} \) be entire in \((z_1, \ldots, z_n)\) and converge in a domain \( G_{\rho} : z \in \mathbb{C}^n; |z|^2 < R^2, R > 0 \) is a fixed positive real. Then the series \( u(z,t) = \sum_{k=0}^\infty a_kR_k(z,t) \) converges for all \( n \)-complex variables \((z_1, \ldots, z_n)\) and real \( t \) and uniformly so in compact subsets of \((z,t)\) space.

Bragg and Dettman [2] proved the following theorem.

**Theorem 1.3.** Let \( \phi(x) = \sum_{k=0}^\infty a_kx^{2k} \) be analytic in \((x_1^2, \ldots, x_n^2)\) and converge in a domain \( D \) that includes the origin. Then the series \( \sum_{k=0}^\infty a_kP_k(x,t) \) converges to an analytic solution of the problem (1.1) replacing \( z \) by \( x \), at least in region \( S \) where \( S \) is defined by \( (x,t) \in S \) if and only if
\[ |x_1| + |t|, \ldots, |x_n| + |t|, (x_{m+1}^2 + t^2)^{1/2}, \ldots, (x_n^2 + t^2)^{1/2} \in D. \quad (1.6) \]
We shall proceed to the complex transformation of above Theorem A in the following manner.

Let \((z_1, \ldots, z_n)\) be an element of \(\mathbb{C}^n\) and \(\mathbb{R}^{2n}\), the space of real coordinates. The transformation from real to the complex coordinates are given by \(x_k = \frac{z_k + \overline{z}_k}{2}\), \(y_k = \frac{z_k - \overline{z}_k}{2i}\). We equip \(\mathbb{C}^n\) with the Euclidean metric of \(\mathbb{R}^{2n}\);

\[
d s^2 = \sum_{k=1}^{n} (dx_k^2 + dy_k^2) = \sum_{k=1}^{n} dz_k.d\overline{z}_k.
\]

Let \(z_k\) be a point on the domain \(G_R\) for which \(|a_k R_k(z_k, 0)| = \sup_{z_k \in G_R} |a_k R_k(z_k, 0)| = C_k\). By a rotation, we can assume that \(z_k^2 = (x_k^2, 0, \ldots, 0)\). If \(\tilde{f}(w) = f(w^2, 0, \ldots, 0)\) and \(\tilde{f}(w) = \sum_{k=0}^{\infty} a_k w^{2k}\) is the Taylor series expansion of \(\tilde{f}\) at the origin, then \(|a_k R_k(z_k^2)| = C_k\) and therefore we have the following theorem.

**Theorem 1.4.** Let \(\phi(z) = \sum_{k=0}^{\infty} a_k z^{2k}\) be entire in \((z_1^2, \ldots, z_n^2)\) and converge in a domain \(G_R\) that includes the origin. Then the series \(u(z, t) = \sum_{||k||=0}^{\infty} a_k R_k(z, t)\) converges to an entire solution of the problem \([1.1]\) at least in a region \(S\) where \(S\) is defined by \((z, t) \in S\) if and only if

\[
|z_1| + |t|, \ldots, |z_m| + |t|, (z_{m+1}^2 + t^2)^{1/2}, \ldots, (z_n^2 + t^2)^{1/2} \in G_R.
\]

Let \(\phi(z) = \sum_{k=0}^{\infty} a_k jk^{2k}\) be the power series expansion of the function \(\phi(z)\). Then the maximum modulus of \(u(z, t)\) and \(\phi(z)\) are defined as in complex function theory \([12]\) pp. 129, 132,

\[
M_{f,G}(R) = \max_{z \in G_R} |f(z)|, \quad M_{u,S}(R) = \max_{(z, t) \in S} |u(z, t)|.
\]

Following the usual definitions of order and type of an entire function of \(n\)-complex variables \((z_1^2, \ldots, z_n^2)\), the order \(\rho\) and type \(T\) of \(u(z, t)\) are defined as in \([3]\)

\[
\rho(u) = \limsup_{R \to \infty} \frac{\log \log M_{u,S}(R)}{\log R}, \quad (1.7)
\]

\[
T(u) = \limsup_{R \to \infty} \frac{\log M_{u,S}(R)}{R^\rho(u)}. \quad (1.8)
\]

In this paper we characterize the order, lower order, type and lower type of entire function solutions of problem \([1.1]\) in terms of a set \(\{R_k(z, t)\}\) of multionomials for \(n \geq 2\). Multinomials of this type have been constructed by Miles and Yong \([12]\) when \(z = x\) and \(m = n\) or \(m = 0\). In these cases \([1.1]\) reduces to either the generalized Euler-Poisson-Darboux or the generalized Beltrami equation. Gilbert and Howard \([5, 6]\) discussed analyticity properties of solutions of special cases of \([1.1]\). Bragg and Dettman obtained representation of analytic solutions of problem \([1.1]\) for \(z = x\) in terms of these multinomials for \(n \geq 2\) \([2]\) and for \(n = 1\) in \([3]\). It has been found \([2]\) that \(R_k(x, t)\), \(n \geq 2\), can be expressed as a convolution of \(n\) polynomials \(R_k(x, t), j = 1, \ldots, n\). For \(n = 1\) the corresponding \(R_k(x, t)\) are defined in terms of Jacobi polynomials. The Growth estimates for the solutions of \([1.1]\) in terms of multinomials \(R_k(z, t)\) for \(n \geq 2\) then permit the obtaining of global region of convergence from acknowledge of singularities of the given data function \(\phi(z)\). It should be noted that the function \(\phi(z)\) is the analytic continuation of its restriction to the axis of symmetry; i.e., \(\phi(z) = u(z, 0)\). Using various
techniques, the characterizations of order and type of entire function solutions of similar problems were obtained by McCoy [13, 14] Kumar [8, 9, 10] and others for \( n = 1 \). However, none of them have considered the case for \( n \geq 2 \).

2. Auxiliary Results

In this section we shall prove some auxiliary results which will be used in the sequel.

**Lemma 2.1.** If \( u(z,t) = \sum_{\|k\|=0}^{\infty} a_k R_k(z,t) \) is an entire function solution of problem \( \{1.1\} \) in terms of a set \( \{R_k(z,t)\} \) of monomials corresponding to given data function \( \phi(z) = \sum_{\|k\|=0}^{\infty} a_k z^{2k} \) in \( \{1.1\} \), then \( \phi \) and \( \phi^\ast \) are also entire functions of \( n \)-complex variables \( (z_1, \ldots, z_n) \). Further,

\[
[N(\varepsilon)]^{-1} M_{\phi,G}(R) \leq M_{u,S}(\varepsilon^{-1}R) \leq CM_{\phi^\ast,G}(R)
\]

(2.1)

where

\[
\phi^\ast(z) = \sum_{\|k\|=0}^{\infty} |a_k| \left\{ \prod_{j=1}^{n} k_j^{p_j} \right\} z_1^{2k_1}, \ldots, z_n^{2k_n},
\]

\[
N(\varepsilon) = \sup \{N(\varepsilon e^{i\theta}, \xi) : 0 \leq \theta \leq 2\pi, -1 \leq \xi \leq 1, 0 < \varepsilon < 1\}
\]

and \( C \) is a constant.

**Proof.** From Theorem 1.1 and 1.2, bearing in mind with the relation of [2, (3.1)], we obtain

\[
|u(z,t)| \leq \sum_{\|k\|=0}^{\infty} |a_k| \left\{ \Gamma \left( \frac{a+1}{2n} \right) \right\} \frac{n 2^m K^{n-m}}{\pi^{m/2}} \left\{ \prod_{j=1}^{m} k_j \right\} \frac{\Gamma(k_j + (a_j+1)/2)}{\Gamma(k_j - 1/2)}
\]

\[
\times \{|z_j| + |t|\}^{2k_j} \left\{ \prod_{j=m+1}^{n} \frac{k_j^{q_j} k_j!}{\Gamma((k_j) + (a+1)/2)} \right\}
\]

where \( q_j = \max \{(a_j - 1)/2, (a+1)/2n\} - 1, -1/2, j = m+1, \ldots, n \).

Using the relation \( \Gamma(x+a)/\Gamma x \sim x^a \) as \( x \to \infty \), we have

\[
\frac{\Gamma(k_j + (a_j+1)/2)}{\Gamma(k_j - 1/2)} \sim (k_j - 1/2)^{(a_j+1)/2}, \quad \frac{k_j^{q_j} k_j!}{\Gamma((k_j) + (a+1)/2)} \sim k_j^{q_j+1} (k_j)^{(a+1)/2n}
\]

and we see that there exist constants \( C, p_1, \ldots, p_n \) with \( p_j = p_j(a_j), j = 1, \ldots, m \) and \( p_j = p_j(a_j, a, n) \) for \( j = m+1, \ldots, n \) such that

\[
|u(z,t)| \leq \sum_{\|k\|=0}^{\infty} |a_k| C \left\{ \prod_{j=1}^{n} k_j^{p_j} \right\} (|z_1| + |t|)^{2k_1} \ldots (|z_m| + |t|)^{2k_m}
\]

\[
\times (z_{m+1}^2 + t^2)^{k_{m+1}} \ldots (z_n^2 + t^2)^{k_n}
\]

(2.2)

Now, \( |\phi(z)| \leq \sum_{\|k\|=0}^{\infty} |a_k| |z_1|^{2k_1} \ldots |z_n|^{2k_n} \), the series \( \{2.2\} \) converges for \( z \in G_R \).

But for \( z \in G_R \), the series

\[
\sum_{\|k\|=0}^{\infty} |a_k| \left\{ \prod_{j=1}^{n} k_j^{p_j} \right\} |z_1|^{2k_1} \ldots |z_n|^{2k_n}
\]
also converges. By Theorem 1.2, if \( \phi(z) \) is entire in \((z_1^2, \ldots, z_n^2)\), then \( u(z,t) \) converges to an entire solution of problem (1.1). We see that

\[
\lim_{\|k\|\to\infty} |a_k| \left| \prod_{j=1}^{n} k_j^{p_j} \right|^{1/|2k|} = \lim_{\|k\|\to\infty} |a_k|^{1/|2k|} = 0.
\]

Hence both \( \phi \) and \( \phi^* \) are entire.

Using (2.2) we obtain

\[
M_{u,S}(R) \leq C \sum_{\|k\|=0}^{\infty} |a_k| \left\{ \prod_{j=1}^{n} k_j^{p_j} \right\} R^{2k} = CM_{\phi^*,G}(R) \quad (2.3)
\]

where

\[
\phi^*(z) = \sum_{\|k\|=0}^{\infty} |a_k| \left\{ \prod_{j=1}^{n} k_j^{p_j} \right\} z_1^{2k_1} \cdots z_n^{2k_n}.
\]

Now for reverse relation, we have

\[
\phi(z) = \sum_{\|k\|=0}^{\infty} a_k z_1^{2k_1} \cdots z_n^{2k_n}
\]

\[
|\phi(z)| \leq \sum_{\|k\|=0}^{\infty} |a_k| \left\{ \prod_{j=1}^{n} k_j^{p_j} \right\} |z_1|^{2k_1} \cdots |z_n|^{2k_n}
\]

\[
= \sum_{\|k\|=0}^{\infty} |a_k| \left\{ \prod_{j=1}^{n} k_j^{p_j} \right\} \left| z_1 + |t| \right|^{2k_1} \cdots \left| z_m + |t| \right|^{2k_m}
\]

\[
\times \left[ z_{m+1}^{2} + t^{2} \right]^{1/2} \cdots \left[ z_{n}^{2} + t^{2} \right]^{1/2}.
\]

This relation is valid globally, and leads to the estimates

\[
|\phi(z)| \leq M_{u,S}(R)N(\varepsilon, \varepsilon) = \left| z \right|/R \leq \max_{1 \leq j \leq n} \left( \frac{|z_j|}{R_j} \right)^2,
\]

\[
N(\varepsilon) = \sup \{|N(\varepsilon \cos \theta, \varepsilon \sin \theta)| : 0 \leq \theta \leq 2\pi, -1 \leq \xi \leq 1\}.
\]

For \( z = \varepsilon \Re e^{i\theta} \) real, \( 0 < \varepsilon < 1 \), we have

\[
M_{\phi,G}(\varepsilon R) \leq M_{u,S}(R)N(\varepsilon)
\]

or

\[
[N(\varepsilon)]^{-1} M_{\phi,G}(R) \leq M_{u,S}(\varepsilon^{-1} R).
\]

Combining (2.3) and (2.4) we obtain (2.1). \( \square \)

**Lemma 2.2.** Let \( u(z,t) \) be an entire function solution of (1.1) in terms of a set \( \{ R_k(z,t) \} \) of multionomials corresponding to given data function \( \phi(z) \) in (1.1). Then the orders and types of \( u(z,t) \) and \( \phi \) respectively are identical.
Proof. Let \( \phi(z) = \sum_{|k| = 0}^{\infty} a_k z^k \) be an entire function of order \( \rho(\phi) \) and type \( T(\phi) \). Then it is well known [7, Thm. 1] that
\[
\rho(\phi) = \limsup_{|k| \to \infty} \frac{||2k|| \log ||k||}{-\log |a_k|}, \tag{2.5}
\]
\[
(\varepsilon \rho(\phi) T(\phi))^{1/\rho(\phi)} = \limsup_{||k|| \to \infty} \left\{ ||2k||^{1/\rho(\phi)} ||a_k| d_k(G) \right\}. \tag{2.6}
\]
Hence for the function \( \phi^*(z) = \sum_{|k| = 0}^{\infty} |a_k| \left\{ \prod_{j=1}^{n} k_j^{p_j} \right\} z^k \), we have
\[
\frac{1}{\rho(\phi^*)} = \liminf_{||k|| \to \infty} \frac{\log ||a_k| \prod_{j=1}^{n} k_j^{p_j} |^{-1}}{2||k|| \log ||k||}
\]
\[
= \liminf_{||k|| \to \infty} \frac{\log |a_k|^{-1} - \log \prod_{j=1}^{n} k_j^{p_j}}{2||k|| \log ||k||}
\]
\[
= \liminf_{||k|| \to \infty} \frac{\log |a_k|^{-1}}{2||k|| \log ||k||}.
\]
Hence \( \rho(\phi) = \rho(\phi^*) \). Since \( \phi \) and \( \phi^* \) have same order, using (2.6) we can easily show that \( T(\phi) = T(\phi^*) \).

Now using the relation (2.4) with the definitions of order and type given by (1.7) and (1.8), the proof is complete. \( \square \)

Lemma 2.3. If \( |a_k|/|a_{k'}|, ||k'|| = ||k|| + 1, \) forms a non-decreasing function of \( k \) then \( |\beta_k|/|\beta_{k'}| \) also forms a non-decreasing function of \( k \), where
\[
\beta_k = a_k \left\{ \Gamma \left( \frac{a + 1}{2n} \right) \right\}^{-2} \left\{ \prod_{j=1}^{m} k_j (k_j - 1/2)^{(\alpha_j + 2)/2} \right\} \times \left\{ \prod_{j=m+1}^{n} k_j^{(q_j + 1 + (a+1)/2n)} \right\}. \tag{2.7}
\]
Proof. We have
\[
\frac{|\beta_k|}{|\beta_{k'}|} = \frac{a_k \left\{ \Gamma \left( \frac{a + 1}{2n} \right) \right\}^{-2} \left\{ \prod_{j=1}^{m} k_j (k_j - 1/2)^{(\alpha_j + 2)/2} \right\}}{a_{k+1} \left\{ \Gamma \left( \frac{a+1}{2n} \right) \right\}^{-2} \left\{ \prod_{j=1}^{m} k_j (k_j - 1/2)^{(\alpha_j + 2)/2} \right\}}
\times \frac{\left\{ \prod_{j=m+1}^{n} k_j^{q_j + 1 + (a+1)/2n} \right\}}{a_{k+1} \left\{ \prod_{j=m+1}^{n} k_j^{q_j + 1 + (a+1)/2n} \right\}}
\times \frac{\left\{ \prod_{j=m+1}^{n} k_j^{q_j + 1 + (a+1)/2n} \right\}}{a_{k+1} \left\{ \prod_{j=m+1}^{n} k_j^{q_j + 1 + (a+1)/2n} \right\}}
\]
\[
= \frac{a_k}{a_{k+1}} \frac{\prod_{j=1}^{m} k_j (k_j - 1/2)^{(\alpha_j + 2)/2} \left\{ \prod_{j=m+1}^{n} k_j^{q_j + 1 + (a+1)/2n} \right\}}{\prod_{j=1}^{m} (k_j + 1)(k_j + 1/2)^{(\alpha_j + 2)/2} \left\{ \prod_{j=m+1}^{n} (k_j + 1)(q_j + 1 + (a+1)/2n) \right\}}.
\]
Let
\[
G(x) = \frac{\prod_{j=1}^{m} x_j (x_j - 1/2)^{(\alpha_j + 2)/2} \prod_{j=m+1}^{n} x_j^{(q_j + 1 + (a+1)/2n)}}{\prod_{j=1}^{m} (x_j + 1)(x_j + 1/2)^{(\alpha_j + 2)/2} \prod_{j=m+1}^{n} (x_j + 1)(q_j + 1 + (a+1)/2n)}.
\]
\[ \log G(x) = \sum_{j=1}^{m} \log|x_j(x_j - 1/2)^{(\alpha_j + 2)/2}| + \sum_{j=m+1}^{n} \log x_j^{(q_j + 1 + (a+1)/2n)} - \sum_{j=1}^{m} \log(x_j + 1)(x_j + \frac{1}{2})^{(\alpha_j + 2)/2} - \sum_{j=m+1}^{n} \log(x_j + 1)^{(q_j + 1 + (a+1)/2n)} \]

By logarithmic differentiation, we obtain
\[ \frac{G'(x)}{G(x)} = \sum_{j=1}^{m} \left( \frac{1}{x_j} + \frac{(\alpha_j + 2)}{2(x_j - \frac{1}{2})} \right) + \sum_{j=m+1}^{n} \frac{q_j + 1 + \frac{(a+1)}{2n}}{x_j} - \sum_{j=1}^{m} \frac{1}{x_j} + \frac{(\alpha_j + 2)}{2(x_j + \frac{1}{2})} - \sum_{j=m+1}^{n} \frac{q_j + 1 + \frac{(a+1)}{2n}}{x_j + 1}. \]

Let
\[ t(x_j) = \sum_{j=1}^{m} \frac{1}{x_j} + \frac{(\alpha_j + 2)}{2(x_j - \frac{1}{2})} + \sum_{j=m+1}^{n} \frac{q_j + 1 + \frac{(a+1)}{2n}}{x_j}. \]

Then \( t(x_j) - t(x_{j+1}) > 0 \) for any \( x_j > 0 \). Hence \( t(x_j) \) is decreasing function and subsequently \( G'(x_j) > 0 \) for \( x_j > 0 \). Hence \( |\beta_k|/|\beta_{k'}| \) is non-decreasing if \( |a_k|/|a_{k'}| \) is non-decreasing.

\section*{3. Main Results}

**Theorem 3.1.** Let \( u(z, t) \) be an entire function converges to solution of problem \((1.1)\) corresponding to given data function \( \phi(z) \) in \((1.1)\) having order \( \rho(u) \). Then
\[ \rho(u) = \limsup_{\|k\| \to \infty} \frac{\|2k\| \log \|k\|}{-\log |\beta_k|} \quad (3.1) \]

where \( \beta_k \) is given by \((2.7)\).

**Proof.** It is well known \cite{7} Thm. 1 that if \( f(z) = \sum_{\|k\|=0}^{\infty} a_k z^{2k} \) be an entire function of order \( \rho(f) \) then
\[ \rho(f) = \limsup_{\|k\| \to \infty} \frac{\|2k\| \log \|k\|}{-\log |a_k|}. \quad (3.2) \]

Hence for the function \( u(z, 0) = \sum_{\|k\|=0}^{\infty} \beta_k z_1^{2k_1} \ldots z_n^{2k_n} \), we have

\[
\frac{1}{\rho(u)} = \liminf_{\|k\| \to \infty} \frac{-\log |\beta_k|}{\|2k\| \log \|2k\|} = \liminf_{\|k\| \to \infty} \frac{\log |a_k|^{-1} - \log \left\{ \Gamma \left( \frac{(a+1)}{2n} \right) \right\} \left( n 2^n K^{n-m} \sum_{j=1}^{\infty} \prod_{j=1}^{n} k_j^{p_j} \right) }{\|2k\| \log \|2k\|} = \liminf_{\|k\| \to \infty} \frac{\log |a_k|^{-1} - \log \left\{ \Gamma \left( \frac{(a+1)}{2n} \right) \right\} \left( n 2^n K^{n-m} \sum_{j=1}^{\infty} \prod_{j=1}^{n} k_j^{p_j} \right) }{\log \|2k\|} = \liminf_{\|k\| \to \infty} \frac{\log |a_k|^{-1}}{\|2k\| \log \|2k\|}.
\]

Now using \((3.2)\) for data function \( \phi(z) \), we get the required results.
Theorem 3.2. Let \( u(z,t) \) be an entire function converges to solution of (1.1) corresponding to given data function \( \phi(z) \) in (1.1) having type \( T(u) \). Then
\[
(\varepsilon \rho(u) T(u))^{1/\rho(u)} = \limsup_{\|k\| \to \infty} \left\{ \|2k\|^{1/\rho(u)} \left[ |\beta_k| d_k(G) \right]^{1/\|2k\|} \right\}, \quad (0 < \rho(u) < \infty).
\]

Proof. For an entire function \( f(z) = \sum_{\|k\|=0}^{\infty} a_k z^{2k} \), Gol'dberg [7, Thm. 1] obtained type in terms of the coefficients of its Taylor series expansion as
\[
(\varepsilon \rho(f) T(f))^{1/\rho(f)} = \limsup_{\|k\| \to \infty} \left\{ \|2k\|^{1/\rho(f)} \left[ |a_k| d_k(G) \right]^{1/\|2k\|}, \quad (0 < \rho(f) < \infty) \right\}. \tag{3.3}
\]
It can be seen that
\[
\left[ |\beta_k| d_k(G) \right]^{1/\|2k\|} \to \left[ |a_k| d_k(G) \right]^{1/\|2k\|} \quad \text{as} \quad \|k\| \to \infty. \tag{3.4}
\]
Hence the result follows by using (3.3) for data function \( \phi(z) \) and taking into account the equation (3.4). \( \square \)

In analogy with the definitions of order \( \rho(u) \) and type \( T(u) \), we define lower order \( \lambda(u) \) and lower type \( t(u) \) as
\[
\lambda(u) = \liminf_{R \to \infty} \frac{\log \log M_{u,S}(R)}{\log R}, \quad t(u) = \liminf_{R \to \infty} \frac{\log M_{u,S}(R)}{R^{t(u)}}, \quad 0 < \rho(u) < \infty.
\]

Theorem 3.3. Let \( u(z,t) \) be an entire function converges to the problem (1.1) corresponding to data function \( \phi(z) \) in (1.1) having lower order \( \lambda(u) \). Then
\[
\lambda(u) \geq \liminf_{\|k\| \to \infty} \frac{\|2k\| \log \|2k\|}{-\log |\beta_k|}. \tag{3.5}
\]
Also if \( |\beta_k|/|\beta_k'|, \) where \( \|k'\| = \|k\| + 1 \), is a non-decreasing function of \( k \), then equality holds in (3.5).

Proof. For entire function \( f(z) = \sum_{\|k\|=0}^{\infty} a_k z^{2k} \ldots z^{2k_n} \), if \( |a_k|/|a_k'| \) forms a non-decreasing function of \( k \) then we have [1, Thm. 1]
\[
\lambda(f) = \liminf_{\|k\| \to \infty} \frac{\|2k\| \log \|2k\|}{\log |a_k'|^{-1}}. \tag{3.6}
\]
Let \( |\beta_k|/|\beta_k'| \) forms a non-decreasing function of \( k \) for \( k > k_0 \). Applying Lemma 2.3 and (3.6) to \( u(z,0) = \sum_{\|k\|=0}^{\infty} \beta_k z^{2k_1} \ldots z^{2k_n} \), we obtain
\[
\frac{1}{\lambda(u)} = \limsup_{\|k\| \to \infty} \frac{\log |a_k|^{-1} - \log \left( \left| C \prod_{j=1}^{n} k_j^{p_j} \right| \right)}{\|2k\| \log \|2k\|} = \limsup_{\|k\| \to \infty} \frac{\log |a_k|^{-1}}{\|2k\| \log \|2k\|}
\]
Then \( \lambda(u) = \lambda(\phi). \) \( \square \)

In a similar manner we can prove the following theorem.

Theorem 3.4. Let \( u(z,t) \) be an entire function converging to a solution of (1.1) corresponding to data function \( \phi(z) \) in (1.1) having lower type \( t(u) \). Then
\[
t(u) \geq \liminf_{\|k\| \to \infty} \frac{\|2k\|}{\varepsilon \rho(u) |\beta_k|^{t(u)/\|2k\|}}. \tag{3.7}
\]
Also, if \( |\beta_k|/|\beta_k'|, \) where \( \|k'\| = \|k\| + 1 \), is a non-decreasing function of \( k > k_0 \), then equality holds in (3.7).
References


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