

## LINEAR SECOND-ORDER PROBLEMS WITH STURM-LIOUVILLE-TYPE MULTI-POINT BOUNDARY CONDITIONS

BRYAN P. RYNNE

ABSTRACT. We consider the eigenvalue problem for the equation  $-u'' = \lambda u$  on  $(-1, 1)$ , together with general Sturm-Liouville-type, multi-point boundary conditions at  $\pm 1$ . We show that the basic spectral properties of this problem are similar to those of the standard Sturm-Liouville problem with separated boundary conditions. In particular, for each integer  $k \geq 0$  there exists a unique, simple eigenvalue  $\lambda_k$  whose eigenfunctions have ‘oscillation count’ equal to  $k$ .

Similar multi-point problems have been considered before for Dirichlet-type or Neumann-type multi-point boundary conditions, or a mixture of these. Different oscillation counting methods have been used in each of these cases. A new oscillation counting method is used here which unifies and extends all the results for these special case to the general Sturm-Liouville-type boundary conditions.

### 1. INTRODUCTION

We consider the linear eigenvalue problem consisting of the equation

$$-u'' = \lambda u, \quad \text{on } (-1, 1), \quad (1.1)$$

where  $\lambda \in \mathbb{R}$ , together with the general multi-point boundary conditions

$$\alpha_0^\pm u(\pm 1) + \beta_0^\pm u'(\pm 1) = \sum_{i=1}^{m^\pm} \alpha_i^\pm u(\eta_i^\pm) + \sum_{i=1}^{m^\pm} \beta_i^\pm u'(\eta_i^\pm), \quad (1.2)$$

where  $m^\pm \geq 1$  are integers,  $\alpha_0^\pm, \beta_0^\pm \in \mathbb{R}$ , and, for each  $i = 1, \dots, m^\pm$ , the numbers  $\alpha_i^\pm, \beta_i^\pm \in \mathbb{R}$ , and  $\eta_i^\pm \in [-1, 1]$ , with  $\eta_i^\pm \neq \pm 1$ . We write  $\alpha^\pm := (\alpha_1^\pm, \dots, \alpha_{m^\pm}^\pm) \in \mathbb{R}^{m^\pm}$ , and similarly for  $\beta^\pm, \eta^\pm$ . The notation  $\alpha^\pm = 0$  or  $\beta^\pm = 0$ , will mean the zero vector in  $\mathbb{R}^{m^\pm}$ , as appropriate. Naturally, an *eigenvalue* is a number  $\lambda$  for which (1.1)-(1.2), has a non-trivial solution  $u$  (an *eigenfunction*). The *spectrum*,  $\sigma$ , is the set of eigenvalues. Although the boundary conditions (1.2) are non-local, for ease of discussion we will usually say that the condition with superscript  $\pm$  holds ‘at the end point  $\pm 1$ ’.

---

2000 *Mathematics Subject Classification*. 34B05, 34B10, 34B24, 34B25.

*Key words and phrases*. Second order ordinary differential equations; multi-point boundary conditions; Sturm-Liouville problems.

©2012 Texas State University - San Marcos.

Submitted October 28, 2011. Published August 21, 2012.

Throughout we will suppose that the following conditions hold:

$$\alpha_0^\pm \geq 0, \quad \alpha_0^\pm + |\beta_0^\pm| > 0, \quad (1.3)$$

$$\pm\beta_0^\pm \geq 0, \quad (1.4)$$

$$\left(\frac{\sum_{i=1}^{m^\pm} |\alpha_i^\pm|}{\alpha_0^\pm}\right)^2 + \left(\frac{\sum_{i=1}^{m^\pm} |\beta_i^\pm|}{\beta_0^\pm}\right)^2 < 1, \quad (1.5)$$

with the convention that if any denominator in (1.5) is zero then the corresponding numerator must also be zero, and the corresponding fraction is omitted from (1.5) (by (1.3), at least one denominator is nonzero in each condition). The condition (1.3) simply ensures that the boundary conditions at  $\pm 1$  actually involve the values  $u(\pm 1)$  or  $u'(\pm 1)$ . We will describe the motivation and consequences of (1.4) and (1.5) further here, and also in the following sections.

When  $\alpha^\pm = \beta^\pm = 0$  the multi-point boundary conditions (1.2) reduce to standard (*single-point*) separated conditions at  $x = \pm 1$ , and the overall multi-point problem (1.1)-(1.2) reduces to a separated, linear Sturm-Liouville problem. Thus, we will term the conditions (1.2) *Sturm-Liouville-type* boundary conditions. The spectral properties of the separated problem are of course well known, see for example [3], but the spectral properties of the above general multi-point problem have not previously been obtained. Indeed, it is only recently that the basic spectral properties of any multi-point problems have been obtained, and these were obtained under more restrictive assumptions on the boundary conditions.

Boundary value problems with multi-point boundary conditions have been extensively studied recently, see for example, [1, 4, 5, 6, 7, 9, 10, 12, 13, 14, 15, 16, 17], and the references therein. Many of these papers consider the problem on the interval  $(0, 1)$ , and impose a single-point Dirichlet or Neumann condition at the end-point  $x = 0$ , and a multi-point condition at  $x = 1$ . In our notation, these particular single-point conditions correspond to the special cases  $\beta_0^- = 0$  or  $\alpha_0^- = 0$ , respectively (as well as  $\alpha^- = \beta^- = 0$ ), so of course are covered by our results here. We have used the interval  $(-1, 1)$  in order to simplify the notation for problems with multi-point boundary conditions at both end-points — our results are, of course, independent of the interval on which the problem is posed. Problems with a single-point boundary condition at one end-point can often be treated using shooting methods (starting at the end with the single-point condition) and so are considerably simpler to deal with than problems having multi-point boundary conditions at both end-points (for which shooting is not possible). Problems with multi-point conditions at both end-points have been considered in [5, 7, 9, 13, 14] (and in many references therein — the bibliography in [9] is particularly extensive).

The papers [13] and [14] discussed the following particular special cases, or *types*, of multi-point boundary conditions:

$$\text{Dirichlet-type: } \sum_{i=1}^{m^\pm} |\alpha_i^\pm| < 1 = \alpha_0^\pm, \quad \beta_0^\pm = 0, \quad \beta^\pm = 0; \quad (1.6)$$

$$\text{Neumann-type: } \alpha_0^\pm = 0, \quad \alpha^\pm = 0, \quad \sum_{i=1}^{m^\pm} |\beta_i^\pm| < 1 = \beta_0^\pm. \quad (1.7)$$

This terminology is motivated by observing that a Dirichlet-type (respectively Neumann-type) condition reduces to a single-point Dirichlet (respectively Neumann) condition when  $\alpha = 0$  (respectively  $\beta = 0$ ). The case of a Dirichlet-type condition at one end point and a Neumann-type condition at the other end point was also discussed in [14], where such conditions were termed *mixed*. Clearly, the hypotheses (1.6) and (1.7) are special cases of the general hypothesis (1.5), and in these cases (1.4) can be attained simply by multiplying the boundary condition at  $x = -1$  by  $-1$ , so (1.4) is trivial. Hence, our results here will unify and generalise all the results in [13] and [14].

It was shown in [13] and [14] that the spectra of these particular boundary value problems have many of the ‘standard’ properties of the spectrum of the separated Sturm-Liouville problem, specifically:

- ( $\sigma$ -a)  $\sigma$  is a strictly increasing sequence of real eigenvalues  $\lambda_k$ ,  $k = 0, 1, \dots$ ;
- ( $\sigma$ -b)  $\lim_{k \rightarrow \infty} \lambda_k = \infty$ ;

for each  $k \geq 0$ :

- ( $\sigma$ -c)  $\lambda_k$  has geometric multiplicity 1;
- ( $\sigma$ -d) the eigenfunctions of  $\lambda_k$  have an ‘oscillation count’ equal to  $k$ .

In the separated problem the oscillation count referred to in property ( $\sigma$ -d) is simply the number of interior (*nodal*) zeros of an eigenfunction. However, in the multi-point problem it was found in [13] and [14] that this method of counting eigenfunction oscillations no longer yields property ( $\sigma$ -d), and alternative, slightly ad hoc, methods were adopted, with different approaches being used for different types of problem. We will discuss this further below, and a more detailed discussion is given in Section 9.4 of [14]. Suffice it to say, for now, that the eigenfunction oscillation count we adopt here, based on a Prüfer angle approach (see Section 4.1), extends and unifies the disparate approaches adopted in [13] and [14].

It was also shown in [13] and [14] that, in order to obtain the spectral properties ( $\sigma$ -a)-( $\sigma$ -d), the conditions (1.6) and (1.7) are optimal for the Dirichlet-type and Neumann-type conditions respectively, in the sense that, in either of these cases, if the inequality  $< 1$  in (1.6) or (1.7) is relaxed to  $< 1 + \epsilon$ , for any  $\epsilon > 0$ , then  $\sigma$  need not have all the properties ( $\sigma$ -a)-( $\sigma$ -d). For the general Sturm-Liouville-type boundary conditions (1.2) it will be shown here that if (1.4) and (1.5) hold then  $\sigma$  has the properties ( $\sigma$ -a)-( $\sigma$ -d), and if either (1.4) or (1.5) do not hold then  $\sigma$  need not have all these properties.

**Remark 1.1.** (i) Changing the length of the interval on which we consider the problem rescales the coefficients  $\beta_0^\pm, \beta^\pm$ , but not the coefficients  $\alpha_0^\pm, \alpha^\pm$ . Such a change should not affect our hypotheses on the coefficients, and indeed the condition (1.5) is invariant with respect to such a rescaling. Thus, the form of condition (1.5) seems natural in this respect.

(ii) In the separated case (that is, when  $\alpha^\pm = \beta^\pm = 0$ ) the sign condition (1.4) ensures that  $\lambda_0 > 0$  (except in the Neumann case, when  $\lambda_0 = 0$ ), and if this sign condition does not hold then negative eigenvalues may exist. It will be shown below that this is also true for the above Sturm-Liouville-type boundary conditions (assuming that (1.3) and (1.5) hold); it will also be shown that negative eigenvalues may have geometric multiplicity 2. Of course, this cannot happen in the separated problem due to uniqueness of the solutions for initial value problems associated with (1.1). Hence, the full set of ‘standard’ properties ( $\sigma$ -a)-( $\sigma$ -d) need not hold if

the sign condition (1.4) is not satisfied.

(iii) In principle, we should consider the possibility of complex eigenvalues, especially as the problem is not ‘self-adjoint’ (without defining this precisely). Indeed, if we did not impose the condition (1.5) then complex eigenvalues could in fact occur. However, with this condition it can be shown that all eigenvalues must be real — the proof is very similar to the proof of Lemma 4.9 below, which shows that under our hypotheses the eigenvalues are positive. In the light of this we will simply take it for granted throughout the paper that all our coefficients, functions and function spaces are real.

(iv) We primarily consider the spectral properties  $(\sigma\text{-a})$ - $(\sigma\text{-d})$  because of their potential applications to nonlinear problems (many of the cited references use eigenvalue properties to deal with nonlinear problems, using relatively standard arguments such as Rabinowitz’ global bifurcation theory). Of course, there are many other linear spectral properties that could be investigated, such as eigenfunction expansions (the problem is not self-adjoint, so this would not be trivial). However, for brevity, we will omit any discussion of nonlinear problems or other linear properties here.

(v) Boundary conditions having a more general non-local dependence on the function  $u$  than the finite sums of values at points in the interval  $(-1, 1)$  (as in (1.2)) have also been considered recently by several authors, see for example [15] and the references therein. These papers have considered Dirichlet-type and Neumann-type boundary conditions in which the finite summations have been replaced with Lebesgue-Stieltjes integrals, see [15] for further details (finite summations can be obtained simply by using step functions in Lebesgue-Stieltjes integrals, so such integral conditions generalise the finite summation conditions). The methods and results below can readily be extended to deal with such integral formulations of the boundary conditions — the only significant additional step required is dealing with the necessary measure and integration theory. These measure-theoretic details are described, for Dirichlet-type and Neumann-type conditions, in [5]. Since this step is relatively routine we will avoid all such measure-theoretic difficulties here by simply considering the finite summation conditions (1.2).

**1.1. Plan of the paper.** The paper is organised as follows. In Section 2 we introduce various function spaces, and then use these to define an operator realization of the multi-point problem, and state the main properties of this operator. In Section 3 we prove an existence and uniqueness result for a problem consisting of equation (1.1) together with a single, multi-point, boundary condition. This problem could be regarded as a multi-point analogue of the usual initial value problem for equation (1.1). We also give some counter examples which show that this uniqueness result can fail in the multi-point setting when  $\lambda < 0$ . As mentioned in Remark 1.1-(ii), the uniqueness result for this ‘multi-point, initial value problem’ then implies the simplicity of the eigenvalues of (1.1), (1.2), in the usual manner, and the loss of this uniqueness can result in the existence of eigenvalues having geometric multiplicity 2. In particular, this shows the necessity of the sign condition (1.4) if we wish to obtain all the properties  $(\sigma\text{-a})$ - $(\sigma\text{-d})$ .

Our main results are obtained in Section 4. In Section 4.1 we describe a Prüfer angle method of counting the oscillations of the eigenfunctions, and we then use this technique in Section 4.2 to obtain our main results regarding the properties of the spectrum. We also show that this Prüfer angle construction generalises and

unifies the various oscillation counting methods used in [13] and [14] in the Dirichlet-type, Neumann-type and mixed cases respectively. In Section 4.3 we show that, under suitable additional hypotheses, the principal eigenfunction is positive. In Section 4.4 we reinterpret the eigenvalues as the characteristic values of the inverse operator constructed in Section 2, and show that these characteristic values have algebraic multiplicity 1; this result then yields the value of the topological degree of an associated linear operator. In Section 4.5 we give some counter examples to show the necessity of the hypothesis (1.5).

**1.2. Some further notation.** Clearly, the eigenvalues  $\lambda_k$  (and other objects to be introduced below) depend on the values of the coefficients  $\alpha_0^\pm, \beta_0^\pm, \alpha^\pm, \beta^\pm, \eta^\pm$ , but in general we regard these coefficients as fixed, and omit them from our notation. However, at certain points of the discussion it will be convenient to regard some, or all, of these coefficients as variable, and to indicate the dependence of various functions on these coefficients. To do this concisely we will write:

$$\alpha_0 := (\alpha_0^-, \alpha_0^+) \in \mathbb{R}^2 \text{ (for given numbers } \alpha_0^\pm \in \mathbb{R}\text{);}$$

$$\alpha := (\alpha^-, \alpha^+) \in \mathbb{R}^{m^-+m^+} \text{ (for given coefficient vectors } \alpha^\pm \in \mathbb{R}^{m^\pm}\text{);}$$

and similarly for  $\beta_0, \beta, \eta$ . We also define  $\mathbf{0} := (0, 0) \in \mathbb{R}^{m^-+m^+}$ . We may then write, for example,  $\lambda_k(\alpha, \beta)$  to indicate the dependence of  $\lambda_k$  on  $(\alpha, \beta)$ .

In most of the paper we will regard  $(\alpha_0, \beta_0)$  as fixed, but at some points in the discussion it will be convenient to allow  $(\alpha, \beta)$  to vary, so long as the conditions (1.3)-(1.5) continue to hold. To describe this we define the following sets, for any  $(\alpha_0, \beta_0) \in \mathbb{R}^4$  satisfying (1.3) and (1.4):

$$\mathcal{B}(\alpha_0^\pm, \beta_0^\pm) := \{(\alpha^\pm, \beta^\pm) \in \mathbb{R}^{2m^\pm} : (\alpha_0^\pm, \beta_0^\pm, \alpha^\pm, \beta^\pm) \text{ satisfies (1.5)}\},$$

$$\mathcal{B}(\alpha_0, \beta_0) := \{(\alpha, \beta) \in \mathbb{R}^{2(m^-+m^+)} : (\alpha_0, \beta_0, \alpha, \beta) \text{ satisfies (1.5)}\}$$

(so  $\mathcal{B}(\alpha_0, \beta_0)$  is isomorphic to  $\mathcal{B}(\alpha_0^-, \beta_0^-) \times \mathcal{B}(\alpha_0^+, \beta_0^+)$ ); we also define the set

$$\mathcal{B} := \{(\alpha_0, \beta_0, \alpha, \beta) \in \mathbb{R}^{2(2+m^-+m^+)} : (1.3)-(1.5) \text{ hold}\}.$$

At some points, when dealing with individual boundary conditions, it will be convenient to let  $\nu$  denote one of the signs  $\{\pm\}$ , in which case, for a function  $u$ , the notation  $u(\nu)$  will denote the value of  $u$  at the corresponding end point  $\pm 1$ .

## 2. AN OPERATOR REALISATION OF THE MULTI-POINT PROBLEM

For any integer  $n \geq 0$ , let  $C^n[-1, 1]$  denote the usual Banach space of  $n$ -times continuously differentiable functions on  $[-1, 1]$ , with the usual sup-type norm, denoted by  $|\cdot|_n$ . A suitable space in which to search for solutions of (1.1), incorporating the boundary conditions (1.2), is the space

$$X := \{u \in C^2[-1, 1] : u \text{ satisfies (1.2)}\},$$

$$\|u\|_X := |u|_2, \quad u \in X.$$

Letting  $Y := C^0[-1, 1]$ , with the norm  $\|\cdot\|_Y := |\cdot|_0$ , we now define an operator  $\Delta : X \rightarrow Y$  by

$$\Delta u := u'', \quad u \in X.$$

By the definition of the spaces  $X, Y$ , the operator  $\Delta$  is a well-defined, bounded, linear operator, and the eigenvalue problem (1.1)-(1.2) can be rewritten in the form  $-\Delta(u) = \lambda u, u \in X$ . We will consider the eigenvalue problem in Section 4.2 below, for now we will consider the invertibility of  $\Delta$ .

In the Neumann-type case (that is, when  $\alpha_0^\pm = 0$ ) it is clear that any constant function  $c$  lies in  $X$ , and  $\Delta c = 0$ , so  $\Delta$  cannot be invertible. Thus, to obtain invertibility it is necessary to exclude the Neumann-type case. In view of the assumption (1.3), we can achieve this by imposing the further condition

$$\alpha_0^- + \alpha_0^+ > 0. \quad (2.1)$$

The following theorem shows that this condition is sufficient to ensure invertibility of  $\Delta$ .

**Theorem 2.1.** *Suppose that (1.3)–(1.5) and (2.1) hold. Then  $\Delta : X \rightarrow Y$  has a bounded inverse.*

*Proof.* We will show that the equation

$$\Delta u = h, \quad h \in Y, \quad (2.2)$$

has a unique solution for all  $h \in Y$ . Following the proof of Theorem 3.1 in [13] (which considers Dirichlet-type conditions and constructs a solution of (2.2) via a compact integral operator) shows that it suffices to prove the uniqueness of the solutions of (2.2). To prove this we observe that any solution  $u_0$  of (2.2) with  $h = 0$  must have the form  $u_0(x) = c_0 + c_1x$ , for some  $(c_0, c_1) \in \mathbb{R}^2$ , and substituting  $u_0$  into the boundary conditions (1.2) yields the pair of equations

$$c_0 \left( \alpha_0^\pm - \sum_{i=1}^{m^\pm} \alpha_i^\pm \right) + c_1 \left( \beta_0^\pm - \sum_{i=1}^{m^\pm} \beta_i^\pm \pm \alpha_0^\pm - \sum_{i=1}^{m^\pm} \alpha_i^\pm \eta_i^\pm \right) = 0. \quad (2.3)$$

It now follows from (1.3)–(1.5) that

$$\alpha_0^\pm - \sum_{i=1}^{m^\pm} \alpha_i^\pm \geq 0, \quad \pm \left( \beta_0^\pm - \sum_{i=1}^{m^\pm} \beta_i^\pm \pm \alpha_0^\pm - \sum_{i=1}^{m^\pm} \alpha_i^\pm \eta_i^\pm \right) > 0,$$

and it follows from (2.1) that at least one of the left hand inequalities here is strict. These sign properties now ensure that the determinant associated with the pair of equations (2.3) is non-zero, so that  $(c_0, c_1) = (0, 0)$  is the unique solution of (2.3). This proves the desired uniqueness result for (2.2), and hence proves the theorem.  $\square$

In applications, continuity properties of the inverse operator  $\Delta^{-1}$  with respect to the various parameters in the problem are important. We will describe one such result — other such results could be obtained in a similar manner.

**Corollary 2.2.** *The operator  $\Delta(\alpha_0, \beta_0, \alpha, \beta)^{-1} : Y \rightarrow C^2[-1, 1]$  depends continuously on  $(\alpha_0, \beta_0, \alpha, \beta) \in \mathcal{B} \setminus \{(\alpha_0, \beta_0, \alpha, \beta) : \alpha_0^- + \alpha_0^+ = 0\}$  (with respect to the usual topology for bounded linear operators).*

*Proof.* The functions  $\Phi^\pm$  in the construction of  $\Delta(\alpha_0, \beta_0, \alpha, \beta)^{-1}$  in the proof of Theorem 2.1 in [13] are continuous with respect to  $(\alpha_0, \beta_0, \alpha, \beta)$ , so the result follows immediately from that proof.  $\square$

**Remark 2.3.** We have used the spaces  $C^n[-1, 1]$ ,  $n = 0, 2$ , to define the operator  $\Delta$ , and Theorem 2.1 showed that the resulting operator is invertible. This is the function space setting that we will use here. However, one could also use a Sobolev space setting to define a similar operator as follows. For arbitrary fixed  $q \geq 1$ , let

$$\tilde{Y} := L^q(-1, 1), \quad \tilde{X} := \{u \in W^{2,q}[-1, 1] : u \text{ satisfies (1.2)}\}.$$

Then  $\tilde{\Delta} : \tilde{X} \rightarrow \tilde{Y}$  can be defined in the obvious manner, and a similar proof to that of Theorem 2.1 shows that  $\tilde{\Delta}$  is invertible.

### 3. PROBLEMS WITH A SINGLE BOUNDARY CONDITION

In this section we consider the following problem with a single, multi-point boundary condition,

$$-u'' = \lambda u, \quad \text{on } \mathbb{R}, \quad (3.1)$$

$$\alpha_0 u(\eta_0) + \beta_0 u'(\eta_0) = \sum_{i=1}^m \alpha_i u(\eta_i) + \sum_{i=1}^m \beta_i u'(\eta_i), \quad (3.2)$$

where  $m \geq 1$ ,  $\alpha_0, \beta_0, \eta_0 \in \mathbb{R}$ , and  $\alpha, \beta, \eta \in \mathbb{R}^m$ . The conditions (1.3) and (1.5) have obvious analogues in the current setting, simply by omitting the superscripts  $\pm$ , which we will use without further comment, while the condition (1.4) has no analogue here and  $\beta_0$  may have either sign. For any  $(\alpha_0, \beta_0)$  satisfying (1.3) we let  $\mathcal{B}(\alpha_0, \beta_0)$  denote the set of  $(\alpha, \beta) \in \mathbb{R}^{2m}$  satisfying (1.5).

**Theorem 3.1.** *Suppose that  $(\alpha_0, \beta_0, \alpha, \beta)$  satisfies (1.3) and (1.5), and  $\lambda \geq 0$ . Then the set of solutions of (3.1), (3.2), is one-dimensional.*

*Proof.* If  $\lambda = 0$  then any solution of (3.1) has the form of  $u_0$  used in the proof of Theorem 2.1, and substituting  $u_0$  into (3.2) yields a linear equation relating the coefficients  $c_0, c_1$ . A similar argument to the proof of Theorem 2.1 now shows that the set of solutions of this equation is one-dimensional.

Now suppose that  $\lambda > 0$ . For any  $s > 0$ ,  $\theta \in \mathbb{R}$ , we define  $w(s, \theta) \in C^1(\mathbb{R})$  by

$$w(s, \theta)(x) := \sin(sx + \theta), \quad x \in \mathbb{R}. \quad (3.3)$$

Clearly, any solution of (3.1) must have the form  $u = Cw(s, \theta)$ , with  $s = \lambda^{1/2}$  and suitable  $C$ ,  $\theta \in \mathbb{R}$ . For the rest of this proof we regard  $\theta$ ,  $\alpha$ ,  $\beta$  as variable, but all the other parameters and coefficients will be regarded as fixed and omitted from the notation when this is convenient. Defining  $\Gamma : \mathbb{R} \times \mathbb{R}^{2m} \rightarrow \mathbb{R}$  by

$$\begin{aligned} \Gamma(\theta, \alpha, \beta) := & \alpha_0 \sin(s\eta_0 + \theta) + s\beta_0 \cos(s\eta_0 + \theta) - \sum_{i=1}^m \alpha_i \sin(s\eta_i + \theta) \\ & - s \sum_{i=1}^m \beta_i \cos(s\eta_i + \theta), \end{aligned}$$

it is clear that  $\Gamma$  is  $C^1$ , and substituting (3.3) into (3.2) shows that  $w(s, \theta)$  satisfies (3.1), (3.2) if and only if

$$\Gamma(\theta, \alpha, \beta) = 0. \quad (3.4)$$

Hence, it suffices to consider the set of solutions of (3.4).

Next, by definition, for any  $(\alpha, \beta) \in \mathbb{R}^{2m}$  the function  $\Gamma(\cdot, \alpha, \beta)$  is  $\pi$ -antiperiodic, so to prove the theorem it suffices to show that if  $(\alpha, \beta) \in \mathcal{B}(\alpha_0, \beta_0)$  then  $\Gamma(\cdot, \alpha, \beta)$  has exactly one zero in the interval  $[0, \pi)$  (by  $\pi$ -antiperiodicity, other zeros of  $\Gamma(\cdot, \alpha, \beta)$  do not contribute distinct solutions of (3.1), (3.2)). We will prove this by a continuation argument.

We first observe that if  $(\alpha, \beta) = (0, 0)$  then  $\Gamma(\cdot, 0, 0)$  has exactly 1 zero in  $[0, \pi)$  and this zero is simple. To extend this property to  $(\alpha, \beta) \neq (0, 0)$  we will require the following lemma ( $\Gamma_\theta$  will denote the partial derivative of  $\Gamma$  with respect to  $\theta$ ).

**Lemma 3.2.** *Suppose that  $(\alpha_0, \beta_0, \alpha, \beta)$  satisfies (1.3) and (1.5), and  $\lambda > 0$ . Then*

$$\Gamma(\theta, \alpha, \beta) = 0 \implies \Gamma_\theta(\theta, \alpha, \beta) \neq 0.$$

*Proof.* Suppose, on the contrary, that

$$\Gamma(\theta, \alpha, \beta) = \Gamma_\theta(\theta, \alpha, \beta) = 0, \quad (3.5)$$

for some  $\theta \in \mathbb{R}$  and  $(\alpha, \beta) \in \mathcal{B}(\alpha_0, \beta_0)$ . We now regard  $(\theta, \alpha, \beta)$  as fixed, and write

$$S(\eta) := \sin(s\eta + \theta), \quad C(\eta) := \cos(s\eta + \theta).$$

With this notation, equations (3.5) become

$$\alpha_0 S(\eta_0) + s\beta_0 C(\eta_0) = \sum_{i=1}^m (\alpha_i S(\eta_i) + s\beta_i C(\eta_i)), \quad (3.6)$$

$$\alpha_0 C(\eta_0) - s\beta_0 S(\eta_0) = \sum_{i=1}^m (\alpha_i C(\eta_i) - s\beta_i S(\eta_i)). \quad (3.7)$$

By (1.5) we can choose  $b_0 \in [0, \pi/2]$  such that, with  $C_b := \cos b_0$ ,  $S_b := \sin b_0$ ,

$$\sum_{i=1}^m |\alpha_i| \leq C_b \alpha_0, \quad \sum_{i=1}^m |\beta_i| \leq S_b |\beta_0|, \quad (3.8)$$

with at least one strict inequality in (3.8).

Now suppose that  $\beta_0 \geq 0$ . Elementary operations on (3.6), (3.7) now yield

$$\begin{aligned} & C_b \alpha_0 + S_b s \beta_0 \\ &= \sum_{i=1}^m \alpha_i \left( C_b (S(\eta_0) S(\eta_i) + C(\eta_0) C(\eta_i)) + S_b (C(\eta_0) S(\eta_i) - S(\eta_0) C(\eta_i)) \right) \\ & \quad + s \sum_{i=1}^m \beta_i \left( C_b (S(\eta_0) C(\eta_i) - C(\eta_0) S(\eta_i)) + S_b (C(\eta_0) C(\eta_i) + S(\eta_0) S(\eta_i)) \right) \\ &= \sum_{i=1}^m \alpha_i \left( C_b \cos s(\eta_0 - \eta_i) - S_b \sin s(\eta_0 - \eta_i) \right) \\ & \quad + s \sum_{i=1}^m \beta_i \left( C_b \sin s(\eta_0 - \eta_i) + S_b \cos s(\eta_0 - \eta_i) \right) \\ &= \sum_{i=1}^m \alpha_i \cos s(b_0 + \eta_0 - \eta_i) + s \sum_{i=1}^m \beta_i \sin s(b_0 - \eta_0 + \eta_i) \\ &\leq \sum_{i=1}^m |\alpha_i| + s \sum_{i=1}^m |\beta_i| < C_b \alpha_0 + S_b s \beta_0, \end{aligned}$$

by (3.8). This contradiction shows that (3.5) cannot hold, and so proves the lemma, when  $\beta_0 \geq 0$ . If  $\beta_0 < 0$  then we simply replace  $C_b \alpha_0 + S_b s \beta_0$  with  $C_b \alpha_0 - S_b s \beta_0$  in the above calculation to obtain a similar contradiction, which completes the proof of Lemma 3.2.  $\square$

Now, since the set  $\mathcal{B}(\alpha_0, \beta_0)$  is connected it follows from continuity, together with Lemma 3.2, the implicit function theorem and the  $\pi$ -antiperiodicity of  $\Gamma(\cdot, \alpha, \beta)$ , that  $\Gamma(\cdot, \alpha, \beta)$  has exactly 1 (simple) zero in  $[0, \pi)$  for all  $(\alpha, \beta) \in \mathcal{B}(\alpha_0, \beta_0)$ . This completes the proof of Theorem 3.1.  $\square$



For Dirichlet-type and Neumann-type problems, Theorem 3.1 was proved in [13] and [14], respectively. An adaptation of the proof of Lemma 3.2 also yields the following result, which will be crucial below.

**Lemma 3.3.** *Suppose that  $\lambda > 0$  and  $(\alpha_0, \beta_0, \alpha, \beta)$  satisfies (1.3) and (1.5). If  $u$  is a non-trivial solution of (3.1), (3.2) then*

$$\lambda\beta_0u(\eta_0) - \alpha_0u'(\eta_0) \neq 0. \quad (3.9)$$

*Proof.* The argument is similar to the proof of Lemma 3.2, and we use the notation from there. In particular, we suppose that  $u$  has the form of  $w$  given in (3.3), so that (3.2) takes the form (3.6), and to obtain a contradiction we suppose that (3.9) fails, that is, with this form of  $u$ ,

$$s\beta_0S(\eta_0) - \alpha_0C(\eta_0) = 0. \quad (3.10)$$

Multiplying (3.6) by  $S(\eta_0)$  and  $C(\eta_0)$ , and using (3.10), yields respectively

$$\begin{aligned} \alpha_0 &= S(\eta_0) \sum_{i=1}^m (\alpha_i S(\eta_i) + \beta_i s C(\eta_i)), \\ s\beta_0 &= C(\eta_0) \sum_{i=1}^m (\alpha_i S(\eta_i) + \beta_i s C(\eta_i)). \end{aligned}$$

If  $\beta_0 \geq 0$  then combining these inequalities and using (3.8) yields

$$\begin{aligned} C_b\alpha_0 + S_b s\beta_0 &= (C_b S(\eta_0) + S_b C(\eta_0)) \sum_{i=1}^m (\alpha_i S(\eta_i) + \beta_i s C(\eta_i)) \\ &< C_b\alpha_0 + S_b s\beta_0, \end{aligned}$$

which is the desired contradiction in this case. If  $\beta_0 < 0$  then we simply replace  $C_b\alpha_0 + S_b s\beta_0$  with  $C_b\alpha_0 - S_b s\beta_0$  in the preceding calculation to obtain a similar contradiction. This completes the proof of Lemma 3.3.  $\square$

We also have the following immediate application of Theorem 3.1 to the eigenvalue problem.

**Corollary 3.4.** *Suppose that  $(\alpha_0^\pm, \beta_0^\pm, \alpha^\pm, \beta^\pm)$  satisfy (1.3) and (1.5). Then any eigenvalue  $\lambda > 0$  of (1.1), (1.2), has geometric multiplicity one.*

**3.1. Counter examples.** The following example shows that if  $\lambda < 0$  then Theorem 3.1 need not hold.

**Example 3.5.** Consider (3.1) with  $\lambda = -1$ , together with the boundary condition

$$u(-1) + u'(-1) = \alpha_1 u(0) + \beta_2 u'(1), \quad (3.11)$$

that is, with  $\alpha_0 = \beta_0 = 1$ ,  $\beta_1 = \alpha_2 = 0$  and  $\eta_0 = -1$ ,  $\eta_1 = 0$ ,  $\eta_2 = 1$ ; we will choose  $\alpha_1$  and  $\beta_2$  below. The general solution of equation (3.1) is  $u(x) = c_+ e^x + c_- e^{-x}$ , for arbitrary  $(c_+, c_-) \in \mathbb{R}^2$ , and substituting this solution into the boundary condition (3.11) yields the equation

$$c_+(2 - \alpha_1 e - \beta_2 e^2) - c_-(\alpha_1 e^{-1} - \beta_2 e^{-2}) = 0. \quad (3.12)$$

Now, setting

$$\alpha_1 = \frac{2}{e(e^2 + 1)}, \quad \beta_2 = \frac{2}{e^2 + 1},$$

we see that  $(\alpha_0, \beta_0, \alpha, \beta)$  satisfies (1.5), and (3.12) holds for all  $(c_+, c_-) \in \mathbb{R}^2$ . Hence, the solution set of this boundary value problem is two-dimensional, and so Theorem 3.1 does not hold in this case.  $\square$

Example 3.5 can be extended to the eigenvalue problem to show that Corollary 3.4 need not hold for negative eigenvalues.

**Example 3.6.** Consider the multi-point eigenvalue problem consisting of equation (3.1) together with the pair of boundary conditions

$$u(\pm 1) \mp u'(\pm 1) = \alpha_1 u(0) \mp \beta_2 u'(\mp 1), \quad (3.13)$$

with  $\alpha_1$  and  $\beta_2$  as in Example 3.5. It can be verified (as in Example 3.5) that  $\lambda = -1$  is an eigenvalue of this boundary value problem with geometric multiplicity two. Hence, Corollary 3.4 need not hold for negative eigenvalues. We observe that both sets of boundary condition coefficients in this problem satisfy (1.5), but of course the sign condition (1.4) does not hold (which allows the negative eigenvalue).  $\square$

The final example in this section shows that if  $\lambda < 0$  then Theorem 3.1 need not hold, even with a Dirichlet-type boundary condition (that is, with  $\beta_0 = 0$  and  $\beta = 0$ ). However, this example is not relevant to the eigenvalue problem since negative eigenvalues do not occur with Dirichlet-type boundary conditions (also, in this example  $\eta_1 < \eta_0 < \eta_2$ , which is not consistent with the distribution of these points in the eigenvalue problem).

**Example 3.7.** Consider (3.1) with  $\lambda = -1$ , together with the boundary condition

$$u(0) = \frac{e(e^2 - 1)}{e^4 - 1} (u(-1) + u(1)). \quad (3.14)$$

It can be verified that (1.5) again holds, and for arbitrary  $(c_+, c_-) \in \mathbb{R}^2$ , the function  $u(x) = c_+ e^x + c_- e^{-x}$  satisfies both (3.1) and (3.14), that is the solution set of this boundary value problem is again two-dimensional.  $\square$

#### 4. THE STRUCTURE OF $\sigma$

In this section we discuss the structure of the spectrum of the multi-point eigenvalue problem (1.1)-(1.2), which we can rewrite as

$$-\Delta u = \lambda u, \quad u \in X. \quad (4.1)$$

We will show that  $\sigma$  has the properties ( $\sigma$ -a)-( $\sigma$ -d) described in the introduction, that is, the multi-point spectrum has similar properties to the spectrum of the standard Sturm-Liouville problem with separated boundary conditions. In particular, we will obtain a characterisation of the eigenvalues in terms of an oscillation count of the corresponding eigenfunctions, as in the property ( $\sigma$ -d) in the introduction.

The standard method of counting the oscillations of the eigenfunctions of separated problems is by counting the number of (nodal) zeros in the interval  $(-1, 1)$ , and it is well known that this approach yields property ( $\sigma$ -d) in this case. Unfortunately, this need not be true for the multi-point boundary conditions. This was first observed in [12], in the case of a problem with a single-point Dirichlet condition at one end point and a multi-point Dirichlet-type condition at the other end point. For such a problem it was shown that, for  $k \geq 0$ , if  $u_k$  is an eigenfunction corresponding to  $\lambda_k$  then  $u_k$  could have either  $k$  or  $k + 1$  zeros in  $(-1, 1)$ , whereas  $u'_k$  has exactly  $k + 1$  zeros in  $(-1, 1)$  (these zeros of  $u'_k$  were termed ‘bumps’

in [12]). The results of [12] were then extended to a similar  $p$ -Laplacian problem in [4], and  $p$ -Laplacian problems with multi-point Dirichlet-type conditions at both end points in [13]. Thus, in the Dirichlet-type case, using nodal zeros to count the eigenfunction oscillations fails, and in fact the oscillations are best described by counting bumps (and by starting the enumeration of the eigenvalues/eigenfunctions at  $k = 1$ , that is, the first eigenfunction has a single bump).

However, it was then shown in [14] that counting bumps fails in the case of Neumann and mixed boundary conditions, and in fact in [12, 13, 14] a different oscillation counting procedure was adopted for each of these three types of boundary conditions, and each of these procedures could fail when applied to the other problems. To deal with the general Sturm-Liouville-type boundary conditions here we will use a Prüfer angle technique to characterise the oscillation count of the eigenfunctions. This technique will unify and extend the various types of oscillation count used previously in [12, 13, 14].

In view of this we begin with a preliminary section discussing a Prüfer angle method of defining an oscillation count for the multi-point problem. We then use this oscillation count to describe the multi-point spectrum.

**4.1. Prüfer angles and oscillation count.** The Prüfer angle is a standard technique in the theory of ordinary differential equations, although there are slight variations in the precise definitions and functions used. The basic formulation is described in [3, Chapter 8] (although the terminology ‘Prüfer angle’ is not used in [3]). However, a more general formulation is described in [2, Section 2] (in a  $p$ -Laplacian context), together with some remarks about various ‘modified Prüfer angle’ formulations, and their history. In fact, we will adopt the form of the angle used in [2, Lemma 2.5], which was used earlier by Elbert (see Remark 4.7 below for the reason for our use of this formulation). We will then see that, in contrast to the separated case, the multi-point boundary conditions (1.2) do not determine the exact values of the Prüfer angle at the end points  $\pm 1$ , but instead they place bounds on these angles.

We will give a full description of our constructions and results relating to the boundary conditions (1.2) but, for brevity, we will not describe the basic details of the Prüfer angle technique here but simply refer to [2] and [3].

Let  $C_s^1[-1, 1]$  denote the set of functions  $u \in C^1[-1, 1]$  having only simple zeros (that is,  $|u(x)| + |u'(x)| > 0$  for all  $x \in [-1, 1]$ ). For any  $\lambda > 0$  and  $u \in C_s^1[-1, 1]$ , we define a ‘modified’ Prüfer angle function  $\omega_{(\lambda, u)} \in C^0[-1, 1]$  by

$$\omega_{(\lambda, u)}(-1) \in [0, \pi), \quad \omega_{(\lambda, u)}(x) := \tan^{-1} \frac{\lambda^{1/2} u(x)}{u'(x)}, \quad x \in [-1, 1] \quad (4.2)$$

(when  $u'(x) = 0$  the value of  $\omega_{(\lambda, u)}(x)$  is defined by continuity). We note that the standard Prüfer angle does not have the factor  $\lambda^{1/2}$  in the definition. Geometrically, for each  $x \in [-1, 1]$  we can regard  $\omega_{(\lambda, u)}(x)$  as the angle between the vectors  $(u'(x), \lambda^{1/2} u(x))$  and  $(1, 0)$  in  $\mathbb{R}^2$ , defined to vary continuously with respect to  $x$  (so  $\omega_{(\lambda, u)}(x)$  need not lie within  $[0, \pi/2]$ , or even within  $[0, 2\pi]$ ). Clearly, if  $u$  is a non-trivial solution of the differential equation (1.1) then  $u \in C_s^1[-1, 1]$ , so  $\omega_{(\lambda, u)}$  is well defined.

From now on we suppose that (1.3) and (1.4) hold, and we also define the angles

$$\omega_{\lambda,0}^- := -\tan^{-1} \frac{\lambda^{1/2}\beta_0^-}{\alpha_0^-} \in [0, \pi/2], \quad \omega_{\lambda,0}^+ := -\tan^{-1} \frac{\lambda^{1/2}\beta_0^+}{\alpha_0^+} \in [\pi/2, \pi],$$

where the permissible ranges chosen here for the values of  $\omega_{\lambda,0}^\pm$  are consistent with the sign conditions (1.4). Geometrically,  $\omega_{\lambda,0}^\pm$  are the angles between the vectors  $(\alpha_0^\pm, -\lambda^{1/2}\beta_0^\pm)$  and  $(1, 0)$ .

4.1.1. *Suppose that  $(\alpha, \beta) = (\mathbf{0}, \mathbf{0})$ .* In this case the boundary conditions (1.2) reduce to the separated conditions

$$\lambda^{-1/2}(u'(\pm 1), \lambda^{1/2}u(\pm 1)) \cdot (\lambda^{1/2}\beta_0^\pm, \alpha_0^\pm) = \alpha_0^\pm u(\pm 1) + \beta_0^\pm u'(\pm 1) = 0 \quad (4.3)$$

(where the left hand side is the usual dot product of the vectors). That is, a function  $u \in C^1[-1, 1]$  satisfies (1.2) if and only if

$$(u'(\pm 1), \lambda^{1/2}u(\pm 1)) \text{ is perpendicular to } (\lambda^{1/2}\beta_0^\pm, \alpha_0^\pm). \quad (4.4)$$

Since the vectors  $(\alpha_0^\pm, -\lambda^{1/2}\beta_0^\pm)$  and  $(\lambda^{1/2}\beta_0^\pm, \alpha_0^\pm)$  are perpendicular, we see that  $u$  satisfies (4.4) if and only if

$$(u'(\pm 1), \lambda^{1/2}u(\pm 1)) \text{ is parallel to } (\alpha_0^\pm, -\lambda^{1/2}\beta_0^\pm), \quad (4.5)$$

which is equivalent to

$$\omega_{(\lambda,u)}(\pm 1) = \omega_{\lambda,0}^\pm \pmod{\pi}. \quad (4.6)$$

Standard Sturm-Liouville theory for the separated boundary conditions (4.3) now yields the following properties of the spectrum, see Theorem 2.1 in [3, Chapter 8] (and the proof of this theorem).

**Theorem 4.1.** *Suppose that  $(\alpha, \beta) = (\mathbf{0}, \mathbf{0})$ . Then  $\sigma$  consists of a strictly increasing sequence of real eigenvalues  $\lambda_k^0 \geq 0$ ,  $k = 0, 1, \dots$ . For each  $k \geq 0$ :*

- (a)  $\lambda_k^0$  has geometric multiplicity one;
- (b)  $\lambda_k^0$  has an eigenfunction  $u_k^0$  whose Prüfer angle  $\omega_k^0 := \omega_{(\lambda_k^0, u_k^0)}$  satisfies

$$\omega_k^0(-1) = \omega_{\lambda,0}^-, \quad \omega_k^0(1) = \omega_{\lambda,0}^+ + k\pi. \quad (4.7)$$

**Remark 4.2.** By definition, for any  $u \in C_s^1[-1, 1]$ ,

$$\begin{aligned} u(x) = 0 &\iff \omega_{(\lambda,u)}(x) = 0 \pmod{\pi}, \\ u'(x) = 0 &\iff \omega_{(\lambda,u)}(x) = \frac{\pi}{2} \pmod{\pi}. \end{aligned}$$

In addition, it can be verified that if  $u$  satisfies the differential equation (1.1), with  $\lambda > 0$ , then

$$u(x)u'(x) = 0 \implies \omega'_{(\lambda,u)}(x) > 0,$$

so it follows from (4.7) that, for all  $k \geq 0$ , the eigenfunction  $u_k^0$  has exactly  $k$  zeros in the interval  $(-1, 1)$ ; this is the usual ‘oscillation count’ for the standard, separated, Sturm-Liouville problem. Thus the oscillation count of the eigenfunctions of the separated problem can be described by the Prüfer angle, and this count is encapsulated in (4.7).

4.1.2. Suppose that  $(\mathbf{0}, \mathbf{0}) \neq (\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathcal{B}(\boldsymbol{\alpha}_0, \boldsymbol{\beta}_0)$ . In this case the eigenfunctions need not satisfy (4.5)-(4.7) — to provide a replacement for these formulae we first prove the following lemma.

**Lemma 4.3.** *Suppose that  $u$  is an eigenfunction, with eigenvalue  $\lambda > 0$ . Then*

$$\omega_{(\lambda, u)}(\pm 1) - \omega_{\lambda, 0}^{\pm} \neq \frac{\pi}{2} \pmod{\pi}. \quad (4.8)$$

*Proof.* It follows from the definitions of  $\omega_{(\lambda, u)}$  and the angles  $\omega_{\lambda, 0}^{\pm}$  that

$$\omega_{(\lambda, u)}(\pm 1) - \omega_{\lambda, 0}^{\pm} = \frac{\pi}{2} \pmod{\pi} \iff \lambda \beta_0^{\pm} u(\pm 1) - \alpha_0^{\pm} u'(\pm 1) = 0,$$

so the result follows from Lemma 3.3 (by putting  $\eta_0 = \pm 1$ , etc.).  $\square$

The geometrical interpretation of (4.8) is:

$$(u'(\pm 1), \lambda^{1/2} u(\pm 1)) \text{ is not perpendicular to } (\alpha_0^{\pm}, -\lambda^{1/2} \beta_0^{\pm}). \quad (4.9)$$

Thus we see that going from separated to multi-point boundary conditions has relaxed the ‘strictly parallel’ condition (4.5), holding in the separated case, to the ‘not perpendicular’ condition (4.9), holding in the multi-point case.

Motivated by Theorem 4.1 and Lemma 4.3, we introduce some further notation.

**Definition 4.4.** For  $k \geq 0$ ,  $P_k^+$  will denote the set of  $(\lambda, u) \in (0, \infty) \times C_s^1[-1, 1]$  for which the Prüfer angle  $\omega_{(\lambda, u)}$  satisfies

$$|\omega_{(\lambda, u)}(-1) - \omega_{\lambda, 0}^-| < \pi/2, \quad |\omega_{(\lambda, u)}(1) - \omega_{\lambda, 0}^+ - k\pi| < \pi/2; \quad (4.10)$$

also,  $P_k^- := -P_k^+$  and  $P_k := P_k^- \cup P_k^+$ .

The sets  $P_k^{\pm}$ ,  $k \geq 0$ , are open, disjoint subsets of  $(0, \infty) \times C^1[-1, 1]$ , and they will be used to count eigenfunction oscillations in Theorem 4.8 below. In fact, the results of Theorem 4.8 below will demonstrate that, for general  $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \neq (\mathbf{0}, \mathbf{0})$ , the conditions (4.8) and (4.10) are suitable replacements for conditions (4.6) and (4.7) respectively. As a preliminary to this we observe that the above definitions, together with Corollary 3.4 and Lemma 4.3 yield the following result.

**Corollary 4.5.** *Suppose that  $u$  is an eigenfunction, with eigenvalue  $\lambda > 0$ . Then:*

- (a)  $\lambda$  has geometric multiplicity 1;
- (b)  $(\lambda, u) \notin \partial P_l$ , for any  $l \geq 0$ ;
- (c) there exists  $k \geq 0$  such that  $(\lambda, u) \in P_k$ .

Motivated by Corollary 4.5 we define the sets

$$\sigma_k := \{\lambda \in \sigma : \text{for any eigenfunction } u \text{ of } \lambda, (\lambda, u) \in P_k\}, \quad k \geq 0.$$

By Corollary 4.5, we have  $\sigma = \cup_{k \geq 0} \sigma_k$ .

**Remark 4.6.** In [13] and [14] certain subsets of  $C_s^1[-1, 1]$ , denoted  $T_k$  and  $S_k$ , were used to count oscillations in the Dirichlet-type and Neumann-type cases respectively. It follows from the results in Remark 4.2 and the definitions of  $T_k$  and  $S_k$  in [13, Section 2.2] and [14, Section 2.2] that, for each integer  $k \geq 0$ :

- Neumann-type case:  $\omega_{\lambda, 0}^{\pm} = \frac{\pi}{2}$  and  $(\lambda, u) \in P_k \implies u$  has exactly  $k$  zeros in  $(-1, 1)$  and  $u \in S_k$ ;
- Dirichlet-type case:  $\omega_{\lambda, 0}^- = 0$ ,  $\omega_{\lambda, 0}^+ = \pi$  and  $(\lambda, u) \in P_k \implies u'$  has exactly  $k + 1$  zeros in  $(-1, 1)$  and  $u \in T_{k+1}$ .

Hence, in the Dirichlet-type and Neumann-type cases respectively, the sets  $P_k$  used here are analogous to the sets  $(0, \infty) \times T_{k+1}$  and  $(0, \infty) \times S_k$ , and we see that using the sets  $P_k$  to count the eigenfunction oscillations extends the oscillation counting methods used in the above special cases to the general Sturm-Liouville-type boundary conditions considered here.

**Remark 4.7.** The above constructions depended on (3.9), via Lemma 4.3, and the occurrence of the term  $\lambda^{1/2}$  in (3.9) dictated that the term  $\lambda^{1/2}$  should appear in the definition of the Prüfer angle. This is why we have used the ‘modified’ Prüfer angle here.

**4.2. The structure of  $\sigma$ .** We can now prove the following theorem for general  $(\alpha, \beta)$ , which extends Theorem 4.1 to the general multi-point Sturm-Liouville problem.

**Theorem 4.8.** *Suppose that (1.3)–(1.5) hold. Then  $\sigma$  consists of a strictly increasing sequence of real eigenvalues  $\lambda_k \geq 0$ ,  $k = 0, 1, \dots$ , such that  $\lim_{k \rightarrow \infty} \lambda_k = \infty$ . For each  $k \geq 0$ :*

- (a)  $\lambda_k$  has geometric multiplicity 1;
- (b)  $\lambda_k$  has an eigenfunction  $u_k$  such that  $(\lambda_k, u_k) \in P_k^+$ .

*In the Neumann-type case  $\lambda_0 = 0$ , while if (2.1) holds then  $\lambda_0 > 0$ .*

*Proof.* We will prove a series of results regarding the eigenvalues and eigenfunctions, which culminate in the proof of the theorem. The fact that the eigenvalues have geometric multiplicity 1 has already been proved in Corollary 3.4.

**Lemma 4.9.** *If  $\lambda$  is an eigenvalue then  $\lambda \geq 0$ . If (2.1) holds then  $\lambda > 0$ .*

*Proof.* Suppose that  $\lambda < 0$  and define  $s := \sqrt{-\lambda}$ . Then any eigenfunction  $u$  has the form  $u(x) = c_+ e^{sx} + c_- e^{-sx}$ , for some  $(c_+, c_-) \in \mathbb{R}^2$ , and we see from this that  $\max |u|$  and  $\max |u'|$  must both be attained at the same end point, say at  $x = 1$ . Hence,  $u(1)$  and  $u'(1)$  have the same sign. By (1.4),  $\beta_0^+ \geq 0$ , so by (1.2) and (1.5),

$$\begin{aligned} \alpha_0^+ |u|_0 + \beta_0^+ |u'|_0 &= |\alpha_0^+ u(1) + \beta_0^+ u'(1)| \\ &\leq |u|_0 \sum_{i=1}^{m^+} |\alpha_i^+| + |u'|_0 \sum_{i=1}^{m^+} |\beta_i^+| \\ &< \alpha_0^+ |u|_0 + \beta_0^+ |u'|_0, \end{aligned}$$

and this contradiction proves the first part of the lemma. Next, if (2.1) holds then it follows from Theorem 2.1 that  $\lambda \neq 0$ , which completes the proof.  $\square$

**Remark 4.10.** It is well known that if the sign conditions (1.4) do not hold then Lemma 4.9 need not be true, even in the separated case. For example, if

$$\alpha_0^\pm = \pm \epsilon, \quad \alpha = 0, \quad \beta_0^\pm = \pm 1, \quad \beta = 0.$$

The properties of the spectrum in the Neumann-type case have been proved in [14], so from now on in the proof we will suppose that (2.1) holds. Thus, by Theorem 2.1 and Lemma 4.9, if  $\lambda$  is an eigenvalue with eigenfunction  $u$ , then  $\lambda > 0$  and we may suppose that  $\lambda = s^2$ ,  $u = w(s, \theta)$ , for suitable  $s > 0$ ,  $\theta \in \mathbb{R}$  (up to a

scaling of the eigenfunction), where  $w(s, \theta)$  was defined in (3.3). Defining functions  $\Gamma^\pm : (0, \infty) \times \mathbb{R} \times \mathbb{R}^{2(m^- + m^+)} \rightarrow \mathbb{R}$  by

$$\Gamma^\pm(s, \theta, \alpha^\pm, \beta^\pm) := \alpha_0^\pm \sin(\pm s + \theta) + s\beta_0^\pm \cos'(\pm s + \theta) - \sum_{i=1}^{m^\pm} \alpha_i^\pm \sin(s\eta_i^\pm + \theta) - s \sum_{i=1}^{m^\pm} \beta_i^\pm \cos(s\eta_i^\pm + \theta),$$

and substituting  $w(s, \theta)$  into (1.2) shows that  $\lambda = s^2$  is an eigenvalue if and only if the pair of equations

$$\Gamma^\pm(s, \theta, \alpha^\pm, \beta^\pm) = 0 \tag{4.11}$$

holds, for some  $\theta \in \mathbb{R}$ . Hence, it suffices to consider the set of solutions of (4.11).

We will now prove Theorem 4.8 by continuation with respect to  $(\alpha, \beta)$ , away from  $(\alpha, \beta) = (\mathbf{0}, \mathbf{0})$ , where the required information on the solutions of (4.11) follows from the standard theory of the separated problem in Theorem 4.1. For reference, we state this in the following lemma.

**Lemma 4.11.** *Suppose that  $(\alpha, \beta) = (\mathbf{0}, \mathbf{0})$ . For each  $k = 0, 1, \dots$ , if we write  $s_k^0 := (\lambda_k^0)^{1/2}$  (where  $\lambda_k^0$  is as in Theorem 4.1), then there exists a unique  $\theta_k^0 \in [0, \pi)$  such that  $(s_k^0, \theta_k^0)$  satisfies (4.11).*

Of course, by the periodicity properties of  $\Gamma^\pm$  with respect to  $\theta$ , there are other solutions of (4.11) (with  $(\alpha, \beta) = (\mathbf{0}, \mathbf{0})$ ) than those in Lemma 4.11, but these do not yield distinct solutions of the eigenvalue problem (4.1). In fact, to remove these extra solutions and to reduce the domain of  $\theta$  to a compact set, from now on we will regard  $\theta$  as lying in the circle obtained from the interval  $[0, 2\pi]$  by identifying the points 0 and  $2\pi$ , which we denote by  $S^1$ , and we regard the domain of the functions  $\Gamma^\pm$  as  $(0, \infty) \times S^1 \times \mathcal{B}(\alpha_0^\pm, \beta_0^\pm)$ .

We now consider (4.11) when  $(\alpha, \beta) \neq (\mathbf{0}, \mathbf{0})$ . The following proposition provides some information on the signs of the partial derivatives  $\Gamma_s^\nu, \Gamma_\theta^\nu$  at the zeros of  $\Gamma^\nu$ .

**Lemma 4.12.** *Suppose that  $\nu \in \{\pm\}$  and  $(\alpha^\nu, \beta^\nu) \in \mathcal{B}(\alpha_0^\nu, \beta_0^\nu)$ . Then*

$$\Gamma^\nu(s, \theta, \alpha^\nu, \beta^\nu) = 0 \implies \nu \Gamma_s^\nu(s, \theta, \alpha^\nu, \beta^\nu) \Gamma_\theta^\nu(s, \theta, \alpha^\nu, \beta^\nu) > 0. \tag{4.12}$$

*Proof.* By a similar proof to that of Lemma 3.2 it can be shown that

$$\Gamma^\nu(s, \theta, \alpha^\nu, \beta^\nu) = 0 \implies \Gamma_s^\nu(s, \theta, \alpha^\nu, \beta^\nu) \Gamma_\theta^\nu(s, \theta, \alpha^\nu, \beta^\nu) \neq 0. \tag{4.13}$$

We now regard  $(s, \theta, \alpha^\nu, \beta^\nu)$  as fixed, and consider the equation

$$G(\tilde{\theta}, t) := \Gamma^\nu(s, \tilde{\theta}, t\alpha^\nu, t\beta^\nu) = 0, \quad (\tilde{\theta}, t) \in S^1 \times [0, 1]. \tag{4.14}$$

It is clear that if  $t \in [0, 1]$  then  $(t\alpha^\nu, t\beta^\nu) \in \mathcal{B}(\alpha_0^\nu, \beta_0^\nu)$ , so by (4.13),

$$G(\theta, 1) = 0 \quad \text{and} \quad G(\tilde{\theta}, t) = 0 \implies G_{\tilde{\theta}}(\tilde{\theta}, t) \neq 0. \tag{4.15}$$

Hence, by (4.15), the implicit function theorem, and the compactness of  $S^1$ , there exists a  $C^1$  solution function  $t \rightarrow \tilde{\theta}(t) : [0, 1] \rightarrow S^1$ , for (4.14) such that

$$\tilde{\theta}(1) = \theta, \quad \Gamma^\nu(s, \tilde{\theta}(t), t\alpha^\nu, t\beta^\nu) = 0, \quad t \in [0, 1]$$

(the local existence of this solution function, near  $t = 1$ , is trivial; standard arguments show that its domain can be extended to include the interval  $[0, 1]$  — see the proof of part (b) of Lemma 4.13 below for a similar argument).

Next, by the definition of  $\Gamma^\nu$ , (4.12) holds at  $(s, \tilde{\theta}(0), 0, 0)$  and hence, by (4.13) and continuity, (4.12) holds at  $(s, \tilde{\theta}(t), t\alpha^\nu, t\beta^\nu)$  for all  $t \in [0, 1]$ . In particular,

putting  $t = 1$  shows that (4.12) holds at  $(s, \theta, \alpha^\nu, \beta^\nu)$ , which completes the proof of Lemma 4.12.  $\square$

We now return to the pair of equations (4.11). To solve these using the implicit function theorem we define the Jacobian determinant

$$J(s, \theta, \alpha, \beta) := \begin{vmatrix} \Gamma_s^-(s, \theta, \alpha^-, \beta^-) & \Gamma_\theta^-(s, \theta, \alpha^-, \beta^-) \\ \Gamma_s^+(s, \theta, \alpha^+, \beta^+) & \Gamma_\theta^+(s, \theta, \alpha^+, \beta^+) \end{vmatrix},$$

for  $(s, \theta, \alpha, \beta) \in (0, \infty) \times S^1 \times \mathcal{B}(\alpha_0, \beta_0)$ . It follows from the sign properties of  $\Gamma_s^\pm, \Gamma_\theta^\pm$  proved in Lemma 4.12 that

$$\Gamma^+(s, \theta, \alpha^+, \beta^+) = \Gamma^-(s, \theta, \alpha^-, \beta^-) = 0 \implies J(s, \theta, \alpha, \beta) \neq 0, \tag{4.16}$$

and hence we can solve (4.11) for  $(s, \theta)$ , as functions of  $(\alpha, \beta)$ , in a neighbourhood of an arbitrary solution of (4.11).

Now suppose that  $(s, \theta, \alpha, \beta) \in (0, \infty) \times S^1 \times \mathcal{B}(\alpha_0, \beta_0)$  is an arbitrary (fixed) solution of (4.11). By (4.16) and the implicit function theorem there exists a maximal open interval  $\tilde{I}$  containing 1 and a  $C^1$  solution function

$$t \rightarrow (\tilde{s}(t), \tilde{\theta}(t)) : \tilde{I} \rightarrow (0, \infty) \times S^1,$$

such that

$$(\tilde{s}(1), \tilde{\theta}(1)) = (s, \theta), \quad \Gamma^\pm(\tilde{s}(t), \tilde{\theta}(t), t\alpha^\pm, t\beta^\pm) = 0, \quad t \in \tilde{I}.$$

Furthermore, by Corollary 4.5 and continuity, there exists an integer  $\tilde{k} \geq 0$  such that

$$(\tilde{s}(t)^2, w(\tilde{s}(t), \tilde{\theta}(t))) \in P_{\tilde{k}}, \quad t \in \tilde{I}. \tag{4.17}$$

**Lemma 4.13.** (a) *There exists constants  $C, \delta > 0$  such that  $\delta \leq \tilde{s}(t) \leq C, t \in \tilde{I};$*   
 (b)  $0 \in \tilde{I}$ .

*Proof.* (a) From the form of  $w(s, \theta)$ , there exists  $C > 0$  such that if  $s \geq C$  then  $(s^2, w(s, \theta)) \notin P_{\tilde{k}}$ , for any  $\theta \in S^1$ . Hence, by (4.17),  $\tilde{s}(t) \leq C$  for any  $t \in \tilde{I}$ . Now suppose that the lower bound  $\delta > 0$  does not exist, so that we may choose a sequence  $t_n \in \tilde{I}, n = 1, 2, \dots$ , with  $\tilde{s}(t_n) \rightarrow 0$ . Writing  $\tilde{s}_n := s(t_n), \tilde{\theta}_n := \theta(t_n)$  and  $\tilde{w}_n := w(\tilde{s}_n, \tilde{\theta}_n), n = 1, 2, \dots$ , it is clear that, as  $n \rightarrow \infty$ ,

$$|\tilde{w}'_n|_0 = O(\tilde{s}_n) \quad \text{and} \quad |\tilde{w}_n - c_\infty|_0 \rightarrow 0,$$

for some constant  $c_\infty$  (after taking a subsequence if necessary, and regarding  $c_\infty$  as an element of  $C^0[-1, 1]$ ). We now consider various cases.

Suppose that  $c_\infty \neq 0$ . By (2.1),  $\alpha_0^\nu \neq 0$  for some  $\nu \in \{\pm\}$ , and the corresponding boundary condition (1.2) yields

$$0 = \alpha_0^\nu \tilde{w}_n(\nu) - \sum_{i=1}^{m^\nu} \alpha_i^\nu \tilde{w}_n(\eta_i^\nu) + O(\tilde{s}_n) \rightarrow c_\infty \left( \alpha_0^\nu - \sum_{i=1}^{m^\nu} \alpha_i^\nu \right),$$

which contradicts (1.5), and so proves the existence of  $\delta > 0$  in this case.

Now suppose that  $c_\infty = 0$ . Without loss of generality we also suppose that  $\tilde{\theta}_n \searrow 0$  (after taking a subsequence if necessary) and so, for all  $n$  sufficiently large,  $|\tilde{w}_n|_0$  is attained at the end point  $x = 1$ .



Suppose that  $\alpha_0^+ \neq 0$ . By the definition of  $\tilde{w}_n$ , we obtain from (1.2)

$$\tilde{s}_n \left( \alpha_0^+ - \sum_{i=1}^{m^+} \alpha_i^+ \eta_i^+ + \beta_0^+ - \sum_{i=1}^{m^+} \beta_i^+ \right) + \tilde{\theta}_n \left( \alpha_0^+ - \sum_{i=1}^{m^+} \alpha_i^+ \right) = O(\tilde{s}_n^3 + \tilde{\theta}_n^3),$$

but, by (1.3)-(1.5), the terms in the brackets on the left hand side are strictly positive, so this is contradictory when  $n$  is sufficiently large.

Suppose that  $\alpha_0^+ = 0$ , and so  $\beta_0^+ > 0$  (by (1.3), (1.4)). Dividing (1.2) by  $\tilde{s}_n$  and letting  $n \rightarrow \infty$  yields

$$0 = s_n^{-1} \left( \beta_0^+ \tilde{w}'_n(1) - \sum_{i=1}^{m^+} \beta_i^+ \tilde{w}'_n(\eta_i^+) \right) \rightarrow \beta_0^+ - \sum_{i=1}^{m^+} \beta_i^+ > 0,$$

by (1.5), which is again contradictory. This completes the proof of part (a) of Lemma 4.13.

(b) Suppose that  $0 \notin \tilde{I}$ , and let  $\hat{t} = \inf\{t \in \tilde{I} \mid t \geq 0\}$ . By part (a) of the lemma, there exists a sequence  $t_n \in \tilde{I}$ ,  $n = 1, 2, \dots$ , and a point  $(\hat{s}, \hat{\theta}) \in (0, \infty) \times S^1$ , such that

$$\lim_{n \rightarrow \infty} t_n = \hat{t}, \quad \lim_{n \rightarrow \infty} (\tilde{s}(t_n), \tilde{\theta}(t_n)) = (\hat{s}, \hat{\theta}).$$

Clearly, the point  $(\hat{s}, \hat{\theta}, \hat{t}\alpha, \hat{t}\beta)$  satisfies (4.11) so, by the above results, the solution function  $(\tilde{s}, \tilde{\theta})$  extends to an open neighbourhood of  $\hat{t}$ , which contradicts the choice of  $\hat{t}$  and the maximality of the interval  $\tilde{I}$ . □

For any given  $(\alpha, \beta) \in \mathcal{B}(\alpha_0, \beta_0)$  the above arguments have shown that:

- (a) any solution  $(s, \theta, \alpha, \beta) \in (0, \infty) \times S^1 \times \mathcal{B}(\alpha_0, \beta_0)$  of (4.11) can be continuously connected to exactly one of the solutions  $\{(s_k^0, \theta_k^0, \mathbf{0}, \mathbf{0}) : k \geq 0\}$ .

Similar arguments show that:

- (b) any solution  $\{(s_k^0, \theta_k^0, \mathbf{0}, \mathbf{0}) : k \geq 0\}$  can be continuously connected to exactly one solution, say  $(s_k(\alpha, \beta), \theta_k(\alpha, \beta), \alpha, \beta) \in (0, \infty) \times S^1 \times \mathcal{B}(\alpha_0, \beta_0)$ , of (4.11).

Hence, for each  $k \geq 0$ , we obtain the eigenvalue and eigenfunction

$$(\lambda_k(\alpha, \beta), u_k(\alpha, \beta)) := (s_k(\alpha, \beta)^2, w(s_k(\alpha, \beta), \theta_k(\alpha, \beta))) \in P_k,$$

and we see that there is no eigenvalue  $\tilde{\lambda} \neq \lambda_k(\alpha, \beta)$ , with eigenfunction  $\tilde{u}$ , for which  $(\tilde{\lambda}, \tilde{u}) \in P_k$ .

Next, by Theorem 4.1,  $s_k^0 = s_k^0(\mathbf{0}, \mathbf{0}) < s_{k+1}^0 = s_{k+1}^0(\mathbf{0}, \mathbf{0})$  and by Theorem 3.1,  $s_k(\alpha, \beta) \neq s_{k+1}(\alpha, \beta)$  for any  $(\alpha, \beta) \in \mathcal{B}(\alpha_0, \beta_0)$ , so it follows from the continuation construction that  $s_k(\alpha, \beta) < s_{k+1}(\alpha, \beta)$  for all  $(\alpha, \beta) \in \mathcal{B}(\alpha_0, \beta_0)$ .

Finally, for fixed  $(\alpha, \beta)$ , the fact that  $(\lambda_k(\alpha, \beta), u_k(\alpha, \beta)) \in P_k$ , for  $k \geq 1$ , shows that as  $k \rightarrow \infty$  the oscillation count tends to  $\infty$ , so by standard properties of the differential equation (1.1) we must have  $\lim_{k \rightarrow \infty} \lambda_k = \infty$ . This concludes the proof of Theorem 4.8. □

The implicit function theorem construction of  $\lambda_k$  and  $u_k$  in the proof of Theorem 4.8 also imply continuity properties which will be useful below, so we state these in the following corollary (continuity of  $u_k$  will be in the space  $C^0[-1, 1]$ , although stronger results could easily be obtained).

**Corollary 4.14.** *For each  $k \geq 0$ :  $\lambda_k \in \mathbb{R}$  and  $u_k \in C^0[-1, 1]$  depend continuously on  $(\alpha_0, \beta_0, \alpha, \beta, \eta) \in \mathcal{B} \times (-1, 1]^{m^-} \times [-1, 1]^{m^+}$ .*

**4.3. Positivity of the principal eigenfunction.** In many applications it is important to know that the *principal eigenfunction*  $u_0$  is positive. Thus we will now consider conditions which ensure that this is true. Simple examples show that  $u_0$  may not be positive if the vectors  $\alpha^\pm$  contain negative components, but the following theorem shows that  $u_0$  will be positive if  $\alpha^\pm \geq 0$  (with these inequalities holding componentwise).

**Theorem 4.15.** *Suppose that (1.3)–(1.5) hold, and  $\alpha^\pm \geq 0$ . Then:*

- (a)  $u_0 > 0$  on  $(-1, 1)$ ;
- (b) if  $\beta_0^\nu \neq 0$ , for some  $\nu \in \{\pm\}$ , then  $u_0(\nu) > 0$ .

*Proof.* By standard Sturm-Liouville theory the result is true when  $(\alpha, \beta) = (\mathbf{0}, \mathbf{0})$  (part (a) is standard and (b) follows immediately since, under the stated hypotheses,  $u_0(\nu) = 0 \Rightarrow u'_0(\nu) = 0$ , and an eigenfunction cannot have a double zero). Now suppose that both  $\beta_0^\pm \neq 0$ . If the result fails then, by using a limiting argument in the construction of the eigenvalues by continuation from  $(\alpha, \beta) = (\mathbf{0}, \mathbf{0})$  in the proof of Theorem 4.8, we can show that there exists some  $(\alpha, \beta) \in \mathcal{B}(\alpha_0, \beta_0)$ , with  $\alpha^\pm \geq 0$ , such that the principal eigenfunction  $u_0(\alpha, \beta) \geq 0$  satisfies:

- (1)  $u_0(\alpha, \beta) > 0$  on  $(-1, 1)$  (since  $u_0(\alpha, \beta)$  cannot have a double zero);
- (2)  $u_0(\alpha, \beta)(\nu) = 0$ , and hence  $|u'_0(\alpha, \beta)(\nu)| = |u'_0(\alpha, \beta)|_0$ , for some  $\nu \in \{\pm\}$ .

Now, by (1.2)–(1.5)

$$\begin{aligned} 0 &= \beta_0^\nu u'_0(\alpha, \beta)(\nu) - \sum_{i=1}^{m^\nu} \alpha_i^\nu u_0(\alpha, \beta)(\eta_i^\nu) - \sum_{i=1}^{m^\nu} \beta_i^\nu u'_0(\alpha, \beta)(\eta_i^\nu) \\ &\leq -|u'_0(\alpha, \beta)|_0 \left( |\beta_0^\nu| - \sum_{i=1}^{m^\nu} |\beta_i^\nu| \right) < 0, \end{aligned}$$

and this contradiction shows that this case cannot occur.

Next, suppose that one, or both, of  $\beta_0^\pm = 0$ . We replace the coefficients  $\beta_0^\pm$  by  $\beta_0^\pm \pm 1/n$ ,  $n = 1, 2, \dots$ , and then let  $n \rightarrow \infty$ . By the result just proved, each of the corresponding principal eigenfunctions, say  $u_{0,n} \geq 0$ , have the properties (a) and (b), and so by Corollary 4.14 the limiting eigenfunction, say  $u_{0,\infty} \geq 0$ , satisfies (a), and we can now prove that  $u_{0,\infty}$  satisfies (b) by the same calculation as before.  $\square$

**4.4. Algebraic multiplicity.** Throughout this section we will suppose that (2.1) holds so that, by Theorem 2.1,  $\Delta$  has an inverse operator  $\Delta^{-1} : Y \rightarrow X$  (see Remark 4.18 below for some comments on the Neumann-type case, when (2.1) does not hold). We can also regard this inverse as an operator  $\Delta^{-1} : Y \rightarrow Y$ , which we will denote as  $\Delta_Y^{-1}$ . Since  $X$  is compactly embedded into  $Y$ ,  $\Delta_Y^{-1}$  is compact (indeed, this compactness together with the fact that  $\Delta_Y^{-1}$  maps  $Y$  into itself is the motivation for introducing  $\Delta_Y^{-1}$ ). Now, the eigenvalue problem (4.1) is equivalent to the equation

$$(I_Y + \lambda \Delta_Y^{-1})u = 0, \quad u \in Y, \tag{4.18}$$

where  $I_Y$  denotes the identity on  $Y$ . Hence, each eigenvalue  $\lambda_k$ ,  $k = 0, 1, \dots$ , can be regarded as a characteristic value of  $-\Delta_Y^{-1}$ . As usual, we define the algebraic multiplicity of the characteristic value  $\lambda_k$  to be

$$\dim \cup_{j=1}^\infty N((I_Y + \lambda_k \Delta^{-1})^j)$$

(where  $N$  denotes null-space).

**Lemma 4.16.** *For each  $k \geq 0$  the algebraic multiplicity of the characteristic value  $\lambda_k$  of  $-\Delta_Y^{-1}$  is equal to 1.*

*Proof.* The proof is again by continuation with respect to  $(\alpha, \beta)$ , so we now write  $\Delta_Y^{-1}(\alpha, \beta)$  and  $\lambda_k(\alpha, \beta)$ , for  $(\alpha, \beta) \in \mathcal{B}(\alpha_0, \beta_0)$ . When  $(\alpha, \beta) = (\mathbf{0}, \mathbf{0})$  it is easy to see that the algebraic multiplicity of  $\lambda_k(\mathbf{0}, \mathbf{0})$  is equal to 1 (this case corresponds to the standard Sturm-Liouville problem). Next, it was shown in Corollaries 2.2 and 4.14 that  $\Delta_Y^{-1}(\alpha, \beta)$  and  $\lambda_k(\alpha, \beta)$  depend continuously on  $(\alpha, \beta)$ , and Theorem 4.8 shows that as  $(\alpha, \beta)$  varies over  $\mathcal{B}(\alpha_0, \beta_0)$ , eigenvalues with different  $k$  never meet. Hence, by the results in [8, Ch. 2, Sec. 5], the algebraic multiplicity of  $\lambda_k(\alpha, \beta)$  is constant for  $(\alpha, \beta) \in \mathcal{B}(\alpha_0, \beta_0)$  (the discussion in [8, Ch. 2, Sec. 5] is in finite dimensions but, as noted there, the results extend to bounded operators in infinite dimensions). This proves the result.  $\square$

**Remark 4.17.** In the case  $\alpha^- = 0$ ,  $\alpha^+ > 0$ ,  $\beta^\pm = 0$ , Lemma 4.16 was proved directly in [17, Lemma 2.6] and [12, Lemma 3.8] (that is, without relying on perturbation theory for linear operators), but it seems to be difficult to extend this proof to the general case. This result was extended to general Dirichlet-type and Neumann-type problems in [13] and [14] respectively.

**Remark 4.18.** For simplicity we have excluded the Neumann-type case from this section, since in this case the operator  $\Delta$  does not have an inverse. Of course, one could consider the operator  $\Delta - \mu I_Y$ , with  $\mu > 0$ ; it can be shown that this operator has an inverse, which is compact (as a mapping into  $Y$ ), that is,  $\Delta$  has compact resolvent. We could then obtain similar results to those above. However, this would entail considerable additional notational complexity, and the Neumann-type case was treated in detail in [14], so we will simply omit this case here.

**4.5. Counter examples.** In this section we will show that Theorem 4.8 need not be true if (1.5) does not hold, and that the condition (1.5) is, in some sense, optimal for the validity of Theorem 4.8. In fact, for the Dirichlet-type problem, it was shown in [12, Examples 3.5, 3.6] that if  $\sum_{i=1}^{m^\pm} |\alpha_i^\pm| = \alpha_0^\pm$  then we may have an eigenvalue/eigenfunction pair  $(\lambda, u) \in \partial P_k$ , for some  $k$  (in the present notation) while if  $\sum_{i=1}^{m^\pm} |\alpha_i^\pm| > \alpha_0^\pm$  then we may have  $\sigma_k = \emptyset$  for a finite, but arbitrarily large, set of integers  $k$ , that is, the corresponding eigenvalues  $\lambda_k$  may be ‘missing’ from the sequence of eigenvalues constructed in Theorem 4.8. Similar examples were constructed for the Neumann-type case in [14, Examples 4.17, 4.18]. These examples show that condition (1.5) is optimal in the cases where one or other of the fractions on the left hand side of (1.5) is absent. Thus it seems of interest to also show that (1.5) is optimal when both fractions are present. The following example will do this when these fractions are nonzero and equal to each other. More precisely, in this case we will show that if the number 1 on the right hand side of (1.5) is increased by an arbitrarily small amount then Theorem 4.8 need not hold, and arbitrarily many eigenvalues may be ‘missing’.

For notational simplicity we will consider the problem on the interval  $(0, 1)$ , with a standard Dirichlet condition at  $x = 0$ , and the following multi-point condition at  $x = 1$

$$\alpha_0 u(1) + \beta_0 u'(1) = \alpha_1 u(\eta_1) + \beta_2 u'(\eta_2). \quad (4.19)$$

For any eigenvalue  $\lambda = s^2 > 0$  the corresponding eigenfunction must have the form  $C \sin sx$ ,  $C \in \mathbb{R}$ . Hence, defining  $\Gamma : \mathbb{R} \rightarrow \mathbb{R}$  by

$$\Gamma(s) := \alpha_0 \sin s + s\beta_0 \cos s - \alpha_1 \sin s\eta_1 - s\beta_2 \cos s\eta_2, \quad s \in \mathbb{R},$$

it is clear that  $\lambda = s^2$  is an eigenvalue iff  $\Gamma(s) = 0$ , and also, for any integer  $k \geq 0$ ,  $\lambda \in \sigma_k \implies \lambda \in [(k-2)\pi, (k+2)\pi]$ .

To construct our counter example we will show that with a suitable choice of the coefficients in the boundary condition (4.19) there exists a ‘long’ interval  $I$  such that if  $s \in I$  then  $\Gamma(s) \neq 0$ , that is,  $s^2$  cannot be an eigenvalue. This will show that  $\sigma_k = \emptyset$  for a range of values of  $k$ .

Choose a ‘large’ integer  $k_0$  (we will be more specific below), and set:

$$\epsilon = \frac{10}{k_0}, \quad s(\gamma) = (1 + \gamma\epsilon)k_0\pi, \quad \gamma \in [-1, 1].$$

Hence, as  $\gamma$  varies over the interval  $[-1, 1]$ , the number  $s(\gamma)$  varies over the interval

$$I_{k_0} := [(k_0 - 10)\pi, (k_0 + 10)\pi].$$

We also set:

$$\begin{aligned} \alpha_0 &= 1, & \beta_0 &= \frac{1}{k_0\pi}, \\ \alpha_1 &= \frac{1 + \epsilon}{\sqrt{2}}, & \beta_2 &= \frac{1}{k_0\pi} \frac{1 + \epsilon}{\sqrt{2}}, \\ \eta_1 &= \frac{1}{2k_0}, & \eta_2 &= \frac{1}{k_0}. \end{aligned}$$

Simple estimates now show that if  $\epsilon$  is sufficiently small (that is, if  $k_0$  is sufficiently large) then, for  $\gamma \in [-1, 1]$ ,

$$\begin{aligned} \Gamma(s(\gamma)) &\leq \sqrt{2} + \epsilon - \frac{1 + \epsilon}{\sqrt{2}} \left( \sin \frac{\pi}{2}(1 + \epsilon) - (1 + \epsilon) \cos \pi(1 + \epsilon) \right) \\ &\leq \sqrt{2} + \epsilon - \sqrt{2}(1 + \epsilon)(1 - \epsilon/14) \\ &< \epsilon \left( 1 - \frac{13\sqrt{2}}{14} + O(\epsilon) \right) \\ &< 0. \end{aligned}$$

This shows that there is no eigenvalue  $\lambda = s^2$  with  $s \in I_{k_0}$ , that is,  $\sigma_k = \emptyset$  if  $k \in [k_0 - 7, k_0 + 7]$ . Clearly, there is nothing special about the number 10 in this example, so in fact we could construct an example for which  $\sigma_k = \emptyset$  for an arbitrarily long succession of integers  $k$ . Also, since

$$\frac{\alpha_1}{\alpha_0} = \frac{\beta_2}{\beta_0} = \frac{1 + \epsilon}{\sqrt{2}},$$

and  $\epsilon$  is arbitrarily small, we see that if the number 1 in condition (1.5) is increased by an arbitrarily small amount then Theorem 4.8 need not hold.

#### REFERENCES

- [1] C. Bai, J. Fang; Existence of multiple positive solutions for nonlinear  $m$ -point boundary value problems, *J. Math. Anal. Appl.* **281** (2003), 76–85.
- [2] P. Binding, P. Drábek; Sturm-Liouville theory for the  $p$ -Laplacian, *Studia Sci. Math. Hungar.* **40** (2003), 375–396.

- [3] E. A. Coddington, N. Levinson; *Theory of Ordinary Differential Equations*, McGraw-Hill, New York (1955).
- [4] N. Dodds, B. P. Rynne; Spectral properties and nodal solutions for second-order,  $m$ -point,  $p$ -Laplacian boundary value problems, *Topol. Methods Nonlinear Anal.* **32** (2008), 21–40.
- [5] F. Genoud, B. P. Rynne; Some recent results on the spectrum of multi-point eigenvalue problems for the  $p$ -Laplacian, *Commun. Appl. Anal.* **15** (2011), 413–434.
- [6] M. García-Huidobro, Ch. P. Gupta, R. Manásevich; Some multipoint boundary value problems of Neumann-Dirichlet type involving a multipoint  $p$ -Laplace like operator, *J. Math. Anal. Appl.* **333** (2007), 247–264.
- [7] C. P. Gupta; A non-resonant generalized multi-point boundary-value problem of Dirichlet type involving a  $p$ -Laplacian type operator, *Proceedings of the Sixth Mississippi State-UBA Conference on Differential Equations and Computational Simulations, 127–139, Electron. J. Differ. Equ. Conf., 15, Southwest Texas State Univ., San Marcos, TX, 2007.*
- [8] T. Kato; *Perturbation Theory for Linear Operators*, Springer, 1984.
- [9] Y. Liu; Non-homogeneous boundary-value problems of higher order differential equations with  $p$ -Laplacian, *Electron. J. Differential Equations* **2008**, No. 22.
- [10] R. Ma, D. O'Regan; Nodal solutions for second-order  $m$ -point boundary value problems with nonlinearities across several eigenvalues, *Nonlinear Anal.* **64** (2006), 1562–1577.
- [11] P. H. Rabinowitz; Some global results for nonlinear eigenvalue problems, *J. Funct. Analysis* **7** (1971), 487–513
- [12] B. P. Rynne; Spectral properties and nodal solutions for second-order,  $m$ -point, boundary value problems, *Nonlinear Analysis* **67** (2007), 3318–3327.
- [13] B. P. Rynne; Spectral properties of second-order, multi-point,  $p$ -Laplacian boundary value problems, *Nonlinear Analysis* **72** (2010), 4244–4253.
- [14] B. P. Rynne; Spectral properties of  $p$ -Laplacian problems with Neumann and mixed-type multi-point boundary conditions, *Nonlinear Analysis* **74** (2010), 1471–1484.
- [15] J. R. L. Webb, G. Infante; Positive solutions of nonlocal boundary value problems: a unified approach, *J. London Math. Soc.* **74** (2006), 673–693.
- [16] J. R. L. Webb, K. Q. Lan; Eigenvalue criteria for existence of multiple positive solutions of nonlinear boundary value problems of local and nonlocal type, *Topol. Methods Nonlinear Anal.* **27** (2006), 91–115.
- [17] X. Xu; Multiple sign-changing solutions for some  $m$ -point boundary-value problems, *Electron. J. Differential Equations* **89** (2004).

BRYAN P. RYNNE

DEPARTMENT OF MATHEMATICS AND THE MAXWELL INSTITUTE FOR MATHEMATICAL SCIENCES,  
HERIOT-WATT UNIVERSITY, EDINBURGH EH14 4AS, SCOTLAND

*E-mail address:* `bryan@ma.hw.ac.uk`