MULTIPLE POSITIVE SOLUTIONS FOR SECOND-ORDER
THREE-POINT BOUNDARY-VALUE PROBLEMS WITH SIGN
CHANGING NONLINEARITIES

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ABSTRACT. In this article, we study the second-order three-point boundary-value problem
\[ u''(t) + a(t)u'(t) + f(t, u) = 0, \quad 0 \leq t \leq 1, \]
\[ u'(0) = 0, \quad u(1) = \alpha u(\eta), \]
where \( 0 < \eta < 1, \alpha \in C([0,1]), (-\infty, 0) \) and \( f \) is allowed to change sign.
We show that there exist two positive solutions by using Leggett-Williams fixed-point theorem.

1. INTRODUCTION

The study of multi-point boundary-value problems for linear second-order ordinary differential equations was initiated by Kiguradze and Lomtatidze [11], Lomtatidze [15], Il’in and Moviseev [9, 10], Agarwal and Kiguradze [1], Lomtatidze and Malaguti [16]. Motivated by the study of [9, 10], Gupta [6] studied certain three-point boundary-value problems for nonlinear ordinary differential equations. Since then, more general nonlinear multi-point boundary-value problems have been studied by several authors. We refer the reader to Gupta [7], Li, Liu and Jia [13], Liu [14], Ma [17, 18] for some references along this line. Using results from fixed point theory, such as the fixed-point theorems by Banach, Krasnosel’skii, Leggett-Williams etc., in studying second-order dynamic systems is a standard and useful tool (see, e.g. [2, 3, 4]).

Recently, Ma [17] studied the three-point boundary-value problem (BVP)
\[ u''(t) + a(t)f(u) = 0, \quad 0 \leq t \leq 1, \]
\[ u(0) = 0, \quad u(1) = \alpha u(\eta), \]
where \( 0 < \eta < 1, \alpha \) is a positive constant, \( a \in C([0,1]), f \in C([0, +\infty), [0, +\infty]) \) and there exists \( x_0 \in (0, 1) \) such that \( a(x_0) > 0 \). Author got the existence and multiplicity of positive solutions theorems under the condition that \( f \) is either superlinear or sublinear by using Krasnoselskii’s fixed point theorem.
In 2001, Ma[18] considered m-point boundary-value problem
\[ u''(t) + h(t)f(u) = 0, \quad 0 \leq t \leq 1, \]
\[ u(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} \beta_i u(\xi_i), \]
where \( \beta_i > 0 \) (\( i = 1, 2, \ldots, m - 2 \)), \( 0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1 \), \( h \in C([0,1], [0, +\infty)) \) and \( f \in C([0, +\infty), [0, +\infty)) \). Author established the existence of positive solutions under the condition that \( f \) is either superlinear or sublinear.

In [13], the authors studied the three-point boundary-value problem
\[ u''(t) + a(t)u'(t) + \lambda f(t, u) = 0, \quad 0 \leq t \leq 1, \]
\[ u'(0) = 0, \quad u(1) = \alpha u(\eta), \]
where \( 0 < \eta < 1 \), \( \alpha \) is a positive constant, \( a \in C([0,1], (-\infty, 0)) \), \( f \in C([0,1] \times \mathbb{R}^+, \mathbb{R}) \) and there exists \( M > 0 \) such that \( f(t,u) \geq -M \) for \((t,u) \in [0,1] \times \mathbb{R}^+ \). They obtained the existence of one positive solution by using Krasnoselskii’s fixed point theorem.

Motivated by the results mentioned above, in this paper, we study the existence of positive solutions of three-point boundary-value problem
\[ u''(t) + a(t)u'(t) + f(t, u) = 0, \quad 0 \leq t \leq 1, \]
\[ u'(0) = 0, \quad u(1) = \alpha u(\eta), \quad (1.1) \]
where \( 0 < \alpha, \eta < 1 \), \( a \in C([0,1], (-\infty, 0)) \) and \( f \) is allowed to change sign. We show that there exist two positive solutions by using Leggett-Williams fixed-point theorem. Our ideas are similar those used in [13], but a little different. By applying Leggett-Williams fixed-point theorem, we get the new results, which are different from the previous results and the conditions are easy to be checked. In particular, we do not need that \( f \) be either superlinear or sublinear which was required in [7, 13, 17, 18].

In the rest of the paper, we make the following assumptions
\( (H1) \quad 0 < \alpha, \eta < 1; \)
\( (H2) \quad a \in C([0,1], (-\infty, 0)); \)
\( (H3) \quad f : [0,1] \times \mathbb{R}^+ \rightarrow \mathbb{R} \) is continuous and there exists \( M > 0 \) such that \( f(t,u) \geq -M \) for \((t,u) \in [0,1] \times \mathbb{R}^+ \).

By a positive solution of \( (1.1) \), we understand a function \( u \) which is positive on \((0,1)\) and satisfies the differential equations as well as the boundary conditions in \( (1.1) \).

2. Preliminaries

In this section, we give some definitions and lemmas.

**Definition 2.1.** Let \( E \) be a real Banach space. A nonempty closed set \( P \subset E \) is said to be a cone provided that
\( (i) \quad u \in P \) and \( a \geq 0 \) imply \( au \in P \);
\( (ii) \quad u, -u \in P \) implies \( u = 0 \);
\( (iii) \quad u, v \in P \) implies \( u + v \in P \).
Theorem 2.3. Assume $E$ be a real Banach space, $P \subset E$ be a cone. Let $A : \overline{P} \to P$ be completely continuous and $\alpha$ be a nonnegative concave functional on $P$ such that $\alpha(y) \leq \|y\|$, for $y \in \overline{P}_c$. Suppose that there exist $0 < a < b < d < c$ such that

(i) $\{ y \in P(\alpha, b, d) : \alpha(y) > b \} \neq \emptyset$ and $\alpha(Ay) > b$, for all $y \in P(\alpha, b, d)$;
(ii) $\|Ay\| < a$, for all $\|y\| \leq a$;
(iii) $\alpha(Ay) > b$ for all $y \in P(\alpha, b, c)$ with $\|Ay\| > d$.

Then $A$ has at least three fixed points $y_1, y_2, y_3$ satisfying

$$\|y_1\| < a, \quad b < \alpha(y_2), \quad \|y_3\| > a, \quad \alpha(y_3) < b.$$  

Lemma 2.4. Assume that (H1), (H2) hold. Then for any $y \in C[0,1]$ the BVP

$$u''(t) + a(t)u'(t) + y(t) = 0, \quad 0 \leq t \leq 1,$$

$$u'(0) = 0, \quad u(1) = \alpha u(\eta),$$  

has unique solution

$$u(t) = -\int_0^t \left( \frac{1}{p(s)} \int_0^s p(\tau)y(\tau)d\tau \right)ds + \frac{1}{1-\alpha} \int_0^t \left( \frac{1}{p(s)} \int_0^s p(\tau)y(\tau)d\tau \right)ds \right. \left. - \frac{\alpha}{1-\alpha} \int_0^\eta \left( \frac{1}{p(s)} \int_0^s p(\tau)y(\tau)d\tau \right)ds,$$

where $p(t) = \exp \left( \int_0^t a(\tau)d\tau \right)$.

Lemma 2.5. Assume that (H1), (H2) hold. Let $y \in C[0,1]$ and $y(t) \geq 0$ for all $t \in [0,1]$, then the unique solution of

$$u''(t) + a(t)u'(t) + y(t) = 0, \quad 0 \leq t \leq 1,$$

$$u'(0) = 0, \quad u(1) = \alpha u(\eta),$$  

satisfies

$$\min_{t \in [0,1]} u(t) \geq \gamma \|u\|,$$

where $\gamma = \frac{\alpha(1-\eta)}{1-\alpha\eta}$.

Lemma 2.7. Let $\omega$ be the unique solution of the initial-value problem

$$u''(t) + a(t)u'(t) + 1 = 0, \quad 0 \leq t \leq 1,$$

$$u'(0) = 0, \quad u(1) = \alpha u(\eta).$$  

Then $\omega(t) \leq \Gamma \gamma$ for $t \in [0,1]$, where $\gamma = (\alpha(1-\eta))/(1-\alpha\eta)$ and

$$\Gamma = \left( \int_0^\eta \left( \frac{1}{p(s)} \int_0^s p(\tau)d\tau \right)ds + \frac{1}{1-\alpha} \int_0^1 \left( \frac{1}{p(s)} \int_0^s p(\tau)d\tau \right)ds \right)^{-\frac{1}{\gamma}},$$
3. Main Results

For convenience, we let \( \gamma = \frac{\alpha(1-n)}{1-\alpha n} \),
\[
l = \int_0^1 \left( \frac{1}{p(s)} \int_0^s p(\tau) d\tau \right) ds, \quad h = \frac{1}{1-\alpha} \int_0^1 \left( \frac{1}{p(s)} \int_0^s p(\tau) d\tau \right) ds.
\]
Let \( E = C[0,1] \), then \( E \) is Banach space, with the norm \( \|u\| = \sup_{t \in [0,1]} |u(t)| \). We define a cone in \( E \) by
\[
P = \{ u \in E : u \geq 0, \min_{t \in [0,1]} u(t) \geq \gamma \|u\| \}.
\]

Our main results are the following theorems.

**Theorem 3.1.** Suppose conditions (H1)--(H3) hold and there exist positive constants \( e, b, c, N \) with \( \Gamma < e < e + M \Gamma \gamma < b < \gamma^2 c, \frac{1}{\gamma} < N < \frac{e}{\Gamma} \) such that
\[
(A1) \ f(t,u) < \frac{e}{N} - M \text{ for } t \in [0,1], 0 \leq u \leq e; \\
(A2) \ f(t,u) \geq \frac{b}{N} N - M \text{ for } t \in [0,1], b - M \Gamma \gamma \leq u \leq b; \\
(A3) \ f(t,u) \leq \frac{b}{N} - M \text{ for } t \in [0,1], 0 \leq u \leq c,
\]
where the number \( \Gamma \) is defined in Lemma 2.7. Then (1.1) has at least two positive solutions.

**Proof.** Let \( \omega \) be a solution of (2.3) and \( z = M \omega \). By Lemma 2.7 we have \( z(t) = M \omega(t) \leq M \Gamma \gamma < e \gamma \). It is easy to see that (1.1) has a positive solution \( u \) if and only if \( u + z = \pi \) is a solution of the boundary-value problem
\[
u''(t) + a(t)u'(t) = -g(t, u - z), \quad 0 \leq t \leq 1,  \\
u'(0) = 0, \quad u(1) = \alpha u(\eta), \quad (3.1)
\]
and \( \pi > z \) for \( t \in (0,1) \), where \( g : [0,1] \times R \rightarrow [0, +\infty) \) is defined by
\[
g(t,y) = \begin{cases} f(t,y) + M, & (t,y) \in [0,1] \times [0, +\infty), \\ f(t,0) + M, & (t,y) \in [0,1] \times (-\infty, 0). \end{cases}
\]

For \( v \in P \), define the operator
\[
Tv(t) = -\int_0^t \left( \frac{1}{p(s)} \int_0^s p(\tau) g(\tau, v(\tau) - z(\tau)) d\tau \right) ds \\
+ \frac{1}{1-\alpha} \int_0^1 \left( \frac{1}{p(s)} \int_0^s p(\tau) g(\tau, v(\tau) - z(\tau)) d\tau \right) ds \\
- \frac{\alpha}{1-\alpha} \int_0^\eta \left( \frac{1}{p(s)} \int_0^s p(\tau) g(\tau, v(\tau) - z(\tau)) d\tau \right) ds,
\]
where \( p \) is defined in Lemma 2.4. By Lemmas 2.4, 2.5 and 2.6, we can check \( T(P) \subseteq P \). It is easy to check \( T \) is completely continuous by Arzela-Ascoli theorem.

In the following, we show that all the conditions of Theorem 2.3 are satisfied. Firstly, we define the nonnegative, continuous concave functional \( \alpha : P \rightarrow [0, \infty) \) by
\[
\alpha(v) = \min_{t \in [0,1]} v(t),
\]
Obviously, for every \( v \in P \), \( \alpha(v) \leq \|v\| \).
We first show that \( T(\mathcal{P}_c) \subseteq \mathcal{P}_c \). Let \( v \in \mathcal{P}_c \) and \( t \in [0,1] \) be arbitrary. When \( v(t) \geq z(t) \), we have \( 0 \leq v(t) - z(t) \leq v(t) \leq c \) and thus \( g(t, v(t) - z(t)) = f(t, v(t) - z(t)) + M \geq 0 \). By (A3) we have

\[
g(t, v(t) - z(t)) \leq \frac{c}{h}.
\]

When \( v(t) < z(t) \), we have \( g(t, v(t) - z(t)) = f(t, 0) + M \geq 0 \). Again by (A3) we have

\[
g(t, v(t) - z(t)) \leq \frac{c}{h}.
\]

Therefore, we have proved that, if \( v \in \mathcal{P}_c \), then \( g(t, v(t) - z(t)) \leq \frac{c}{h} \) for \( t \in [0,1] \). Then,

\[
\|Tv\| = Tv(0) = \frac{1}{1 - \alpha} \int_0^1 \frac{1}{p(s)} \int_0^s p(\tau) g(\tau, v(\tau) - z(\tau)) d\tau ds
\]

\[
- \frac{\alpha}{1 - \alpha} \int_0^1 \frac{1}{p(s)} \int_0^s p(\tau) g(\tau, v(\tau) - z(\tau)) d\tau ds
\]

\[
\leq \frac{1}{1 - \alpha} \int_0^1 \frac{1}{p(s)} \int_0^s p(\tau) g(\tau, v(\tau) - z(\tau)) d\tau ds
\]

\[
\leq \frac{c}{h} \frac{1}{1 - \alpha} \int_0^1 \frac{1}{p(s)} \int_0^s p(\tau) d\tau ds = c.
\]

Thus \( Tv \in \mathcal{P}_c \). Therefore, we have \( T(\mathcal{P}_c) \subseteq \mathcal{P}_c \). Especially, if \( v \in \mathcal{P}_c \), then assumption (A1) yields \( g(t, v(t) - z(t)) \leq \frac{c}{h} \) for \( t \in [0,1] \). So, we have \( T : \mathcal{P}_c \to \mathcal{P}_c \), i.e., the assumption (ii) of Theorem 2.3 holds.

To verify condition (i) of Theorem 2.3, let \( v(t) = \frac{b}{\gamma t} \), then \( v \in \mathcal{P}_c \), \( \alpha(v) = b/\gamma^2 > b \). That is \( \{v \in P(\alpha, b, \frac{b}{\gamma}, T(\mathcal{P}_c)) : \alpha(v) > b\} \neq \emptyset \). Moreover, if \( v \in P(\alpha, b, \frac{b}{\gamma}, T(\mathcal{P}_c)) \), then \( \alpha(v) \geq b \), so \( b \leq \|v\| \leq \frac{b}{\gamma} \). Thus, \( 0 < b - MT\gamma \leq v(t) - z(t) \leq v(t) \leq \frac{b}{\gamma} \), \( t \in [0,1] \).

From assumption (A2) we obtain \( g(t, v(t) - z(t)) \geq \frac{b}{T} N \) for \( t \in [0,1] \). By the definition of \( \alpha \) and above-proved inclusion \( T(\mathcal{P}_c) \subseteq \mathcal{P}_c \), we have

\[
\alpha(Tv) = \min_{t \in [0,1]} Tv(t) \geq \|Tv\| = \gamma Tv(0)
\]

\[
= \gamma \left( \frac{1}{1 - \alpha} \int_0^1 \frac{1}{p(s)} \int_0^s p(\tau) g(\tau, v(\tau) - z(\tau)) d\tau ds \right)
\]

\[
- \frac{\alpha}{1 - \alpha} \int_0^1 \frac{1}{p(s)} \int_0^s p(\tau) g(\tau, v(\tau) - z(\tau)) d\tau ds
\]

\[
\geq \gamma \left( \frac{1}{1 - \alpha} \int_0^1 \frac{1}{p(s)} \int_0^s p(\tau) g(\tau, v(\tau) - z(\tau)) d\tau ds \right)
\]

\[
- \frac{\alpha}{1 - \alpha} \int_0^1 \frac{1}{p(s)} \int_0^s p(\tau) g(\tau, v(\tau) - z(\tau)) d\tau ds
\]

\[
= \gamma \int_0^1 \frac{1}{p(s)} \int_0^s p(\tau) g(\tau, v(\tau) - z(\tau)) d\tau ds
\]

\[
\geq \frac{b}{T} N \int_0^1 \frac{1}{p(s)} \int_0^s p(\tau) d\tau ds
\]

\[
= \gamma Nb > b.
\]

Therefore, condition (i) of Theorem 2.3 is satisfied with \( d = b/\gamma^2 \).
Finally, we address condition (iii) of Theorem 2.3. For this we choose \( v \in P(\alpha, b, c) \) with \( \|Tv\| > b/\gamma^2 \). Then from above-proved inclusion \( T(P) \subseteq P \), we have
\[
\alpha(Tv) = \min_{t \in [a, 1]} Tv(t) \geq \gamma \|Tv\| \geq \frac{b}{\gamma} > b.
\]
Hence, condition (iii) of Theorem 2.3 holds with \( \|Tv\| > b/\gamma^2 \).

To sum up, all the hypotheses of Theorem 2.3 are satisfied. Hence \( T \) has at least three positive fixed points \( v_1, v_2 \) and \( v_3 \) such that
\[
\|v_1\| < e, \quad b < \alpha(v_2), \quad \|v_3\| > e, \quad \alpha(v_3) < b.
\]
Further, \( u_i = v_i - z \) \((i = 1, 2, 3)\) are solutions of (3.1). Moreover,
\[
v_2(t) \geq \gamma \|v_2\| \geq \gamma \alpha(v_2) > \gamma b > \gamma MT \geq z(t), \quad t \in [0, 1],
\]
\[
v_3(t) \geq \gamma \|v_3\| > \gamma e > \gamma MT \geq z(t), \quad t \in [0, 1].
\]
So \( u_2 = v_2 - z, u_3 = v_3 - z \) are two positive solutions of (1.1). This completes the proof. \( \square \)

**Theorem 3.2.** Suppose (H1)–(H3) hold, and there exist positive constants \( a_i, b_i, N \) with \( MT < a_i < a_i + \Gamma \gamma < b_i < \gamma^2 a_{i+1}, \frac{1}{\gamma} < N < \frac{a_i+1}{b_i}, \) \((i = 1, 2, \ldots, n-1)\) such that
\[
(A4) \quad f(t, u) < \frac{a_i}{M} - M \text{ for } t \in [0, 1], \quad 0 \leq u \leq a_i \quad (i = 1, 2, \ldots, n);
\]
\[
(A5) \quad f(t, u) \geq \frac{b_i}{\gamma} M - M \text{ for } t \in [0, 1], \quad 0 \leq u \leq \frac{b_i}{\gamma} \quad (i = 1, 2, \ldots, n-1).
\]
Then, (1.1) has at least \( 2(n-1) \) positive solutions.

**Proof.** When \( n = 2 \), the assumptions of Theorem 3.1 hold (with \( c = a_2 \)), so we can get at least two positive solutions \( u_2 \) and \( u_3 \) such that \( a_1 < u_2 \leq a_2 \) and \( u_3 \leq a_2 \). Following the identical fashion, by the induction method we immediately complete the proof. \( \square \)

Our results are different from those in [13], in particular, the following condition that was used in [13], is not needed in this article
\[
\lim_{u \to \infty} \frac{f(t, u)}{u} = +\infty \text{ uniformly on } [0, 1].
\]

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