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ROBUSTNESS OF A NONUNIFORM (μ, ν) TRICHOTOMY IN BANACH SPACES

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Abstract. In this article, we consider the robustness of a nonuniform (μ, ν) trichotomy in Banach spaces, in the sense that the existence of such a trichotomy for a given linear equation persists under sufficiently small linear perturbations. The continuous dependence with the perturbation of the constants in the notion of trichotomy is studied, and the related robustness of strong (μ, ν) trichotomy is also presented.

1. Introduction

Exponential trichotomy is the most complex asymptotic property of dynamical systems arising from the central manifold theory. When asymptotic behavior around the equilibrium point of a dynamical system is controlled by either the attraction of the stable manifold or the repulsion of the unstable manifold, exponential dichotomy describes a rather idealistic situation where the solution is either exponentially stable on the stable subspaces or exponentially unstable on the unstable subspaces. When asymptotic behavior is described through the splitting of the main space into stable, unstable and central subspaces at each point from the flows domain, exponential trichotomy reflects a deeper analysis of the behavior of solutions of dynamical systems. The conception of trichotomy was first introduced by Sacker and Sell [28]. They described SS-trichotomy for linear differential systems by linear skew-product flows. Later, Elaydi and Hájek [18, 19] gave the notions of exponential trichotomy for differential systems and for nonlinear differential systems, respectively. These notions are stronger notions than SS-trichotomy. Recently, Barreira and Valls [6] considered a general concept of nonuniform exponential trichotomy, from which can see exponential trichotomy as a special case of the nonuniform exponential trichotomy. For more information about exponential trichotomy we refer the reader to [23].

The notion of exponential trichotomy plays a central role in the study of center manifolds, it is one of the powerful tools in the analysis of the asymptotic behavior of dynamical systems. When a linear dynamics possesses no unstable directions, all solutions converge exponentially to the center manifold, and thus the stability of the zero solution under sufficiently small perturbations is completely determined by the

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behavior on any center manifold. The study of center manifolds can be traced back to the works of Pliss [27] and Kelley [22]. A very detailed exposition in the case of autonomous equations is given [29], See also [30] for the case of infinite-dimensional systems. We refer the reader to [14, 15, 29] for more details and further references.

Inspired both in the classical notion of exponential dichotomy and in the notion of nonuniform hyperbolic trajectory introduced by Pesin in [25, 26], Barreira and Valls have introduced the notion of nonuniform exponential dichotomies and have developed the corresponding theory in a systematic way [2, 7]. See also the book [4] for details. In particular, the results proved by Barreira and Valls can be regarded as a nice contribution to the nonuniform hyperbolicity theory. We refer to [1] for a detailed exposition of the nonuniform hyperbolicity theory.

Furthermore, general nonuniform dichotomies have been studied, which extend the notion of nonuniform exponential dichotomy in various ways. In [11, 12], Bento and Silva considered the nonuniform polynomial dichotomy, the existence of smooth stable manifolds in Banach spaces for sufficiently small perturbations of nonuniform polynomial dichotomy was obtained. Nonuniform (μ,ν) dichotomy was studied in [9, 10], Barreira et al [9] have established robustness of this dichotomy, this general nonuniform dichotomies and local stable manifolds was given in [10]. A similar dichotomy for the discrete case was discussed in [13, 31]

In [6], Barreira and Valls have introduced the so-called nonuniform exponential trichotomy. Robustness of such a nonuniform exponential trichotomy was established, which means a nonuniform exponential trichotomy defined by a nonautonomous linear equation

$$x' = A(t)x \tag{1.1}$$

in a Banach space, persists under sufficiently small linear perturbations in the equation

$$x' = [A(t) + B(t)]x. \tag{1.2}$$

In [8], they considered a linear equations (1.1) that may exhibit stable, unstable and central behaviors in different directions, with respect to arbitrary asymptotic rates of the form $e^{c\rho(t)}$ determined by an arbitrary function $\rho(t)$ instead of the usual exponential behavior e^{ct}. They proposed a $\rho-$nonuniform exponential trichotomy and consider the Lyapunov functions for the trichotomy.

In the present paper, our main objective is to consider the general case of nonuniform (μ,ν) trichotomy for an arbitrary nonautonomous linear dynamics, and establish the robustness of the nonuniform (μ,ν) trichotomy in Banach spaces, based on [9] for nonuniform (μ,ν) dichotomy. This means that such a trichotomy persists under sufficiently small linear perturbations. Precisely, the perturbed equation (1.2) admits a nonuniform (μ,ν) trichotomy if the same happens for (1.1) for any sufficiently small perturbations $B(t)$. We also establish the continuous dependence with the perturbation of the constants in the notion of trichotomy and robustness of strong nonuniform (μ,ν) trichotomy. We note that the notion of nonuniform (μ,ν) trichotomy is also an elaboration of the notion of nonuniform (μ,ν) dichotomy.

We remark that the study of robustness in the case of uniform exponential behavior has a long history. Early it was discussed by Perron [24], Coppel [16]. For more recent work, we refer to [20, 21] and the references therein for uniform exponential behavior. We refer to [2, 4, 5, 14] for the study of robustness in the setting of a nonuniform exponential behavior. A trichotomy for the discrete case was discussed in [3, 17]. We emphasize that the trichotomy considered in this paper is more general, this may seem a somewhat formal generalization of the notion of nonuniform exponential trichotomy in [6, 8]. Particularly in view of the applications, it is important to look for more general notions. Moreover, due to the central role played by the notion of trichotomy, most importantly in the theory of center manifolds which are crucial in the study of the asymptotic behavior of trajectories, it is also helpful to understand how trichotomies vary under perturbations.

The remaining part of this paper is organized as follows. Section 2 is a preliminary for our main results. In Section 3, we establish the robustness of nonuniform (μ,ν) trichotomies. The robustness of strong (μ,ν) trichotomies is presented in Section 4.

2. Preliminaries

We say that an increasing function $\mu:\mathbb{R}^+\to[1,+\infty)$ is a growth rate if

$$\mu(0)=1 \quad\text{and}\quad \lim_{t\to+\infty}\mu(t)=+\infty.$$

Let X be a Banach space and denote by $\mathcal{B}(X)$ the space of bounded linear operators acting on X. Given a continuous function $A:\mathbb{R}^+\to\mathcal{B}(X)$. We assume that each solution of (1.1) is global and denote the evolution operator associated with (1.1) by $T(t,s)$; i.e., the linear operator such that

$$T(t,s)x(s)=x(t),\quad t,s>0,$$

where $x(t)$ is any solution of (1.1). Clearly, $T(t,t)=\mathrm{Id}$ and

$$T(t,\tau)T(\tau,s)=T(t,s),\quad t,\tau,s>0.$$

Definition 2.1. [9] We say that equation (1.1) admits a nonuniform (μ,ν) dichotomy in $\mathbb{R}^+$ if there exist projections $P(t):X\to X$ for each $t>0$ satisfying

$$T(t,s)P(s)=P(t)T(t,s),\quad t\ge s, \tag{2.1}$$

and there exist constants $\alpha,\beta,D>0$ $\varepsilon\ge 0$ and two continuously differentiable growth rates μ,ν such that

$$\|T(t,s)P(s)\|\le D\Big(\frac{\mu(t)}{\mu(s)}\Big)^{-\alpha}\nu^{\varepsilon}(s),\quad t\ge s, \tag{2.2}$$

$$\|T(t,s)Q(s)\|\le D\Big(\frac{\mu(s)}{\mu(t)}\Big)^{-\beta}\nu^{\varepsilon}(s),\quad s\ge t, \tag{2.3}$$

where $Q(t)=\mathrm{Id}-P(t)$ for each $t>0$. When $\varepsilon=0$, we say that (1.1) has a uniform (μ,ν) dichotomy or simply a (μ,ν) dichotomy.

For the convenience of the reader, we recall the following result about robustness of a nonuniform (μ,ν) dichotomy obtained in [9]. Set

$$\begin{gathered}\tilde\alpha=\frac{(\alpha-\beta)+\sqrt{(\alpha+\beta)^2-4\delta D(\alpha+\beta)}}{2}\\ \tilde D_1=\frac{D}{1-\delta D/(\beta+\tilde\alpha)},\quad \tilde D_2=\frac{D}{1-\delta D/(\alpha+\tilde\alpha)},\quad \tilde D=\max\{\tilde D_1,\tilde D_2\},\end{gathered}\tag{2.4}$$

$$\delta<\min\Big\{\frac{\alpha+\beta}{4D},\frac{\alpha\beta}{2D(\alpha+\beta)},\frac{\tilde\alpha+\beta}{D},\frac{\tilde\alpha+\alpha}{D},\frac{1}{4D\tilde D}\Big\}. \tag{2.5}$$

We denote the evolution operator associated to equation (1.2) by $\hat T(t,s)$

Lemma 2.2 ([9]). *Let $A, B : \mathbb{R}^+ \to \mathcal{B}(X)$ be continuous functions such that equation* (1.1) *admits a nonuniform (μ,ν) dichotomy in $\mathbb{R}^+$ with $\varepsilon < \min\{\alpha,\beta\}$. Moreover, assume that $B(t)$ satisfies*

$$\|B(t)\| \le \delta\nu^{-\varepsilon}(t)\frac{\mu'(t)}{\mu(t)}, \quad t \ge 0 \tag{2.6}$$

with (2.5)*, then* (1.2) *admits a nonuniform (μ,ν) dichotomy in $\mathbb{R}^+$ with the projections $\hat{P}(t), \hat{Q}(t)$ such that for each $t, s \in \mathbb{R}^+$:*

$$\hat{P}(t) = \hat{T}(t,0)\hat{P}(0)\hat{T}(0,t), \quad \hat{Q}(t) = \hat{T}(t,0)\hat{Q}(0)\hat{T}(0,t), \tag{2.7}$$

$$\|\hat{T}(t,s)|\operatorname{Im}\hat{P}(s)\| \le \tilde{D}\Big(\frac{\mu(t)}{\mu(s)}\Big)^{-\tilde{\alpha}}\nu^{\varepsilon}(s), \quad t \ge s, \tag{2.8}$$

$$\|\hat{T}(t,s)|\operatorname{Im}\hat{Q}(s)\| \le \tilde{D}\Big(\frac{\mu(s)}{\mu(t)}\Big)^{-\tilde{\alpha}}\nu^{\varepsilon}(s), \quad s \ge t. \tag{2.9}$$

Suppose further that growth rates satisfy $\mu \ge \nu$ and

$$\|B(t)\| \le \delta\nu^{-2\varepsilon}(t)\big\{(\alpha+\tilde{\alpha})\frac{\mu'(t)}{\mu(t)} + \varepsilon\frac{\nu'(t)}{\nu(t)}\big\}, \quad t \ge 0, \tag{2.10}$$

then equation (1.2) *admits a nonuniform (μ,ν) dichotomy in $\mathbb{R}^+$ with the constants α, D, ε replaced respectively by $\tilde{\alpha}, 4D\tilde{D}, 2\varepsilon$ in* (2.2)*,* (2.3) *and*

$$\|\hat{P}(t)\| \le 4D\nu^{\varepsilon}(t), \quad \|\hat{Q}(t)\| \le 4D\nu^{\varepsilon}(t). \tag{2.11}$$

Definition 2.3. We say that (1.1) admits a nonuniform (μ,ν) trichotomy in I if there exist projections $P(t), Q(t), R(t) : X \to X$ for each $t \in I$ such that

$$T(t,s)P(s) = P(t)T(t,s), \quad T(t,s)Q(s) = Q(t)T(t,s), \quad T(t,s)R(s) = R(t)T(t,s) \tag{2.12}$$

and

$$P(t) + Q(t) + R(t) = \mathrm{Id} \tag{2.13}$$

for every $t, s \in I$, and there exist constants

$$0 \le \eta < \alpha, \quad 0 \le \xi < \beta, \quad \varepsilon \ge 0, \quad D \ge 1 \tag{2.14}$$

such that for every $t, s \in I$ with $t \ge s$ we have:

$$\|T(t,s)P(s)\| \le D\Big(\frac{\mu(t)}{\mu(s)}\Big)^{-\alpha}\nu^{\varepsilon}(s), \tag{2.15}$$

$$\|T(t,s)R(s)\| \le D\Big(\frac{\mu(t)}{\mu(s)}\Big)^{\xi}\nu^{\varepsilon}(s), \tag{2.16}$$

$$\|T(t,s)^{-1}Q(t)\| \le D\Big(\frac{\mu(t)}{\mu(s)}\Big)^{-\beta}\nu^{\varepsilon}(t), \tag{2.17}$$

$$\|T(t,s)^{-1}R(t)\| \le D\Big(\frac{\mu(t)}{\mu(s)}\Big)^{\eta}\nu^{\varepsilon}(t), \tag{2.18}$$

a We notice that setting $t = s$ in (2.15) and (2.17) we obtain

$$P(t) \le D\nu^{\varepsilon}(t), \quad Q(t) \le D\nu^{\varepsilon}(t), \quad R(t) \le D\nu^{\varepsilon}(t), \tag{2.19}$$

for every $t \in I$. When $\varepsilon = 0$, we say that (1.1) admits a uniform (μ,ν) trichotomy.

The following is an example with a nonuniform (μ,ν) trichotomy which can not be uniform.

Example 2.4. Given $\varepsilon > 0$, $\alpha > 0$, μ and ν are arbitrary differentiable growth rates, consider the differential equation in $\mathbb{R}^3$ given by

$$\begin{aligned} \dot{x} &= \Big(\frac{-\alpha\mu'(t)}{\mu(t)} + \frac{\varepsilon\nu'(t)}{2\nu(t)}(\cos t - 1) - \frac{\varepsilon}{2}\log\nu(t)\sin t\Big)x \\ \dot{y} &= 0 \\ \dot{z} &= \Big(\frac{\alpha\mu'(t)}{\mu(t)} - \frac{\varepsilon\nu'(t)}{2\nu(t)}(\cos t - 1) + \frac{\varepsilon}{2}\log\nu(t)\sin t\Big)z. \end{aligned} \tag{2.20}$$

It is easy to verify that (2.20) has the evolution operator

$$\begin{aligned} T(t,s)(x,y,z) &= (X(t,s)x, Y(t,s)y, Z(t,s)z) \\ &= (T(t,s)P(s)x, T(t,s)R(s)y, T(t,s)Q(s)z), \end{aligned}$$

where

$$X(t,s) = \Big(\frac{\mu(t)}{\mu(s)}\Big)^{-\alpha}\exp\Big(\frac{\varepsilon}{2}\log\nu(t)(\cos t - 1) - \frac{\varepsilon}{2}\log\nu(s)(\cos s - 1)\Big),$$

$$Y(t,s) = 1,$$

$$Z(t,s) = X(s,t) = \Big(\frac{\mu(t)}{\mu(s)}\Big)^{\alpha}\exp\Big(-\frac{\varepsilon}{2}\log\nu(t)(\cos t - 1) + \frac{\varepsilon}{2}\log\nu(s)(\cos s - 1)\Big)$$

One can easily verify that

$$\begin{gathered} \|T(t,s)P(s)\| = \|X(t,s)\| \le \Big(\frac{\mu(t)}{\mu(s)}\Big)^{-\alpha}\nu^{\varepsilon}(s), \\ \|T(t,s)R(s)\| = \|Y(t,s)\| = 1 \le \nu^{\varepsilon}(s), \\ \|T(t,s)^{-1}Q(t)\| = \|Z(t,s)^{-1}\| \le \Big(\frac{\mu(t)}{\mu(s)}\Big)^{-\alpha}\nu^{\varepsilon}(t), \\ \|T(t,s)^{-1}R(t)\| = \|Y(t,s)^{-1}\| = 1 \le \nu^{\varepsilon}(t), \end{gathered}$$

This shows that (2.20) admits a nonuniform (μ,ν) trichotomy in $\mathbb{R}^+$. Moreover, if we take $t = 2k\pi$ and $s = (2k-1)\pi$, $k \in \mathbb{N}$, then

$$\|X(t,s)\| = \Big(\frac{\mu(t)}{\mu(s)}\Big)^{-\alpha}\nu^{\varepsilon}(s),$$

which ensures us that the nonuniform part can not removed when $\varepsilon > 0$.

3. Robustness in semi-infinite intervals

Theorem 3.1. *Let $A, B : I \to B(X)$ be continuous functions in an interval $I = [0, +\infty)$ such that (1.1) admits a nonuniform (μ,ν) trichotomy with $\mu \ge \nu$ in I satisfying*

$$\varepsilon < \min\{(\alpha - \eta)/2, (\beta - \xi)/2\}, \tag{3.1}$$

and assume that $B(t)$ satisfies (2.10) and (2.11) with (2.5), then (1.2) admits a nonuniform (μ,ν) trichotomy in $[0,+\infty)$; i.e.

(i) there exist projections $\hat{P}(t), \hat{Q}(t)$ and $\hat{R}(t)$ for $t \in I$ satisfying (2.12) and (2.13) for every $t, s \in I$;

(ii) for every $t, s \in I$ with $t \ge s$, the corresponding estimates to the ones in (2.15)–(2.19) are valid with constants $\alpha, \beta, \xi, \eta, \varepsilon, D$ replaced respectively by

$$\hat{\alpha} = (\alpha + \eta)/2 + L((\alpha - \eta)/2), \quad \hat{\beta} = (\beta + \xi)/2 + L((\beta - \xi)/2),$$

$$\hat{\xi} = (\beta+\xi)/2 - L((\beta-\xi)/2), \quad \hat{\eta} = (\alpha+\eta)/2 - L((\alpha-\eta)/2),$$
$$\hat{\varepsilon} = 3\varepsilon, \quad \hat{D} = \max\big\{\frac{D}{1-\delta D/(\hat{\alpha}-\eta)}, \frac{D}{1-\delta D/(\hat{\beta}-\xi)}\big\},$$

where $L(x) = x\sqrt{1-2\delta D/x}$,

Proof. Let $x(t) = T(t,s)x(s)$ be a solution of (1.1). We consider the change of variables $y(t) = x(t)\mu^k(t)$, where $k = (\alpha+\eta)/2$. Then $y(t)$ satisfies the linear equation

$$y' = [A(t) + k\frac{\mu'(t)}{\mu(t)}]y. \tag{3.2}$$

Denoting by $T_k(t,s)$ its evolution operator we have

$$T_k(t,s) = T(t,s)\Big(\frac{\mu(t)}{\mu(s)}\Big)^k. \tag{3.3}$$

Since (1.1) admits a nonuniform (μ,ν) trichotomy in I , we conclude that (3.2) admits a nonuniform (μ,ν) dichotomy in I with $\alpha_1 = \beta_1 = (\alpha-\eta)/2$, and projections $P_1(t) = P(t)$ and $Q_1(t) = Q(t) + R(t)$ for each $t \in I$. It follows from Lemma (2.2) that the equation

$$y' = [A(t) + k\frac{\mu'(t)}{\mu(t)} + B(t)]y \tag{3.4}$$

admits a nonuniform (μ,ν) dichotomy, say with projections $\hat{P}_1(t)$ and $\hat{Q}_1(t)$. In particular, the linear subspaces $\hat{E}_1(t) = \hat{P}_1(t)(X)$ and $\hat{F}_1(t) = \hat{Q}_1(t)(X)$ satisfy

$$\hat{E}_1(t) \oplus \hat{F}_1(t) = X. \tag{3.5}$$

Now we consider a second change of variables $z(t) = x(t)\mu^{k'}(t)$, where $k' = -(\beta+\xi)/2$. Then $z(t)$ satisfies the linear equation

$$z' = [A(t) + k'\frac{\mu'(t)}{\mu(t)}]z, \tag{3.6}$$

and denoting by $T_{k'}(t,s)$ its evolution operator we have

$$T_{k'}(t,s) = T(t,s)\Big(\frac{\mu(t)}{\mu(s)}\Big)^{k'}. \tag{3.7}$$

Since (1.1) admits a nonuniform (μ,ν) trichotomy in I , we conclude that (3.6) admits a nonuniform (μ,ν) dichotomy in I with $\alpha_2 = \beta_2 = (\beta-\xi)/2$, and projections $P_2(t) = P(t)$ and $Q_2(t) = Q(t) + R(t)$ for each $t \in I$. It follows from Lemma (2.2) that the equation

$$z' = [A(t) + k'\frac{\mu'(t)}{\mu(t)} + B(t)]y, \tag{3.8}$$

admits a nonuniform (μ,ν) dichotomy, say with projections $\hat{P}_2(t)$ and $\hat{Q}_2(t)$. In particular, the linear subspaces $\hat{E}_2(t) = \hat{P}_2(t)(X)$ and $\hat{F}_2(t) = \hat{Q}_2(t)(X)$ satisfy

$$\hat{E}_2(t) \oplus \hat{F}_2(t) = X. \tag{3.9}$$

We also consider the evolution operators in (3.4) and (3.8), namely

$$\hat{T}_k(t,s) = \Big(\frac{\mu(t)}{\mu(s)}\Big)^k\hat{T}(t,s) \quad \text{and} \quad \hat{T}_{k'}(t,s) = \Big(\frac{\mu(t)}{\mu(s)}\Big)^{k'}\hat{T}(t,s). \tag{3.10}$$

$\square$

In the following, we firstly consider the relationship of these linear subspaces and projections.

Lemma 3.2. *For every $t \in I$ we have*

$$\hat{E}_1(t) \subset \hat{E}_2(t) \quad \text{and} \quad \hat{F}_2(t) \subset \hat{F}_1(t).$$

Proof. Set

$$U(x) = \limsup_{t \to +\infty} \frac{\ln \|\hat{T}_k(t,s)x\|}{\ln \mu(t)}. \tag{3.11}$$

If there exists $x \in \hat{E}_1(t) \backslash \hat{E}_2(t)$, then we write $x = y + z$ with $y \in \hat{E}_2(t)$ and $z \in \hat{F}_2(t)$. Since $x \in \hat{E}_1(t)$, by Lemma (2.2) we have

$$\|\hat{T}_k(t,s)x\| \le \tilde{D}\Big(\frac{\mu(t)}{\mu(s)}\Big)^{-L(\alpha_1)} \nu^{\varepsilon}(s)\|x\|. \tag{3.12}$$

and hence

$$U(x) \le -L(\alpha_1) < 0 \tag{3.13}$$

where $L(x) = x\sqrt{1 - 2\delta D/x}$. Moreover, since $x \in \hat{E}_1(t)\backslash \hat{E}_2(t)$, $y \in \hat{E}_2(t)$, we have $z \neq 0$, and hence

$$U(x) = \max\{U(y), U(z)\} = U(z) = \limsup_{t \to +\infty} \frac{\ln \|\hat{T}_k(t,s)z\|}{\ln \mu(t)}.$$

Since $z \in \hat{F}_2(t)$, for $t \ge s$ we have

$$\begin{aligned}
\|\hat{T}_k(t,s)z\| &= \Big(\frac{\mu(t)}{\mu(s)}\Big)^{(k-k')} \|\hat{T}_{k'}(t,s)z\| \\
&\ge \frac{1}{\tilde{D}}\|z\|\Big(\frac{\mu(t)}{\mu(s)}\Big)^{(k-k'+L(\alpha_2))} \nu^{-\varepsilon}(t) \\
&\ge \frac{1}{\tilde{D}}\|z\|\Big(\frac{\mu(t)}{\mu(s)}\Big)^{(k-k'+L(\alpha_2))} \frac{1}{\mu^{\alpha_2+L(\alpha_2)}(t)}.
\end{aligned}$$

So

$$U(x) = (k - k' - \alpha_2) = \frac{\alpha + \eta + 2\xi}{2} > 0. \tag{3.14}$$

This contradicts the inequality (3.13). Therefore, $\hat{E}_1(t) \subset \hat{E}_2(t)$. In a similar manner, we obtain that $\hat{F}_2(t) \subset \hat{F}_1(t)$ for each $t \in I$. $\square$

By Lemma (3.2), we can prove the following two Lemmas. These proofs are similar to those in [6].

Lemma 3.3. *For every $t \in I$ we have*

$$(\hat{E}_2(t) \cap \hat{F}_1(t)) \oplus \hat{E}_1(t) \oplus \hat{F}_2(t) = X. \tag{3.15}$$

Proof. Since $\hat{E}_1(t) \oplus \hat{F}_1(t) = X$, we have

$$(\hat{E}_2(t) \cap \hat{E}_1(t)) \oplus (\hat{E}_2(t) \cap \hat{F}_1(t)) = \hat{E}_2(t).$$

According to Lemma (3.2), $\hat{E}_1(t) \subset \hat{E}_2(t)$, we have

$$\hat{E}_2(t) \cap \hat{E}_1(t) = \hat{E}_1(t),$$

and hence

$$\hat{E}_1(t) \oplus (\hat{E}_2(t) \cap \hat{F}_1(t)) = \hat{E}_2(t).$$

Then we obtain

$$(\hat{E}_2(t)\cap\hat{F}_1(t))\oplus\hat{E}_1(t)\oplus\hat{F}_2(t)=\hat{E}_2(t)\cap\hat{F}_2(t)=X.$$

□

Lemma 3.4. *For every $t\in I$ we have*

$$\hat{P}_1(t)\hat{Q}_2(t)=\hat{Q}_2(t)\hat{P}_1(t)=0. \tag{3.16}$$

Proof. According to Lemma (3.2), $\hat{E}_1(t)\subset\hat{E}_2(t)$, $\hat{F}_2(t)\subset\hat{F}_1(t)$, for each $x\in X$ we have

$$\begin{gathered}\hat{Q}_2(t)x\in\hat{F}_2(t)\subset\hat{F}_1(t),\\ \hat{P}_1(t)x\in\hat{E}_2(t)\subset\hat{E}_2(t).\end{gathered}$$

Therefore,

$$\begin{gathered}\hat{P}_1(t)\hat{Q}_2(t)x\in\hat{P}_1(t)\hat{F}_1(t)=\hat{P}_1(t)\operatorname{Im}\hat{Q}_1(t)=\{0\},\\ \hat{Q}_2(t)\hat{P}_1(t)x\in\hat{Q}_2(t)\hat{E}_1(2)=\hat{Q}_2(t)\operatorname{Im}\hat{P}_2(t)=\{0\}\,.\end{gathered}$$

□

We proceed with the proof of Theorem (3.1). Set

$$\hat{P}(t)=\hat{P}_1(t),\quad \hat{Q}(t)=\hat{Q}_2(t),\quad \hat{R}(t)=\operatorname{Id}-\hat{P}_1(t)-\hat{Q}_2(t).$$

In view of Lemma (2.2), we have

$$\hat{T}_k(t,s)\hat{P}(s)=\hat{P}(t)\hat{T}_k(t,s),\quad \hat{T}_{k'}(t,s)\hat{Q}(s)=\hat{Q}(t)\hat{T}_{k'}(t,s),$$

according to (3.10), we obtain

$$\hat{T}(t,s)\hat{P}(s)=\hat{P}(t)\hat{T}(t,s),\quad \hat{T}(t,s)\hat{Q}(s)=\hat{Q}(t)\hat{T}(t,s).$$

This implies that

$$\hat{T}(t,s)\hat{R}(s)=\hat{R}(t)\hat{T}(t,s).$$

Since the operators $\hat{P}(t)$ and $\hat{Q}(t)$ are projections,

$$\hat{P}(t)=\hat{P}(t)^2,\quad \hat{Q}(t)=\hat{Q}(t)^2,$$

and by Lemma (3.4) we have

$$\begin{aligned}\hat{R}(t)^2&=(\operatorname{Id}-\hat{P}_1(t)-\hat{Q}_2(t))^2\\ &=\operatorname{Id}-2\hat{P}_1(t)-2\hat{Q}_2(t)+\hat{P}_1(t)^2+\hat{Q}_2(t)^2+\hat{P}_1(t)\hat{Q}_2(t)+\hat{Q}_2(t)\hat{P}_1(t)\\ &=\operatorname{Id}-\hat{P}_1(t)-\hat{Q}_2(t)=\hat{R}(t).\end{aligned}$$

Now we consider the subspaces

$$\hat{E}(t)=\hat{P}(t)(X),\quad \hat{F}(t)=\hat{Q}(t)(X)\quad \hat{G}(t)=\hat{R}(t)(X). \tag{3.17}$$

since $\hat{E}_1(t)=\hat{P}(t)(X)$ and $\hat{F}_2(t)=\hat{Q}(t)(X)$, We have

$$\hat{E}(t)=\hat{E}_1(t),\quad \hat{F}(t)=\hat{F}_2(t). \tag{3.18}$$

by Lemma (3.3), the image of $\operatorname{Id}-\hat{P}(t)-\hat{Q}(t)=\hat{R}(t)$ is $\hat{E}_2(t)\cap\hat{F}_1(t)$. So

$$\hat{G}(t)=\hat{R}(t)(X)=\hat{E}_2(t)\cap\hat{F}_1(t). \tag{3.19}$$

Furthermore, according to (2.11)

$$\|\hat{R}(t)\|=\|Id-\hat{P}_1(t)-\hat{Q}_2(t)\|\le 1+8D\nu^{\varepsilon}(t)\le(1+8D)\nu^{\varepsilon}(t). \tag{3.20}$$

By Lemma (2.2), since $\hat{P}(t) = \hat{P}_1(t)$, for every $t \geq s$ we have

$$\begin{aligned} \|\hat{T}(t,s)|\hat{E}(s)\| &= \|\hat{T}_k(t,s)\big(\frac{\mu(t)}{\mu(s)}\big)^{-k}|\hat{E}_1(s)\| \\ &\leq D_1\big(\frac{\mu(t)}{\mu(s)}\big)^{-(k+L(\alpha_1))}\nu^{2\varepsilon}(s) \end{aligned}$$

for some constant $D_1 > 0$. Similarly, since $\hat{Q}(t) = \hat{Q}_2(t)$ for every $t \geq s \geq 0$ we have

$$\begin{aligned} \|\hat{T}(t,s)^{-1}|\hat{F}(t)\| &= \|\hat{T}_{k'}(t,s)^{-1}\big(\frac{\mu(t)}{\mu(s)}\big)^{k'}|\hat{F}_2(t)\| \\ &\leq D_2\Big(\frac{\mu(t)}{\mu(s)}\Big)^{(k'-L(\alpha_2))}\nu^{2\varepsilon}(t) \end{aligned}$$

for some constant $D_2 > 0$. Furthermore, by (3.18), for every $t \geq s$ we have

$$\begin{aligned} \|\hat{T}(t,s)|\hat{R}(s)\| &\leq \|\hat{T}(t,s)|\hat{G}(s)\| \cdot \|\hat{R}(s)\| \\ &= \|\hat{T}(t,s)|(\hat{E}_2(s) \cap \hat{F}_1(s))\| \cdot \|\hat{R}(s)\| \\ &\leq \|\hat{T}(t,s)|\hat{E}_2(s)\| \cdot \|\hat{R}(s)\| \\ &= D_2\Big(\frac{\mu(t)}{\mu(s)}\Big)^{-k'}\|\hat{T}_{k'}(t,s)|\hat{E}_2(s)\| \cdot \|\hat{R}(s)\| \\ &\leq (1+8D)D_2\Big(\frac{\mu(t)}{\mu(s)}\Big)^{(-k'-L(\alpha_2))}\nu^{3\varepsilon}(s). \end{aligned} \tag{3.21}$$

The last inequality follows from Lemma (2.2) and (3.20), on the other hand, again by (3.18), for every $t \geq s \geq 0$ we have

$$\begin{aligned} \|\hat{T}(t,s)|^{-1}\hat{R}(t)\| &\leq \|\hat{T}(t,s)^{-1}|\hat{G}(t)\| \cdot \|\hat{R}(s)\| \\ &= \|\hat{T}(t,s)^{-1}|\hat{F}_1(t)\| \cdot \|\hat{R}(t)\| \\ &= \Big(\frac{\mu(t)}{\mu(s)}\Big)^{k}\|\hat{T}_{k'}^{-1}(t,s)|\hat{F}_1(t)\| \cdot \|\hat{R}(t)\| \\ &\leq (1+8D)D_1\Big(\frac{\mu(t)}{\mu(s)}\Big)^{(k-L(\alpha_1))}\nu^{3\varepsilon}(t). \end{aligned} \tag{3.22}$$

The last inequality also follows from Lemma (2.2) and (3.20). This shows that (1.2) admits a nonuniform (μ,ν) trichotomy in $[0,+\infty)$.

4. Robustness of strong (μ,ν) trichotomy

We can also consider a stronger version of (μ,ν) trichotomy and establish a corresponding robustness result. Namely, we say that (1.1) admits a strong nonuniform (μ,ν) trichotomy in I if it admits a nonuniform (μ,ν) trichotomy in I and there exist constants $a \geq \alpha$ and $b \geq \beta$ such that for every $t, s \in I$ with $t \geq s$ we have

$$\|T(t,s)^{-1}P(t)\| \leq D\Big(\frac{\mu(t)}{\mu(s)}\Big)^{a}\nu^{\varepsilon}(t), \tag{4.1}$$

$$\|T(t,s)Q(s)\| \leq D\Big(\frac{\mu(t)}{\mu(s)}\Big)^{b}\nu^{\varepsilon}(s), \tag{4.2}$$

We recall that (1.1) is said to admit a strong nonuniform (μ, ν) dichotomy if it admits a nonuniform (μ, ν) dichotomy and there exists $a > \max\{\alpha, \beta\}$ such that for every $t, s \in I$ with $t \geq s$ we have

$$\|T(t,s)^{-1}P(t)\| \leq D\Big(\frac{\mu(t)}{\mu(s)}\Big)^{a}\nu^{\varepsilon}(t), \tag{4.3}$$

$$\|T(t,s)Q(s)\| \leq D\Big(\frac{\mu(t)}{\mu(s)}\Big)^{a}\nu^{\varepsilon}(s), \tag{4.4}$$

Lemma 4.1 ([9]). *Let $A, B : R^+ \to B(X)$ be continuous functions such that* (1.1) *admits a nonuniform (μ, ν) dichotomy in R^+ with $\mu \geq \nu$ satisfying $\epsilon < \min\{\alpha.\beta\}$ and assume that $B(t)$ satisfies* (2.10) *with* (2.5)*, then* (1.2) *admits a strong nonuniform (μ, ν) dichotomy in $\mathbb{R}^+$.*

Theorem 4.2. *Let $A, B : \mathbb{R}^+ \to B(X)$ be continuous functions such that* (1.1) *admits a nonuniform (μ, ν) trichotomy in R^+ with $\mu \geq \nu$ satisfying* (3.1) *and assume that $B(t)$ satisfies* (2.10) *with* (2.5)*, then* (1.2) *admits a strong nonuniform (μ, ν) trichotomy in R^+.*

Proof. Following the proof of Theorem (3.1), We consider the projections $\hat{P}(t) = \hat{P}_1(t)$ and $\hat{Q}(t) = \hat{Q}_2(t)$. According to lemma (4.1), for every $t \geq s$ we have

$$\|\hat{T}(t,s)^{-1}\hat{P}(t)\| = \Big(\frac{\mu(t)}{\mu(s)}\Big)^{k}\|\hat{T}_k(t,s)^{-1}\hat{P}(t)\| \leq 4D\tilde{D}\Big(\frac{\mu(t)}{\mu(s)}\Big)^{k+L(\alpha_1)}\nu^{3\varepsilon}(t),$$

$$\|\hat{T}(t,s)\hat{Q}(s)\| = \Big(\frac{\mu(t)}{\mu(s)}\Big)^{-k'}\|\hat{T}_{k'}(t,s)^{-1}\hat{Q}(s)\| \leq 4D\tilde{D}\Big(\frac{\mu(t)}{\mu(s)}\Big)^{-k'+L(\alpha_2)}\nu^{3\varepsilon}(s),$$

where $k = \alpha + \eta$, $k' = -(\beta + \xi)$, so (1.2) admits a strong nonuniform (μ, ν) trichotomy in $\mathbb{R}^+$. □

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