Electronic Journal of Differential Equations, Vol. 2012 (2012), No. 156, pp. 1–21.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

EXISTENCE OF ALMOST PERIODIC SOLUTIONS FOR SEMILINEAR STOCHASTIC EVOLUTION EQUATIONS DRIVEN BY FRACTIONAL BROWNIAN MOTION

PAUL H. BEZANDRY

Abstract. This article concerns the existence of almost periodic solutions to a class of abstract stochastic evolution equations driven by fractional Brownian motion in a separable real Hilbert space. Under some sufficient conditions, we establish the existence and uniqueness of a pth-mean almost periodic mild solution to those stochastic differential equations.

1. Introduction

Let $(\mathbb{K}, \|\cdot\|, \langle\cdot\rangle)$ and $(\mathbb{H}, \|\cdot\|, \langle\cdot\rangle)$ be real separable Hilbert spaces and $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space. We denote by $\mathcal{L}(\mathbb{H})$ the Banach algebra of all linear bounded operators on $\mathbb{H}$ and by $\mathbb{L}_2 = L_2(\mathbb{K}; \mathbb{H})$ the space of all Hilbert-Schmidt operators acting between $\mathbb{K}$ and $\mathbb{H}$ equipped with the Hilbert-Schmidt norm $\|\cdot\|_{\mathbb{L}_2}$.

Recall that a Wiener process $\{\mathbb{W}(t),\ t \in \mathbb{R}\}$ defined on $(\Omega, \mathcal{F}, \mathbf{P})$ with values in $\mathbb{K}$ can be obtained as follows: let $\{\mathbb{W}_i(t),\ t \in \mathbb{R}_+\}$, $i = 1, 2$, be independent $\mathbb{K}$-valued Wiener processes, then

$$\mathbb{W}(t) = \begin{cases} \mathbb{W}_1(t) & \text{if } t \geq 0, \\ \mathbb{W}_2(-t) & \text{if } t \leq 0 \end{cases}$$

is a Wiener process with $\mathbb{R}$ as time parameter. We let $\mathcal{F}_t = \sigma\{\mathbb{W}(s),\ s \leq t\}$.

Let $\mathbb{K}_0$ be an arbitrary separable Hilbert space and $\mathbb{L}_2^0 = L_2(\mathbb{K}_0; \mathbb{H})$ which is a separable Hilbert space with respect to the Hilbert-Schmidt norm $\|\cdot\|_{\mathbb{L}_2^0}$.

We are concerned with the class of semilinear stochastic differential equations in a real separable Hilbert space $\mathbb{H}$ driven by fractional Brownian motion (fBm) and Wiener process of the general form

$$dX(t) = A(t)X(t) + F(t, X(t))\,dt + G(t, X(t))\,d\mathbb{W}(t) + \Phi(t)\,dB^H(t), \quad t \in \mathbb{R}. \tag{1.1}$$ C1

Here, $(A(t))_{t\in\mathbb{R}}$ is a family of densely defined closed linear operators satisfying Acquistapace-Terreni conditions; $F : \mathbb{R} \times \mathbb{H} \to \mathbb{H}$; $G : \mathbb{R} \times \mathbb{H} \to \mathbb{L}_2^0$; $\Phi : \mathbb{R} \to \mathbb{L}_2$; $\{B^H(t) : t \in \mathbb{R}\}$ is a cylindrical fractional Brownian motion with Hurst parameter

---

2000 *Mathematics Subject Classification.* 60H05, 60H15, 34G20, 43A60.
Key words and phrases. Stochastic differential equation; stochastic processes; almost periodic; Wiener process.

Submitted May 4, 2012. Published September 7, 2012.

$H \in (1/2, 1)$ (Section 2); and $\{\mathbb{W}(t) : t \in \mathbb{R}\}$ is a standard cylindrical Wiener process on $\mathbb{K}_0$. We assume that the processes $\mathbb{W}$ and B^H are independent.

Note that $\Phi(\cdot)$ is assumed to be deterministic. The case where $\Phi(\cdot)$ is random is complicated and not treated in this article.

Stochastic evolution equations (SEEs) of type (1.1) have been studied by many authors, mostly in the case where the last term on the right-hand side of (1.1) is zero or coefficients are deterministic or linear. The main difficulty is due to the fact that fBm is neither a Markov process nor a semimartingale, except for $H = \frac{1}{2}$ (in which case B^H becomes a standard Brownian motion), thus the usual stochastic calculus does not apply. For values of the Hurst parameter $H > \frac{1}{2}$ - the regular case - integrals of Young's type and fractional calculus techniques have been considered [37]. However, for $H < \frac{1}{2}$ this approach fails. As a result, the study of the SEE depends largely on the definitions of the stochastic integrals involved and the results vary.

There are essentially two different methods to define stochastic integrals with respect to fBm:

(i) A path-wise approach that uses the Hölder continuity properties of the sample paths, developed from the works by Ciesielski, Kerkyacharian and Roynette [9] and Zähle [37].
(ii) The stochastic calculus of variations (Malliavin calculus) for the fBm introduced by Dereusefond and Üstünel in [13].

Recently, the existence of almost periodic or pseudo almost periodic solutions to some stochastic differential equations has been considerably investigated in lots of publication [11, 5, 6, 7, 15, 16, 21, 17, 27] because of its significance and applications in physics, mechanics, and mathematical biology.

In this paper, we establish the existence and uniqueness of a pth-mean almost periodic mild solution for the stochastic evolution equation (1.1) with almost periodic coefficients. The proof of our main result, Theorem 4.4, is essentially based on the stochastic calculus of variation (Section 2), Itô stochastic calculus, the use of Proposition 3.15 (below), and the techniques developed by Da Prato and Tudor [11, Proposition 4.4] adapted to our case in order to handle the last two terms of the right-hand side of (1.1) effectively.

The rest of the paper is organized as follows. In Section 2, we briefly revisit some basic facts regarding evolution families and fractional Brownian motion. Basic definitions and results on the concept of almost periodic stochastic processes are given in Section 3. Finally, in Section 4, we give some sufficient conditions for the existence and uniqueness of a pth-mean almost periodic solution to the stochastic evolution equation (1.1).

2. Preliminaries

In this section, $(\mathbb{B}, \|\cdot\|)$ denotes a separable Banach space. For a linear operator A on a Banach space $\mathbb{B}$, we denote the resolvent set of A by $\rho(A)$ and the resolvent $(\lambda - A)^{-1}$ by $R(\lambda, A)$. If $(\mathbb{B}_1, \|\cdot\|_{\mathbb{B}_1}), (\mathbb{B}_2, \|\cdot\|_{\mathbb{B}_2})$ are Banach spaces, then the notation $\mathcal{L}(\mathbb{B}_1, \mathbb{B}_2)$ stands for the Banach space of bounded linear operators from $\mathbb{B}_1$ into $\mathbb{B}_2$. When $\mathbb{B}_1 = \mathbb{B}_2$, this is simply denoted $\mathcal{L}(\mathbb{B}_1)$.

2.1. Evolution families.

A set $\mathcal{U} = \{U(t, s) : t \geq s,\ t, s \in \mathbb{R}\}$ of bounded linear operators on a Banach space $\mathbb{B}$ is called an *evolution family* if

(a) $U(t,s)U(s,r) = U(t,r)$, $U(s,s) = I$ if $r \le s \le t$;
(b) $(t,s) \to U(t,s)x$ is strongly continuous for $t > s$.

We say that an evolution family $\mathcal{U}$ has an *exponential dichotomy* (or is *hyperbolic*) if there are projections $P(t)$ $(t \in \mathbb{R})$, being uniformly bounded and strongly continuous in t and constants $\delta > 0$ and $N \ge 1$ such that

(1) $U(t,s)P(s) = P(t)U(t,s)$;
(2) the restriction $U_Q(t,s) : Q(s)\mathbb{B} \to Q(t)\mathbb{B}$ of $U(t,s)$ is invertible (we then set $\widetilde{U}_Q(s,t) := U_Q(t,s)^{-1}$); and
(3) $\|U(t,s)P(s)\| \le Ne^{-\delta(t-s)}$ and $\|\widetilde{U}_Q(s,t)Q(t)\| \le Ne^{-\delta(t-s)}$ for $t \ge s$ and $t,s \in \mathbb{R}$.

Here and below we let $Q(\cdot) = I - P(\cdot)$. If $P(t) = I$ for $t \in \mathbb{R}$, then U is *exponentially stable*. The evolution family is called *exponentially bounded* if there are constants $M > 0$ and $\gamma \in \mathbb{R}$ such that $\|U(t,s)\| \le Me^{\gamma(t-s)}$ for $t \ge s$.

In the present work, we study operators $A(t)$, $t \in \mathbb{R}$, on a Hilbert space $\mathbb{H}$ subject to the following hypothesis introduced by Acquistapace and Terreni in [1].

There exist constants $\lambda_0 \ge 0$, $\theta \in (\frac{\pi}{2}, \pi)$, L, $K \ge 0$, and μ, $\nu \in (0,1]$ with $\mu + \nu > 1$ such that

$$\Sigma_\theta \cup \{0\} \subset \rho(A(t) - \lambda_0), \quad \|R(\lambda, A(t) - \lambda_0)\| \le \frac{K}{1 + |\lambda|} \tag{2.1}$$

AT1

and

$$\|(A(t) - \lambda_0)R(\lambda, A(t) - \lambda_0)\big[R(\lambda_0, A(t)) - R(\lambda_0, A(s))\big]\| \le L|t-s|^\mu |\lambda|^{-nu}, \tag{2.2}$$

AT2

for t, $s \in \mathbb{R}$, $\lambda \in \Sigma_\theta := \big\{\lambda \in \mathbb{C} - \{0\} : |\arg \lambda| \le \theta\big\}$.

This assumption implies that there exists a unique evolution family $\mathcal{U}$ on $\mathbb{H}$ such that $(t,s) \to U(t,s) \in \mathcal{L}(\mathbb{H})$ is continuous for $t > s$, $U(\cdot,s) \in C^1((s,\infty), \mathcal{L}(\mathbb{H}))$, $\partial_t U(t,s) = A(t)U(t,s)$, and

$$\|A(t)^k U(t,s)\| \le C(t-s)^{-k} \tag{2.3}$$

w1

for $0 < t - s \le 1$, $k = 0, 1$, $0 \le \alpha < \mu$, $x \in D((\lambda_0 - A(s))^\alpha)$, and a constant C depending only on the constants in (2.1)-(2.2). Moreover, $\partial_s^+ U(t,s)x = -U(t,s)A(s)x$ for $t > s$ and $x \in D(A(s))$ with $A(s)x \in \overline{D(A(s))}$. We say that $A(\cdot)$ generates $\mathcal{U}$. Note that $\mathcal{U}$ is exponentially bounded by (2.3) with $k = 0$.

This setting requires some estimates related to $U(t,s)$. For that, we introduce the interpolation spaces for $A(t)$. We refer the reader to the excellent books [20] and [26] for proofs and further information on these interpolation spaces.

Let A be a sectorial operator on $\mathbb{B}$ (for that, in (2.1)-(2.2), replace $A(t)$ with A) and let $\alpha \in (0,1)$. Define the real interpolation space

$$\mathbb{B}_\alpha^A := \Big\{x \in \mathbb{B} : \|x\|_\alpha^A := \sup_{r>0} \|r^\alpha\big(A - \delta_0\big)R\big(r, A - \delta_0\big)x\| < \infty\Big\},$$

which, by the way, is a Banach space when endowed with the norm $\|\cdot\|_\alpha^A$. For convenience we further write

$$\mathbb{B}_0^A := \mathbb{B}, \quad \|x\|_0^A := \|x\|, \quad \mathbb{B}_1^A := D(A), \quad \|x\|_1^A := \|(\delta_0 - A)x\|.$$

Moreover, let $\hat{\mathbb{B}}^A := \overline{D(A)}$ of $\mathbb{B}$. We have the following continuous embedding

$$D(A) \hookrightarrow \mathbb{B}_\beta^A \hookrightarrow D((\delta_0 - A)^\alpha) \hookrightarrow \mathbb{B}_\alpha^A \hookrightarrow \hat{\mathbb{B}}^A \hookrightarrow \mathbb{B}, \tag{2.4}$$

embeddings1

for all $0 < \alpha < \beta < 1$, where the fractional powers are defined in the usual way.

In general, $D(A)$ is not dense in the spaces $\mathbb{B}_\alpha^A$ and $\mathbb{B}$. However, we have the following continuous injection

$$\mathbb{B}_\beta^A \hookrightarrow \overline{D(A)}^{\|\cdot\|_\alpha^A} \tag{2.5}$$ closure

for $0 < \alpha < \beta < 1$.

Given the family of linear operators $A(t)$ for $t \in \mathbb{R}$, satisfying (2.1)-(2.2), we set

$$\mathbb{B}_\alpha^t := \mathbb{B}_\alpha^{A(t)}, \quad \hat{\mathbb{B}}^t := \hat{\mathbb{B}}^{A(t)}$$

for $0 \le \alpha \le 1$ and $t \in \mathbb{R}$, with the corresponding norms. Then the embedding in (2.4) holds with constants independent of $t \in \mathbb{R}$. These interpolation spaces are of class $\mathcal{J}_\alpha$ ([26, Definition 1.1.1]) and hence there is a constant $c(\alpha)$ such that

$$\|y\|_\alpha^t \le c(\alpha)\|y\|^{1-\alpha}\|A(t)y\|^\alpha, \quad y \in D(A(t)). \tag{2.6}$$ J

We have the following fundamental estimates for the evolution family $U(t,s)$.

pes **Proposition 2.1.** [3] *Suppose the evolution family $\mathcal{U} = \{U(t,s),\, t \ge s\}$ has exponential dichotomy. For $x \in \mathbb{B}$, $0 \le \alpha \le 1$ and $t > s$, the following hold:*

(i) *There is a constant $c(\alpha)$, such that*

$$\|U(t,s)P(s)x\|_\alpha^t \le c(\alpha)e^{-\frac{\delta}{2}(t-s)}(t-s)^{-\alpha}\|x\|. \tag{2.7}$$ eq1.1

(ii) *There is a constant $m(\alpha)$, such that*

$$\|\widetilde{U}_Q(s,t)Q(t)x\|_\alpha^s \le m(\alpha)e^{-\delta(t-s)}\|x\|. \tag{2.8}$$ eq2.1

For additional details on evolution families, we refer the reader to the book by Lunardi [26].

2.2. **Fractional Brownian Motion.** For the convenience for the reader we recall briefly here some of the basic results of fractional Brownian motion calculus. For details of this section, we refer the reader to [8, 12, 14, 19, 23] and the references therein.

A standard fractional Brownian motion (fBm) $\{\beta^H(t), t \in \mathbb{R}\}$ with Hurst parameter $H \in (0,1)$ is a Gaussian process with continuous sample paths such that $\mathbf{E}\left[\beta^H(t)\right] = 0$ and

$$\mathbf{E}[\beta^H(t)\beta^H(s)] = \frac{1}{2}\big(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\big) \tag{2.9}$$ A30

for s, $t \in \mathbb{R}$. It is clear that for $H = 1/2$, the process is a standard Brownian motion. In this paper, it is assumed that $H \in (\frac{1}{2}, 1)$.

The fBm has stationary increments: for any $s \in \mathbb{R}$, $\{\beta^H(t+s) - \beta^H(s)\}_{t\in\mathbb{R}}$ and $\{\beta^H(t)\}_{t\in\mathbb{R}}$ have the same law, and is self-similar: for any $\alpha > 0$, $\beta^H(\alpha t)$ has the same law as $\alpha^H\beta^H(t)$. From (2.9) one can deduce that $\mathbf{E}\big|\beta^H(t)-\beta^H(s)\big|^2 = \big|t-s\big|^{2H}$ and, as a consequence, the trajectories of β^H are almost surely locally α-Hölder continuous for all $\alpha \in (0,H)$. In addition, for $H > \frac{1}{2}$, the increments are positively correlated, and for $H < 1/2$, they are negatively correlated.

This process was introduced by Kolmogorov in [25] and later studied by Mandelbrot and Van Ness in [30]. Its self-similar and long-range dependence (if $H > \frac{1}{2}$) properties (that is, if we put $r(n) = \operatorname{cov}(\beta^H(1), \beta^H(n+1) - \beta^H(n))$, then $\sum_{n=1}^\infty r(n) = \infty$) make this process a useful driving noise in models arising in physics, telecommunication networks, finance and other fields.

Fix a Hurst constant H, $\frac{1}{2} < H < 1$. Define

$$\phi(s,t) = H(2H-1)|s-t|^{2H-2}; \quad s,\ t \in \mathbb{R}\,. \tag{2.10}$$ A1

The function ϕ is called fractional kernel.

Let $\mathbb{K}$ be a real separable Hilbert space and let $\mathcal{Q}$ be a self-adjoint and positive operator on $\mathbb{K}$ ($\mathcal{Q} = \mathcal{Q}^\star > 0$). It is typical and usually convenient to assume moreover that $\mathcal{Q}$ is nuclear ($\mathcal{Q} \in \mathcal{L}_1(\mathbb{K})$). In this case it is well known that $\mathcal{Q}$ admits a sequence $(\lambda_n)_{n\geq 0}$ of eigenvalues with $\lambda_n > 0$ converging to zero and $\sum_{n\geq 0} \lambda_n < \infty$. The following definition provides an infinite-dimensional analogue of the definition of a fractional Brownian motion in a finite-dimensional space with Hurst parameter $H \in (0,1)$.

Definition 2.2. A $\mathbb{K}$-valued Gaussian process $\{B^H(t), t \in \mathbb{R}\}$ on $(\Omega, \mathcal{F}, \mathbf{P})$ is called a *fractional Brownian motion of $\mathcal{Q}$-covariance type with Hurst parameter $H \in (0,1)$* (or, more simply, a fractional $\mathcal{Q}$-Brownian motion with Hurst parameter H) if

(1) $\mathbf{E}\big[B^H(t)\big] = 0$ for all $t \in \mathbb{R}$,
(2) $\operatorname{cov}(B^H(t), B^H(s)) = \frac{1}{2}\big(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\big)\mathcal{Q}$, for all $t \in \mathbb{R}$
(3) $\{B^H(t), t \in \mathbb{R}\}$ has $\mathbb{K}$-valued, continuous sample paths a.s.-$\mathbf{P}$,

where $\operatorname{cov}(X,Y)$ denotes the *covariance operator* for the Gaussian random variables X and Y and $\mathbf{E}$ stands for the mathematical expectation on $(\Omega, \mathcal{F}, \mathbf{P})$.

The existence of a fractional $\mathcal{Q}$-Brownian motion is given in the following proposition

Proposition 2.3. *Let $H \in (0,1)$ be fixed and $\mathcal{Q}$ be a linear operator such that $\mathcal{Q} = \mathcal{Q}^\star$ and $\mathcal{Q} \in \mathcal{L}_1(\mathbb{K})$, where $\mathcal{L}_1(\mathbb{K})$ denotes the space of trace class operators on $\mathbb{K}$. Then there is a fractional $\mathcal{Q}$-Brownian motion with Hurst parameter H.*

A fractional Brownian motion of $\mathcal{Q}$-covariance type can be defined directly by the infinite series

$$B^H(t) := \sum_{n=1}^{\infty} \sqrt{\lambda_n}\beta_n^H(t)e_n \tag{2.11}$$ A35

where $(e_n, n \in \mathbb{N})$ be an orthonormal basis in $\mathbb{K}$ consisting of eigenvectors of $\mathcal{Q}$ and $\{\lambda_n, n \in \mathbb{N}\}$ be a corresponding sequence of eigenvalues of $\mathcal{Q}$ such that $\mathcal{Q}e_n = \lambda_n e_n$ for all $n \in \mathbb{N}$.

Analogically to a standard cylindrical Wiener processes in a Hilbert space, we will define a standard cylindrical fractional Brownian motion in a Hilbert space $\mathbb{K}$ by the formal series

$$B^H(t) := \sum_{n=1}^{\infty} \beta_n^H(t)e_n\,, \tag{2.12}$$ A36

where $\{e_n, n \in \mathbb{N}\}$ is a complete orthonormal basis in $\mathbb{K}$ and $\{\beta_n^H(t), n \in \mathbb{N}, t \in \mathbb{R}\}$ is a sequence of independent, real-valued standard fractional Brownian motions each with the same Hurst parameter $H \in (0,1)$. It is well known that the infinite series (2.12) does not converge in $L^2(\Omega, \mathbb{K})$ so $B^H(t)$ is not well defined $\mathbb{K}$-valued random variable. However, it is easy to verify (see [31]) that for any Hilbert space $\mathbb{K}_1$ such that $\mathbb{K} \hookrightarrow \mathbb{K}_1$ and the embedding is a Hilbert-Schmidt operator, the series (2.12) defines a $\mathbb{K}_1$-valued random variable and $\{B^H(t), t \in \mathbb{R}\}$ is a $\mathbb{K}_1$-valued fractional Brownian motion of $\mathcal{Q}$-covariance type.

Next, we outline the discussion leading to the definition of the stochastic integral of the form

$$\int_{T_1}^{T_2} g(t)\, dB^H(t)\,, \tag{2.13}$$ A38

where T_1, $T_2 \in \mathbb{R}$, $T_1 < T_2$, is defined for $g : [T_1, T_2] \to \mathcal{L}(\mathbb{K}, \mathbb{H})$ where $\mathcal{L}(\mathbb{K}, \mathbb{H})$ is a family of bounded linear operators from $\mathbb{K}$ to $\mathbb{H}$. The function g is assumed to be deterministic.

In the sequel, we will consider only $H \in (1/2, 1)$. The integral (2.13) is an $\mathbb{H}$-valued random variable that is independent of the choice of K_1. We need the following lemma.

A40 **Lemma 2.4** ([31])**.** *If $p > 1/H$, then for a $\varphi \in L^p([T_1, T_2], \mathbb{R})$ the following inequality is satisfied*

$$\int_{T_1}^{T_2}\int_{T_1}^{T_2} \varphi(u)\varphi(v)\phi(u-v)\, du\, dv \le C_{T_1,T_2} \big|\varphi\big|^2_{L^p([T_1,T_2];\mathbb{R})}$$

for some $C_{T_1,T_2} > 0$ that only depends on T_1 and T_2. The function ϕ is defined as in (2.10)*.*

The stochastic integral

$$\int_{T_1}^{T_2} g(t)\, d\beta^H(t) \tag{2.14}$$ A39

is defined for $g \in L^p([T_1, T_2], \mathbb{H})$, where $\{\beta^H(t), t \in [T_1, T_2]\}$ is a scalar fractional Brownian motion.

Let $\mathcal{E}$ be the family of $\mathbb{H}$-valued step functions; that is,

$$\Big\{ g : g(s) = \sum_{i=1}^{n-1} g_i \chi_{[t_i,t_{i+1})}(s), T_1 = t_1 < t_2 < \dots < t_n = T_2 \\ \text{and } g_i \in \mathbb{H} \text{ for } i \in \{1, \dots, n-1\}\Big\}.$$

For $g \in \mathcal{E}$, define the stochastic integral (2.14) as

$$\int_{T_1}^{T_2} g(t)\, d\beta^H(t) := \sum_{i=1}^{n-1} g_i(\beta^H(t_{i+1}) - \beta^H(t_i))$$

The expectation of this random variable is zero and the second moment is

$$\mathbf{E}\Big\| \int_{T_1}^{T_2} g(t)\, d\beta^H(t)\Big\|^2_{\mathbb{H}} = \int_{T_1}^{T_2} \langle g(u), g(v)\rangle_{\mathbb{H}} \phi(u-v)\, du\, dv\,.$$

By Lemma 2.4, it follows that

$$\mathbf{E}\Big\| \int_{T_1}^{T_2} g(t)\, d\beta^H(t)\Big\|^2_{\mathbb{H}} \le C_{T_1,T_2,p}\Big(\int_{T_1}^{T_2} \|g(s)\|^p_{\mathbb{H}}\, ds\Big)^{2/p}.$$

for some constant $C_{T_1,T_2,p}$ that only depends on T_1, T_2, and p. By this inequality, the stochastic integral can be uniquely extended from $\mathcal{E}$ to $L^p([T_1, T_2], \mathbb{H})$, because $\mathcal{E}$ is dense in $L^p([T_1, T_2], \mathbb{H})$.

Now we define the stochastic integral

$$\int_{T_1}^{T_2} g(t)\, dB^H(t) \tag{2.15}$$ A42

for a $\mathbb{K}$-valued standard cylindrical fractional Brownian motion and for $g : [T_1, T_2] \to \mathbb{L}_2$.

Let $p > 1/H$ be arbitrary but fixed. We will assume that for each $x \in \mathbb{K}$, $g(\cdot)x \in L^p([T_1, T_2]; \mathbb{H})$ and that

$$\int_{T_1}^{T_2} \int_{T_1}^{T_2} \|g(s)\|_{\mathbb{L}_2} \|g(r)\|_{\mathbb{L}_2} \phi(r-s)\, drds < \infty\,, \tag{2.16}$$

A41

where ϕ is given by (2.10).

We define the integral (2.15) as

$$\int_{T_1}^{T_2} g(t)\, dB^H(t) := \sum_{n=1}^{\infty} \int_{T_1}^{T_2} g(t) e_n \, d\beta_n^H(t) \tag{2.17}$$

A43

where $(e_n, n \in \mathbb{N})$ and $(\beta_n^H(\cdot), n \in \mathbb{N})$ are given in the definition of a standard fractional Brownian motion (2.12). Since $g(\cdot)e_n \in L^p([T_1, T_2], \mathbb{H})$ for each $n \in \mathbb{N}$, the terms in series (2.17) are well defined as stated above. The sequence of random variables $\{\int_{T_1}^{T_2} g(t) e_n \, d\beta_n^H(t),\ n \in \mathbb{N}\}$ are clearly mutually independent Gaussian random variables. Since

$$\begin{aligned}
\mathbf{E}\Big\| \int_{T_1}^{T_2} g(t)\, d\mathbb{B}^H(t) \Big\|_{\mathbb{H}}^2 &= \sum_{n=1}^{\infty} \mathbf{E}\Big\| \int_{T_1}^{T_2} g(t) e_n \, d\beta_n^H(t) \Big\|_{\mathbb{H}}^2 \\
&= \sum_{n=1}^{\infty} \int_{T_1}^{T_2} \int_{T_1}^{T_2} \langle g(s)e_n, g(r)e_n \rangle_{\mathbb{H}} \phi(r-s)\, drds \\
&\le \int_{T_1}^{T_2} \int_{T_1}^{T_2} \|g(s)\|_{\mathbb{L}_2} \|g(r)\|_{\mathbb{L}_2} \phi(r-s)\, drds < \infty\,,
\end{aligned}$$

the series in (2.17) is a $\mathbb{H}$-valued Gaussian random variable.

3. Almost periodic stochastic processes

For the reader's convenience, we review some basic definitions and results for the notion of almost periodicity.

3.1. **Almost periodic functions.** Let $x : \mathbb{R} \to \mathbb{B}$ be a continuous function. For a sequence $\alpha = \{\alpha_n\}$ in $\mathbb{R}$, the notation $T_\alpha x = y$ means that for each $t \in \mathbb{R}$, $\lim_{n\to\infty} x(t + \alpha_n) = y(t)$.

Definition 3.1. A continuous function $x : \mathbb{R} \to \mathbb{B}$ is said to be (Bohr) *almost periodic* if for each $\varepsilon > 0$ there exists $l(\varepsilon) > 0$ such that any interval of length $l(\varepsilon)$ contains at least a number τ for which

$$\sup_{t\in\mathbb{R}} \|x(t+\tau) - x(t)\| < \varepsilon.$$

We have the following characterization of almost periodicity.

IJK **Proposition 3.2.** *Let $x : \mathbb{R} \to \mathbb{B}$ be a continuous function. Then the following statements are equivalent:*

(i) *x is (Bohr) almost periodic.*
(ii) *(Bochner) For every sequence $\alpha' = \{\alpha'_n\} \subset \mathbb{R}$ there exists a subsequence $\alpha = \{\alpha_n\} \subset \{\alpha'_n\}$ and a continuous function $y : \mathbb{R} \to \mathbb{B}$ such that $T_\alpha x = y$ pointwise.*

(iii) *For every pair of sequences* (α'_n) *and* (β'_n)*, there exist subsequences* $\boldsymbol{\alpha} = (\alpha_n) \subset (\alpha'_n)$ *and* $\boldsymbol{\beta} = (\beta_n) \subset (\beta'_n)$ *respectively, with the same indexes such that* $T_\alpha T_\beta x = T_{\alpha+\beta} x$ *pointwise.*

Definition 3.3. A function $f : \mathbb{R} \times \mathbb{B}_1 \to \mathbb{B}_2$, $(t, x) \mapsto f(t, x)$, which is jointly continuous, is said to *almost periodic* in $t \in \mathbb{R}$ uniformly in $x \in \mathbb{K}$ ($\mathbb{K} \subset \mathbb{B}_1$ being a compact subspace) if for any $\varepsilon > 0$, there exists $l(\varepsilon, \mathbb{K}) > 0$ such that any interval of length $l(\varepsilon, \mathbb{K})$ contains at least a number τ for which

$$\sup_{t\in\mathbb{R}} \|f(t + \tau, x) - f(t, x)\|_{\mathbb{B}_2} < \varepsilon$$

for each x in $\mathbb{K}$.

Almost periodic stochastic processes. For a random variable $X : (\Omega, \mathcal{F}, \mathbf{P}) \to \mathbb{B}$, we shall denote by $\mathbf{P} \circ X^{-1}$ its distribution and its expectation denoted by $\mathbf{E}[X]$ is defined as

$$\mathbf{E}[X] = \int_\Omega X(\omega) d\mathbf{P}(\omega)\,.$$

For $p \geq 2$, the collection of all strongly measurable, p^{th} or p-th integrable $\mathbb{B}$-valued random variables, denoted by $L^p(\Omega, \mathbb{B})$, is a Banach space equipped with norm

$$\|X\|_{L^p(\Omega,\mathbb{B})} = (\mathbf{E}\|X\|^p)^{1/p}\,.$$

Before we give the definition of almost periodicity in distribution we recall the following definition:

Let us denote by $\mathcal{P}(\mathbb{B})$ the set of all probability measures on $\mathcal{B}(\mathbb{B})$ the σ-Borel algebra of $\mathbb{B}$. We shall denote by $C(\mathbb{R}; \mathbb{B})$ the class of all continuous functions from $\mathbb{R}$ to $\mathbb{B}$, and by $C_b(\mathbb{B})$ the class of all continuous functions $f : \mathbb{B} \to \mathbb{R}$ with $\|f\|_\infty := \sup_{t\in\mathbb{R}} |f(t)| < \infty$.

For $f \in C_b(\mathbb{B})$,

$$\|f\|_L = \sup\Big\{\frac{|f(u) - f(v)|}{\|u - v\|}; u \neq v\Big\},$$
$$\|f\|_{BL} = \max\{\|f\|_\infty, \|f\|_L\}\,.$$

For μ and $\nu \in \mathcal{P}(\mathbb{B})$, we define

$$d_{BL}(\mu, \nu) = \sup\Big\{\Big|\int_{\mathbb{B}} f\, d(\mu - \nu)\Big| : \|f\|_{BL} \leq 1\Big\}\,.$$

The metric d_{BL} on $\mathcal{P}(\mathbb{B})$ is complete and generates the weak topology (see [18]).

From now on $\mathcal{P}(\mathbb{B})$ is endowed with the metric d_{BL}.

def3.4 **Definition 3.4.** A stochastic process X is *almost periodic in distribution* if the mapping $t \mapsto \hat{\mu}(t) = \mathbf{P} \circ X(t + \cdot)^{-1}$ from $\mathbb{R}$ to $\mathcal{P}(C(\mathbb{R}; \mathbb{B}))$ is almost periodic.

def3.5 **Definition 3.5.** A stochastic process X is said to be *almost periodic in probability* if for any $\varepsilon > 0$ and $\eta > 0$ there exists $l = l(\varepsilon, \eta) > 0$ such that any interval of length l contains at least a number τ for which

$$\sup_{t\in\mathbb{R}} \mathbf{P}\{\|X(t + \tau) - X(t)\| > \eta\} \leq \varepsilon.$$

def3.6 **Definition 3.6.** A stochastic process $X : \mathbb{R} \to L^p(\Omega; \mathbb{B})$ is said to be *continuous in pth mean* whenever

$$\lim_{t\to s} \mathbf{E}\|X(t) - X(s)\|^p = 0.$$

KK **Definition 3.7.** A continuous stochastic process $X : \mathbb{R} \to L^p(\Omega; \mathbb{B})$ is said to be *pth-mean almost periodic* if for each $\varepsilon > 0$ there exists $l(\varepsilon) > 0$ such that any interval of length $l(\varepsilon)$ contains at least a number τ for which

$$\sup_{t\in\mathbb{R}} \mathbf{E}\|X(t+\tau) - X(t)\|^p < \varepsilon.$$

The collection of all stochastic processes $X : \mathbb{R} \to L^p(\Omega; \mathbb{B})$ which are pth-mean almost periodic is then denoted by $AP(\mathbb{R}; L^p(\Omega; \mathbb{B}))$.

The next lemma provides with some properties of the pth-mean almost periodic processes.

PH **Lemma 3.8.** [5] *If X belongs to $AP(\mathbb{R}; L^p(\Omega; \mathbb{B}))$, then*

(i) *the mapping $t \to \mathbf{E}\|X(t)\|^p$ is uniformly continuous;*
(ii) *there exists a constant $M > 0$ such that $\mathbf{E}\|X(t)\|^p \le M$, for all $t \in \mathbb{R}$.*

Let $\mathrm{UCB}(\mathbb{R}; L^p(\Omega; \mathbb{B}))$ denote the collection of all stochastic processes $X : \mathbb{R} \to L^p(\Omega; \mathbb{B})$, which are uniformly continuous and bounded. It is then easy to check that $\mathrm{UCB}(\mathbb{R}; L^p(\Omega; \mathbb{B}))$ is a Banach space when it is equipped with the norm:

$$\|X\|_\infty = \sup_{t\in\mathbb{R}} (\mathbf{E}\|X(t)\|^p)^{1/p}.$$

Lemma 3.9. [5] *$AP(\mathbb{R}; L^p(\Omega; \mathbb{B})) \subset UCB(\mathbb{R}; L^p\Omega; \mathbb{B}))$ is a closed subspace.*

In view of the above, the space $AP(\mathbb{R}; L^p(\Omega; \mathbb{B}))$ of pth-mean almost periodic processes equipped with the norm $\|\cdot\|_\infty$ is a Banach space.

VV **Proposition 3.10.** [4] *If X is pth-mean almost periodic, then it is almost periodic in probability. Conversely, if X is almost periodic in probability and the family $\{\|X(t)\|^p,\ t \in \mathbb{R}\}$ is uniformly integrable, then X is pth-mean almost periodic.*

Let $\boldsymbol{\alpha} = \{\alpha_n\}$ and denote $T_{\boldsymbol{\alpha}} X(\omega, t) := \lim_{n\to\infty} X(\omega, t + \alpha_n)$ for each $\omega \in \Omega$ and each $t \in \mathbb{R}$ if it exists.

def3.11 **Definition 3.11.** A stochastic process X satisfies *Bochner's almost sure uniform double sequence criterion* if, for every pair of sequences (α'_n) and (β'_n), there exists a measurable subset $\Omega_1 \subset \Omega$ with $\mathbf{P}(\Omega_1) = 1$ and there exist subsequences $\boldsymbol{\alpha} = (\alpha_n) \subset (\alpha'_n)$ and $\boldsymbol{\beta} = (\beta_n) \subset (\beta'_n)$ respectively, with the same indexes (independent of ω) such that, for every $t \in \mathbb{R}$,

$$T_\alpha T_\beta X(\omega, t) = T_{\alpha+\beta} X(\omega, t), \quad \forall \omega \in \Omega_1 .$$

(In this case, Ω_1 depends on the pair of sequences (α'_n) and (β'_n).)

EE **Proposition 3.12** ([4])**.** *The following properties of X are equivalent:*

(i) *X satisfies Bochner's almost sure uniform double sequence criterion.*
(ii) *X is almost periodic in probability.*

Propositions 3.10 and 3.12 give us the following property.

CC **Proposition 3.13** ([4])**.** *If X satisfies Bochner's almost sure uniform double sequence criterion and the family $\{\|X(t)\|^p,\ t \in \mathbb{R}\}$ is uniformly integrable, then X is pth-mean almost periodic.*

DD **Proposition 3.14** ([4])**.** *If X is almost periodic in distribution, then X satisfies Bochner's almost sure uniform double sequence criterion*

Combining Proposition 3.12, 3.13, and 3.14, we obtain the following important property.

FF **Proposition 3.15.** *If X is almost periodic in distribution and the family $\{\|X(t)\|^p, t \in \mathbb{R}\}$ is uniformly integrable, then X is pth-mean almost periodic.*

AB **Theorem 3.16** ([5])**.** *Let $F : \mathbb{R} \times \mathbb{B}_1 \to \mathbb{B}_2$, $(t, x) \mapsto F(t, x)$ be an almost periodic function in $t \in \mathbb{R}$ uniformly in $x \in \mathbb{K}$ ($\mathbb{K} \subset \mathbb{B}_1$ being a compact subspace). Suppose that F is Lipschitz in the following sense:*

$$\mathbf{E}\|F(t, Y) - F(t, Z)\|^p_{\mathbb{B}_2} \le M\mathbf{E}\|Y - Z\|^p_{\mathbb{B}_1}$$

for all $Y, Z \in L^p(\Omega; \mathbb{B}_1)$ and for each $t \in \mathbb{R}$, where $M > 0$. Then for any pth-mean almost periodic process $\Phi : \mathbb{R} \to L^p(\Omega; \mathbb{B}_1)$, the stochastic process $t \mapsto F(t, \Phi(t))$ is pth-mean almost periodic.

4. Main Result

Throughout this paper, we require the following assumptions:

(H0) The family of operators $A(t)$ satisfies Acquistpace-Terreni conditions and the evolution family $\mathcal{U} = \{U(t, s), t \ge s\}$ associated with $A(t)$ is exponentially stable, that is, there exist constant M, $\delta > 0$ such that

$$\|U(t, s)\| \le Me^{-\delta(t-s)}$$

for all $t \ge s$;

(H1) The function $F : \mathbb{R} \times \mathbb{H} \to \mathbb{H}$, $(t, x) \mapsto F(t, x)$ is almost periodic in $t \in \mathbb{R}$ uniformly in $x \in \mathcal{O}$ ($\mathcal{O} \subset \mathbb{H}$ being a compact subspace). Moreover, F is Lipschitz in the following sense: there exists $K > 0$ for which

$$\mathbf{E}\|F(t, X) - F(t, Y)\|^p \le K\mathbf{E}\|X - Y\|^p$$

for all $X, Y \in L^p(\Omega; \mathbb{H})$ and $t \in \mathbb{R}$;

(H2) The function $G : \mathbb{R} \times \mathbb{H} \to \mathbb{L}^0_2$, $(t, x) \mapsto G(t, x)$ is almost periodic in $t \in \mathbb{R}$ uniformly in $x \in \mathcal{O}'$ ($\mathcal{O}' \subset \mathbb{H}$ being a compact subspace). In addition, G satisfies the following properties:
  - (i) $\sup_{t\in\mathbb{R}} \mathbf{E}\|G(t, X)\|^{2p}_{\mathbb{L}^0_2} < \infty$ for all $X \in L^p(\Omega, \mathbb{H})$;
  - (ii) G is Lipschitz in the following sense: there exists $K' > 0$ for which

$$\mathbf{E}\|G(t, X) - G(t, Y)\|^p_{\mathbb{L}^0_2} \le K'\mathbf{E}\|X - Y\|^p$$

for all $X, Y \in L^p(\Omega; \mathbb{H})$ and $t \in \mathbb{R}$;

(H3) The function $\Phi : \mathbb{R} \to \mathbb{L}_2$, $t \mapsto \Phi(t)$ is almost periodic.

To study (1.1) we need the following lemma which can be seen as an immediate consequence of [29, Proposition 4.4].

C **Lemma 4.1.** *Suppose $A(t)$ satisfies the 'Acquistapace-Terreni' conditions, $U(t, s)$ is exponentially stable and $R(\lambda_0, A(\cdot)) \in AP(\mathbb{R}; \mathcal{L}(\mathbb{H}))$. Let $h > 0$. Then, for any $\varepsilon > 0$, there exists $l(\varepsilon) > 0$ such that every interval of length l contains at least a number τ with the property that*

$$\|U(t + \tau, s + \tau) - U(t, s)\| \le \varepsilon\, e^{-\frac{\delta}{2}(t-s)}$$

for every t, s with $|t - s| \ge h$.

rmk **Remark 4.2.** Lemma 4.1 implies that for every sequence $\alpha' = \{\alpha'_n\} \subset \mathbb{R}$ there exists a subsequence $\alpha = \{\alpha_n\} \subset \alpha'$ and a operator $\tilde{U}(\cdot,\cdot)$ such that

$$\lim_{n\to\infty} U(t+\alpha_n, s+\alpha_n) = \tilde{U}(t,s)$$

for every t, s with $|t-s| \geq h$.

In the rest of the paper, let us assume that $B^H = \{B^H_t, t \in \mathbb{R}\}$ is a cylindrical fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$ and with values in $\mathbb{K}$, and that $\mathbb{W} = \{\mathbb{W}(t), t \in \mathbb{R}\}$ is a standard cylindrical Wiener process on $\mathbb{K}_0$, independent of B^H. For each $t \in \mathbb{R}$, we denote $\mathcal{F}_t$ the σ-field generated by the random variables $\{B^H(s), \mathbb{W}(s), s \in [0,t]\}$ and the $\mathbf{P}$-null sets. In addition to the natural filtration $\{\mathcal{F}_t, t \in \mathbb{R}\}$ we will consider bigger filtration $\{\mathcal{G}_t, t \in \mathbb{R}\}$ such that

(1) $\{\mathcal{G}_t\}$ is right-continuous and $\mathcal{G}_0$ contains the $\mathbf{P}$-null sets;
(2) B^H is $\mathcal{G}_0$-measurable and $\mathbb{W}$ is a $\mathcal{G}_t$-Brownian motion.

Note that $\hat{\mathcal{F}}_t \subset \mathcal{G}_t$, where $\hat{\mathcal{F}}_t$ is the σ-field generated by the random variables $\{B^H, \mathbb{W}(s), s \in [0,t]\}$ and the $\mathbf{P}$-null sets.

We consider mild solutions of (1.1) in the following sense.

def4.3 **Definition 4.3.** A mild solution of the stochastic differential equation (1.1) is a triple $\big((X, B^H, \mathbb{W}), (\Omega, \mathcal{F}, \mathbf{P}), \{\mathcal{G}_t, t \in \mathbb{R}\}\big)$, where

(1) $(\Omega, \mathcal{F}, \mathbf{P})$ is a complete probability space, $\{\mathcal{G}_t\}$ is a right-continuous filtration such that $\mathcal{G}_0$ contains the $\mathbf{P}$- null sets.
(2) $\mathbb{W}$ is a $\mathcal{G}_t$- Brownian motion.
(3) B^H is a fractional Brownian of Hurst parameter H which is $\mathcal{G}_0$- measurable.
(4) $(X, B^H, \mathbb{W})$ satisfies the equation

$$\begin{aligned} X(t) = X(s) &+ \int_s^t U(t,r)F(r,X(r))\,dr + \int_s^t U(t,r)G(r,X(r))\,d\mathbb{W}(r) \\ &+ \int_s^t U(t,r)\Phi(r)\,dB^H(r), \quad \text{a.s.}\mathbf{P}, \end{aligned} \tag{4.1}$$ C3

for all $t \geq s$ for each $s \in \mathbb{R}$.

Note that the first integral on the right-hand side of (4.1) is taken in the Bochner sense, the second integral is interpreted in the Itô sense, and the third is defined in Section 2. Also, all integrals making up the fixed point operator are defined in terms of the given Wiener process $\mathbb{W}$ and fractional Brownian motion B^H, and the unique fixed point solution will be a mild solution, which is 'strong in the probabilistic sense'.

Now, we are ready to present our main result.

m **Theorem 4.4.** *Under assumptions* (H0)–(H3)*, Equation* (1.1) *has a unique pth-mean almost periodic mild solution, which can be explicitly expressed as*

$$\begin{aligned} X(t) = &\int_{-\infty}^t U(t,s)F(s,X(s))\,ds + \int_{-\infty}^t U(t,s)G(s,X(s))\,d\mathbb{W}(s) \\ &+ \int_{-\infty}^t U(t,s)\Phi(s)\,dB^H(s), \quad a.s.\mathbf{P}, \end{aligned}$$

for each $t \in \mathbb{R}$ whenever

$$\Theta := M^p\Big(\frac{K}{\delta^p} + C_p\, K'\big(\frac{p-2}{p\delta}\big)^{\frac{p-2}{2}}\big(\frac{1}{p\delta}\big)\Big) < 1\,,$$

for $p > 2$ and

$$\Theta := M^2\Big(2\frac{K}{\delta^2} + \frac{K'}{\delta}\Big) < 1$$

forp $= 2$.

Proof. First of all, note that

$$\begin{aligned} X(t) &= \int_{-\infty}^t U(t,s)F(s,X(s))\,dr + \int_{-\infty}^t U(t,s)G(s,X(s))\,d\mathbb{W}(s) \\ &\quad + \int_{-\infty}^t U(t,s)\Phi(s)\,dB^H(s), \quad \text{a.s. } \mathbf{P} \end{aligned}$$

is well-defined and satisfies

$$\begin{aligned} X(t) &= X(s) + \int_s^t U(t,r)F(r,X(r))\,dr + \int_s^t U(t,r)G(r,X(r))\,d\mathbb{W}(r) \\ &\quad + \int_s^t U(t,r)\Phi(r)\,dB^H(r), \quad \text{a.s. } \mathbf{P} \end{aligned}$$

ab for all $t \geq s$ for each $s \in \mathbb{R}$, and hence X given by (4.1) is a mild solution to (1.1).

Define $\Lambda X(t) = \Gamma_1 X(t) + \Gamma_2 X(t)$, where

$$\begin{gathered} \Gamma_1 X(t) := \int_{-\infty}^t U(t,\sigma)\varphi X(\sigma) d\sigma, \\ \Gamma_2 X(t) := \int_{-\infty}^t U(t,\sigma)\psi X(\sigma) d\mathbb{W}(\sigma) + \int_{-\infty}^t U(t,\sigma)\Phi(\sigma) dB^H(\sigma)\,, \end{gathered}$$

with $\varphi X(t) = F(t, X(t))$ and $\psi X(t) = G(t, X(t))$.

To prove Theorem 4.4 we need the following key lemmas.

m1 **Lemma 4.5.** *Assume that the hypotheses* (H0)–(H1) *hold. Then $\Gamma_1 X(\cdot)$ is pth-mean almost periodic.*

Proof. We need to show that $\Gamma_1 X(\cdot)$ is pth-mean almost periodic whenever X is. Indeed, assuming that X is p-th mean almost periodic and using assumption (H1), Theorem 3.16, and Lemma 4.1, given $\varepsilon > 0$, one can find $l_\varepsilon > 0$ such that any interval of length l_ε contains at least τ with the property that

$$\|U(t+\tau, s+\tau) - U(t,s)\| \leq \varepsilon e^{-\frac{\delta}{2}(t-s)}$$

for all $t - s \geq \varepsilon$, and

$$\mathbf{E}\|\varphi X(\sigma+\tau) - \varphi X(\sigma)\|^p < \eta$$

for each $\sigma \in \mathbb{R}$, where $\eta(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Moreover, it follows from Lemma 3.8 (ii) that there exists a positive constant K_1 such that

$$\sup_{\sigma \in \mathbb{R}} \mathbf{E}\|\varphi X(\sigma)\|^p \leq K_1\,.$$

Now, using assumption (H0) and Hölder's inequality, we obtain

$$\mathbf{E}\|\Gamma_1 X(t+\tau) - \Gamma_1 X(t)\|^p$$

$$
\begin{aligned}
&\le 3^{p-1}\mathbf{E}\Big[\int_0^\infty \|U(t+\tau,t+\tau-s)\|\,\|\varphi X(t+\tau-s)-\varphi X(t-s)\|ds\Big]^p\\
&\quad + 3^{p-1}\mathbf{E}\Big[\int_\varepsilon^\infty \|U(t+\tau,t+\tau-s)-U(t,t-s)\|\,\|\varphi X(t-s)\|ds\Big]^p\\
&\quad + 3^{p-1}\mathbf{E}\Big[\int_0^\varepsilon \|U(t+\tau,t+\tau-s)-U(t,t-s)\|\,\|\varphi X(t-s)\|ds\Big]^p\\
&\le 3^{p-1}M^p\mathbf{E}\Big[\int_0^\infty e^{-\delta s}\|\varphi X(t+\tau-s)-\varphi X(t-s)\|ds\Big]^p\\
&\quad + 3^{p-1}\varepsilon^p\mathbf{E}\Big[\int_\varepsilon^\infty e^{-\frac{\delta}{2}s}\|\varphi X(t-s)\|ds\Big]^p + 3^{p-1}M^p\mathbf{E}\Big[\int_0^\varepsilon 2e^{-\delta s}\|\varphi X(t-s)\|ds\Big]^p\\
&\le 3^{p-1}M^p\Big(\int_0^\infty e^{-\delta s}\,ds\Big)^{p-1}\Big(\int_0^\infty e^{-\delta s}\mathbf{E}\|\varphi X(t+\tau-s)-\varphi X(t-s)\|^p ds\Big)\\
&\quad + 3^{p-1}\varepsilon^p\Big(\int_0^\infty e^{-\delta s}\,ds\Big)^{p-1}\Big(\int_\varepsilon^\infty e^{-\frac{\delta ps}{2}}\mathbf{E}\|\varphi X(t-s)\|^p ds\Big)\\
&\quad + 6^{p-1}M^p\Big(\int_0^\varepsilon e^{-\delta s}\,ds\Big)^{p-1}\Big(\int_0^\varepsilon e^{-\frac{\delta ps}{2}}\mathbf{E}\|\varphi X(t-s)\|^p ds\Big)\\
&\le 3^{p-1}M^p\Big(\int_0^\infty e^{-\delta s}\,ds\Big)^{p}\sup_{s\in\mathbb{R}}\mathbf{E}\|\varphi X(t+\tau-s)-\varphi X(t-s)\|^p\\
&\quad + 3^{p-1}\varepsilon^p\Big(\int_\varepsilon^\infty e^{-\delta s}\,ds\Big)^{p}\sup_{s\in\mathbb{R}}\mathbf{E}\|\varphi X(t-s)\|^p\\
&\quad + 6^{p-1}M^p\Big(\int_0^\varepsilon e^{-\delta s}\,ds\Big)^{p}\sup_{s\in\mathbb{R}}\mathbf{E}\|\varphi X(t-s)\|^p\\
&\le 3^{p-1}M^p\Big(\frac{1}{\delta^p}\Big)\eta + 3^{p-1}M^pK_1\Big(\frac{1}{\delta^p}\Big)\varepsilon^p + 6^{p-1}M^p\varepsilon^pK_1\varepsilon^p,
\end{aligned}
$$

which implies that $\Gamma_1X(\cdot)$ is pth-mean almost periodic. □

The next lemma concerns $\Gamma_2X(\cdot)$. For that, let us fix $h>0$ and write $\Gamma_2X(t)$ as

$$\Gamma_2X(t)=\Gamma_{21}^hX(t)+\Gamma_{22}^hX(t)\,,$$

where

$$\Gamma_{21}^hX(t):=\int_{t-h}^t U(t,\sigma)\psi X(\sigma)d\mathbb{W}(\sigma)+\int_{t-h}^t U(t,\sigma)\Phi(\sigma)dB^h(\sigma)$$

and

$$\Gamma_{22}^hX(t):=\int_{-\infty}^{t-h} U(t,\sigma)\psi X(\sigma)d\mathbb{W}(\sigma)+\int_{-\infty}^{t-h} U(t,\sigma)\Phi(\sigma)dB^H(\sigma)\,.$$

m2 **Lemma 4.6.** *Let us assume that* (H0)–(H2) *are satisfied. The following holds.*

(i) *Let $\alpha\in(0,1/2-1/p)$ if $p>2$ and $\alpha\in(0,1/2)$ if $p=2$. The family $\{\|\Gamma_{22}^hX(t)\|_\alpha^p, t\in\mathbb{R}\}$ is uniformly integrable. In particular the family of distributions $\{\mathbf{P}\circ\big[\Gamma_{22}^hX(t)\big]^{-1}, t\in\mathbb{R}\}$ is tight.*
(ii) *$\Gamma_{22}^hX(\cdot)$ is almost periodic in distribution.*
(iii) *$\Gamma_{22}^hX(\cdot)$ is pth-mean almost periodic.*
(iv) *$\Gamma_{21}^hX(\cdot)$ is pth-mean almost periodic.*

Proof. (i) We split the proof of (i) in two cases: $p > 2$ and $p = 2$. We start with the case where $p > 2$. For that, we use the following theorem due to de la Vallée-Poussin.

$\boxed{\mathtt{R}}$ **Theorem 4.7.** *The family* $\{X(t), t \in \mathbb{R}\}$ *of real random variables is uniformly integrable if and only if there exists a nonnegative increasing convex function* $\Psi(\cdot)$ *on* $[0,\infty)$ *such that* $\lim_{x\to\infty} \frac{\Psi(x)}{x} = \infty$ *and* $\sup_{t\in\mathbb{R}} \mathbf{E}\big[\Psi(|X(t)|)\big] < \infty$.

To show the uniform integrability of the family $\{\|\Gamma_{22}^h X(t)\|_\alpha^p, t \in \mathbb{R}\}$, it suffices, by Theorem 4.7, to show that

$$\sup_{t\in\mathbb{R}} \mathbf{E}\|\Gamma_{22}^h X(t)\|_\alpha^{2p} < \infty\,.$$

To this end, we use the factorization formula of the stochastic convolution integral

$$\Gamma_{22}^h X(t) = \frac{\sin(\pi\xi)}{\pi}\big[R_\xi^h \mathbb{S}_\psi + R_\xi^h \mathbb{S}_\Phi\big](t) \quad \text{a.s.}$$

where

$$(R_\xi^h \mathbb{S}_\psi)(t) = \int_{-\infty}^{t-h} (t-\sigma)^{\xi-1} U(t,s)\mathbb{S}_\psi(s)\,ds$$

and

$$(R_\xi^h \mathbb{S}_\Phi)(t) = \int_{-\infty}^{t-h} (t-\sigma)^{\xi-1} U(t,s)\mathbb{S}_\Phi(s)\,ds$$

with

$$\begin{aligned}
\mathbb{S}_\psi(s) &= \int_{-\infty}^{s} (s-\sigma)^{-\xi} U(s,\sigma)\psi X(\sigma)\,d\mathbb{W}(\sigma)\,,\\
\mathbb{S}_\Phi(s) &= \int_{-\infty}^{s} (s-\sigma)^{-\xi} U(s,\sigma)\Phi(\sigma)\,dB^H(\sigma)\,,
\end{aligned}$$

and ξ satisfying $\alpha + \frac{1}{p} < \xi < \frac{1}{2}$.

We then have

$$\begin{aligned}
&\mathbf{E}\|\Gamma_{22}^h X(t)\|_\alpha^{2p}\\
&\le 2^{2p-1}|\frac{\sin\pi\xi}{\pi}|^{2p}\Big\{\mathbf{E}\Big[\int_{-\infty}^{t-h}(t-s)^{\xi-1}\|U(t,s)\mathbb{S}_\psi(s)\|_\alpha\,ds\Big]^{2p}\\
&\quad + \mathbf{E}\Big[\int_{-\infty}^{t-h}(t-s)^{\xi-1}\|U(t,s)\mathbb{S}_\Phi(s)\|_\alpha\,ds\Big]^{2p}\Big\}\\
&\le 2^{2p-1}M(\alpha)^{2p}\Big|\frac{\sin\pi\xi}{\pi}\Big|^{2p}\Big\{\mathbf{E}\Big[\int_{-\infty}^{t}(t-s)^{\xi-\alpha-1}e^{-\delta(t-s)}\|\mathbb{S}_\psi(s)\|\,ds\Big]^{2p}\\
&\quad + \mathbf{E}\Big[\int_{-\infty}^{t}(t-s)^{\xi-\alpha-1}e^{-\delta(t-s)}\|\mathbb{S}_\Phi(s)\|\,ds\Big]^{2p}\Big\}\\
&\le 2^{2p-1}M(\alpha)^{2p}\Big|\frac{\sin\pi\xi}{\pi}\Big|^{2p}\Big[\Big(\int_{-\infty}^{t}(t-s)^{\frac{2p}{2p-1}(\xi-\alpha-1)}e^{-\delta(t-s)}\,ds\Big)^{2p-1}\\
&\quad \times\Big(\int_{-\infty}^{t}e^{-\delta(t-s)}\mathbf{E}\|\mathbb{S}_\psi(s)\|^{2p}\,ds\Big)\\
&\quad + \Big(\int_{-\infty}^{t}(t-s)^{\frac{2p}{2p-1}(\xi-\alpha-1)}e^{-\delta(t-s)}\,ds\Big)^{2p-1}\Big(\int_{-\infty}^{t}e^{-\delta(t-s)}\mathbf{E}\|\mathbb{S}_\Phi(s)\|^{2p}\,ds\Big)\Big]
\end{aligned}$$

$$\le C_1(\Gamma,\alpha,\xi,\delta,p)\Big[\sup_{s\in\mathbb{R}}\mathbf{E}\|\mathbb{S}_\psi(s)\|^{2p}+\sup_{s\in\mathbb{R}}\mathbf{E}\|\mathbb{S}_\Phi(s)\|^{2p}\Big]\,,$$

where $C_1(\Gamma,\alpha,\xi,\delta,p)$ is a constant depending only on Gamma function $\Gamma(\cdot)$ and constants α, ξ, δ, and p.

Now, let us evaluate $\sup_{s\in\mathbb{R}}\mathbf{E}\|\mathbb{S}_\psi(s)\|^{2p}$ and $\sup_{s\in\mathbb{R}}\mathbf{E}\|\mathbb{S}_\Phi(s)\|^{2p}$. Since

$$\begin{aligned}&\int_{-\infty}^s \mathbf{E}\|(s-\sigma)^{-\xi}U(s,\sigma)\psi X(\sigma)\|^2\,ds\\&\le M^2\int_{-\infty}^s (s-\sigma)^{-2\xi}e^{-2\delta(s-\sigma)}\mathbf{E}\|\psi X(\sigma)\|^2_{\mathbb{L}_2^0}\,ds<\infty\end{aligned}$$

for all $s\in\mathbb{R}$, then by [32, Lemma 2.2]

$$\begin{aligned}\mathbf{E}\|\mathcal{S}_\psi(s)\|^{2p}&\le C_p\mathbf{E}\Big(\int_{-\infty}^s\|(s-\sigma)^{-\xi}U(s,\sigma)\psi X(\sigma)\|^2\,d\sigma\Big)^p\\&\le M^{2p}C_p\mathbf{E}\Big(\int_{-\infty}^s(s-\sigma)^{-2\xi}e^{-2\delta(s-\sigma)}\|\psi X(\sigma)\|^2_{\mathbb{L}_2^0}\,d\sigma\Big)^p\\&\le M^{2p}C_p\Big(\int_{-\infty}^s(s-\sigma)^{-\frac{2p\xi}{p-1}}e^{-2\delta(s-\sigma)}\,d\sigma\Big)^{p-1}\\&\quad\times\Big(\int_{-\infty}^s e^{-2\delta(s-\sigma)}\mathbf{E}\|\psi(\sigma)\|^{2p}_{\mathbb{L}_2^0}\,d\sigma\Big)\\&\le C_2(\Gamma,\xi,\delta,p)\sup_{\sigma\in\mathbb{R}}\mathbf{E}\|\psi(\sigma)\|^{2p}_{\mathbb{L}_2^0}\,,\end{aligned}$$

where $C_2(\Gamma,\xi,\delta,p)$ is a constant depending only on Gamma function $\Gamma(\cdot)$ and constants ξ, δ, and p.

For $\sup_{s\in\mathbb{R}}\mathbf{E}\|\mathbb{S}_\Phi(s)\|^{2p}$, since for every $s\in\mathbb{R}$, $\int_{-\infty}^s(s-\sigma)^{-\xi}U(s,\sigma)\Phi(\sigma)\,dB^H(\sigma)$ is a centered Gaussian random variable and using Kahane-Khintchine inequality, there exists a constant C_p such that

$$\begin{aligned}&\mathbf{E}\|\int_{-\infty}^s(s-\sigma)^{-\xi}U(s,\sigma)\Phi(\sigma)\,dB^H(\sigma)\|^{2p}\\&\le C_p\Big(\mathbf{E}\|\int_{-\infty}^s(s-\sigma)^{-\xi}U(s,\sigma)\Phi(\sigma)\,dB^H(\sigma)\|^2\Big)^p.\end{aligned}$$

Now, write

$$\begin{aligned}&\mathbf{E}\|\int_{-\infty}^s(s-\sigma)^{-\xi}U(s,\sigma)\Phi(\sigma)\,dB^H(\sigma)\|^2\\&=\sum_{n=1}^\infty\mathbf{E}\|\int_{-\infty}^s(s-\sigma)^{-\xi}U(s,\sigma)\Phi(\sigma)e_n\,d\beta_n^H(\sigma)\|^2\,,\end{aligned}$$

where $\{e_n,n\in\mathbb{N}\}$ is a complete orthonormal basis in $\mathbb{K}$ and $\{\beta_n^H(t),n\in\mathbb{N},t\in\mathbb{R}\}$ is a sequence of independent, real-valued standard fractional Brownian motions each with the same Hurst parameter $H\in\frac12,1)$.

Thus, using fractional Itô isometry one can write

$$\begin{aligned}&\mathbf{E}\|\int_{-\infty}^s(s-\sigma)^{-\xi}U(s,\sigma)\Phi(\sigma)\,dB^H(\sigma)\|^2\\&=\sum_{n=1}^\infty\int_{-\infty}^s\int_{-\infty}^s\Big\langle(s-\sigma)^{-\xi}U(s,\sigma)\Phi(\sigma)e_n,(s-r)^{-\xi}U(s,r)\Phi(r)e_n\Big\rangle\end{aligned}$$

$$
\begin{aligned}
&\times H(2H-1)|\sigma-r|^{2H-2}\,d\sigma\,dr\\
&\le H(2H-1)\int_{-\infty}^{s}(s-\sigma)^{-\xi}\Big\{\|U(s,\sigma)\Phi(\sigma)\|\\
&\quad\times\int_{-\infty}^{s}(s-r)^{-\xi}\|U(s,r)\Phi(r)\||\sigma-r|^{2H-2}\,dr\Big\}\,d\sigma\\
&\le H(2H-1)M^2\int_{-\infty}^{s}(s-\sigma)^{-\xi}\Big\{e^{-\delta(s-\sigma)}\|\Phi(\sigma)\|_{\mathbb{L}_2}\\
&\quad\times\int_{-\infty}^{s}(s-r)^{-\xi}e^{-\delta(s-r)}\|\Phi(r)\|_{\mathbb{L}_2}|\sigma-r|^{2H-2}\,dr\Big\}\,d\sigma\,.
\end{aligned}
$$

Since Φ is bounded, one can then conclude that

$$
\begin{aligned}
&\mathbf{E}\|\int_{-\infty}^{s}(s-\sigma)^{-\xi}U(s,\sigma)\Phi(\sigma)\,dB^H(\sigma)\|^2\\
&\le H(2H-1)M^2\Big(\sup_{t\in\mathbb{R}}\|\Phi(t)\|_{\mathbb{L}_2}\Big)^2\int_{-\infty}^{s}(s-\sigma)^{-\xi}e^{-\delta(s-\sigma)}\\
&\quad\times\Big\{\int_{-\infty}^{s}(s-r)^{-\xi}e^{-\delta(s-r)}|\sigma-r|^{2H-2}\,dr\Big\}\,d\sigma\,.
\end{aligned}
$$

Make the following change of variables, $u=s-r$ for the first integral and $v=s-\sigma$ for the second integral. One can then write

$$
\begin{aligned}
&\mathbf{E}\|\int_{-\infty}^{s}(s-\sigma)^{-\xi}U(s,\sigma)\Phi(\sigma)\,dB^H(\sigma)\|^2\\
&\le H(2H-1)M^2\big(\sup_{t\in\mathbb{R}}\|\Phi(t)\|_{\mathbb{L}_2}\big)^2\int_0^\infty v^{-\xi}e^{-\delta v}\Big\{\int_0^\infty u^{-\xi}e^{-\delta u}|u-v|^{2H-2}\,du\Big\}\,dv\\
&\le H(2H-1)M^2\big(\sup_{t\in\mathbb{R}}\|\Phi(t)\|_{\mathbb{L}_2}\big)^2(A_1+A_2)\,,
\end{aligned}
$$

where

$$
\begin{aligned}
A_1&=\int_0^\infty v^{-\xi}e^{-\delta v}\Big\{\int_v^\infty u^{-\xi}e^{-\delta u}(u-v)^{2H-2}\,du\Big\}\,dv,\\
A_2&=\int_0^\infty v^{-\xi}e^{-\delta v}\Big\{\int_0^v u^{-\xi}e^{-\delta u}(v-u)^{2H-2}\,du\Big\}\,dv\,.
\end{aligned}
$$

To evaluate A_1, we make change of variables $w=u-v$ and use the fact that $(w+v)^{-\xi}\le v^{-\xi}$ to obtain

$$
\begin{aligned}
A_1&=\int_0^\infty v^{-\xi}e^{-2\delta v}\Big\{\int_0^\infty (w+v)^{-\xi}e^{-\delta w}w^{2H-2}\,dw\Big\}\,dv\\
&\le\Big(\int_0^\infty v^{-2\xi}e^{-2\delta v}\,dv\Big)\Big(\int_0^\infty w^{2H-2}e^{-\delta w}\,dw\Big)\\
&=\Gamma(1-2\xi)\big(\frac{1}{2\delta}\big)^{1-2\xi}\Gamma(2H-1)\Big(\frac{1}{\delta}\Big)^{2H-1}\,.
\end{aligned}
$$

As to A_2, we first evaluate the integral $\int_0^v u^{-\xi}e^{-\delta u}(v-u)^{2H-2}\,du$. For that, we make change of variables $w=\frac{u}{v}$ to obtain

$$
\int_0^v u^{-\xi}e^{-\delta u}(v-u)^{2H-2}\,du\le\int_0^v u^{-\xi}(v-u)^{2H-2}\,du
$$

$$
\begin{aligned}
&= v^{1-\xi+2H-2}\int_0^1 w^{(1-\xi)-1}(1-w)^{(2H-1)-1}\,dw \\
&= v^{1-\xi+2H-2}\frac{\Gamma(1-\xi)\Gamma(2H-1)}{\Gamma(2H-\xi)}
\end{aligned}
$$

Thus,

$$
\begin{aligned}
A_2 &\le \frac{\Gamma(1-\xi)\Gamma(2H-1)}{\Gamma(2H-\xi)}\int_0^\infty e^{-\delta v}v^{1-2\xi+2H-2}\,dv \\
&= \frac{\Gamma(1-\xi)\Gamma(2H-1)}{\Gamma(2H-\xi)}\Gamma(2H-2\xi)\big(\frac{1}{\delta}\big)^{2H-2\xi}.
\end{aligned}
$$

Combining, we obtain

$$
\mathbf{E}\|\mathbb{S}_\Phi(s)\|^{2p} \le C_3(\Gamma,\xi,\delta,H,p)\sup_{\sigma\in\mathbb{R}}\|\Phi(\sigma)\|_{\mathbb{L}_2}^{2p},
$$

where $C_3(\Gamma,\xi,\delta,H,p)$ is a constant depending only on Gamma function $\Gamma(\cdot)$ and constants α, ξ, δ, H, and p. Thus,

$$
\begin{aligned}
\mathbf{E}\|\Gamma_{22}^h X(t)\|_\alpha^{2p} \le C_1(\Gamma,\alpha,\xi,\delta,p)\Big[&C_2(\Gamma,\xi,\delta,p)\sup_{\sigma\in\mathbb{R}}\mathbf{E}\|\psi(\sigma)\|_{\mathbb{L}_2^0}^{2p} \\
&+ C_3(\Gamma,\xi,\delta,H,p)\sup_{\sigma\in\mathbb{R}}\|\Phi(\sigma)\|_{\mathbb{L}_2}^{2p}\Big] < \infty,
\end{aligned}
$$

and true for any $t\in\mathbb{R}$. For the case $p=2$, a similar computation shows that

$$
\sup_{t\in\mathbb{R}}\mathbf{E}\|\Gamma_{22}^h X(t)\|_\alpha^4 < \infty.
$$

Moreover, using the Chebyshev inequality, one can easily show that the family of distributions $\big\{\mathbf{P}\circ[\Gamma_{22}^h X(t)]^{-1}, t\in\mathbb{R}\big\}$ is tight.

(ii) To show the almost periodicity in distribution of $\Gamma_{22}^h X(\cdot)$, we follow closely the work done by Da Prato and Tudor [11]. To this end, we state without proofs some of their results and adapt them to our case.

m3 **Proposition 4.8.** [11] *Let A, G, Φ, $\{A_n, G_n, \Phi_n\}_{n\in\mathbb{N}}$ satisfy* (H0), (H2), (H3) *with the same constants δ, K'. Let U, U_n be the evolution operators generated by A, A_n, and let $\{\Gamma X(t)\}_{t\in\mathbb{R}}$, $\{\Gamma X_n(t)\}_{t\in\mathbb{R}}$ be the stochastic convolution integrals corresponding to A, G, Φ, and A_n, G_n, Φ_n respectively. Assume in addition that*

(i) *$\lim_{n\to\infty} U_n(t,s)x = U(t,s)x$ for all $x\in\mathbb{H}$ and for every $|t-s|\ge h$.*
(ii) *$\lim_{n\to\infty} G_n(t,x) = G(t,x)$ for all $x\in\mathbb{H}$ and for every $t\in\mathbb{R}$.*
(iii) *$\lim_{n\to\infty}\Phi_n(t) = \Phi(t)$ for every $t\in\mathbb{R}$.*
(iv) *For each $t\in\mathbb{R}$, the family of distributions $\{\mathbf{P}\circ[\Gamma X_n(t)]^{-1}\}_{n\in\mathbb{N}}$ is tight.*

Then

$$
\lim_{n\to\infty} d_{BL}\Big(\mathbf{P}\circ[\Gamma X_n(t+\cdot)]^{-1}, \mathbf{P}\circ[\Gamma X(t+\cdot)]^{-1}\Big) = 0
$$

in $\mathcal{P}\big(C(\mathbb{R};\mathbb{H})\big)$ for all $t\in\mathbb{R}$.

We can now prove (ii). Let $\alpha'=(\alpha')\subset\mathbb{R}$, $\beta'=(\beta')\subset\mathbb{R}$ and by (H2) and (H3), choose common subsequences $\alpha=(\alpha)\subset\alpha'$, $\beta=(\beta_n)\subset\beta'$ such that

$$
T_{\alpha+\beta}\Phi(t) = T_\alpha T_\beta\Phi(t) \quad \text{for each } t\in\mathbb{R}, \tag{4.2}
$$ a0

$$
T_{\alpha+\beta}G(t,x) = T_\alpha T_\beta G(t,x) \tag{4.3}
$$ a1

uniformly on $\mathbb{R}\times\mathcal{O}'$, where $\mathcal{O}'$ is any compact set of $\mathbb{H}$. Also, by Lemma 4.1, we have:

$$\lim_{n\to\infty} U(t+\alpha_n, s+\alpha_n)x = U(t+\sigma_1, s+\sigma_1)x \tag{4.4} \quad \boxed{\texttt{a2}}$$

$$\lim_{n\to\infty} U(t+\beta_n+\sigma_1, s+\beta_n+\sigma_1)x = U(t+\sigma_2+\sigma_1, s+\sigma_2+\sigma_1)x \tag{4.5} \quad \boxed{\texttt{a3}}$$

$$\lim_{n\to\infty} U(t+\alpha_n+\beta_n, s+\alpha_n+\beta_n)x = U(t+\sigma_1+\sigma_2, s+\sigma_1+\sigma_2)x \tag{4.6} \quad \boxed{\texttt{a4}}$$

for all $x\in\mathbb{H}$, for every $|t-s|\ge h$.

By using (4.2)-(4.6), Lemma 4.6 (i), and Proposition 4.8, applied successively to $\{\Gamma^h_{22}X(t+\beta_n)\}_{t\in\mathbb{R}}$, $\{\widetilde{\Gamma^h_{22}}X(t+\alpha_n)\}_{t\in\mathbb{R}}$ ($\{\widetilde{\Gamma^h_{22}}X(\cdot)\}_{t\in\mathbb{R}}$ is the stochastic convolution integral associated with $U(t+\sigma_2+\alpha_n, s+\sigma_2+\alpha_n)$, $T_\beta G$, $T_\beta\Phi$), and $\big\{\Gamma^h_{22}X(t+\alpha_n+\beta_n)\big\}_{t\in\mathbb{R}}$, we obtain common sequences $\alpha''\subset\alpha$, $\beta''\subset\beta$ such that

$$T_{\alpha''+\beta''}\hat{\mu^h}(t+\cdot) = T_{\alpha''}T_{\beta''}\hat{\mu^h}(t+\cdot)$$

for every $t\in\mathbb{R}$. Here, $\hat{\mu^h}(t+\cdot) = \mathbf{P}\circ[\Gamma^h_{22}X(t+\cdot)]^{-1}$. By Proposition 3.2, we deduce that the mapping $\mathbb{R}\to\mathcal{P}(C(\mathbb{R};\mathbb{H})) : t\to\hat{\mu^h}(t+\cdot)$ is almost periodic.

(iii) We now prove the pth-mean almost periodicity of $\Gamma^h_{22}X(\cdot)$. The latter follows immediately from (i), (ii), and Proposition 3.15.

(iv) For this, we use Definition 3.7. Fix $\varepsilon>0$ and choose $h=h(\varepsilon)>0$ such that $h(\varepsilon)\to 0$ as $\varepsilon\to 0$.

$$\begin{aligned}
&\mathbf{E}\|\Gamma^h_{21}X(t+\tau)-\Gamma^h_{21}X(t)\|^p\\
&\le 2^{p-1}\mathbf{E}\Big\|\int_{t+\tau-h}^{t+\tau} U(t+\tau,\sigma)\psi X(\sigma)d\mathbb{W}(\sigma) - \int_{t-h}^{t} U(t,\sigma)\psi X(\sigma)d\mathbb{W}(\sigma)\Big\|^p\\
&\quad + 2^{p-1}\mathbf{E}\Big\|\int_{t+\tau-h}^{t+\tau} U(t+\tau,\sigma)\Phi(\sigma)dB^H(\sigma) - \int_{t-h}^{t} U(t,\sigma)\Phi(\sigma)\,dB^H(\sigma)\Big\|^p\\
&\le 4^{p-1}\Big\{\mathbf{E}\Big\|\int_{t+\tau-h}^{t+\tau} U(t+\tau,\sigma)\psi X(\sigma)d\mathbb{W}(\sigma)\Big\|^p + \mathbf{E}\Big\|\int_{t-h}^{t} U(t,\sigma)\psi X(\sigma)\,d\mathbb{W}(\sigma)\Big\|^p\Big\}\\
&\quad + 4^{p-1}\Big\{\mathbf{E}\Big\|\int_{t+\tau-h}^{t+\tau} U(t+\tau,\sigma)\Phi(\sigma)\,dB^H(\sigma)\Big\|^p + \mathbf{E}\Big\|\int_{t-h}^{t} U(t,\sigma)\Phi(\sigma)\,dB^H(\sigma)\Big\|^p\Big\}\\
&\le 4^{p-1}I_1 + 4^{p-1}I_2\,.
\end{aligned}$$

First, let us evaluate I_1. Since

$$\int_{t-h}^{t}\mathbf{E}\|U(t,\sigma)\psi X(\sigma)\|^2\,d\sigma \le M^2\int_{t-h}^{t} e^{-2\delta(t-\sigma)}\mathbf{E}\|\psi X(\sigma)\|^2_{\mathbb{L}_2^0}\,d\sigma<\infty\,,$$

for each $t\in\mathbb{R}$, the application of [32, Lemma 2.2] gives us

$$\begin{aligned}
I_1 &\le 2^{p-1}C_p\Big\{\mathbf{E}\Big(\int_{t+\tau-h}^{t+\tau}\|U(t+\tau,\sigma)\psi X(\sigma)\|^2 d\sigma\Big)^{p/2}\\
&\quad + \mathbf{E}\Big(\int_{t-h}^{t}\|U(t,\sigma)\psi X(\sigma)\|^2 d\sigma\Big)^{p/2}\Big\}\\
&\le 2^{p-1}M^pC_p\Big\{\mathbf{E}\Big(\int_{t+\tau-h}^{t+\tau} e^{-2\delta(t+\tau-s)}\|\psi X(\sigma)\|^2_{\mathbb{L}_2^0}\,d\sigma\Big)^{p/2}\\
&\quad + \mathbf{E}\Big(\int_{t-h}^{t} e^{-2\delta(t-s)}\|\psi X(\sigma)\|^2 d\sigma\Big)^{p/2}\Big\}
\end{aligned}$$

$$\le 2^p C_p \sup_{s\in\mathbb{R}} \mathbf{E}\|\psi X(s)\|^p_{\mathbb{L}_2^0} h^p\,.$$

A similar computation using Kahana-Khintchine inequality and fractional Ito identity shows that

$$I_2 \le 2^p C_p \sup_{s\in\mathbb{R}} \|\Phi(s)\|^{2p}_{\mathbb{L}_2} h^p\,.$$

Hence, $\Gamma^h_{21} X(\cdot)$ is pth-mean almost periodic. □

In view of Lemmas 4.5 and 4.6 (i)–(iv), it is clear that Λ maps $AP(\mathbb{R}; L^p(\Omega,\mathbb{H}))$ into itself. To complete the proof, it suffices to show that Λ is a contraction.

Let X and Y be in $AP(\mathbb{R}; L^p(\Omega,\mathbb{H}))$. Proceeding as before starting with the case where $p > 2$ and using (H0), an application of Hölder's inequality, [32, Lemma 2.2], followed by (H1) and (H2) gives

$$\begin{aligned}
&\mathbf{E}\|\Lambda X(t) - \Lambda Y(t)\|^p\\
&\le 2^{p-1}\mathbf{E}\Big[\int_{-\infty}^t \|U(t,\sigma)\|\,\|\varphi X(\sigma) - \varphi Y(\sigma)\|\,d\sigma\Big]^p\\
&\quad + 2^{p-1} C_p \mathbf{E}\Big[\int_{-\infty}^t \|U(t,\sigma)\|^2 \|\psi X(\sigma) - \psi Y(\sigma)\|^2_{\mathbb{L}_2^0}\,d\sigma\Big]^{p/2}\\
&\le 2^{p-1} M^p \Big(\int_{-\infty}^t e^{-\delta(t-s)}\Big)^{p-1}\Big(\int_{-\infty}^t e^{-\delta(t-s)} \mathbf{E}\|\varphi X(\sigma) - \varphi Y(\sigma)\|^p\,d\sigma\Big)\\
&\quad + 2^{p-1} C_p \Big(\int_{-\infty}^t e^{-\frac{p}{p-2}\delta(t-s)}\,d\sigma\Big)^{\frac{p-2}{2}}\Big(\int_{-\infty}^t e^{-\frac{p}{2}\delta(t-s)} \mathbf{E}\|\psi X(\sigma) - \psi Y(\sigma)\|^p_{\mathbb{L}_2^0}\,d\sigma\Big)\\
&\le 2^{p-1} M^p K \Big(\int_{-\infty}^t e^{-\delta(t-\sigma)}\,d\sigma\Big)^p \|X - Y\|^p_\infty\\
&\quad + 2^{p-1} C_p M^p K' \Big(\int_{-\infty}^t e^{-\frac{p\delta}{p-2}(t-\sigma)}\,d\sigma\Big)^{\frac{p-2}{2}}\Big(\int_{-\infty}^t e^{-\frac{p\delta}{2}(t-\sigma)}\,d\sigma\Big)\|X - Y\|^p_\infty\\
&= 2^p M^p \Big[K\Big(\frac{1}{\delta^p}\Big) + C_p K'\Big(\frac{p-2}{p\delta}\Big)^{\frac{p-2}{2}}\Big(\frac{1}{p\delta}\Big)\Big]\|X - Y\|^p_\infty = \Theta\cdot\|X - Y\|^p_\infty.
\end{aligned}$$

As to the case $p = 2$, we have

$$\begin{aligned}
&\mathbf{E}\|\Lambda X(t) - \Lambda Y(t)\|^2\\
&\le 2M^2\Big(\int_{-\infty}^t e^{-\delta(t-s)}\,ds\Big)\Big(\int_{-\infty}^t e^{-\delta(t-s)} \mathbf{E}\|\varphi X(s) - \varphi Y(s)\|^2\,ds\Big)\\
&\quad + 2M^2 \int_{-\infty}^t e^{-2\delta(t-s)} \mathbf{E}\|\psi X(s) - \psi Y(s)\|^2_{\mathbb{L}_2^0}\,ds\\
&\le 2M^2\cdot K\Big(\int_{-\infty}^t e^{-\delta(t-s)}\,ds\Big)\Big(\int_{-\infty}^t e^{-\delta(t-s)} \mathbf{E}\|X(s) - Y(s)\|^2\,ds\Big)\\
&\quad + 2M^2\cdot K' \int_{-\infty}^t e^{-2\delta(t-s)} \mathbf{E}\|X(s) - Y(s)\|^2\,ds\\
&\le 2M^2\cdot K\Big(\int_{-\infty}^t e^{-\delta(t-s)}\,ds\Big)^2 \sup_{s\in\mathbb{R}} \mathbf{E}\|X(s) - Y(s)\|^2\Big)\\
&\quad + 2M^2\cdot K'\Big(\int_{-\infty}^t e^{-2\delta(t-s)}\,ds\Big) \sup_{s\in\mathbb{R}} \mathbf{E}\|X(s) - Y(s)\|^2
\end{aligned}$$

$$
\begin{aligned}
&\le 2M^2\Big(\frac{K}{\delta^2}+\frac{K'}{\delta}\Big)\|X-Y\|_\infty^2\\
&\le \Theta\cdot\|X-Y\|_\infty^2\,.
\end{aligned}
$$

Consequently, if $\Theta < 1$, then Λ has a unique fixed-point, which obviously is the unique pth-mean almost periodic solution to (1.1). □

References

AT [1] P. Acquistapace and B. Terreni; A unified approach to abstract linear parabolic equations, Tend. Sem. Mat. Univ. Padova **78** (1987) 47-107.

at [2] L. Arnold and C. Tudor; Stationary and almost periodic solutions of almost periodic affine stochastic differential equations, *Stochastics and Stochastic Reports* **64** (1998), 177-193.

W [3] M. Baroun, S. Boulite, T. Diagana, and L. Maniar; Almost periodic solutions to some semilinear nonautonomous thermoelastic plate equations. *J. Math. Anal. Appl.* **349**(2009), no. 1, 74–84.

BMR [4] F. Bedouhene, O. Mellah, and P. Raynaud de Fitte; Bochner-almost periodicity for stochastic processes. *Stochastic Anal. Appl.*, **30**(2) (2012) :322-342.

BD [5] P. Bezandry and T. Diagana; *Almost periodic stochastic processes*, Springer, 2011

BD1 [6] P. Bezandry and T. Diagana; Existence of almost periodic solutions to some stochastic differential equation. *Applicable Anal.* **87** (2007), No. 7, 817-827.

BD2 [7] P. Bezandry and T. Diagana; Square-mean almost periodic solutions to nonautonomous stochastic differential equation. *Electron. J. Differential Equations*, **2007** (2007), No. 117, 1-10.

BHOS [8] Biagini, F., Hu, Y., Oksendal, B., and Sulem, A.; A stochastic maximum principle for processes driven by fractional Brownian motion, *Stochastic Processes and Their Applications*, **100** (2002), 233-253.

CKR [9] Ciesielski, Z., Kerkyacharian, G., and Roynette, B.; Quelques espapces fonctionels associés à des processes gaussians, *Studia Mathematica*, **107** (2) (1993), 171-204

C [10] C. Corduneanu; *Almost periodic functions*, 2nd Edition, Chelsea-New York, 1989.

da [11] G. Da Prato and C. Tudor; Periodic and almost periodic solutions for semilinear stochastic evolution equations, *Stoch. Anal. Appl.* **13**(1) (1995), 13–33.

DK [12] A Dasgupta and G. Kallianpur; Arbitrage opportunities for a class of Gladyshev processes, *Appl. Math. Optim.* **41** (2000), 377-385.

DU [13] L. Decreusefond and A. S. Üstünel; Stochastic analysis of the fractional Brownian motion, *Potential Anal.* **10** (1998), 177-214.

DU2 [14] L. Decreusefond and A. S. Üstünel; Fractional Brownian motion: theory and applications, *ESAIM Procceedings* **5** (1998), 75-86.

Dia [15] T. Diagana; Weighted pseudo almost periodic functions and applications, *C. R. Acad. Sci. Paris* Ser. I 343, 643-646 (2006).

Dia2 [16] T. Diagana, E. Hernandez, and M. Rabello; Pseudo almost periodic solutions to some nonautomous neutral functional differential equations with unbounded delay, Math. Comput. Modeling 45, 12241-1252 (2007).

D [17] A. Ya. Dorogovtsev and O. A. Ortega, On the existence of periodic solutions of a stochastic equation in a Hilbert space. *Visnik Kiiv. Univ. Ser. Mat. Mekh.* No. **30** (1988), 21-30, 115

Du [18] R. M. Dudley; Real analysis and probability, Wadsworth and Brooks/Cole Math Series, 1989.

DHP [19] T. Duncan, Y. Hu, and B. Pasik-Duncan; Stochastic calculus for fractional Brownian motion I. theory, *SIAM J. Control Optim.* **38** (2000), 582-612

EN [20] K. J. Engel and R. Nagel; *One parameter semigroups for linear evolution equations*, Graduate texts in Mathematics, Springer Verlag 1999.

F [21] A. M. Fink; Almost periodic differential equations, Springer, Berlin (1974).

GN [22] Guerra and Nualart; Stochastic differential equations driven by fractional Brownian motion and standard Brownian motion, *Stochastic Analysis and Applications* **26**, No. 5, 2008, 1053-1075.

HO [23] Y. Hu and B. ksendal; Fractional white noise calculus and applications to finance, *Infinite Dimensional Analysis, quantum Probability and Related Topics* **6** (2003), 1-32.

I [24] A. Ichikawa; Stability of semilinear stochastic evolution equations. *J. Math. Anal. Appl.* **90** (1982), no.1, 12-44.

K [25] T. Kawata; Almost periodic weakly stationary processes. *Statistics and probability: essays in honor of C. R. Rao*, pp. 383–396, North-Holland, Amsterdam-New York, 1982.

Lun [26] A. Lunardi; *Analytic semigroups and optimal regularity in parabolic problems*, PNLDE Vol. *16*, Birkhäauser Verlag, Basel, 1995.

Ksn [27] B. Ksendal; Stochastic differential equations: an introduction with applications (sixth edition), Universitext, Springer-Verlag, Berlin (2003).

K2 [28] A. N. Kolmogorov; Wienershe spiralen and einige andere interessante kurven im Hilbertschen raum, *C. R. (Doklady) Acad. Sci. URSS (NS)* **26** (1940), 115-118

MR [29] L. Maniar and S. Roland; Almost periodicity of inhomogeneous parabolic evolution equations. Lecture Notes in Pure and Appl. Math. Vol. 234, Dekker, New York, 2003, pp. 299-318.

MV [30] B. B. Mandelbrot, and J. W. Van Ness; Fractional Brownian motions, fractional noises and applications, *SIAM Rev.* **10** (4) (1968), 422-437.

P [31] J. Pospísil; On ergodicity of stochastic evolution equations driven by fractional Brownian motion, *Proceedings of the Prague Stochastics*, 2006, 590-599

Se [32] J. Seidler; Da Prato-Zabczyk's maximal inequality revisited I, *Mathematica Bohemica*, **118** (1993), , No.1, 67-106.

S [33] R. J. Swift; Almost periodic harmonizable processes, *Georgian Math. J.* **3** (1996), no. 3, 275–292.

t3 [34] C. Tudor; Almost periodic solutions of affine stochastic evolutions equations, *Stochastics and Stochastics Reports* **38** (1992), 251-266.

Ya1 [35] A. Yagi; Parabolic equations in which the coefficients are generators of infinitely differentiable semigroups II, Funkcial. Ekvac. **33** (1990), 139–150.

Ya2 [36] A. Yagi; Abstract quasilinear evolution equations of parabolic type in Banach spaces, Boll. Un. Mat. Ital. B (7) **5** (1991), 341–368.

Z [37] M. Zähle; Integration with respect to fractal functions and stochastic calculus I., *Prob. Theory Relat. Fields* **111** (1998), 333-374

Paul H. Bezandry
Department of Mathematics, Howard University, Washington, DC 20059, USA
E-mail address: pbezandry@howard.edu