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TRAVELING WAVES AND SPREADING SPEED ON A LATTICE MODEL WITH AGE STRUCTURE

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Abstract. In this article, we study a lattice differential model for a single species with distributed age-structure in an infinite patchy environment. Using method of approaches by Diekmann and Thieme, we develop a comparison principle and construct a suitable sub-solution to the given model, and show that there exists a spreading speed of the system which in fact coincides with the minimal wave speed.

1. Introduction

Assume $u(t,a,x)$ is the population density at time t, age a and spatial location x, and x denotes the point coordinate which may be an integer, in $\mathbb{Z}$, or real number in $\mathbb{R}$. We study the species in a patchy environment with infinite number of patches connected by diffusion of population within the neighboring islands, where we can describe the patches as integer nodes of a one-dimensional lattice. In this case we change x to j, and let $u(t,a,j) = u_j(t,a)$ denote the population density of the species at j-th patch. Let $f(r)$ be a probability density function which specifies the probability of maturing of an individual with age $a \geq r$. This function satisfies $f(0) = 0$, $f(\infty) = 0$ and $\int_0^\infty f(r)dr = 1$. Let $w_j(t)$ denotes the total of mature population at time t and location j:

$$w_j(t) = \int_0^\infty f(r)\Big(\int_r^\infty u_j(t,a)da\Big)dr.$$

Ling [5] derived the lattice model

$$\begin{aligned}\frac{dw_j(t)}{dt} &= D[w_{j+1}(t) + w_{j-1}(t) - 2w_j(t)] - dw_j(t)\\ &\quad + \frac{1}{2\pi}\int_0^\infty e^{-da} f(a) \sum_{l=-\infty}^{\infty} \beta(a,l) b(w_{l+j}(t-a))da, \quad t > 0,\end{aligned} \tag{1.1}$$

where

$$\beta(a,l) = 2\int_0^\pi \cos(l\omega) e^{-4Da\sin^2(\frac{\omega}{2})} d\omega.$$

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Note that this equation has a nonlocal term $\sum_{l=-\infty}^{\infty}\beta(a,l)b(w_{l+j}(t-a))$ and a delay that is continuously distributed and infinite. Ling studied the existence and uniqueness of solutions to (1.1) with an initial value, also discussed the global attractivity of the zero solution, and the existence of wavefronts with speed greater than the spreading speed c_* of traveling wave. Motivated by the method in Diekmann and Thieme [9], in this article, we give a study on the traveling wave and spreading speed for (1.1). More information on the traveling waves for lattice differential systems can be found in [2, 3, 4, 5, 7, 8, 10] and the references therein.

Let $\mathbb{R}_+ := [0,+\infty)$ and $\tilde{f}(d) := \int_0^\infty f(a)e^{-da}da < 1$. We will use the following assumptions:

- (H0) $b(0) = 0$, $b(w) \le b'(0)w$ for $w \ge 0$; $b(w)\tilde{f}(d) < dw$ for $w > 0$, and $b'(0)\tilde{f}(d) < d$.
- (H1) $b(0) = 0$, $b \in C^1(\mathbb{R}_+,\mathbb{R}_+)$, b is non-decreasing function on $[0,K]$ and $b(K)\tilde{f}(d) \le dK$, $|b(u)-b(v)| \le b'(0)|u-v|$ for $u,v \in \mathbb{R}_+$.
- (H2) $b(0) = 0$, b is non-decreasing function on $[0,K]$, $b(w) \le b'(0)w$ for $w \in \mathbb{R}_+$.
- (H3) $b'(0)\tilde{f}(d) > d$, $b(w)\tilde{f}(d) = dw$ admits a positive solution w^+ on $(0,K]$. $b(w)\tilde{f}(d) > dw$ for $0 < w < w^+$; and $b(w)\tilde{f}(d) < dw$ for $w > w^+$.

This article is organized as follows. In Section 2, we introduce some definitions and properties of the characteristic equations. In Section 3, we establish the well-posedness and the comparison principle for (1.1), and obtain our main result on the existence of the spreading speed c_* of traveling wave of (1.1). We also give an estimate for c_* and study the relation between the spreading speed with the minimal wave speed.

2. Preliminaries

A solution $\{w_j(t)\}_{j\in\mathbb{Z}}$ is called a traveling wave of (1.1) provided that it has the form $w_j(t) = \phi(j+ct) = \phi(s)$. A sequence of functions $W(t) = \{w_j(t)\}_{j\in\mathbb{Z}}$ is called isotropic on an interval I if $w_j(t) = w_{-j}(t)$ for $j \in \mathbb{Z}$ and $t \in I$. Define

$$C_K^+(-\infty,T] = \{\phi : \phi \text{ is continuous function defined from } (-\infty,T] \text{ to } [0,K]\}.$$

We need also the following notation.

$$\begin{gathered}
B_N = \{j \in \mathbb{N} : |j| \le N, N \in \mathbb{N}\},\\
w_j(t) = w(t,j) \text{ for } j \in \mathbb{Z}, \quad W(t) = W(t,\cdot) = \{w_j(t)\}_{j\in\mathbb{Z}},\\
\operatorname{supp} W(t,\cdot) = \{j : w(t,j) \ne 0\} \text{ is the support of } W(t,\cdot),\\
W(t) \ge V(t) \text{ if } w_j(t) \ge v_j(t) \text{ for } j \in \mathbb{Z},\\
W(t) \succ V(t) \text{ if } W(t) \ge V(t) \text{ and } w_j(t) > v_j(t) \text{ for } j \in \operatorname{supp} V(t,\cdot).
\end{gathered}$$

A constant $c_* > 0$ is called the spreading speed of (1.1) provided that

$$\lim_{t\to\infty} \sup\{w_j(t) : |j| \ge ct\} = 0 \quad \text{for } c > c_*, \tag{2.1}$$

$$\lim_{t\to\infty} \inf\{w_j(t) : |j| \le ct\} \ge w^+ > 0 \quad \text{for } c \in (0,c_*). \tag{2.2}$$

where $\{w_j(t)\}_{j\in\mathbb{Z}}$ is a solution of (1.1).

Substituting $w_j(t) = \phi(j + ct) = \phi(s)$ into (1.1), we obtain the wave equation

$$\begin{aligned} c\phi'(s) &= D[\phi(s+1) + \phi(s-1) - 2\phi(s)] - d\phi(s) \\ &\quad + \frac{1}{2\pi}\int_0^\infty f(a)e^{-da}\sum_{l=-\infty}^{\infty}\beta(a,l)b(\phi(s+l-ca))da. \end{aligned} \tag{2.3}$$

The following assumption is needed for considering characteristic equation.

- (H4) Assume that for a given $c > 0$, one of the following two conditions is satisfied,
  - (i) For any $\lambda > 0$, $\int_0^\infty f(a)e^{-da}e^{2D(\cosh\lambda-1)a-\lambda ca}da < \infty$ holds.
  - (ii) There has $\lambda_0 > 0$, for any $\lambda < \lambda_0$, $\int_0^\infty f(a)e^{-da}e^{2D(\cosh\lambda-1)a-\lambda ca}da < \infty$ and
$$\lim_{\lambda\to\lambda_0-0}\int_0^\infty f(a)e^{-da}e^{2D(\cosh\lambda-1)a-\lambda ca}da = +\infty.$$

  If case (i) holds, let $\bar\lambda = \bar\lambda(c) = +\infty$; if case (ii) holds, let $\bar\lambda = \bar\lambda(c) = \lambda_0$.

Assume that (H1)-(H4) hold. Then (2.3) has two equilibria $w = 0$ and $w = w^+ > 0$ in $[0, K]$. Denote the characteristic equation of (2.3) at $w^0 := 0$, by $\Delta(\lambda, c) = 0$, we have

$$\Delta(\lambda,c) = -c\lambda + D[e^\lambda + e^{-\lambda} - 2] - d + \frac{b'(0)}{2\pi}\int_0^\infty f(a)e^{-da}\sum_{l=-\infty}^{\infty}\beta(a,l)e^{\lambda l}e^{-\lambda ca}da. \tag{2.4}$$

where

$$\frac{1}{2\pi}\sum_{l=-\infty}^{\infty}\beta(a,l)e^{\lambda l} = \exp\{D[e^{-\lambda} + e^\lambda - 2]a\} = e^{2D(\cosh\lambda-1)a}$$

(see [10]). Simplify (2.4) to obtain

$$\Delta(\lambda,c) := -c\lambda + D[e^\lambda + e^{-\lambda} - 2] - d + b'(0)\int_0^\infty f(a)e^{[-d-c\lambda+2D(\cosh\lambda-1)]a}da = 0. \tag{2.5}$$

From (2.4)-(2.5), it is easy to observe the following fact.

Lemma 2.1. *If b satisfies* (H2)-(H4). *Then there exists a unique pair* (c_*, λ_*) $(c_* > 0, \lambda_* > 0)$ *such that*

- (i) $\Delta(\lambda_*, c_*) = 0$, $\frac{\partial}{\partial\lambda}\Delta(\lambda_*, c_*) = 0$;
- (ii) *for* $0 < c < c_*$ *and any* $\lambda \in (0, \bar\lambda), \Delta(\lambda, c) > 0$;
- (iii) *for* $c > c_*$, *the equation* $\Delta(\lambda, c) = 0$ *has two positive real roots* $0 < \lambda_1 < \lambda_2 < \bar\lambda$, *and there exists* $\epsilon_0 > 0$ *such that for any* $\epsilon \in (0, \epsilon_0)$ *with* $0 < \lambda_1 < \lambda_1 + \epsilon < \lambda_2$, *we have* $\Delta(\lambda_1 + \epsilon, c) < 0$.

We rewrite (2.5) as

$$1 = \frac{1}{\delta + \lambda c}\Big[D(e^\lambda + e^{-\lambda}) + b'(0)\int_0^\infty f(a)e^{-da}e^{2D(\cosh\lambda-1)a-\lambda ca}da\Big] =: L_c(\lambda), \tag{2.6}$$

where $\delta := 2D + d$. Hence c_* can be represented as

$$c_* := \inf\{c > 0 : \text{ there exists some } \lambda \in \mathbb{R}_+, \text{ such that } L_c(\lambda) = 1\}.$$

From Lemma 2.1 we have

$$L_c(\lambda) > 1 \text{ for } \lambda \in (0, \bar\lambda), \text{ and } c \in (0, c_*); \quad L_c(\lambda) < 1 \text{ for } \lambda \in (\lambda_1, \lambda_2) \text{ and } c > c_*.$$

Now we shall show that c_* is the spreading speed of (1.1). Consider the equivalent form

$$\begin{aligned} w_j(t) &= e^{-\delta t} w_j(0) + \int_0^t e^{-\delta(t-s)} \{D[w_{j+1}(s) + w_{j-1}(s)] \\ &\quad + \frac{1}{2\pi}\int_0^\infty f(a)e^{-da}\sum_{l=-\infty}^{\infty} \beta(a,l) b(w_{l+j}(s-a))da\}ds, \quad j \in \mathbb{Z},\ t \geq 0, \\ & w_j(t) = w_j^o(t), \quad j \in \mathbb{Z},\ t \in (-\infty, 0], \end{aligned} \tag{2.7}$$

For any $W^o = \{w_j^o\}_{j\in\mathbb{Z}}$, $w_j^o \in C_K^+(-\infty, 0]$, $w_j^0(0) > 0,\ j \in \mathbb{Z}$, and $T \in [0, \infty]$, define the set

$$\Lambda_T = \{W = \{w_j\}_{j\in\mathbb{Z}} : w_j \in C_K^+(-\infty, T),\ w_j(t) = w_j^o(t) \text{ for } t \in (-\infty, 0]\},$$

Equip Λ_T with the norm

$$\|W\|_\lambda := \sup_{t\in[0,T), j\in\mathbb{Z}} |w_j(t)|e^{-\lambda t}.$$

Therefore, $(\Lambda_T, \|\cdot\|_\lambda)$ is a Banach space. Define the sequence of functions $S^T = \{S_j^T\}_{j\in\mathbb{Z}} \in \Lambda_T$ by

$$S_j^T[W](t) = \begin{cases} e^{-\sigma t} w_j(0) + \int_0^t e^{-\sigma(t-s)} \{D[w_{j+1}(s) + w_{j-1}(s)] \\ +\frac{1}{2\pi}\int_0^\infty f(a)e^{-da}\sum_{l=-\infty}^{\infty} \beta(a,l)b(w_{l+j}(s-a))da\}ds, & j \in \mathbb{Z},\ t \geq 0, \\ w_j^o(t), & j \in \mathbb{Z},\ t < 0. \end{cases}$$

Then $S_j^T[W](t)$ is continuous in $t \in (-\infty, T)$.

Theorem 2.2. *Suppose the initial function $W^o = \{w_j^o\}_{j\in\mathbb{Z}}$ is isotropic on interval $(-\infty, 0]$, $w_j^o \in C_K^+(-\infty, 0]$, $j \in \mathbb{Z}$, and there exists $\bar{N} \in \mathbb{N}$ such that $\operatorname{supp} W^o(t, \cdot) \subseteq B_{\bar{N}}$, $t \in (-\infty, 0]$. Then for any $c > c_*$, (2.1) holds; i.e., $\lim_{t\to\infty} \sup\{w_j(t) |\ |j| \geq ct\} = 0$.*

Proof. Define a sequence of maps by

$$W^{(n)}(t) = S^\infty[W^{(n-1)}](t) \quad \text{for } n \in \mathbb{N},\ t \in \mathbb{R}, \quad W^{(o)}(t) = \{w_j^{(o)}(t)\}_{j\in\mathbb{Z}},$$

$$w_j^{(o)}(t) = \begin{cases} w_j^o(t), & t \in (-\infty, 0], \\ w_j^o(0), & t \in (0, \infty). \end{cases}$$

Then $W^{(o)}(t)$ is isotropic on $\mathbb{R}$, and $\operatorname{supp} W^{(o)}(t, \cdot) \subset B_{\bar{N}}$ for $t \in \mathbb{R}$. Similarly to [5, Theorem 3.1], we obtain a convergent sequence in Λ_∞, which is denoted as $\{W^{(n)}(t)\}$, $t \in [0, \infty)$. Let

$$W(t) = \begin{cases} \lim_{n\to\infty} W^{(n)}(t), & t \in [0, \infty), \\ W^{(o)}(t), & t \in (-\infty, 0]. \end{cases}$$

By Lebesgue's dominated convergence theorem, (2.7) has a solution $W \in \Lambda_\infty$, which is isotropic on $\mathbb{R}$. For any $c_1 > c_*$, let $c_2 \in (c_*, c_1)$. By the assumption on $W^{(o)}$, we choose proper $N \in \mathbb{N}$ such that

$$w_j^{(o)}(t)e^{\lambda(j-c_2t)} \leq Ke^{\lambda N} \quad \text{for } t \geq 0,\ \lambda > 0,\ j \in \mathbb{Z}. \tag{2.8}$$

For $t \geq 0$, by (2.8) we have

$$
\begin{aligned}
& w_j^{(1)}(t)e^{\lambda(j-c_2t)} \\
&= e^{-(\delta+\lambda c_2)t}\Big\{w_j^{(o)}(0)e^{\lambda j} + \int_0^t e^{\delta s} D[w_{j+1}^{(o)}(s)e^{\lambda(j+1)}e^{-\lambda} + w_{j-1}^{(o)}(s)e^{\lambda(j-1)}e^{\lambda}]ds \\
&\quad + \frac{1}{2\pi}\int_0^t e^{\delta s}\int_0^\infty f(a)e^{-da}\sum_{l=-\infty}^{\infty}\beta(a,l)b(w_{l+j}(s-a))e^{\lambda(j+l)}e^{-\lambda l}da\ ds\Big\} \\
&\leq e^{-(\delta+\lambda c_2)t}\Big\{Ke^{\lambda N} + D\int_0^t Ke^{\lambda N}e^{(\delta+\lambda c_2)s}(e^{-\lambda}+e^{\lambda})ds \\
&\quad + b'(0)\Big(\int_0^\infty f(a)e^{-da}e^{2D(cosh\lambda-1)a}da\Big)Ke^{\lambda N}\int_0^t e^{(\delta+\lambda c_2)s}ds\Big\} \\
&= e^{-(\delta+\lambda c_2)t}Ke^{\lambda N}\Big\{1 + \big[D(e^{-\lambda}+e^{\lambda}) \\
&\quad + b'(0)\int_0^\infty f(a)e^{-da}e^{2D(cosh\lambda-1)a}da\big]\int_0^t e^{(\delta+\lambda c_2)s}ds\Big\} \\
&\leq Ke^{\lambda N}[1+L_{c_2}(\lambda)].
\end{aligned}
\tag{2.9}
$$

From the above inequality and by induction, we obtain

$$
w_j^{(n)}(t)e^{\lambda(j-c_2t)} \leq Ke^{\lambda N}[1+L_{c_2}(\lambda)+\dots+(L_{c_2}(\lambda))^n]. \tag{2.10}
$$

Noting $-d+b'(0)\int_0^\infty f(a)e^{-da}da > 0$, we have $L_c(0) > 1$ for $c > 0$. Since $L_c(\lambda) = 1$ has two roots for $c > c_*$, we can choose $\lambda > 0$ such that $L_{c_2}(\lambda) < 1$ for $c_2 > c_*$. Clearly the right side of (2.10) is uniformly bounded for n, thus for every $j \in \mathbb{Z}$,

$$
w_j(t) \leq \frac{Ke^{\lambda N}}{1-L_{c_2}(\lambda)}e^{\lambda(c_2t-j)} \text{ for } t \geq 0.
$$

Since W is isotropic, we have

$$
w_j(t) \leq \frac{Ke^{\lambda N}}{1-L_{c_2}(\lambda)}e^{\lambda(c_2t-|j|)} \text{ for } t \geq 0;
$$

thus,

$$
\sup\{w_j(t)|\ |j| \geq c_1t\} \leq \frac{Ke^{\lambda N}}{1-L_{c_2}(\lambda)}e^{\lambda(c_2-c_1)t} \to 0 \text{ as } t \to \infty.
$$

Hence we obtain $\lim_{t\to\infty}\sup\{w_j(t)|\ |j| \geq c_1t\} = 0$, $c_1 > c_*$. □

3. The spreading speed and minimal speed

For $\Phi \in M_\infty$, $t \geq T > 0$, $j \in \mathbb{Z}$, we define the mapping on $M_\infty = \{\Phi = \{\phi_j\}_{j\in\mathbb{Z}} : \phi_j \in C_K^+(\mathbb{R})\}$ by

$$
\begin{aligned}
E_j^T[\Phi](t) := \int_0^T e^{-\delta s}\{&D[\phi_{j+1}(t-s)+\phi_{j-1}(t-s)] \\
&+ \frac{1}{2\pi}\int_0^\infty f(a)e^{-da}\sum_{l=-\infty}^{\infty}\beta(a,l)b(\phi_{l+j}(t-s-a))da\}ds.
\end{aligned}
$$

Lemma 3.1. *Suppose $\Phi \in M_\infty$ and satisfies the following conditions:*

(i) *for any $t' > 0$, there exists an $N = N(t') \in \mathbb{N}$ such that for any $t \in [0,t']$, $supp\Phi(t,\cdot) \subset B_N$;*

(ii) *if* $\{(t_n, j_n)\}_{n=1}^{\infty} \subset \mathbb{R}_+ \times \mathbb{Z}$, $j_n \in \operatorname{supp}\Phi(t_n, \cdot)$, *and* $\lim_{n\to\infty}(t_n, j_n) = (t_0, j_0)$, *then* $j_0 \in supp\Phi(t_0, \cdot)$.

For such Φ, *assume that*

$$E^T[\Phi](t) \succ \Phi(t) \quad \text{for } t \geq T, \tag{3.1}$$

and the solution of (1.1) *satisfies*

$$W(\bar{t}+t) \succ \Phi(t) \quad \text{for } t \in (-\infty, T] \tag{3.2}$$

for some $\bar{t} \geq 0$. *Then*

$$W(\bar{t}+t) \succ \Phi(t) \quad \text{for } t \in [0, \infty). \tag{3.3}$$

Proof. Let

$$t_0 = \sup\{t \geq T : W(\bar{t}+t) \succ \Phi(t)\} \geq T. \tag{3.4}$$

If $t_0 < \infty$, since $W(t)$ is non-negative, there exists $\{(t_n, j_n)\}_{n=1}^{\infty}$ such that

(a) $t_n \downarrow t_0$, $n \to \infty$,
(b) $j_n \in \operatorname{supp}\Phi(t_n, \cdot)$,
(c) $w_{j_n}(\bar{t}+t_n) \leq \phi_{j_n}(t_n)$.

By assumption (i), $\{j_n\}$ must be bounded. Thus $\{j_n\}$ is composed of finite integers and contains a convergent sub-sequence, which is a constant sequence $\{j_0\}$. From (b) and (c), we know that $j_0 \in supp\Phi(t_0, \cdot)$ and $w_{j_0}(\bar{t}+t_0) \leq \phi_{j_0}(t_0)$. For $t_0 \geq T$ and $\bar{t} \geq 0$, from (2.7) and (3.4) we have

$$\begin{aligned}
w_{j_0}(\bar{t}+t_0) &\geq \int_0^T e^{-\delta s}\{D[w_{j_0+1}(\bar{t}+t_0-s) + w_{j_0-1}(\bar{t}+t_0-s)] \\
&\quad + \frac{1}{2\pi}\int_0^{\infty} f(a)e^{-da}\sum_{l=-\infty}^{\infty}\beta(a,l)b(w_{j_0+l}(\bar{t}+t_0-s-a))da\}ds \\
&\geq \int_0^T e^{-\delta s}\{D[\phi_{j_0+1}(t_0-s) + \phi_{j_0-1}(t_0-s)] \\
&\quad + \frac{1}{2\pi}\int_0^{\infty} f(a)e^{-da}\sum_{l=-\infty}^{\infty}\beta(a,l)b(\phi_{l+j_0}(t_0-s-a))da\}ds \\
&= E_{j_0}^T[\Phi](t_0) > \phi_{j_0}(t_0).
\end{aligned}$$

Since $w_{j_0}(\bar{t}+t_0) \leq \phi_{j_0}(t_0)$, the above inequality is a contradiction. Thus we have $t_0 = \infty$. □

Define $K_c = K_c(h, T, N, \lambda)$ by

$$\begin{aligned}
&K_c(h,T,N,\lambda) \\
&= \int_0^T e^{-(\delta+\lambda c)s}\{D[e^{-\lambda}+e^{\lambda}] + \frac{h}{2\pi}\int_0^T f(a)e^{-da}\sum_{|l|\leq N}\beta(a,l)e^{\lambda l-\lambda ca}da\}ds \\
&= \frac{1-e^{-(\delta+\lambda c)T}}{\delta+\lambda c}\{D[e^{-\lambda}+e^{\lambda}] + \frac{h}{2\pi}\int_0^T f(a)e^{-da}\sum_{|l|\leq N}\beta(a,l)e^{\lambda l-\lambda ca}da\}.
\end{aligned} \tag{3.5}$$

Lemma 3.2. *For any* $c \in (0, c_*)$, *there exist* $h \in (0, b'(0)), T > 0$ *and* $N \in \mathbb{N}$ *such that*

$$K_c(h,T,N,\lambda) > 1 \quad \text{for } \lambda \in \mathbb{R}. \tag{3.6}$$

Proof. From the definition of $K_c(h,T,N,\lambda)$, we have
$$K_c(h,T,N,-\lambda) \geq K_c(h,T,N,\lambda), \quad \lambda \geq 0.$$
We claim that
$$K_c(h,T,N,\lambda) > 1 \quad \text{for } \lambda \geq 0.$$
We first show that there exist $N_0 > 0, \lambda_0 > 0, h_0 \in (0, b'(0))$ and $T_0 > 0$ such that
$$K_c(h,T,N,\lambda) > 1 \quad \text{for } \lambda \geq \lambda_0,\ N \geq N_0,\ h \geq h_0,\ T \geq T_0.$$
However, we can choose proper $N_0 > 0$ and $h_0 \in (0, b'(0))$ such that for all $T > 0$, $N \geq N_0$ and $h \geq h_0$,
$$\frac{h}{2\pi} \int_0^T f(a) e^{-da} \sum_{l=-N}^{N} \beta(a,l) e^{\lambda(l-ca)} da > 0$$
holds uniformly for $\lambda \geq 0$. Since
$$\lim_{\lambda\to\infty} \frac{e^\lambda}{\lambda c_* + \delta} = \infty,$$
we can choose $T_0 > 0$ and $\lambda_0 > 0$ such that
$$1 - e^{-(\lambda c+\delta)T} \geq 1 - e^{-\delta T} \geq 1 - e^{-\delta T_0} > 0,$$
$$\frac{D}{\lambda c + \delta}(1 - e^{-\delta T_0}) e^\lambda > \frac{D}{\lambda_0 c_* + \delta}(1 - e^{-\delta T_0}) e^{\lambda_0} \geq 1,$$
for $T \geq T_0$, $\lambda \geq \lambda_0$. For any $N \geq N_0, T \geq T_0, h \geq h_0$ and $\lambda \geq \lambda_0$, we have
$$K_c(h,T,N,\lambda) > \frac{D}{\lambda_0 c_* + \delta}(1 - e^{-\delta T_0}) e^{\lambda_0} \geq 1.$$
If (3.6) is not true, there exist $\{h_n\}, \{T_n\}, \{\lambda_n\}, \{N_n\}$ such that $h_n \uparrow b'(0)$, $T_n \uparrow \infty$, $N_n \uparrow \infty$, $\{\lambda_n\} \subset [0, \lambda_0]$ and
$$K_c(h_n, T_n, N_n, \lambda_n) \leq 1, \quad n = 1, 2, \dots.$$
Since $\{\lambda_n\}$ is bounded, we choose a convergent sub-sequence $\{\lambda_{n_k}\}$. Obviously $\{\lambda_{n_k}\}$ has a finite limit, denotes as $\tilde{\lambda}$. By Fatou's lemma, we have
$$1 < L_c(\tilde{\lambda}) \leq \liminf_{k\to\infty} K_c(h_{n_k}, T_{n_k}, N_{n_k}, \lambda_{n_k}) \leq 1,$$
which is a contradiction. Hence (3.6) is true. □

Define a function
$$q(y;\omega,\zeta) = \begin{cases} e^{-\omega y}\sin(\zeta y), & y \in [0, \frac{\pi}{\zeta}], \\ 0, & y \in \mathbb{R}/[0, \frac{\pi}{\zeta}]. \end{cases}$$

Lemma 3.3. *Suppose $c \in (0, c_*)$. Then there exist $\zeta_0 > 0$, a continuous function $\omega = \omega(\zeta)$ defined on $[0, \zeta_0]$, and a positive number $\delta_1 \in (0,1)$ such that*
$$\begin{aligned} &\int_0^T e^{-\delta s}\big\{D[q(m+cs+1) + q(m+cs-1)] \\ &+ \frac{h}{2\pi}\int_0^T f(a)e^{-da}\sum_{|l|\leq N}\beta(a,l)q(m+l+cs+ca)da\big\}ds \geq q(m-\delta_1), \end{aligned} \tag{3.7}$$
for $m \in \mathbb{Z}$, where $q(y) = q(y;\omega(\zeta),\zeta)$.

Proof. Define

$$\begin{aligned} L(\lambda) = &\int_0^T e^{-\delta s}\big\{D[e^{-\lambda(cs+1)} + e^{-\lambda(cs-1)}] \\ &+ \frac{h}{2\pi}\int_0^T f(a)e^{-da}\sum_{|l|\le N}\beta(a,l)e^{-\lambda(l+cs+ca)}da\big\}ds, \end{aligned}$$

where T, h, N are defined in Lemma 3.2. By Lemma 3.2, for sufficiently large N,

$$L(\lambda) = K_c(h,T,N,\lambda) > 1 \quad \text{for } \lambda \in \mathbb{R}. \tag{3.8}$$

Let $\lambda = \omega + i\zeta$, then we have

$$L(\lambda)|_{\lambda=\omega+i\zeta} = \operatorname{Re}[L(\lambda)] + i\ \operatorname{Im}[L(\lambda)],$$

where

$$\begin{aligned} &\operatorname{Re}[L(\lambda)] \\ &= D\int_0^T e^{-\delta s}\Big\{e^{-\omega(cs+1)}\cos\zeta(cs+1) + e^{-\omega(cs-1)}\cos\zeta(cs-1)\Big\}ds \\ &\quad + \frac{h}{2\pi}\int_0^T f(a)e^{-da}\sum_{|l|\le N}\beta(a,l)\Big\{\int_0^T e^{-\delta s}e^{-\omega(l+cs+ca)}\cos\zeta(l+cs+ca)ds\Big\}da, \\ &\operatorname{Im}[L(\lambda)] \\ &= -D\int_0^T e^{-\delta s}\Big\{e^{-\omega(cs+1)}\sin\zeta(cs+1) + e^{-\omega(cs-1)}\sin\zeta(cs-1)\Big\}ds \\ &\quad - \frac{h}{2\pi}\int_0^T f(a)e^{-da}\sum_{|l|\le N}\beta(a,l)\Big\{\int_0^T e^{-\delta s}e^{-\omega(l+cs+ca)}\sin\zeta(l+cs+ca)ds\Big\}da. \end{aligned}$$

Since $L''(\lambda) > 0$ and $\lim_{|\lambda|\to\infty} L(\lambda) = \infty$ for $\lambda \in \mathbb{R}$, $L(\lambda)$ attains the minimal value at $\lambda = \theta \in \mathbb{R}$. Thus,

$$\begin{aligned} L'(\theta) = &-D\int_0^T e^{-\delta s}[(cs+1)e^{-\theta(cs+1)} + (cs-1)e^{-\theta(cs-1)}]ds \\ &- \frac{h}{2\pi}\int_0^T f(a)e^{-da}\sum_{|l|\le N}\beta(a,l)\big[\int_0^T e^{-\delta s}(l+cs+ca)e^{-\theta(l+cs+ca)}ds\big]da = 0. \end{aligned}$$

Define a function $H = H(\omega,\zeta)$ by

$$\begin{gathered} H(\omega,\zeta) = \frac{1}{\zeta}\operatorname{Im}[L(\lambda)] \qquad \text{for } \zeta \ne 0, \\ H(\omega,0) = \lim_{\zeta\to 0} H(\omega,\zeta) = L'(\omega). \end{gathered}$$

Obviously $H(\theta,0) = 0$ and $\frac{\partial H}{\partial\omega}(\theta,0) = L''(\theta) > 0$. By implicit function theorem, there exist $\zeta_1 > 0$ and continuous function $\omega = \omega(\zeta), \zeta) \in [0,\zeta_1]$ satisfying $\omega(0) = \theta$, and $H(\omega(\zeta),\zeta) = 0,\ \zeta \in [0,\zeta_1]$. Thus,

$$\operatorname{Im}[L(\lambda)]|_{\lambda=\omega(\zeta)+i\zeta} = 0, \quad \zeta \in [0,\zeta_1]. \tag{3.9}$$

By (3.5) and (3.9), we have

$$\operatorname{Re}[L(\omega+i\zeta)]|_{\omega=\theta,\zeta=0} = L(\theta) > 1.$$

Then there exists $\zeta_2 > 0$ such that

$$\operatorname{Re}[L(\omega(\zeta)+i\zeta)] > 1, \quad \zeta \in [0,\zeta_2]. \tag{3.10}$$

Let $0 < \zeta \le \zeta_0 := \min\{\zeta_1, \zeta_2, \frac{\pi}{N+2c_*T}\}$. For $m \in [0, \frac{\pi}{\zeta}]$, $|l| \le N$ and $a, s \in [0,T]$,

$$-\frac{\pi}{\zeta} < -N \le l \le m + l + cs + ca \le m + l + 2cT < N + 2c_*T + \frac{\pi}{\zeta} \le \frac{2\pi}{\zeta}.$$

Thus,

$$\sin\zeta(m+l+c(s+a)) < 0, \quad \text{for } m+l+c(s+a) \in (-\frac{\pi}{\zeta}, 0) \cup (\frac{\pi}{\zeta}, \frac{2\pi}{\zeta}). \tag{3.11}$$

From the definition of $q(\cdot)$ we obtain

$$\begin{aligned}
&\int_0^T e^{-\delta s}\{D[q(m+cs+1)+q(m+cs-1)] \\
&\quad + \frac{h}{2\pi}\int_0^T f(a)e^{-da}\sum_{|l|\le N}\beta(a,l)q(m+l+cs+ca)da\}ds \\
&\ge D\int_0^T e^{-\delta s}\Big\{e^{-\omega(\zeta)(m+cs+1)}\sin(\zeta(m+cs+1)) \\
&\quad + e^{-\omega(\zeta)(m+cs-1)}\sin(\zeta(m+cs-1))\Big\}ds \\
&\quad + \frac{h}{2\pi}\int_0^T e^{-\delta s}\int_0^T f(a)e^{-da} \\
&\quad \times \sum_{|l|\le N}\beta(a,l)e^{-\omega(\zeta)(m+l+cs+ca)}\sin(\zeta(m+l+cs+ca))da\ ds.
\end{aligned} \tag{3.12}$$

Using $\sin(A+B) = \sin A\cos B + \sin B\cos A$ and (3.10)-(3.12), we have

$$\begin{aligned}
&\int_0^T e^{-\delta s}\{D[q(m+cs+1)+q(m+cs-1)] \\
&\quad + \frac{h}{2\pi}\int_0^T f(a)e^{-da}\sum_{|l|\le N}\beta(a,l)q(m+l+cs+ca)da\}ds \\
&\ge e^{-\omega(\zeta)m}\sin(\zeta m)\operatorname{Re}[L(\lambda)]|_{\lambda=\omega(\zeta)+i\zeta} + e^{-\omega(\zeta)m}\cos(\zeta m)\operatorname{Im}[L(\lambda)]|_{\lambda=\omega(\zeta)+i\zeta} \\
&= e^{-\omega(\zeta)m}\sin(\zeta m) = q(m).
\end{aligned} \tag{3.13}$$

Choose N large enough such that $-N + 2c_*T < 0$, thus (3.12) and (3.13) are strict inequalities on $m \in (0, \frac{\pi}{\zeta})$. Moreover, from (3.11)-(3.12), we know that (3.13) is also a strict inequality for $m = 0$ or $m = \frac{\pi}{\zeta}$. In fact, let $a, s \in [0,T]$, $m = \frac{\pi}{\zeta}$ and $l = N$, then

$$m + l + c(s+a) > \frac{\pi}{\zeta}.$$

Similarly, if $m = 0$ and $l = -N$, then $m + l + c(s+a) < -N + 2c_*T < 0$. Thus for both cases, we have

$$q(m+l+cs+ca) = 0 \quad \text{and} \quad \sin(\zeta(m+l+cs+ca)) < 0,$$

which means (3.13) is a strict inequality for $m = 0$ or $m = \frac{\pi}{\zeta}$. Then for any $m \in [0, \frac{\pi}{\zeta}]$,

$$\begin{aligned} &\int_0^T e^{-\delta s}\{D[q(m+cs+1)+q(m+cs-1)] \\ &+ \frac{h}{2\pi}\int_0^T f(a)e^{-da}\sum_{|l|\le N}\beta(a,l)q(m+l+cs+ca)da\}ds > q(m). \end{aligned} \tag{3.14}$$

If $m \notin [0, \frac{\pi}{\zeta}]$, (3.14) still holds since $q(m) = 0$. From the above discussion, we know that (3.14) holds for $m \in \mathbb{R}$, then (3.7) follows from the continuity consideration. □

Consider the family of functions,

$$\begin{aligned} R(y;\omega,\zeta,\gamma) : &= \max_{\eta \ge -\gamma} q(y+\eta;\omega,\zeta) \\ &= \begin{cases} M, & y \le \gamma+\rho, \\ q(y-\gamma;\omega,\zeta), & \gamma+\rho \le y \le \gamma+\frac{\pi}{\zeta}, \\ 0, & y \ge \gamma+\frac{\pi}{\zeta}, \end{cases} \end{aligned} \tag{3.15}$$

where

$$M = M(\omega,\zeta) := \max\{q(y;\omega,\zeta)|\ 0 \le y \le \frac{\pi}{\zeta}\}. \tag{3.16}$$

We assume M attain the maximum at $\rho = \rho(\omega,\zeta)$. The following lemma gives a sub-solution of (1.1).

Lemma 3.4. *Let $c \in (0, c_*)$ be given, then there exist $T > 0, \zeta > 0, \omega \in \mathbb{R}, \vartheta > 0$ and $\sigma_0 > 0$ such that for $\sigma \in (0, \sigma_0)$ and $t \ge T$, there holds*

$$E^T[\sigma\Phi](t) \succ \sigma\Phi(t) \quad \text{for } t \ge T, \tag{3.17}$$

where $\Phi(t) = \{\phi_j(t)\}_{j\in\mathbb{Z}}, \phi_j(t) = R(|j|;\omega,\zeta,\vartheta+ct)$.

Proof. Let $h \in (0, b'(0)), T > 0, N > 0$ be chosen such that $K_c(h,T,N,\lambda) > 1$ for $\lambda \in \mathbb{R}$. By Lemma 3.3, we can choose $\zeta > 0, \omega = \omega(\zeta)$ and $\delta_1 \in (0,1)$ such that (3.7) holds.

Let σ_h be the smallest positive root of the equation $b(w) = hw$, then $b(w) > hw$ for $w \in (0, \sigma_h)$. Choose $\sigma_0 \in (0, \sigma_h M^{-1})$, where M is defined in (3.16). For $\sigma \in (0, \sigma_0)$ and $t \ge T$, we have

$$\begin{aligned} E_j^T[\sigma\Phi](t) &= \int_0^T e^{-\delta s}\big\{D\sigma[\phi_{j+1}(t-s)+\phi_{j-1}(t-s)] \\ &\quad + \frac{1}{2\pi}\int_0^\infty f(a)e^{-da}\sum_{l=-\infty}^{\infty}\beta(a,l)b(\sigma\phi_{j+l}(t-s-a))da\big\}ds \\ &\ge \int_0^T e^{-\delta s}\big\{D\sigma[\phi_{j+1}(t-s)+\phi_{j-1}(t-s)] \\ &\quad + \frac{1}{2\pi}\int_0^T f(a)e^{-da}\sum_{|l|\le N}\beta(a,l)b(\sigma\phi_{j+l}(t-s-a))da\big\}ds. \end{aligned} \tag{3.18}$$

For any given $\vartheta > 0$, we consider two cases.

Case (i) $|j| \le \vartheta + \rho + c(t-2T) - N$. For $|l| \le N, a, s \in [0,T]$, then

$$|l+j| \le \vartheta + \rho + c(t-2T) \le \vartheta + \rho + c(t-s-a)$$

Since the definition of $E_j^T[\Phi](t)$ and $b(\sigma\phi_{j+l}(t-s-a)) = b(\sigma M) > h\sigma M$, we have

$$\begin{aligned} E_j^T[\sigma\Phi](t) &\ge \big\{2D\sigma M + \frac{1}{2\pi}\int_0^T f(a)e^{-da}\sum_{l=-N}^{N}\beta(a,l)b(\sigma M)da\big\}\int_0^T e^{-\delta s}ds \\ &> \sigma M K_c(h,T,N,0) > \sigma M. \end{aligned} \tag{3.19}$$

Case (ii) $\vartheta + \rho + c(t-2T) - N \le |j| \le \frac{\pi}{\zeta} + \vartheta + ct$. Let $|l| \le N,\ t \ge T$. If $\vartheta \ge \frac{N^2}{2\delta_1} - \rho + cT + N$ (δ_1 is defined in Lemma 3.3), then

$$\begin{aligned} |l+j| = (l^2 + 2lj + j^2)^{1/2} &\le |j| + \frac{lj}{|j|} + \frac{l^2}{2|j|} \\ &\le |j| + \frac{lj}{|j|} + \frac{N^2}{2|j|} \\ &\le |j| + \frac{lj}{|j|} + \frac{N^2}{2(\vartheta+\rho-cT-N)} \le |j| + \frac{lj}{|j|} + \delta_1. \end{aligned}$$

Since $\phi_j(t)$ is non-decreasing for $|j|$, by (3.18) we obtain

$$\begin{aligned} &E_j^T[\sigma\Phi](t) \\ &\ge \int_0^T e^{-\delta s}\Big\{D\sigma[\max_{\eta\ge-\vartheta-c(t-s)} q(|j|+1+\delta_1+\eta) \\ &\quad + \max_{\eta\ge-\vartheta-c(t-s)} q(|j|-1+\delta_1+\eta)] \\ &\quad + \frac{h\sigma}{2\pi}\int_0^T f(a)e^{-da}\sum_{|l|\le N}\beta(a,l)\max_{\eta\ge-\vartheta-c(t-s-a)} q(|j|+l+\delta_1+\eta)da\Big\}ds \\ &= \sigma\int_0^T e^{-\delta s}\Big\{D[\max_{\eta\ge-\vartheta-ct} q(|j|+1+cs+\delta_1+\eta) \\ &\quad + \max_{\eta\ge-\vartheta-ct} q(|j|-1+cs+\delta_1+\eta)] \\ &\quad + \frac{h}{2\pi}\int_0^T f(a)e^{-da}\sum_{|l|\le N}\beta(a,l)\max_{\eta\ge-\vartheta-ct} q(|j|+l+cs+ca+\delta_1+\eta)da\Big\}ds \\ &\ge\sigma\max_{\eta\ge-\vartheta-ct} q(|j|+\eta). \end{aligned}$$

Combining (i) and (ii), we obtain (3.17) and complete the proof. □

The proof of the following lemma is similar to [10, Lemma 5.5], and hence is omitted.

Lemma 3.5. *Assume that $W = \{w_j\}_{j\in\mathbb{Z}}$ is a solution of (1.1), and the following conditions hold:*

(i) *$W^o = \{w_j^o\}_{j\in\mathbb{Z}}$ is isotropic on $(-\infty,0]$, $w_j^o \in C_K^+(-\infty,0]$;*
(ii) *there exists $N_1 \in \mathbb{N}$ such that $\operatorname{supp} W^o(t,\cdot) \subset B_{N_1}$ for $t \in (-\infty,0]$, $w_j^o(0) > 0$ for $j| \le N_1$.*

Then there exists $t_0 > 0$ such that $w_j(t) > 0$ for $t \in [t_0,\infty), j \in \mathbb{Z}$.

Lemma 3.6. *Let $\{Q_n(t,N)\}$ be defined by $Q_1(t,N) \equiv a \in (0,w^+)$,*

$$\begin{aligned} Q_{n+1}(t,N) &= \frac{1}{\delta}\Big[2DQ_n(t,N) \\ &\quad + \frac{1}{2\pi}\int_0^T \{f(a)e^{-da}\sum_{|l|\le N}\beta(a,l)da\}b(Q_n(t,N))\Big](1-e^{-\delta t}) \end{aligned} \tag{3.20}$$

for $n=1,2,\dots$. Then for $\epsilon>0$, there exist $\bar t(\epsilon),\bar N(\epsilon),\bar T(\epsilon)$ and $\bar n(\epsilon)$ such that for any $T\ge \bar T(\epsilon), t\ge \bar t(\epsilon), N\ge \bar N(\epsilon)$ and $n\ge \bar n(\epsilon)$,

$$Q_n(t,N)\ge w^+-\epsilon.$$

Proof. Since

$$\frac{2Dw^+ + b(w^+)\tilde f(d)}{\delta} = w^+,\quad \delta = 2D+d,\quad \tilde f(d)=\int_0^\infty f(a)e^{-da}da,$$
$$0<Q_1(t,N)<w^+,\quad 0<\frac{1}{\delta}(1-e^{-\delta t})<1,\quad 0<\frac{1}{2\pi}\sum_{|l|\le N}\beta(a,l)<1,$$

we have by induction that $0<Q_n(t,N)\le K$ for any $n\in\mathbb{N}$, $t\ge 0$ and $N\in\mathbb{N}$. By (H3), $2Dw+\tilde f(d)b(w)>(2D+d)w$, for $0<w<w^+$. For $\epsilon>0$, we have

$$\sup\Big\{\frac{2Dw+\tilde f(d)b(w)}{(2D+d)w}\mid 0<w\le w^+-\epsilon\Big\}>1.$$

Let $\tilde f_T(d)=\int_0^T f(a)e^{-da}da$. Choose large enough $\alpha(\epsilon)<1$, $\bar T=\bar T(\epsilon)$ such that for $0<w\le w^+-\epsilon, T\ge\bar T$, there holds

$$\alpha(\epsilon)\Big[2Dw+\tilde f_T(d)b(w)\Big]>(2D+d)w. \tag{3.21}$$

Define the sequence:

$$M_1\equiv a,\quad M_{n+1}=\frac{\alpha(\epsilon)}{\delta}\Big[2DM_n+\tilde f_T(d)b(M_n)\Big]\ \text{for } n\ge 2.$$

Obviously,

(i) if $0<M_n\le w^+-\epsilon$, then $M_{n+1}\ge M_n$;
(ii) if $M_n>w^+-\epsilon$, then

$$M_{n+1}>\frac{\alpha(\epsilon)}{\delta}\Big[2D(w^+-\epsilon)+\tilde f_T(d)b(w^+-\epsilon)\Big]\ge w^+-\epsilon.$$

Now we show that $M_n>w^+-\epsilon$ for sufficiently large n. If that is not true, we can assume that $M_n\le w^+-\epsilon$ holds for all n. By (i), we know that $\lim_{n\to\infty}M_n=M\le w^+-\epsilon$ exists and satisfies

$$M=\frac{\alpha(\epsilon)}{\delta}[2DM+b(M)\tilde f_T(d)].$$

which is a contraction to (3.21). Thus there exists $\bar n(\epsilon)>0$ such that $M_n>w^+-\epsilon$ for any $n>\bar n(\epsilon)$.

Let $T\ge\bar T=\bar T(\epsilon)$. We choose $\bar t=\bar t(\epsilon)$ and $\bar N=\bar N(\epsilon)$ such that $1-e^{-\delta\bar t(\epsilon)}\ge\alpha(\epsilon)$ and

$$\frac{1}{2\pi}(1-e^{-\delta\bar t(\epsilon)})\int_0^T\{f(a)e^{-da}\sum_{|l|\le\bar N}\beta(a,l)\}da\ge\alpha(\epsilon)\tilde f_T(d). \tag{3.22}$$

Then $Q_1(t,N) = a \ge M_1$ for $t \ge \bar{t}(\epsilon), T \ge \bar{T}(\epsilon)$ and $N \ge \bar{N}(\epsilon)$. By (3.22) we obtain

$$\begin{aligned}
&Q_{n+1}(t,N)\\
&\ge \frac{1}{\delta}(1-e^{-\delta\bar{t}(\epsilon)})\Big[2DQ_n(t,N) + \frac{b(Q_n(t,N))}{2\pi}\int_0^T \{f(a)e^{-da}\sum_{|l|\le N}\beta(a,l)\}da\Big]\\
&> \frac{1}{\delta}\Big[2D\alpha(\epsilon)Q_n(t,N) + \alpha(\epsilon)\tilde{f}_T(d)b(Q_n(t,N))\Big]\\
&= \frac{\alpha(\epsilon)}{\delta}\Big[2DQ_n(t,N) + \tilde{f}_T(d)b(Q_n(t,N))\Big].
\end{aligned}$$

Using monotonicity of b, we have $Q_n(t,N) \ge M_n \ge w^+ - \epsilon$ for $n > \bar{n}(\epsilon)$. □

Theorem 3.7. *Assume all the conditions for W^o in Lemma 3.5 are satisfied. Then for any $c \in (0, c_*)$, there holds*

$$\lim_{t\to\infty} \inf\{w_j(t) : |j| \le ct\} \ge w^+.$$

Proof. Let $c_1 \in (0, c_*)$, $c_2 \in (c_1, c_*)$. From Lemma 3.4, there exist $T > 0, \zeta > 0, \omega \in \mathbb{R}, \vartheta > 0$ and $\sigma_0 > 0$ such that for $\sigma \in (0, \sigma_0)$ $t \ge T$,

$$E^T[\sigma\Phi](t) \succ \sigma\Phi(t),$$

where $\Phi(t) = \{\phi_j(t)\}_{j\in\mathbb{Z}}, \phi_j(t) := R(|j|;\omega,\zeta,\vartheta + c_2T)$. We can assume $T \ge \bar{T}$, and $\bar{T}$ is defined in Lemma 3.6. From Lemma 3.5, there exists $t_0 > 0$ such that

$$w_j(t) > 0 \quad \text{for } t \in [t_0, t_0 + T],\ j \in \mathbb{Z}.$$

Since $\Phi(t)$ is a bounded function, we can choose $\sigma_1 \in (0, \sigma_0)$ such that

$$\sigma_1 M < w^+, \quad w_j(t_0 + t) > \sigma_1\phi_j(t) \quad \text{for } t \in [0, T],\ j \in \mathbb{Z}.$$

Using the comparison principle (Lemma 3.1), we have

$$w_j(t_0 + t) > \sigma_1\phi_j(t) \quad \text{for } t \in [0,\infty),\ j \in \mathbb{Z}. \tag{3.23}$$

From (3.23) and definition of $\phi_j(t)$, we have

$$w_j(t_0 + t) \ge \sigma_1 M, \quad t \ge 0,\ |j| \le \rho + \vartheta + c_2t. \tag{3.24}$$

By (2.7), we have

$$\begin{aligned}
w_j(t_0+t) \ge \int_0^t e^{-\delta s}\{&D[w_{j+1}(t_0+t-s) + w_{j-1}(t_0+t-s)]\\
&+ \frac{1}{2\pi}\int_0^T f(a)e^{-da}\sum_{|l|\le N}\beta(a,l)b(w_{l+j}(t_0+t-s-a))da\}ds.
\end{aligned}\tag{3.25}$$

Let $a = \sigma_1 M = Q_1(t,N)$, and $Q_n(t,N)$ be defined in Lemma 3.6. From (3.24)-(3.25), we have by induction

$$w_j(t_0 + t) \ge Q_n(t,N), \quad t \ge 0, |j| \le \rho + \vartheta + c_2t - n(N + T).$$

For any $\epsilon > 0$, we choose $\bar{t}(\epsilon), \bar{T}(\epsilon), \bar{N}(\epsilon)$ and $\bar{n}(\epsilon)$ such that

$$w_j(t) \ge w^+ - \epsilon, \quad t \ge t_0 + \bar{t}(\epsilon), \quad |j| \le \rho + \vartheta + c_2(t - t_0) - \bar{n}(\epsilon)(\bar{N}(\epsilon) + \bar{T}(\epsilon)). \tag{3.26}$$

Define

$$t_1 := \max\Big\{t_0 + \bar{t}(\epsilon), \frac{\bar{n}(\epsilon)[\bar{N}(\epsilon) + \bar{T}(\epsilon)] + c_2t_0 - \rho - \vartheta}{c_2 - c_1}\Big\}.$$

Since $c_2 > c_1$ and (3.26), we obtain

$$w_j(t) \geq w^+ - \epsilon \quad \text{for } t \geq t_1, |\, j| \leq c_1 t.$$

Then (2.2) holds. □

The following theorem shows the relation between the minimal wave speed and the spreading speed.

Theorem 3.8. *Assume* (H1)–(H4) *are satisfied. Then lattice system* (1.1) *admits two equilibria, $W = 0$ and $W = w^+ > 0$. Further, for $c \geq c_*$, Equation* (1.1) *has a monotone traveling wave satisfying*

$$\lim_{s \to -\infty} \phi(s) = 0, \quad \lim_{s \to \infty} \phi(s) = w^+. \tag{3.27}$$

For $c \in (0, c_*)$, (1.1) *has no monotone traveling wave satisfying* (3.27).

Proof. From [5, Theorem 5.1], we have that (1.1) admits monotone traveling wave satisfying (3.27) for $c > c_*$, thus we only need to claim the case as $c = c_*$.

Choose a sequence $\{c_n\} \in (c_*, c_* + 1]$ such that $c_{n+1} > c_n$ and $\lim_{n\to\infty} c_n = c_*$. Then the wave equation (2.1) admits a wavefront connecting 0 with w^+, say $\phi_n(j + c_n t)$, which has the speed c_n. It is easy to see $0 < \phi_n(j + c_n t) < w^+$, and

$$\begin{aligned} c\phi_n'(s) &= D[\phi_n(s+1) + \phi_n(s-1) - 2\phi_n(s)] - d\phi_n(s) \\ &\quad + \frac{1}{2\pi} \int_0^\infty f(a) e^{-da} \sum_{l=-\infty}^{\infty} \beta(a,l) b(\phi_n(s + l - c_n a)) da. \end{aligned} \tag{3.28}$$

Since (3.28) is a homogeneous system, from the basis theory of differential equation, we know that a traveling wave of (3.28) is still another traveling wave after sliding. Without generality, we assume $\phi_n(0) = \frac{w^+}{2}$.

Differentiating (3.28) with respect to s, we obtain

$$\begin{aligned} c\phi_n''(s) &= D[\phi_n'(s+1) + \phi_n'(s-1) - 2\phi_n'(s)] - d\phi_n'(s) \\ &\quad + \frac{1}{2\pi} \int_0^\infty f(a) e^{-da} \sum_{l=-\infty}^{\infty} \beta(a,l) \frac{db}{dw}(\phi_n(s + l - c_n a)) \phi_n'(s + l - c_n a) da. \end{aligned} \tag{3.29}$$

From (3.28) and $0 < \phi_n(j + c_n t) < w^+$, there exists M_1, M_2 such that $|\phi_n'(s)| \leq M_1$, $|\phi_n''(s)| \leq M_2$ for $s \in \mathbb{R}$. Thus ϕ_n and ϕ_n' are uniformly bounded, equicontinuous in $\mathbb{R}$. According to Arzela-Ascoli theorem, there has a sub-sequence of c_n, still denoted as c_n, such that $\phi_n(s)$ and $\phi_n'(s)$ are convergent to limits in every bounded and closed subset in $\mathbb{R}$. We denote the limits as $\phi_*(s), \phi_*'(s)$ respectively.

Let $n \to \infty$ in (3.28). By Lebesque's dominated convergence theorem, we have

$$\begin{aligned} c\phi_*'(s) &= D[\phi_*(s+1) + \phi_*(s-1) - 2\phi_*(s)] - d\phi_*(s) \\ &\quad + \frac{1}{2\pi} \int_0^\infty f(a) e^{-da} \sum_{l=-\infty}^{\infty} \beta(a,l) b(\phi_*(s + l - c_* a)) da. \end{aligned} \tag{3.30}$$

Hence $\phi_*(j + c_* t)$ is the traveling wavefront of (1.1) with speed c_* satisfying (3.1).

Now we prove (1.1) admits no traveling wavefront for $c_1 \in (0, c_*)$. Suppose that is not true, and system (1.1) has monotone traveling wave $\phi(s) = \phi(j + c_1 t)$ satisfying (3.27). Thus there exists $s_1 > 0$ such that $\phi(s) > \frac{w^+}{2}$ for $s \geq s_1$. Choose proper initial function: $w_j^o(t) = \phi(j + c_1 t)$, $t \in (-\infty, 0]$, and $\{w_j^o(t)\}_{j\in\mathbb{Z}} \in C_K^+(-\infty, 0]$.

Let $\{w_j(t) = \phi(j + c_1t)\}_{j\in\mathbb{Z}}$ be a solution of (1.1) with initial value $w_j^o(t)$. Noting $\{w_j^o(t)\}_{j\in\mathbb{Z}}$ satisfying conditions in Theorem 3.7, we have

$$\lim_{t\to\infty} \inf\{w_j(t)|\ |j| \le ct\} = \lim_{t\to\infty} \inf\{\phi(j + c_1t)\ |j| \le ct\} \ge w^+ \quad \text{for } c \in (0, c_*).$$

Choose $c_2 \in (c_1, c_*)$, $j = -c_2t$, then

$$\phi(j + c_1t) = \phi((c_1 - c_2)t) \ge w^+ \quad \text{for } t \ge t_1.$$

Let $t \to \infty$, we have

$$\lim_{t\to\infty} \inf\{\phi(j + c_1t)\ |j = -c_2t\} = \lim_{t\to\infty} \inf\{\phi((c_1 - c_2)t)\} \ge w^+,$$

which leads to a contradiction to the first equality in (3.27). Hence (1.1) admits no monotone traveling wave for $c_1 \in (0, c_*)$. □

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