TRAVELING WAVES AND SPREADING SPEED ON A LATTICE MODEL WITH AGE STRUCTURE

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Abstract. In this article, we study a lattice differential model for a single species with distributed age-structure in an infinite patchy environment. Using method of approaches by Diekmann and Thieme, we develop a comparison principle and construct a suitable sub-solution to the given model, and show that there exists a spreading speed of the system which in fact coincides with the minimal wave speed.

1. Introduction

Assume \( u(t, a, x) \) is the population density at time \( t \), age \( a \) and spatial location \( x \), and \( x \) denotes the point coordinate which may be an integer, in \( \mathbb{Z} \), or real number in \( \mathbb{R} \). We study the species in a patchy environment with infinite number of patches connected by diffusion of population within the neighboring islands, where we can describe the patches as integer nodes of a one-dimensional lattice. In this case we change \( x \) to \( j \), and let \( u(t, a, j) = u_j(t, a) \) denote the population density of the species at \( j \)-th patch. Let \( f(r) \) be a probability density function which specifies the probability of maturing of an individual with age \( a \geq r \). This function satisfies \( f(0) = 0 \), \( f(\infty) = 0 \) and \( \int_0^\infty f(r) dr = 1 \). Let \( w_j(t) \) denotes the total of mature population at time \( t \) and location \( j \):

\[
 w_j(t) = \int_0^\infty f(r) \left( \int_r^\infty u_j(t, a) da \right) dr.
\]

Ling \[5\] derived the lattice model

\[
 \frac{dw_j(t)}{dt} = D[w_{j+1}(t) + w_{j-1}(t) - 2w_j(t)] - dw_j(t)
 + \frac{1}{2\pi} \int_0^\infty e^{-da} f(a) \sum_{l=-\infty}^{\infty} \beta(a, l) b(w_{l+j}(t - a)) da, \quad t > 0,
\]

where

\[
 \beta(a, l) = 2 \int_0^\pi \cos(l\omega)e^{-4Da\sin^2(\frac{\omega}{2})} d\omega.
\]

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Note that this equation has a nonlocal term $\sum_{j=-\infty}^{\infty} \beta(a, l)b(w_{t+j}(t-a))$ and a delay that is continuously distributed and infinite. Ling studied the existence and uniqueness of solutions to (1.1) with an initial value, also discussed the global attractivity of the zero solution, and the existence of wavefronts with speed greater than the spreading speed $c_*$ of traveling wave. Motivated by the method in Diekmann and Thieme [9], in this article, we give a study on the traveling wave and spreading speed for (1.1). More information on the traveling waves for lattice differential systems can be found in [2, 3, 4, 5, 7, 8, 10] and the references therein.

Let $\mathbb{R}_+ := [0, +\infty)$ and $f(d) := \int_0^\infty f(a)e^{-da}da < 1$. We will use the following assumptions:

(H0) $b(0) = 0$, $b(w) \leq b'(0)w$ for $w \geq 0$; $b(w)\tilde{f}(d) < dw$ for $w > 0$, and $b'(0)\tilde{f}(d) < d$.

(H1) $b(0) = 0$, $b \in C^1(\mathbb{R}_+, \mathbb{R}_+)$, $b$ is non-decreasing function on $[0, K]$ and $b(K)\tilde{f}(d) \leq dK$, $|b(u) - b(v)| \leq b'(0)|u - v|$ for $u, v \in \mathbb{R}_+$.

(H2) $b(0) = 0$, $b$ is non-decreasing function on $[0, K]$, $b(w) \leq b'(0)w$ for $w \in \mathbb{R}_+$.

(H3) $b'(0)\tilde{f}(d) > d$, $b(w)\tilde{f}(d) = dw$ admits a positive solution $w^+$ on $[0, K]$. $b(w)\tilde{f}(d) > dw$ for $0 < w < w^+$; and $b(w)\tilde{f}(d) < dw$ for $w > w^+$.

This article is organized as follows. In Section 2, we introduce some definitions and properties of the characteristic equations. In Section 3, we establish the well-posedness and the comparison principle for (1.1), and obtain our main result on the existence of the spreading speed $c_*$ of traveling wave of (1.1). We also give an estimate for $c_*$ and study the relation between the spreading speed with the minimal wave speed.

2. Preliminaries

A solution $\{w_j(t)\}_{j \in \mathbb{Z}}$ is called a traveling wave of (1.1) provided that it has the form $w_j(t) = \phi(j + ct) = \phi(s)$. A sequence of functions $W(t) = \{w_j(t)\}_{j \in \mathbb{Z}}$ is called isotropic on an interval $I$ if $w_j(t) = w_{-j}(t)$ for $j \in \mathbb{Z}$ and $t \in I$. Define

$$C^+_K(-\infty, T) = \{\phi: \phi \text{ is continuous function defined from } (-\infty, T) \text{ to } [0, K]\}.$$ 

We need also the following notation.

$$B_N = \{j \in \mathbb{N} : |j| \leq N, N \in \mathbb{N}\},$$

$$w_j(t) = w(t, j) \text{ for } j \in \mathbb{Z}, \quad W(t) = W(t, \cdot) = \{w_j(t)\}_{j \in \mathbb{Z}},$$

$$\text{supp } W(t, \cdot) = \{j : w(t, j) \neq 0\} \text{ is the support of } W(t, \cdot),$$

$$W(t) \geq V(t) \text{ if } w_j(t) \geq v_j(t) \text{ for } j \in \mathbb{Z},$$

$$W(t) > V(t) \text{ if } W(t) \geq V(t) \text{ and } w_j(t) > v_j(t) \text{ for } j \in \text{supp } V(t, \cdot).$$

A constant $c_* > 0$ is called the spreading speed of (1.1) provided that

$$\lim_{t \to \infty} \sup\{w_j(t) : |j| \geq ct\} = 0 \quad \text{for } c > c_*, \quad (2.1)$$

$$\lim_{t \to \infty} \inf\{w_j(t) : |j| \leq ct\} \geq w^+ > 0 \quad \text{for } c \in (0, c_*). \quad (2.2)$$

where $\{w_j(t)\}_{j \in \mathbb{Z}}$ is a solution of (1.1).
Substituting \( w_j(t) = \phi(j + ct) = \phi(s) \) into (1.1), we obtain the wave equation
\[
\phi'(s) = D[\phi(s + 1) + \phi(s - 1) - 2\phi(s)] - d\phi(s)
\]
\[
+ \frac{1}{2\pi} \int_0^\infty f(a)e^{-da} \sum_{l=-\infty}^\infty \beta(a,l)b(\phi(s + l - ca))da.
\]
(2.3)
The following assumption is needed for considering characteristic equation.

(H4) Assume that for a given \( c > 0 \), one of the following two conditions is satisfied,

(i) For any \( \lambda > 0 \),
\[
\int_0^\infty f(a)e^{-da}e^{2D(\cosh \lambda - 1)a - \lambda ca} da < \infty \text{ holds.}
\]
(ii) There has \( \lambda_0 > 0 \), for any \( \lambda < \lambda_0 \),
\[
\int_0^\infty f(a)e^{-da}e^{2D(\cosh \lambda - 1)a - \lambda ca} da < \infty
\]
\[
\lim_{\lambda \rightarrow \lambda_0 - 0} \int_0^\infty f(a)e^{-da}e^{2D(\cosh \lambda - 1)a - \lambda ca} da = +\infty.
\]

If case (i) holds, let \( \bar{\lambda} = \bar{\lambda}(c) = +\infty \); if case (ii) holds, let \( \bar{\lambda} = \bar{\lambda}(c) = \lambda_0 \).

Assume that (H1)-(H4) hold. Then (2.3) has two equilibria \( c > c^* \) and any \( c < c^* \).

We rewrite (2.5) as
\[
\Delta(\lambda, c) := -c\lambda + D[e^\lambda + e^{-\lambda} - 2] - d + \frac{b'(0)}{2\pi} \int_0^\infty f(a)e^{-da} \sum_{l=-\infty}^\infty \beta(a,l)e^{\lambda l}e^{-\lambda ca} da.
\]
(2.4)
where
\[
\frac{1}{2\pi} \sum_{l=-\infty}^\infty \beta(a,l)e^{\lambda l} = \exp\{D[e^{-\lambda} + e^\lambda - 2|a] = e^{2D(\cosh \lambda - 1)a}
\]
\[\text{(see (10))}.\]

Simplify (2.4) to obtain
\[
\Delta(\lambda, c) := -c\lambda + D[e^\lambda + e^{-\lambda} - 2] - d + b'(0) \int_0^\infty f(a)e^{-d - c\lambda + 2D(\cosh \lambda - 1)a} da = 0.
\]
(2.5)
From (2.4)-(2.5), it is easy to observe the following fact.

**Lemma 2.1.** If \( b \) satisfies (H2)-(H4). Then there exists a unique pair \((c_*, \lambda_*)\) \((c_*>0, \lambda_*>0)\) such that

(i) \( \Delta(\lambda_*, c_*) = 0, \quad \frac{\partial}{\partial \lambda} \Delta(\lambda_*, c_*) = 0; \)
(ii) for \( 0 < c < c_* \) and any \( \lambda \in (0, \bar{\lambda}), \Delta(\lambda, c) > 0; \)
(iii) for \( c > c_* \), the equation \( \Delta(\lambda, c) = 0 \) has two positive real roots \( 0 < \lambda_1 < \lambda_2 < \lambda \), and there exists \( \epsilon_0 > 0 \) such that for any \( \epsilon \in (0, \epsilon_0) \) with \( 0 < \lambda_1 < \lambda_1 + \epsilon < \lambda_2 \), we have \( \Delta(\lambda_1 + \epsilon, c) < 0 \).

We rewrite (2.5) as
\[
1 = \frac{1}{\delta + \lambda c} \left[D(e^\lambda + e^{-\lambda}) + b'(0) \int_0^\infty f(a)e^{-da}e^{2D(\cosh \lambda - 1)a - \lambda ca} da\right] := L_c(\lambda),
\]
(2.6)
where \( \delta := 2D + d \). Hence \( c_* \) can be represented as

\( c_* := \inf\{c > 0 : \text{there exists some } \lambda \in \mathbb{R}_+, \text{ such that } L_c(\lambda) = 1\} \).

From Lemma 2.1 we have
\( L_c(\lambda) > 1 \) for \( \lambda \in (0, \bar{\lambda}) \), and \( c \in (0, c_*) \); \( L_c(\lambda) < 1 \) for \( \lambda \in (\lambda_1, \lambda_2) \) and \( c > c_* \).
Now we shall show that \( c_* \) is the spreading speed of (1.1). Consider the equivalent form
\[
 w_j(t) = e^{-\delta t} w_j(0) + \int_0^t e^{-\delta(t-s)} \{ D[w_{j+1}(s) + w_{j-1}(s)] \\
 + \frac{1}{2\pi} \int_0^\infty f(a)e^{-da} \sum_{l=-\infty}^{\infty} \beta(a,l)b(w_{l+j}(s-a))da \} ds, \quad j \in \mathbb{Z}, \ t \geq 0,
\]
\[
 w_j(t) = w_j^o(t), \quad j \in \mathbb{Z}, \ t \in (-\infty, 0],
\]
(2.7)
For any \( W^o = \{ w_j^o \}_{j \in \mathbb{Z}}, \ w_j^o \in C^+_K(-\infty, 0], w_j^o(0) > 0, \ j \in \mathbb{Z}, \) and \( T \in [0, \infty], \) define the set
\[
 \Lambda_T = \{ W = \{ w_j \}_{j \in \mathbb{Z}} : w_j \in C^+_K(-\infty, T), w_j(t) = w_j^o(t) \text{ for } t \in (-\infty, 0] \},
\]
Equip \( \Lambda_T \) with the norm
\[
 ||W||_\lambda := \sup_{t \in [0,T], j \in \mathbb{Z}} |w_j(t)|e^{-\lambda t}.
\]
Therefore, \( (\Lambda_T, \| \cdot \|_\lambda) \) is a Banach space. Define the sequence of functions \( S^T_j \) by
\[
 S^T_j [W](t) = \begin{cases} 
 e^{-\sigma t} w_j(0) + \int_0^t e^{-\sigma(t-s)} \{ D[w_{j+1}(s) + w_{j-1}(s)] \\
 + \frac{1}{2\pi} \int_0^\infty f(a)e^{-da} \sum_{l=-\infty}^{\infty} \beta(a,l)b(w_{l+j}(s-a))da \} ds, & j \in \mathbb{Z}, \ t \geq 0, \\
 w_j^o(t), & j \in \mathbb{Z}, \ t < 0.
\end{cases}
\]
Then \( S^T_j [W](t) \) is continuous in \( t \in (-\infty, T). \)

Theorem 2.2. Suppose the initial function \( W^o = \{ w_j^o \}_{j \in \mathbb{Z}} \) is isotropic on interval \((-\infty, 0], w_j^o \in C^+_K(-\infty, 0], j \in \mathbb{Z}, \) and there exists \( N \in \mathbb{N} \) such that \( \text{supp} W_o(t, \cdot) \subseteq B_N, t \in (-\infty, 0]. \) Then for any \( c > c_*, \ [2.1] \) holds; i.e., \( \lim_{t \to -\infty} \sup \{ w_j(t) \mid |j| \geq ct \} = 0. \)

Proof. Define a sequence of maps by
\[
 W^{(n)}(t) = S^n [W^{(n-1)}](t) \quad \text{for } n \in \mathbb{N}, \ t \in \mathbb{R}, \ W^{(o)}(t) = \{ w_j^{(o)}(t) \}_{j \in \mathbb{Z}}, \ \quad w_j^{(o)}(t) = \begin{cases} 
 w_j^o(t), & t \in (-\infty, 0], \\
 w_j^o(0), & t \in (0, \infty).
\end{cases}
\]
Then \( W^{(o)}(t) \) is isotropic on \( \mathbb{R}, \) and \( \text{supp} W^{(o)}(t, \cdot) \subseteq B_N \) for \( t \in \mathbb{R}. \) Similarly to [5] Theorem 3.1, we obtain a convergent sequence in \( \Lambda_\infty, \) which is denoted as \( \{ W^{(n)}(t) \}, t \in [0, \infty). \) Let
\[
 W(t) = \begin{cases} 
 \lim_{n \to -\infty} W^{(n)}(t), & t \in [0, \infty), \\
 W^{(o)}(t), & t \in (-\infty, 0].
\end{cases}
\]
By Lebesgue’s dominated convergence theorem, \( [2.7] \) has a solution \( W \in \Lambda_\infty, \) which is isotropic on \( \mathbb{R}. \) For any \( c_1 > c_*, \) let \( c_2 \in (c_*, c_1). \) By the assumption on \( W^{(o)}, \) we choose proper \( N \in \mathbb{N} \) such that
\[
 w_j^{(o)}(t)e^{\lambda(j-c_2 t)} \leq Ke^{\lambda N} \quad \text{for } t \geq 0, \ \lambda > 0, \ j \in \mathbb{Z}.
\]
(2.8)
For $t \geq 0$, by (2.8) we have
\[
\begin{align*}
\psi_j(t) & = e^{-(\alpha + \lambda_2)t}\sum_{j=1}^{\infty} \int_0^t e^{\lambda_{j-2} t} \frac{1}{2\pi} \int_0^\infty f(a) e^{-da} \sum_{l=-\infty}^{\infty} \beta(a, l) b(w_{l+j} (s-a)) e^{\lambda_{j+l}} da ds \\
& + e^{-(\alpha + \lambda_2)t} \left\{ Ke^{\lambda N} + D \int_0^t Ke^{\lambda N} e^{(\alpha + \lambda_2) s} e^{-\lambda} da ds \right\} \\
& + b'(0) \left( \int_0^\infty f(a) e^{-da} e^{2D(cosh(\lambda - 1) a) da} Ke^{\lambda N} \right) \\
& + b'(0) \left( \int_0^\infty f(a) e^{-da} e^{2D(cosh(\lambda - 1) a) da} \right) \\
& \leq Ke^{\lambda N} \left[ 1 + \left( \int_0^\infty f(a) e^{-da} e^{2D(cosh(\lambda - 1) a) da} \right) \right].
\end{align*}
\]
(2.9)

From the above inequality and by induction, we obtain
\[
\psi_j(n) \leq Ke^{\lambda N} \left[ 1 + L_{C_2}(\lambda) + \cdots + (L_{C_2}(\lambda))^n \right].
\]
(2.10)

Noting $-d + b'(0) \int_0^\infty f(a) e^{-da} da > 0$, we have $L_{C_2}(0) > 1$ for $c > 0$. Since $L_{C_2}(\lambda) = 1$ has two roots for $c > c_*$, we can choose $\lambda > 0$ such that $L_{C_2}(\lambda) < 1$ for $c_2 > c_*$. Clearly the right side of (2.10) is uniformly bounded for $n$, thus for every $j \in \mathbb{Z}$,
\[
\psi_j(t) \leq \frac{Ke^{\lambda N}}{1 - L_{C_2}(\lambda)} e^{\lambda(c_2 - j)} \text{ for } t \geq 0.
\]

Since $W$ is isotropic, we have
\[
\psi_j(t) \leq \frac{Ke^{\lambda N}}{1 - L_{C_2}(\lambda)} e^{\lambda(c_2 - j)} \text{ for } t \geq 0;
\]
thus,
\[
\sup \{ \psi_j(t) \mid |j| \geq c_1 t \} \leq \frac{Ke^{\lambda N}}{1 - L_{C_2}(\lambda)} e^{\lambda(c_2 - c_1) t} \to 0 \text{ as } t \to \infty.
\]
Hence we obtain $\lim_{t \to \infty} \sup \{ \psi_j(t) \mid |j| \geq c_1 t \} = 0, c_1 > c_*$. \hfill \Box

## 3. The spreading speed and minimal speed

For $\Phi \in M_\infty$, $t \geq T > 0, j \in \mathbb{Z}$, we define the mapping on $M_\infty = \{ \Phi = \{ \phi_j \}_{j \in \mathbb{Z}} : \phi_j \in C^+_K(\mathbb{R}) \}$ by
\[
E_j^T[\Phi](t) := \int_0^T e^{-\delta s} \{ D[\phi_{j+1}(t-s) + \phi_{j-1}(t-s)] \\
+ \frac{1}{2\pi} \int_0^\infty f(a) e^{-da} \sum_{l=-\infty}^{\infty} \beta(a, l) b(\phi_{l+j}(t-s-a))da \} ds.
\]

**Lemma 3.1.** Suppose $\Phi \in M_\infty$ and satisfies the following conditions:

(i) for any $t' > 0$, there exists an $N = N(t') \in \mathbb{N}$ such that for any $t \in [0, t']$, supp$\Phi(t, \cdot) \subset B_N$;
(ii) If \( \{(t_n, j_n)\}_{n=1}^{\infty} \subset \mathbb{R}_+ \times \mathbb{Z} \), \( j_n \in \text{supp} \Phi(t_n, \cdot) \), and \( \lim_{n \to -\infty} (t_n, j_n) = (t_0, j_0) \), then \( j_0 \in \text{supp} \Phi(t_0, \cdot) \).

For such \( \Phi \), assume that
\[
E^T[\Phi](t) \geq \Phi(t) \quad \text{for } t \geq 0, \tag{3.1}
\]
and the solution of \( (1.1) \) satisfies
\[
W(\bar{t} + t) \geq \Phi(t) \quad \text{for } t \in (-\infty, T] \tag{3.2}
\]
for some \( \bar{t} \geq 0 \). Then
\[
W(\bar{t} + t) \geq \Phi(t) \quad \text{for } t \in [0, \infty). \tag{3.3}
\]

Proof. Let
\[
t_0 = \sup \{ t \geq T : W(\bar{t} + t) \geq \Phi(t) \} \geq T. \tag{3.4}
\]
If \( t_0 < \infty \), since \( W(t) \) is non-negative, there exists \( \{(t_n, j_n)\}_{n=1}^{\infty} \) such that
(a) \( t_n \downarrow t_0 \), \( n \to \infty \),
(b) \( j_n \in \text{supp} \Phi(t_n, \cdot) \),
(c) \( w_{j_n}(\bar{t} + t_n) \leq \phi_{j_n}(t_n) \).

By assumption (i), \( \{j_n\} \) must be bounded. Thus \( \{j_n\} \) is composed of finite integers and contains a convergent sub-sequence, which is a constant sequence \( \{j_0\} \). From (b) and (c), we know that \( j_0 \in \text{supp} \Phi(t_0, \cdot) \) and \( w_{j_0}(\bar{t} + t_0) \leq \phi_{j_0}(t_0) \). For \( t_0 \geq T \) and \( \bar{t} \geq 0 \), from \( (2.7) \) and \( (3.4) \), we have
\[
w_{j_0}(\bar{t} + t_0) \geq \int_0^t e^{-\delta s} \left\{ D[w_{j_0+1}(\bar{t} + t_0 - s) + w_{j_0-1}(\bar{t} + t_0 - s)]
+ \frac{1}{2\pi} \int_0^\infty f(a)e^{-da} \sum_{l=-\infty}^{\infty} \beta(a, l)b(w_{j_0+1}(\bar{t} + t_0 - s - a))da \right\} ds
\]
\[
\geq \int_0^T e^{-\delta s} \left\{ D[\phi_{j_0+1}(t_0 - s) + \phi_{j_0-1}(t_0 - s)]
+ \frac{1}{2\pi} \int_0^\infty f(a)e^{-da} \sum_{l=-\infty}^{\infty} \beta(a, l)b(\phi_{l+j_0}(t_0 - s - a))da \right\} ds
\]
\[
= E_{j_0}^T[\Phi](t_0) \geq \phi_{j_0}(t_0).
\]

Since \( w_{j_0}(\bar{t} + t_0) \leq \phi_{j_0}(t_0) \), the above inequality is a contradiction. Thus we have \( t_0 = \infty \).

Define \( K_c = K_c(h, T, N, \lambda) \) by
\[
K_c(h, T, N, \lambda) = \int_0^t e^{-(\delta + \lambda c)s} \left\{ D[e^{-\lambda s} + e^{\lambda s}] + \frac{h}{2\pi} \int_0^T f(a)e^{-da} \sum_{|l| \leq N} \beta(a, l)e^{\lambda l - \lambda ca}da \right\} ds
\]
\[
= 1 - e^{-(\delta + \lambda c)T} \left\{ D[e^{-\lambda s} + e^{\lambda s}] + \frac{h}{2\pi} \int_0^T f(a)e^{-da} \sum_{|l| \leq N} \beta(a, l)e^{\lambda l - \lambda ca}da \right\}. \tag{3.5}
\]

Lemma 3.2. For any \( c \in (0, c_*) \), there exist \( h \in (0, b'(0)) \), \( T > 0 \) and \( N \in \mathbb{N} \) such that
\[
K_c(h, T, N, \lambda) > 1 \quad \text{for } \lambda \in \mathbb{R}. \tag{3.6}
\]
Proof. From the definition of $K_c(h, T, N, \lambda)$, we have

$$K_c(h, T, N, -\lambda) \geq K_c(h, T, N, \lambda), \quad \lambda \geq 0.$$  

We claim that

$$K_c(h, T, N, \lambda) > 1 \quad \text{for} \quad \lambda \geq 0.$$  

We first show that there exist $N_0 > 0, \lambda_0 > 0, h_0 \in (0, b'(0))$ and $T_0 > 0$ such that

$$K_c(h, T, N, \lambda) > 1 \quad \text{for} \quad \lambda \geq \lambda_0, N \geq N_0, h \geq h_0, T \geq T_0.$$  

However, we can choose proper $N_0 > 0$ and $h_0 \in (0, b'(0))$ such that for all $T > 0$, $N \geq N_0$ and $h \geq h_0$,

$$\frac{h}{2\pi} \int_0^T f(a)e^{-da} \sum_{l=-N}^N \beta(a,l)e^{\lambda(l-ca)}da > 0$$

holds uniformly for $\lambda \geq 0$. Since

$$\lim_{\lambda \to \infty} \frac{e^\lambda}{\lambda c_* + \delta} = \infty,$$

we can choose $T_0 > 0$ and $\lambda_0 > 0$ such that

$$1 - e^{-(\lambda c + \delta)T} \geq 1 - e^{-\delta T} \geq 1 - e^{-\delta T_0} > 0,$$

$$\frac{D}{\lambda c + \delta}(1 - e^{-\delta T_0})e^{\lambda} > \frac{D}{\lambda_0 c_* + \delta}(1 - e^{-\delta T_0})e^{\lambda_0} \geq 1,$$

for $T \geq T_0, \lambda \geq \lambda_0$. For any $N \geq N_0, T \geq T_0, h \geq h_0$ and $\lambda \geq \lambda_0$, we have

$$K_c(h, T, N, \lambda) > \frac{D}{\lambda_0 c_* + \delta}(1 - e^{-\delta T_0})e^{\lambda_0} \geq 1.$$  

If (3.6) is not true, there exist $\{h_n\}, \{T_n\}, \{\lambda_n\}, \{N_n\}$ such that $h_n \uparrow b'(0), T_n \uparrow \infty, N_n \uparrow \infty, \lambda_n \subset [0, \lambda_0]$ and

$$K_c(h_n, T_n, N_n, \lambda_n) \leq 1, \quad n = 1, 2, \ldots.$$  

Since $\{\lambda_n\}$ is bounded, we choose a convergent sub-sequence $\{\lambda_{n_k}\}$. Obviously $\{\lambda_{n_k}\}$ has a finite limit, denotes as $\hat{\lambda}$. By Fatou’s lemma, we have

$$1 < L_{\hat{\lambda}} \leq \liminf_{k \to \infty} K_c(h_{n_k}, T_{n_k}, N_{n_k}, \lambda_{n_k}) \leq 1,$$

which is a contradiction. Hence (3.6) is true.  

Define a function

$$q(y; \omega, \zeta) = \begin{cases} e^{-\omega y \sin(\zeta y)}, & y \in [0, \frac{\pi}{\zeta}], \\ 0, & y \in \mathbb{R} \setminus [0, \frac{\pi}{\zeta}]. \end{cases}$$

Lemma 3.3. Suppose $c \in (0, c_*)$. Then there exist $\zeta_0 > 0$, a continuous function $\omega = \omega(\zeta)$ defined on $[0, \zeta_0]$, and a positive number $\delta_1 \in (0, 1)$ such that

$$\int_0^T e^{-\delta s} \{D[q(m + cs + 1) + q(m + cs - 1)] + \frac{h}{2\pi} \int_0^T f(a)e^{-da} \sum_{||l|| \leq N} \beta(a,l)q(m + l + cs + ca)da\}ds \geq q(m - \delta_1),$$

(3.7)

for $m \in \mathbb{Z}$, where $q(y) = q(y; \omega(\zeta), \zeta)$.  

Proof. Define

$$L(\lambda) = \int_0^T e^{-\delta s} \left\{ D[e^{-\lambda(cs+1)} + e^{-\lambda(cs-1)}] \right\} ds + \frac{h}{2\pi} \int_0^T f(a)e^{-\sigma a} \sum_{|l| \leq N} \beta(a,l) e^{-\lambda(l+cs+ca)} ds,$$

where $T, h, N$ are defined in Lemma 3.2. By Lemma 3.2, for sufficiently large $N$,

$$L(\lambda) = K e(h, T, N, \lambda) > 1 \quad \text{for } \lambda \in \mathbb{R}.$$  \hfill (3.8)

Let $\lambda = \omega + i\zeta$, then we have

$$L(\lambda)|_{\lambda=\omega+i\zeta} = \text{Re}[L(\lambda)] + i \text{ Im}[L(\lambda)],$$

where

$$\text{Re}[L(\lambda)] = D \int_0^T e^{-\delta s} \left\{ e^{-\omega(cs+1)} \cos \zeta(cs+1) + e^{-\omega(cs-1)} \cos \zeta(cs-1) \right\} ds + \frac{h}{2\pi} \int_0^T f(a)e^{-\sigma a} \sum_{|l| \leq N} \beta(a,l) \left\{ \int_0^T e^{-\delta s} e^{-\omega(l+cs+ca)} \cos \zeta(l+cs+ca) ds \right\} da,$$

$$\text{Im}[L(\lambda)] = -D \int_0^T e^{-\delta s} \left\{ e^{-\omega(cs+1)} \sin \zeta(cs+1) + e^{-\omega(cs-1)} \sin \zeta(cs-1) \right\} ds - \frac{h}{2\pi} \int_0^T f(a)e^{-\sigma a} \sum_{|l| \leq N} \beta(a,l) \left\{ \int_0^T e^{-\delta s} e^{-\omega(l+cs+ca)} \sin \zeta(l+cs+ca) ds \right\} da.$$

Since $L''(\lambda) > 0$ and $\lim_{|\lambda| \to \infty} L(\lambda) = \infty$ for $\lambda \in \mathbb{R}$, $L(\lambda)$ attains the minimal value at $\lambda = \theta \in \mathbb{R}$. Thus,

$$L'(\theta) = -D \int_0^T e^{-\delta s} [(cs+1)e^{-\theta(cs+1)} + (cs-1)e^{-\theta(cs-1)}] ds - \frac{h}{2\pi} \int_0^T f(a)e^{-\sigma a} \sum_{|l| \leq N} \beta(a,l) \left\{ \int_0^T e^{-\delta s} (l+cs+ca)e^{-\theta(l+cs+ca)} ds \right\} da = 0.$$

Define a function $H = H(\omega, \zeta)$ by

$$H(\omega, \zeta) = \frac{1}{\zeta} \text{ Im}[L(\lambda)] \quad \text{for } \zeta \neq 0,$$

$$H(\omega, 0) = \lim_{\zeta \to 0} H(\omega, \zeta) = L'(\omega).$$

Obviously $H(\theta, 0) = 0$ and $\frac{\partial H}{\partial \omega}(\theta, 0) = L''(\theta) > 0$. By implicit function theorem, there exist $\zeta_1 > 0$ and continuous function $\omega = \omega(\zeta, \zeta) \in [0, \zeta_1]$ satisfying $\omega(0) = \theta$, and $H(\omega(\zeta), \zeta) = 0$, $\zeta \in [0, \zeta_1]$. Thus,

$$\text{Im}[L(\lambda)]|_{\lambda=\omega(\zeta)+i\zeta} = 0, \quad \zeta \in [0, \zeta_1].$$  \hfill (3.9)

By (3.5) and (3.9), we have

$$\text{Re}[L(\omega + i\zeta)]|_{\omega=\theta, \zeta=0} = L(\theta) > 1.$$
Then there exists $\zeta_2 > 0$ such that
\[
\text{Re}[L(\omega(\zeta) + i\zeta)] > 1, \quad \zeta \in [0, \zeta_2].
\] (3.10)

Let $0 < \zeta \leq \zeta_0 := \min\{\zeta_1, \zeta_2, \frac{\pi}{N+2c_sT}\}$. For $m \in [0, \frac{\pi}{\zeta}]$, $|l| \leq N$ and $a, s \in [0, T]$,
\[
-\frac{\pi}{\zeta} < -N \leq l \leq m + l + cs + ca \leq m + l + 2cT < N + 2c_sT + \frac{\pi}{\zeta} \leq \frac{2\pi}{\zeta}.
\]
Thus,
\[
\sin\zeta(m + l + c(s + a)) < 0, \quad \text{for } m + l + c(s + a) \in (-\frac{\pi}{\zeta}, 0) \cup (\frac{\pi}{\zeta}, \frac{2\pi}{\zeta}).
\] (3.11)

From the definition of $q(\cdot)$ we obtain
\[
\begin{align*}
\int_0^T & e^{-\delta s} \{D[q(m + cs + 1) + q(m + cs - 1)] \\
& + \frac{h}{2\pi} \int_0^T f(a)e^{-da} \sum_{|l| \leq N} \beta(a, l)q(m + l + cs + ca)da\} ds \\
& \geq D \int_0^T e^{-\delta s} \left\{e^{-\omega(\zeta)(m+cs+1)}\sin(\zeta(m + cs + 1)) \\
& + e^{-\omega(\zeta)(m+cs-1)}\sin(\zeta(m + cs - 1))\right\} ds \\
& + \frac{h}{2\pi} \int_0^T e^{-\delta s} \int_0^T f(a)e^{-da} \\
& \times \sum_{|l| \leq N} \beta(a, l)e^{-\omega(\zeta)(m+l+cs+ca)}\sin(\zeta(m + l + cs + ca))da\} ds.
\end{align*}
\] (3.12)

Using $\sin(A + B) = \sin A \cos B + \sin B \cos A$ and (3.10)-(3.12), we have
\[
\begin{align*}
\int_0^T & e^{-\delta s} \{D[q(m + cs + 1) + q(m + cs - 1)] \\
& + \frac{h}{2\pi} \int_0^T f(a)e^{-da} \sum_{|l| \leq N} \beta(a, l)q(m + l + cs + ca)da\} ds \\
& \geq e^{-\omega(\zeta)m}\sin(\zeta m)\text{Re}[L(\lambda)|_{\lambda=\omega(\zeta)+i\zeta}] + e^{-\omega(\zeta)m}\cos(\zeta m)\text{Im}[L(\lambda)|_{\lambda=\omega(\zeta)+i\zeta} \\
& = e^{-\omega(\zeta)m}\sin(\zeta m) = q(m).
\end{align*}
\] (3.13)

Choose $N$ large enough such that $-N + 2c_sT < 0$, thus (3.12) and (3.13) are strict inequalities on $m \in (0, \frac{\pi}{\zeta})$. Moreover, from (3.11)-(3.12), we know that (3.13) is also a strict inequality for $m = 0$ or $m = \frac{\pi}{\zeta}$. In fact, let $a, s \in [0, T]$, $m = \frac{\pi}{\zeta}$ and $l = N$, then
\[
m + l + c(s + a) > \frac{\pi}{\zeta}.
\]
Similarly, if $m = 0$ and $l = -N$, then $m + l + c(s + a) < -N + 2c_sT < 0$. Thus for both cases, we have
\[
q(m + l + cs + ca) = 0 \quad \text{and} \quad \sin(\zeta(m + l + cs + ca)) < 0,
\]
Lemma 3.4. Let sub-solution of (1.1). We assume \( \Phi(\cdot) \) where \( \sigma \)

\[
\int_0^T e^{-\delta s} \{D[q(m + cs + 1) + q(m + cs - 1)]
+ \frac{h}{2\pi} \int_0^T f(a)e^{-da} \sum_{|l| \leq N} \beta(a,l)q(m + l + cs + ca)da\} ds > q(m).
\]

(3.14)

If \( m \notin [0, \frac{\pi}{\zeta}] \), (3.14) still holds since \( q(m) = 0 \). From the above discussion, we know that (3.14) holds for \( m \in \mathbb{R} \), then (3.7) follows from the continuity consideration.

Consider the family of functions,

\[
R(y; \omega, \zeta, \gamma) := \max_{\eta \geq -\gamma} q(y + \eta; \omega, \zeta)
\]

\[
= \begin{cases} 
M, & y \leq \gamma + \rho, \\
qu(y - \gamma; \omega, \zeta), & \gamma + \rho \leq y \leq \gamma + \frac{\pi}{\zeta}, \\
0, & y \geq \gamma + \frac{\pi}{\zeta},
\end{cases}
\]

(3.15)

where

\[
M = M(\omega, \zeta) := \max\{q(y; \omega, \zeta)| 0 \leq y \leq \frac{\pi}{\zeta}\}.
\]

(3.16)

We assume \( M \) attain the maximum at \( \rho = \rho(\omega, \zeta) \). The following lemma gives a sub-solution of (1.1).

Lemma 3.4. Let \( c \in (0, c_*) \) be given, then there exist \( T > 0, \zeta > 0, \omega \in \mathbb{R}, \vartheta > 0 \) and \( \sigma_0 > 0 \) such that for \( \sigma \in (0, \sigma_0) \) and \( t \geq T \), there holds

\[
E^T[\sigma \Phi](t) > \sigma \Phi(t) \quad \text{for } t \geq T,
\]

(3.17)

where \( \Phi(t) = \{\phi(t)\}_{j \in \mathbb{Z}}, \phi(t) = R([j]; \omega, \zeta, \vartheta + ct) \).

Proof. Let \( h \in (0, b'(0)), T > 0, N > 0 \) be chosen such that \( K_c(h, T, N, \lambda) > 1 \) for \( \lambda \in \mathbb{R} \). By Lemma 3.3, we can choose \( \zeta > 0, \omega = \omega(\zeta) \) and \( \delta_1 \in (0, 1) \) such that (3.7) holds.

Let \( \sigma_h \) be the smallest positive root of the equation \( b(w) = hw \), then \( b(w) > hw \) for \( w \in (0, \sigma_h) \). Choose \( \sigma_0 \in (0, \sigma_h M^{-1}) \), where \( M \) is defined in (3.16). For \( \sigma \in (0, \sigma_0) \) and \( t \geq T \), we have

\[
E^T[\sigma \Phi](t) = \int_0^T e^{-\delta s} \{D[\phi_{j+1}(t - s) + \phi_{j-1}(t - s)]
+ \frac{1}{2\pi} \int_0^T f(a)e^{-da} \sum_{|l| \leq N} \beta(a,l)b(\sigma \phi_{j+1}(t - s - a))da\} ds
\]

(3.18)

\[
\geq \int_0^T e^{-\delta s} \{D[\phi_{j+1}(t - s) + \phi_{j-1}(t - s)]
+ \frac{1}{2\pi} \int_0^T f(a)e^{-da} \sum_{|l| \leq N} \beta(a,l)b(\sigma \phi_{j+1}(t - s - a))da\} ds.
\]

For any given \( \vartheta > 0 \), we consider two cases.
Case (i) $|j| \leq \vartheta + \rho + c(t - 2T) - N$. For $|l| \leq N, a, s \in [0, T]$, then

$$|l + j| \leq \vartheta + \rho + c(t - 2T) \leq \vartheta + \rho + c(t - s - a)$$

Since the definition of $E^T_j[a\Phi](t)$ and $b(\sigma\phi_{j+1}(t - s - a)) = b(\sigma M) > h\sigma M$, we have

$$E^T_j[a\Phi](t) \geq \frac{2D\sigma M + \frac{1}{2\pi}}{2\pi} \int_0^T f(a)e^{-da} \sum_{l=-N}^N \beta(a, l)b(\sigma M)da \int_0^T e^{-\delta s}ds$$

$$> \sigma MK_c(h, T, N, 0) > \sigma M. \tag{3.19}$$

Case (ii) $\vartheta + \rho + c(t - 2T) - N \leq |j| \leq \frac{\vartheta}{2} + \vartheta + ct$. Let $|l| \leq N, \ t \geq T$. If $\vartheta \geq \frac{N^2}{2\pi} - \rho + cT + N \ (\delta_1 \text{ is defined in Lemma } 3.3)$, then

$$|l + j| = (l^2 + 2lj + j^2)^{1/2} \leq |j| + \frac{lj}{|j|} + \frac{N^2}{2|j|}$$

$$\leq |j| + \frac{lj}{|j|} + \frac{N^2}{2(\vartheta + \rho - cT - N)} \leq |j| + \frac{lj}{|j|} + \delta_1.$$ 

Since $\phi_j(t)$ is non-decreasing for $|j|$, by (3.18) we obtain

$$E^T_j[a\Phi](t)$$

$$\geq \int_0^T e^{-\delta s}\left\{D\sigma\left[\max_{\eta \geq -\vartheta - c(t-s)} q(|j| + 1 + \delta_1 + \eta)\right.ight.$$

$$+ \left.\max_{\eta \geq -\vartheta - c(t-s)} q(|j| - 1 + \delta_1 + \eta)\right]$$

$$+ \frac{h\sigma}{2\pi} \int_0^T f(a)e^{-da} \sum_{l \leq N} \beta(a, l) \max_{\eta \geq -\vartheta - c(t-s-a)} q(|j| + l + \delta_1 + \eta)da\right\}ds$$

$$= \sigma \int_0^T e^{-\delta s}\left\{D\left[\max_{\eta \geq -\vartheta - ct} q(|j| + 1 + cs + \delta_1 + \eta)\right.ight.$$

$$+ \left.\max_{\eta \geq -\vartheta - ct} q(|j| - 1 + cs + \delta_1 + \eta)\right]$$

$$+ \frac{h}{2\pi} \int_0^T f(a)e^{-da} \sum_{l \leq N} \beta(a, l) \max_{\eta \geq -\vartheta - ct} q(|j| + l + cs + ca + \delta_1 + \eta)da\right\}ds$$

$$\geq \sigma \max_{\eta \geq -\vartheta - ct} q(|j| + \eta).$$

Combining (i) and (ii), we obtain (3.17) and complete the proof. \hfill \Box

The proof of the following lemma is similar to [10, Lemma 5.5], and hence is omitted.

**Lemma 3.5.** Assume that $W = \{w_j\}_{j \in \mathbb{Z}}$ is a solution of (1.1), and the following conditions hold:

(i) $W^o = \{w_j^o\}_{j \in \mathbb{Z}}$ is isotropic on $(-\infty, 0], w_j^o \in C_K^+(\varnothing, 0]$;

(ii) there exists $N_1 \in \mathbb{N}$ such that $\text{supp } W^o(t, \cdot) \subset B_{N_1}$ for $t \in (-\infty, 0], w_j^o(0) > 0$ for $j \leq N_1$.

Then there exists $t_0 > 0$ such that $w_j(t) > 0$ for $t \in [t_0, \infty), j \in \mathbb{Z}$.
Lemma 3.6. Let \( \{Q_n(t,N)\} \) be defined by \( Q_1(t,N) = a \in (0, w^+) \),
\[
Q_{n+1}(t,N) = \frac{1}{\delta} \left[ 2DQ_n(t,N) \right] + \frac{1}{2\pi} \int_0^T \left\{ f(a)e^{-da} \sum_{|l| \leq N} \beta(a,l)da \left( Q_n(t,N) \right) \right\} (1 - e^{-\delta t}) \tag{3.20}
\]
for \( n = 1, 2, \ldots \). Then for \( \epsilon > 0 \), there exist \( \bar{t}(\epsilon), \bar{N}(\epsilon), \bar{T}(\epsilon) \) and \( \bar{n}(\epsilon) \) such that for any \( T \geq \bar{T}(\epsilon), t \geq \bar{t}(\epsilon), N \geq \bar{N}(\epsilon) \) and \( n \geq \bar{n}(\epsilon) \),
\[
Q_n(t,N) \geq w^+ - \epsilon.
\]

Proof. Since
\[
\frac{2Dw^+ + b(w^+)\bar{f}(d)}{\delta} = w^+, \quad \delta = 2D + d, \quad \bar{f}(d) = \int_0^\infty f(a)e^{-da}da,
\]
we have by induction that \( 0 < Q_1(t,N) < w^+, \ 0 < \frac{1}{\delta}(1 - e^{-\delta t}) < 1, \ 0 < \frac{1}{2\pi} \sum_{|l| \leq N} \beta(a,l) < 1 \),
we have by induction that \( 0 < Q_n(t,N) \leq K \) for any \( n \in \mathbb{N}, t \geq 0 \) and \( N \in \mathbb{N} \). By (H3), \( 2Dw + \bar{f}(d)b(w) > (2D + d)w \), for \( 0 < w < w^+ \). For \( \epsilon > 0 \), we have
\[
\sup \left\{ \frac{2Dw + \bar{f}(d)b(w)}{(2D + d)w} \mid 0 < w \leq w^+ - \epsilon \right\} > 1.
\]
Let \( \bar{f}_T(d) = \int_0^T f(a)e^{-da}da \). Choose large enough \( \alpha(\epsilon) < 1, \ \bar{T} = \bar{T}(\epsilon) \) such that for \( 0 < w \leq w^+ - \epsilon, T \geq \bar{T} \), there holds
\[
\alpha(\epsilon) \left[ 2Dw + \bar{f}_T(d)b(w) \right] > (2D + d)w. \tag{3.21}
\]
Define the sequence:
\[
M_1 \equiv a, \quad M_{n+1} = \frac{\alpha(\epsilon)}{\delta} \left[ 2DM_n + \bar{f}_T(d)b(M_n) \right] \text{ for } n \geq 2.
\]
Obviously,
(i) if \( 0 < M_n \leq w^+ - \epsilon \), then \( M_{n+1} \geq M_n \);
(ii) if \( M_n > w^+ - \epsilon \), then
\[
M_{n+1} > \frac{\alpha(\epsilon)}{\delta} \left[ 2D(w^+ - \epsilon) + \bar{f}_T(d)b(w^+ - \epsilon) \right] \geq w^+ - \epsilon.
\]
Now we show that \( M_n > w^+ - \epsilon \) for sufficiently large \( n \). If that is not true, we can assume that \( M_n \leq w^+ - \epsilon \) holds for all \( n \). By (i), we know that \( \lim_{n \to \infty} M_n = M \leq w^+ - \epsilon \) exists and satisfies
\[
M = \frac{\alpha(\epsilon)}{\delta} (2DM + b(M)\bar{f}_T(d)).
\]
which is a contraction to (3.21). Thus there exists \( \bar{n}(\epsilon) > 0 \) such that \( M_n > w^+ - \epsilon \) for any \( n > \bar{n}(\epsilon) \).

Let \( T \geq \bar{T} = \bar{T}(\epsilon) \). We choose \( \bar{t} = \bar{t}(\epsilon) \) and \( \bar{N} = \bar{N}(\epsilon) \) such that \( 1 - e^{-\delta \bar{t}(\epsilon)} \geq \alpha(\epsilon) \) and
\[
\frac{1}{2\pi} (1 - e^{-\delta \bar{t}(\epsilon)}) \int_0^T \left\{ f(a)e^{-da} \sum_{|l| \leq \bar{N}} \beta(a,l) \right\} da \geq \alpha(\epsilon) \bar{f}_T(d). \tag{3.22}
\]
Then \( Q_1(t, N) = a \geq M_1 \) for \( t \geq \tilde{t}(\epsilon), T \geq \tilde{T}(\epsilon) \) and \( N \geq \tilde{N}(\epsilon) \). By (3.22) we obtain
\[
Q_{n+1}(t, N) \geq \frac{1}{\delta}(1 - e^{-\delta t(\epsilon)}) \left[ 2DQ_n(t, N) + \frac{b(Q_n(t, N))}{2\pi} \int_0^T \{ f(a)e^{-da} \sum_{|l| \leq N} \beta(a, l) \} da \right] \\
> \frac{1}{\delta} \left[ 2D\alpha(t)Q_n(t, N) + \alpha(t)\tilde{f}_T(d)b(Q_n(t, N)) \right] \\
= \frac{\alpha(t)}{\delta} \left[ 2DQ_n(t, N) + \tilde{f}_T(d)b(Q_n(t, N)) \right].
\]

Using monotonicity of \( b \), we have \( Q_n(t, N) \geq M_n \geq w^+ - \epsilon \) for \( n > \bar{n}(\epsilon) \). \( \square \)

**Theorem 3.7.** Assume all the conditions for \( W^0 \) in Lemma 3.5 are satisfied. Then for any \( c \in (0, c_*) \), there holds
\[
\lim_{t \to \infty} \inf \{ w_j(t) : |j| \leq ct \} \geq w^+
\]

**Proof.** Let \( c_1 \in (0, c_*) \). From Lemma 3.4 there exist \( T > 0, \zeta > 0, \omega \in \mathbb{R}, \vartheta > 0 \) and \( \sigma_0 > 0 \) such that for \( \sigma \in (0, \sigma_0) \), \( t \geq T \),
\[
E^T[\sigma \Phi](t) > \sigma \Phi(t),
\]
where \( \Phi(t) = \{ \phi_j(t) \}_{j \in \mathbb{Z}}, \Phi(t) := R([j]; \omega, \zeta, \vartheta + c_2T) \). We can assume \( T \geq \bar{T} \), and \( \bar{T} \) is defined in Lemma 3.6. From Lemma 3.5 there exists \( t_0 > 0 \) such that
\[
w_j(t_0) > 0 \quad \text{for} \quad t \in [t_0, t_0 + T], \quad j \in \mathbb{Z}.
\]
Since \( \Phi(t) \) is a bounded function, we can choose \( \sigma_1 \in (0, \sigma_0) \) such that
\[
\sigma_1M < w^+, \quad w_j(t_0 + t) > \sigma_1 \phi_j(t) \quad \text{for} \quad t \in [0, T], \quad j \in \mathbb{Z}.
\]

Using the comparison principle (Lemma 3.1), we have
\[
w_j(t_0 + t) > \sigma_1 \phi_j(t) \quad \text{for} \quad t \in [0, \infty), \quad j \in \mathbb{Z}.
\]
From (3.23) and definition of \( \phi_j(t) \), we have
\[
w_j(t_0 + t) \geq \sigma_1 M, \quad t \geq 0, \quad |j| \leq \rho + \vartheta + c_2t.
\]

By (2.27), we have
\[
w_j(t_0 + t) \geq \int_0^t e^{-\delta s} \left[ D[w_j+1(t_0 + t - s) + w_j-1(t_0 + t - s)] \right] \\
+ \frac{1}{2\pi} \int_0^T f(a)e^{-da} \sum_{|l| \leq N} \beta(a, l)b(w_{j+1}(t_0 + t - s) - a)da ds
\]
Let \( a = \sigma_1 M = Q_1(t, N) \), and \( Q_n(t, N) \) be defined in Lemma 3.6. From (3.24) and (3.25), we have by induction
\[
w_j(t_0 + t) \geq Q_n(t, N), \quad t \geq 0, |j| \leq \rho + \vartheta + c_2t - n(N + T).
\]

For any \( \epsilon > 0 \), we choose \( \tilde{t}(\epsilon), \tilde{T}(\epsilon), \tilde{N}(\epsilon) \) and \( \tilde{n}(\epsilon) \) such that
\[
w_j(t) \geq w^+ - \epsilon, \quad t \geq t_0 + \tilde{t}(\epsilon), \quad |j| \leq \rho + \vartheta + c_2(t - t_0) - \tilde{n}(\epsilon)(\tilde{N}(\epsilon) + \tilde{T}(\epsilon)).
\]
Define
\[
t_1 := \max \left\{ t_0 + \tilde{t}(\epsilon), \frac{\tilde{n}(\epsilon)(\tilde{N}(\epsilon) + \tilde{T}(\epsilon)) + c_2t_0 - \rho - \vartheta}{c_2 - c_1} \right\}.
\]
Since \( c_2 > c_1 \) and \([3.26]\), we obtain
\[
w_j(t) \geq w^+ - \epsilon \quad \text{for } t \geq t_1, |j| \leq c_1 t.
\]
Then \([2.2]\) holds.

The following theorem shows the relation between the minimal wave speed and the spreading speed.

**Theorem 3.8.** Assume (H1)--(H4) are satisfied. Then lattice system \([1.1]\) admits two equilibria, \( W = 0 \) and \( W = w^+ > 0 \). Further, for \( c \geq c_* \), Equation \([1.1]\) has a monotone traveling wave satisfying
\[
\lim_{s \to -\infty} \phi(s) = 0, \quad \lim_{s \to \infty} \phi(s) = w^+.
\]
\((3.27)\)

For \( c \in (0, c_*) \), \([1.1]\) has no monotone traveling wave satisfying \((3.27)\).

**Proof.** From \([5\,\text{Theorem 5.1}]\), we have that \([1.1]\) admits monotone traveling wave satisfying \((3.27)\) for \( c > c_* \), thus we only need to claim the case as \( c = c_* \).

Choose a sequence \( \{c_n\} \in (c_*, c_* + 1) \) such that \( c_{n+1} > c_n \) and \( \lim_{n \to \infty} c_n = c_* \). Then the wave equation \([2.1]\) admits a wavefront connecting 0 with \( w^+ \), say \( \phi_n(j + c_n t) \), which has the speed \( c_n \). It is easy to see \( 0 < \phi_n(j + c_n t) < w^+ \), and
\[
c_n \phi_n'(s) = D[\phi_n(s + 1) + \phi_n(s - 1) - 2\phi_n(s)] - d\phi_n(s)
\]
\[+ \frac{1}{2\pi} \int_0^\infty f(a)e^{-da} \sum_{l=-\infty}^{\infty} \beta(a, l) b(\phi_n(s + l - c_n a))da.
\]
\((3.28)\)

Since \((3.28)\) is a homogeneous system, from the basis theory of differential equation, we know that a traveling wave of \((3.28)\) is still another traveling wave after sliding.

Without generality, we assume \( \phi_n(0) = \frac{w^+}{2} \).

Differentiating \((3.28)\) with respect to \( s \), we obtain
\[
c_n \phi_n''(s) = D[\phi_n'(s + 1) + \phi_n'(s - 1) - 2\phi_n'(s)] - d\phi_n'(s)
\]
\[+ \frac{1}{2\pi} \int_0^\infty f(a)e^{-da} \sum_{l=-\infty}^{\infty} \beta(a, l) \frac{db}{dw}(\phi_n(s + l - c_n a))\phi_n'(s + l - c_n a)da.
\]
\((3.29)\)

From \((3.28)\) and \( 0 < \phi_n(j + c_n t) < w^+ \), there exists \( M_1, M_2 \) such that \( |\phi_n'(s)| \leq M_1, \ |\phi_n''(s)| \leq M_2 \) for \( s \in \mathbb{R} \). Thus \( \phi_n \) and \( \phi_n' \) are uniformly bounded, equi-continuous in \( \mathbb{R} \). According to Arzela-Ascoli theorem, there has a sub-sequence of \( c_n \), still denoted as \( c_n \), such that \( \phi_n(s) \) and \( \phi_n'(s) \) are convergent to limits in every bounded and closed subset in \( \mathbb{R} \). We denote the limits as \( \phi_*(s), \phi_*'(s) \) respectively.

Let \( n \to \infty \) in \((3.28)\). By Lebesque’s dominated convergence theorem, we have
\[
c \phi_*'(s) = D[\phi_*(s + 1) + \phi_*(s - 1) - 2\phi_*(s)] - d\phi_*(s)
\]
\[+ \frac{1}{2\pi} \int_0^\infty f(a)e^{-da} \sum_{l=-\infty}^{\infty} \beta(a, l) b(\phi_*(s + l - c_* a))da.
\]
\((3.30)\)

Hence \( \phi_*(j + c_* t) \) is the traveling wavefront of \([1.1]\) with speed \( c_* \) satisfying \((3.1)\).

Now we prove \([1.1]\) admits no traveling wavefront for \( c_1 \in (0, c_*) \). Suppose that is not true, and system \([1.1]\) has monotone traveling wave \( \phi(s) = \phi(j + c_1 t) \) satisfying \((3.27)\). Thus there exists \( s_1 > 0 \) such that \( \phi(s) > \frac{w^+}{2} \) for \( s \geq s_1 \). Choose proper initial function: \( w_0^j(t) = \phi(j + c_1 t), \ t \in (-\infty, 0], \) and \( \{w_0^j(t)\}_{j \in \mathbb{Z}} \in C_K^+(-\infty, 0]. \)
Let \( \{ w_j(t) = \phi(j + c_1 t) \}_{j \in \mathbb{Z}} \) be a solution of (1.1) with initial value \( w_0^j(t) \). Noting \( \{ w_0^j(t) \}_{j \in \mathbb{Z}} \) satisfying conditions in Theorem 3.7, we have

\[
\liminf_{t \to \infty} \{ w_j(t) \mid |j| \leq ct \} = \liminf_{t \to \infty} \{ \phi(j + c_1 t) \mid |j| \leq ct \} \geq w^+ \quad \text{for} \quad c \in (0, c^*_1).
\]

Choose \( c_2 \in (c_1, c^*_1) \), \( j = -c_2 t \), then

\[
\phi(j + c_1 t) = \phi((c_1 - c_2) t) \geq w^+ \quad \text{for} \quad t \geq t_1.
\]

Let \( t \to \infty \), we have

\[
\liminf_{t \to \infty} \{ \phi(j + c_1 t) \mid j = -c_2 t \} = \liminf_{t \to \infty} \{ \phi((c_1 - c_2) t) \} \geq w^+,
\]

which leads to a contradiction to the first equality in (3.27). Hence (1.1) admits no monotone traveling wave for \( c_1 \in (0, c^*_1) \). □

References


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