Electronic Journal of Differential Equations, Vol. 2012 (2012), No. 172, pp. 1–14.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

STABILITY FOR LINEAR NEUTRAL INTEGRO-DIFFERENTIAL EQUATIONS WITH VARIABLE DELAYS

ABDELOUAHEB ARDJOUNI, AHCENE DJOUDI, IMENE SOUALHIA

Abstract. In this article we study a linear neutral integro-differential equation with variable delays and give suitable conditions to obtain asymptotic stability of the zero solution, by means of fixed point technique. An asymptotic stability theorem with a necessary and sufficient condition is proved, which improves and generalizes previous results due to Burton [5], Becker and Burton [4] and Jin and Luo [15]. We provide an example that illustrates our results.

1. Introduction

Without doubt, Lyapunov's direct method has been, for more than 100 years, the main tool for investigating the stability properties of a wide variety of ordinary, functional, partial differential and integro-differential equations. Nevertheless, the application of this method to problems of stability in differential and integro-differential equations with delays has encountered serious obstacles if the delays are unbounded or if the equation has unbounded terms [6]–[8]. In recent years, several investigators have tried stability by using a new technique. Particularly, Burton, Furumochi, Becker and others began a study in which they noticed that some of these difficulties vanish or might be overcome by means of fixed point theory (see [1]–[18], [20]). The fixed point theory does not only solve the problem on stability but has other significant advantage over Lyapunov's. The conditions of the former are often averages but those of the latter are usually pointwise (see [6]).

In this article we consider the linear neutral integro-differential equation with variable delays

$$x'(t) = -\sum_{j=1}^{N} \int_{t-\tau_j(t)}^{t} a_j(t,s)x(s)ds + \sum_{j=1}^{N} c_j(t)x'(t-\tau_j(t)), \tag{1.1}$$

with the initial condition

$$x(t) = \psi(t) \quad \text{for } t \in [m(0), 0],$$

where $\psi \in C([m(0), 0], \mathbb{R})$ and

$$m_j(0) = \inf\{t - \tau_j(t),\ t \ge 0\}, \quad m(0) = \min\{m_j(0),\ 1 \le j \le N\}.$$

2000 *Mathematics Subject Classification.* 34K20, 34K30, 34K40.
Key words and phrases. Fixed point; stability; integro-differential equation; variable delay.

Submitted June 20, 2012. Published October 12, 2012.

Here $C(S_1, S_2)$ denotes the set of all continuous functions $\varphi : S_1 \to S_2$ with the supremum norm $\|\cdot\|$. Throughout this paper we assume that $a_j \in C(\mathbb{R}^+ \times [m(0), \infty), \mathbb{R})$, $c_j \in C^1(\mathbb{R}^+, \mathbb{R})$, and $\tau_j \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $t - \tau_j(t) \to \infty$ as $t \to \infty$ for $j = 1, 2, \dots, N$.

Special cases of equation (1.1) have been investigated by many authors. For example, Burton in [5], Becker and Burton in [4], Jin and Luo in [15] studied the equation

$$x'(t) = -\int_{t-\tau_1(t)}^{t} a_1(t,s)x(s)ds, \tag{1.2}$$

and proved the following theorems, respectively,

Theorem 1.1 ([5]). *Suppose that $\tau_1(t) = r$ and there exists a constant $\alpha < 1$ such that*

$$2\int_{t-r}^{t} |A(t,s)|ds \le \alpha \quad \text{for all } t \ge 0, \tag{1.3}$$

$$\int_0^t A(s,s)ds \to \infty \quad \text{as } t \to \infty, \tag{1.4}$$

where

$$A(t,s) = \int_{t-s}^{r} a_1(u+s,s)du \quad \text{with } A(t,t) = \int_0^r a_1(u+t,t)du.$$

Then the zero solution of (1.2) is asymptotically stable.

Theorem 1.2 ([4]). *Suppose that τ_1 is differentiable, $t-\tau_1(t)$ is strictly increasing, and there exist constants $k \ge 0$, $\alpha \in (0,1)$ such that for $t \ge 0$,*

$$-\int_0^t G(s,s)ds \le k, \tag{1.5}$$

$$\int_{t-\tau_1(t)}^{t} |G(t,s)|ds + \int_0^t e^{-\int_s^t G(u,u)du}|G(s,s)|\Big(\int_{s-\tau_1(s)}^{s} |G(s,u)|du\Big)ds \le \alpha, \tag{1.6}$$

with

$$G(t,s) = \int_t^{f(s)} a_1(u,s)du, \quad G(t,t) = \int_t^{f(t)} a_1(u,t)du,$$

where f is the inverse function of $t-\tau_1(t)$. Then for each continuous initial function $\psi : [m_1(0), 0] \to \mathbb{R}$, there is a unique continuous function $x : [m_1(0), \infty) \to \mathbb{R}$ with $x(t) = \psi(t)$ on $[m_1(0), 0]$ that satisfies (1.2) on $[0, \infty)$. Moreover, x is bounded on $[m_1(0), \infty)$. Furthermore, the zero solution of (1.2) is stable at $t = 0$. If, in addition,

$$\int_0^t G(s,s)ds \to \infty \quad \text{as } t \to \infty, \tag{1.7}$$

then $x(t) \to 0$ as $t \to \infty$.

Theorem 1.3 ([15]). *Let τ_1 be differentiable. Suppose that there exist constants $k \ge 0$, $\alpha \in (0,1)$ and a function $h_1 \in C(\mathbb{R}^+, \mathbb{R})$ such that for $t \ge 0$,*

$$-\int_0^t h_1(s)ds \le k, \tag{1.8}$$

$$
\begin{aligned}
&\int_{t-\tau_1(t)}^{t} |h_1(s) + B_1(t,s)| ds \\
&+ \int_0^t e^{-\int_s^t h_1(u)du} |h_1(s)| \Big(\int_{s-\tau_1(s)}^{s} |h_1(u) + B_1(s,u)| du \Big) ds \qquad (1.9)\\
&+ \int_0^t e^{-\int_s^t h_1(u)du} |h_1(s-\tau_1(s)) + B_1(s, s-\tau_1(s))| |1-\tau_1'(s)| \le \alpha,
\end{aligned}
$$

where

$$
B_1(t,s) = \int_t^s a_1(u,s)du \quad \text{with } B_1(t, t-\tau_1(t)) = \int_t^{t-\tau_1(t)} a_1(u, t-\tau_1(t))du.
$$

Then for each continuous initial function $\psi : [m_1(0), 0] \to \mathbb{R}$, there is an unique continuous function $x : [m_1(0), \infty) \to \mathbb{R}$ with $x(t) = \psi(t)$ on $[m_1(0), 0]$ that satisfies (1.2) *on $[0, \infty)$. Moreover, x is bounded on $[m_1(0), \infty)$. Furthermore, the zero solution of* (1.2) *is stable at $t = 0$. If, in addition,*

$$
\int_0^t h_1(s)ds \to \infty \quad \text{as } t \to \infty, \qquad (1.10)
$$

then $x(t) \to 0$ as $t \to \infty$.

Remark 1.4. The result by Becker and Burton in Theorem 1.2 requires that $t - \tau_1(t)$ be strictly increasing. In Theorem 1.3, this condition is clearly removed. Also, the conditions of stability in Theorem 1.3 are less restrictive than Theorem 1.2. Thus, Theorem 1.3 improves Theorems 1.1 and 1.2.

Our objective here is to improve Theorem 1.3 and extend it to investigate a wide class of linear integro-differential equation with variable delays of neutral type presented in (1.1). To do this we define a suitable continuous function H (see Theorem 2.2 below) and find conditions for H, with no need of further assumptions on the inverse of delays $t - \tau_j(t)$, so that for a given continuous initial function ψ a mapping P for (1.1) is constructed in such a way to map a complete metric space S_ψ in itself and in which P possesses a fixed point. This procedure will enable us to establish and prove an asymptotic stability theorem for the zero solution of (1.1) with a necessary and sufficient condition and with less restrictive conditions. The results obtained in this paper improve and generalize the main results in [4, 5, 15]. We provide an example to illustrate our claim.

2. Main results

For each $\psi \in C([m(0), 0], \mathbb{R})$, a solution of (1.1) through $(0, \psi)$ is a continuous function $x : [m(0), \sigma) \to \mathbb{R}$ for some positive constant $\sigma > 0$ such that x satisfies (1.1) on $[0, \sigma)$ and $x = \psi$ on $[m(0), 0]$. We denote such a solution by $x(t) = x(t, 0, \psi)$. From the existence theory we can conclude that for each $\psi \in C([m(0), 0], \mathbb{R})$, there exists a unique solution $x(t) = x(t, 0, \psi)$ of (1.1) defined on $[0, \infty)$. We define $\|\psi\| = \max\{|\psi(t)| : m(0) \le t \le 0\}$. Stability definitions may be found in [6], for example.

Our aim here is to generalize Theorem 1.3 to equation (1.1) by giving a necessary and sufficient condition for asymptotic stability of the zero solution.

It is known that studying the stability of an equation using a fixed point technic involves the construction of a suitable fixed point mapping. This can be an arduous task. So, to construct our mapping, we begin by transforming (1.1) to a more tractable, but equivalent, equation, which we then invert to obtain an equivalent integral equation from which we derive a fixed point mapping. After that, we define a suitable complete space, depending on the initial condition, so that the mapping is a contraction. Using Banach's contraction mapping principle, we obtain a solution for this mapping, and hence a solution for (1.1), which is asymptotically stable.

First, we have to transform (1.1) into an equivalent equation that possesses the same basic structure and properties to which we apply the variation of parameters to define a fixed point mapping.

Lemma 2.1. *Equation* (1.1) *is equivalent to*

$$\begin{aligned} x'(t) &= \sum_{j=1}^{N} B_j(t, t-\tau_j(t))(1-\tau_j'(t))x(t-\tau_j(t)) \\ &\quad + \sum_{j=1}^{N} \frac{d}{dt}\int_{t-\tau_j(t)}^{t} B_j(t,s)x(s)ds + \sum_{j=1}^{N} c_j(t)x'(t-\tau_j(t)), \end{aligned} \tag{2.1}$$

where

$$B_j(t,s) = \int_t^s a_j(u,s)du \quad \text{and} \quad B_j(t,t-\tau_j(t)) = \int_t^{t-\tau_j(t)} a_j(u,t-\tau_j(t))du.$$

Proof. Differentiating the integral term in (2.1), we obtain

$$\begin{aligned} &\frac{d}{dt}\int_{t-\tau_j(t)}^{t} B_j(t,s)x(s)ds \\ &= B_j(t,t)x(t) - B_j(t,t-\tau_j(t))(1-\tau_j'(t))x(t-\tau_j(t)) + \int_{t-\tau_j(t)}^{t} \frac{\partial}{\partial t}B_j(t,s)x(s)ds. \end{aligned}$$

Substituting this into (2.1), it follows that (2.1) is equivalent to (1.1) provided B_j satisfies the following conditions

$$B_j(t,t) = 0 \quad \text{and} \quad \frac{\partial}{\partial t}B_j(t,s) = -a_j(t,s). \tag{2.2}$$

This euqality implies

$$B_j(t,s) = -\int_0^t a_j(u,s)du + \phi(s), \tag{2.3}$$

for some function ϕ, and $B_j(t,s)$ must satisfy

$$B_j(t,t) = -\int_0^t a_j(u,t)du + \phi(t) = 0.$$

Consequently,

$$\phi(t) = \int_0^t a_j(u,t)du.$$

Substituting this into (2.3), we obtain

$$B_j(t,s) = -\int_0^t a_j(u,s)du + \int_0^s a_j(u,s)du = \int_t^s a_j(u,s)du.$$

This definition of B_j satisfies (2.2). Consequently, (1.1) is equivalent to (2.1). $\square$

Theorem 2.2. *Suppose that τ_j is twice differentiable and $\tau_j'(t) \neq 1$ for all $t \in \mathbb{R}^+$, and there exist continuous functions $h_j : [m_j(0), \infty) \to \mathbb{R}$ for $j = 1, 2, \dots, N$ and a constant $\alpha \in (0,1)$ such that for $t \geq 0$*

$$\liminf_{t\to\infty} \int_0^t H(s)ds > -\infty, \tag{2.4}$$

and

$$\begin{aligned} &\sum_{j=1}^N \Big|\frac{c_j(t)}{1-\tau_j'(t)}\Big| + \sum_{j=1}^N \int_{t-\tau_j(t)}^t |h_j(s) + B_j(t,s)|ds \\ &+ \sum_{j=1}^N \int_0^t e^{-\int_s^t H(u)du} |[h_j(s-\tau_j(s)) + B(s, s-\tau_j(s))](1-\tau_j'(s)) - r_j(s)|ds \\ &+ \sum_{j=1}^N \int_0^t e^{-\int_s^t H(u)du} |H(s)| \Big(\int_{s-\tau_j(s)}^s |h_j(u) + B_j(s,u)|du\Big) ds \leq \alpha, \end{aligned} \tag{2.5}$$

where

$$H(t) = \sum_{j=1}^N h_j(t), \quad r_j(t) = \frac{[c_j(t)H(t) + c_j'(t)](1-\tau_j'(t)) + c_j(t)\tau_j''(t)}{(1-\tau_j'(t))^2},$$

and

$$B_j(t,s) = \int_t^s a_j(u,s)du \quad \textit{with} \quad B_j(t, t-\tau_j(t)) = \int_t^{t-\tau_j(t)} a_j(u, t-\tau_j(t))du.$$

Then the zero solution of (1.1) is asymptotically stable if and only if

$$\int_0^t H(s)ds \to \infty \quad \textit{as } t \to \infty. \tag{2.6}$$

Proof. First, suppose that (2.6) holds. We set

$$K = \sup_{t\geq 0}\{e^{-\int_0^t H(s)ds}\}. \tag{2.7}$$

Let $\psi \in C([m(0), 0], \mathbb{R})$ be fixed and define

$S_\psi := \{\varphi \in C([m(0), \infty), \mathbb{R}) : \varphi(t) = \psi(t) \text{ for } t \in [m(0), 0] \text{ and } \varphi(t) \to 0 \text{ as } t \to \infty\}.$

Endowed with the supremum norm $\|\cdot\|$; that is, for $\phi \in S_\psi$,

$$\|\phi\| := \sup\{|\phi(t)| : t \in [m(0), \infty)\}.$$

In other words, we carry out our investigations in the complete metric space (S_ψ, ρ) where ρ is supremum metric

$$\rho(x,y) := \sup_{t\geq m(0)} |x(t) - y(t)| = \|x - y\|, \quad \text{for } x, y \in S_\psi.$$

Rewrite (1.1) in the following equivalent form

$$\begin{aligned} x'(t) = &\sum_{j=1}^N B_j(t, t-\tau_j(t))(1-\tau_j'(t))x(t-\tau_j(t)) \\ &+ \sum_{j=1}^N \frac{d}{dt}\int_{t-\tau_j(t)}^t B_j(t,s)x(s)ds + \sum_{j=1}^N c_j(t)x'(t-\tau_j(t)) \end{aligned} \tag{2.8}$$

Multiplying both sides of (2.8) by $\exp\big(\int_0^t H(u)du\big)$ and integrating with respect to s from 0 to t, we obtain

$$\begin{aligned}
x(t) = \psi(0)e^{-\int_0^t H(u)du} &+ \int_0^t e^{-\int_s^t H(u)du} \sum_{j=1}^N h_j(s)x(s)ds \\
&+ \int_0^t e^{-\int_s^t H(u)du} \sum_{j=1}^N \frac{d}{ds}\int_{s-\tau_j(s)}^s B_j(s,u)x(u)du \\
&+ \int_0^t e^{-\int_s^t H(u)du} \sum_{j=1}^N B_j(s,s-\tau_j(s))(1-\tau_j'(s))x(s-\tau_j(s))ds \\
&+ \int_0^t e^{-\int_s^t H(u)du} \sum_{j=1}^N c_j(s)x'(s-\tau_j(s))ds.
\end{aligned}$$

Thus,

$$\begin{aligned}
x(t) = \psi(0)e^{-\int_0^t H(u)du} &+ \sum_{j=1}^N \int_0^t e^{-\int_s^t H(u)du} h_j(s)x(s)ds \\
&+ \sum_{j=1}^N \int_0^t e^{-\int_s^t H(u)du} \frac{d}{ds}\int_{s-\tau_j(s)}^s B_j(s,u)x(u)du \\
&+ \sum_{j=1}^N \int_0^t e^{-\int_s^t H(u)du} B_j(s,s-\tau_j(s))(1-\tau_j'(s))x(s-\tau_j(s))ds \\
&+ \sum_{j=1}^N \int_0^t e^{-\int_s^t H(u)du} c_j(s)x'(s-\tau_j(s))ds.
\end{aligned}$$

Performing an integration by parts, we obtain

$$\begin{aligned}
x(t) &= \psi(0)e^{-\int_0^t H(u)du} + \sum_{j=1}^N \int_0^t e^{-\int_s^t H(u)du} d\Big(\int_{s-\tau_j(s)}^s [h_j(u)+B_j(s,u)]x(u)du\Big) \\
&\quad + \sum_{j=1}^N \int_0^t e^{-\int_s^t H(u)du} [h_j(s-\tau_j(s))+B_j(s,s-\tau_j(s))] \\
&\quad \times (1-\tau_j'(s))x(s-\tau_j(s))ds + \sum_{j=1}^N \int_0^t \frac{c_j(s)}{1-\tau_j'(s)} e^{-\int_s^t H(u)du} dx(s-\tau_j(s)) \\
&= \Big(\psi(0) - \sum_{j=1}^N \frac{c_j(0)}{1-\tau_j'(0)}\psi(-\tau_j(0)) - \sum_{j=1}^N \int_{-\tau_j(0)}^0 [h_j(s)+B_j(0,s)]\psi(s)ds\Big) \\
&\quad \times e^{-\int_0^t H(u)du} \\
&+ \sum_{j=1}^N \frac{c_j(t)}{1-\tau_j'(t)} x(t-\tau_j(t)) + \sum_{j=1}^N \int_{t-\tau_j(t)}^t [h_j(s)+B_j(t,s)]x(s)ds \\
&\quad + \sum_{j=1}^N \int_0^t e^{-\int_s^t H(u)du} \{[h_j(s-\tau_j(s))+B_j(s,s-\tau_j(s))]
\end{aligned}$$

$$
\begin{aligned}
&\times (1-\tau_j'(s)) - r_j(s)\}x(s-\tau_j(s))ds \\
&-\sum_{j=1}^N \int_0^t e^{-\int_s^t H(u)du} H(s)\Big(\int_{s-\tau_j(s)}^s [h_j(u)+B_j(s,u)]x(u)du\Big)ds.
\end{aligned}
$$

Now use this equality to define the operator $P : S_\psi \to S_\psi$ by $(P\varphi)(t) = \psi(t)$ if $t \in [m(0), 0]$ and for $t \geq 0$ we let

$$
\begin{aligned}
(P\varphi)(t) = &\Big(\psi(0) - \sum_{j=1}^N \frac{c_j(0)}{1-\tau_j'(0)}\psi(-\tau_j(0)) \\
&- \sum_{j=1}^N \int_{-\tau_j(0)}^0 [h_j(s)+B_j(0,s)]\psi(s)ds\Big)e^{-\int_0^t H(u)du} \\
&+ \sum_{j=1}^N \frac{c_j(t)}{1-\tau_j'(t)}\varphi(t-\tau_j(t)) + \sum_{j=1}^N \int_{t-\tau_j(t)}^t [h_j(s)+B_j(t,s)]\varphi(s)ds \\
&+ \sum_{j=1}^N \int_0^t e^{-\int_s^t H(u)du}\{[h_j(s-\tau_j(s)) + B_j(s,s-\tau_j(s))] \\
&\times (1-\tau_j'(s)) - r_j(s)\}\varphi(s-\tau_j(s))ds \\
&- \sum_{j=1}^N \int_0^t e^{-\int_s^t H(u)du} H(s)\Big(\int_{s-\tau_j(s)}^s [h_j(u)+B_j(s,u)]\varphi(u)du\Big)ds.
\end{aligned} \tag{2.9}
$$

It is clear that $(P\varphi) \in C([m(0),\infty),\mathbb{R})$. We will show that $(P\varphi)(t) \to 0$ as $t \to \infty$. To this end, denote the five terms on the right hand side of (2.9) by $I_1, I_2, \dots I_5$, respectively. It is obvious that the first term I_1 tends to zero as $t \to \infty$, by condition (2.6). Also, due to the facts that $\varphi(t) \to 0$ and $t - \tau_j(t) \to \infty$ for $j = 1, 2, \dots, N$ as $t \to \infty$, the second term I_2 in (2.9) tends to zero as $t \to \infty$. What is left to show is that each of the remaining terms in (2.9) go to zero at infinity.

Let $\varphi \in S_\psi$ be fixed. For a given $\varepsilon > 0$, we choose $T_0 > 0$ large enough such that $t - \tau_j(t) \geq T_0$, $j = 1, 2, \dots, N$, implies $|\varphi(s)| < \varepsilon$ if $s \geq t - \tau_j(t)$. Therefore, the third term I_3 in (2.9) satisfies

$$
\begin{aligned}
|I_3| &= |\sum_{j=1}^N \int_{t-\tau_j(t)}^t [h_j(s)+B_j(t,s)]\varphi(s)ds| \\
&\leq \sum_{j=1}^N \int_{t-\tau_j(t)}^t |h_j(s)+B_j(t,s)||\varphi(s)|ds \\
&\leq \varepsilon \sum_{j=1}^N \int_{t-\tau_j(t)}^t |h_j(s)+B_j(t,s)|ds \leq \alpha\varepsilon < \varepsilon.
\end{aligned}
$$

Thus, $I_3 \to 0$ as $t \to \infty$. Now consider I_4. For the given $\varepsilon > 0$, there exists a $T_1 > 0$ such that $s \geq T_1$ implies $|\varphi(s-\tau_j(s))| < \varepsilon$ for $j = 1, 2, \dots, N$. Thus, for $t \geq T_1$, the term I_4 in (2.9) satisfies

$$
|I_4| = \Big|\sum_{j=1}^N \int_0^t e^{-\int_s^t H(u)du}\big\{[h_j(s-\tau_j(s)) + B_j(s,s-\tau_j(s))](1-\tau_j'(s)) - r_j(s)\big\}
$$

$$
\begin{aligned}
&\times \varphi(s-\tau_j(s))ds\Big| \\
&\le \sum_{j=1}^{N}\int_0^{T_1} e^{-\int_s^t H(u)du}\big|[h_j(s-\tau_j(s))+B_j(s,s-\tau_j(s))](1-\tau_j'(s))-r_j(s)\big| \\
&\quad |\varphi(s-\tau_j(s))|ds \\
&\quad + \sum_{j=1}^{N}\int_{T_1}^{t} e^{-\int_s^t H(u)du}|[h_j(s-\tau_j(s))+B_j(s,s-\tau_j(s))](1-\tau_j'(s))-r_j(s)\big| \\
&\quad |\varphi(s-\tau_j(s))|ds \\
&\le \sup_{\sigma\ge m(0)}|\varphi(\sigma)|\sum_{j=1}^{N}\int_0^{T_1} e^{-\int_s^t H(u)du}\big|[h_j(s-\tau_j(s))+B_j(s,s-\tau_j(s))] \\
&\quad \times (1-\tau_j'(s))-r_j(s)\big|ds \\
&\quad + \varepsilon\sum_{j=1}^{N}\int_{T_1}^{t} e^{-\int_s^t H(u)du}|[h_j(s-\tau_j(s))+B_j(s,s-\tau_j(s))](1-\tau_j'(s))-r_j(s)|ds.
\end{aligned}
$$

By (2.6), we can find $T_2 > T_1$ such that $t \ge T_2$ implies

$$
\begin{aligned}
&\sup_{\sigma\ge m(0)}|\varphi(\sigma)|\sum_{j=1}^{N}\int_0^{T_1} e^{-\int_s^t H(u)du}|[h_j(s-\tau_j(s))+B_j(s,s-\tau_j(s))] \\
&\times (1-\tau_j'(s))-r_j(s)|ds \\
&= \sup_{\sigma\ge m(0)}|\varphi(\sigma)|e^{-\int_{T_2}^t H(u)du}\sum_{j=1}^{N}\int_0^{T_1} e^{-\int_s^{T_2} H(u)du}|[h_j(s-\tau_j(s))+B_j(s,s-\tau_j(s))] \\
&\quad \times (1-\tau_j'(s))-r_j(s)|ds < \varepsilon.
\end{aligned}
$$

Now, apply (2.5) to have $|I_4| < \varepsilon + \alpha\varepsilon < 2\varepsilon$. Thus, $I_4 \to 0$ as $t \to \infty$. Similarly, by using (2.5), then, if $t \ge T_2$ then term I_5 in (2.9) satisfies

$$
\begin{aligned}
|I_5| &= \Big|\sum_{j=1}^{N}\int_0^{t} e^{-\int_s^t H(u)du}H(s)\Big(\int_{s-\tau_j(s)}^{s}[h_j(u)+B_j(s,u)]\varphi(u)du\Big)ds\Big| \\
&\le \sup_{\sigma\ge m(0)}|\varphi(\sigma)|e^{-\int_{T_2}^t H(u)du}\sum_{j=1}^{N}\int_0^{T_1} e^{-\int_s^{T_2} H(u)du}|H(s)| \\
&\quad \times \Big(\int_{s-\tau_j(s)}^{s}|h_j(u)+B_j(s,u)|du\Big)ds \\
&\quad + \varepsilon\sum_{j=1}^{N}\int_{T_1}^{t} e^{-\int_s^t H(u)du}|H(s)|\Big(\int_{s-\tau_j(s)}^{s}|h_j(u)+B_j(s,u)|du\Big)ds \\
&< \varepsilon + \alpha\varepsilon < 2\varepsilon.
\end{aligned}
$$

Thus, $I_5 \to 0$ as $t \to \infty$. In conclusion $(P\varphi)(t) \to 0$ as $t \to \infty$, as required. Hence P maps S_ψ into S_ψ. Also, by condition (2.5), P is a contraction mapping with contraction constant α. Indeed, for $\phi, \eta \in S_\psi$ and $t > 0$

$$
|(P\varphi)(t) - (P\eta)(t)|
$$

$$
\begin{aligned}
&\le \sum_{j=1}^{N} \big|\frac{c_j(t)}{1-\tau_j'(t)}\big| |\varphi(t-\tau_j(t)) - \eta(t-\tau_j(t))| \\
&\quad + \sum_{j=1}^{N} \int_{t-\tau_j(t)}^{t} |h_j(s) + B_j(t,s)||\varphi(s) - \eta(s)| ds \\
&+ \sum_{j=1}^{N} \int_0^t e^{-\int_s^t H(u)du} |[h_j(s-\tau_j(s)) + B_j(s, s-\tau_j(s))] \\
&\quad \times (1-\tau_j'(s)) - r_j(s)||\varphi(s-\tau_j(s)) - \eta(s-\tau_j(s))| ds \\
&\quad + \sum_{j=1}^{N} \int_0^t e^{-\int_s^t H(u)du} H(s) \Big(\int_{s-\tau_j(s)}^{s} |h_j(u) + B_j(s,u)||\varphi(u) - \eta(u)| du \Big) ds \\
&\le \Big(\sum_{j=1}^{N} \big|\frac{c_j(t)}{1-\tau_j'(t)}\big| + \sum_{j=1}^{N} \int_{t-\tau_j(t)}^{t} |h_j(s) + B_j(t,s)| ds \\
&\quad + \sum_{j=1}^{N} \int_0^t e^{-\int_s^t H(u)du} |[h_j(s-\tau_j(s)) + B_j(s, s-\tau_j(s))](1-\tau_j'(s)) - r_j(s)| ds \\
&\quad + \sum_{j=1}^{N} \int_0^t e^{-\int_s^t H(u)du} H(s) \Big(\int_{s-\tau_j(s)}^{s} |h_j(u) + B_j(s,u)| du \Big) ds \Big) \|\varphi - \eta\|.
\end{aligned}
$$

By condition (2.5), P is a contraction mapping with constant α. By the contraction mapping principle (Smart [19, p. 2]), P has a unique fixed point x in S_ψ which is a solution of (1.1) with $x(t) = \psi(t)$ on $[m(0), 0]$ and $x(t) = x(t, 0, \psi) \to 0$ as $t \to \infty$.

To obtain the asymptotic stability, we need to show that the zero solution of (1.1) is stable. Let $\varepsilon > 0$ be given and choose $\delta > 0$ ($\delta < \varepsilon$) satisfying $2\delta K + \alpha\varepsilon < \varepsilon$. If $x(t) = x(t, 0, \psi)$ is a solution of (1.1) with $\|\psi\| < \delta$, then $x(t) = (Px)(t)$ defined in ((2.9). We claim that $|x(t)| < \varepsilon$ for all $t \ge t_0$. Notice that $|x(s)| < \varepsilon$ on $[m(0), 0]$. If there exists $t^* > 0$ such that $|x(t^*)| = \varepsilon$ and $|x(s)| < \varepsilon$ for $m(0) \le s < t^*$, then it follows from (2.9) that

$$
\begin{aligned}
&|x(t^*)| \\
&\le \|\psi\| \Big(1 + \sum_{j=1}^{N} \big|\frac{c_j(0)}{1-\tau_j'(0)}\big| + \sum_{j=1}^{N} \int_{-\tau_j(0)}^{0} |h_j(s) + B_j(0,s)| ds \Big) e^{-\int_0^{t^*} H(u)du} \\
&\quad + \varepsilon \sum_{j=1}^{N} \big|\frac{c_j(t^*)}{1-\tau_j'(t^*)}\big| + \varepsilon \sum_{j=1}^{N} \int_{t^*-\tau_j(t^*)}^{t^*} |h_j(s) + B_j(t^*,s)| ds \\
&\quad + \varepsilon \int_{t_0}^{t^*} e^{-\int_s^{t^*} H(u)du} |[h_j(s-\tau_j(s)) + B_j(s, s-\tau_j(s))](1-\tau_j'(s)) - r_j(s)| ds \\
&\quad + \varepsilon \sum_{j=1}^{N} \int_{t_0}^{t^*} e^{-\int_s^{t^*} H(u)du} |H(s)| \Big(\int_{s-\tau_j(s)}^{s} |h_j(u) + B_j(s,u)| du \Big) ds \\
&\le 2\delta K + \alpha\varepsilon < \varepsilon,
\end{aligned}
$$

which contradicts the definition of t^*. Thus, $|x(t)| < \varepsilon$ for all $t \geq 0$, and the zero solution of (1.1) is stable. This shows that the zero solution of (1.1) is asymptotically stable if (2.6) holds.

Conversely, suppose (2.6) fails. Then, by (2.4) there exists a sequence $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$ such that $\lim_{n\to\infty} \int_0^{t_n} H(u)du = l$ for some $l \in \mathbb{R}$. We may also choose a positive constant J satisfying

$$-J \leq \int_0^{t_n} H(u)du \leq J,$$

for all $n \geq 1$. To simplify our expressions, we define

$$\begin{aligned}\omega(s) &= \sum_{j=1}^{N} |[h_j(s-\tau_j(s)) + B_j(s, s-\tau_j(s))](1-\tau_j'(s)) - r_j(s)| \\ &\quad + \sum_{j=1}^{N} |H(s)|(\int_{s-\tau_j(s)}^{s} |h_j(u) + B(s,u)|du),\end{aligned}$$

for all $s \geq 0$. By (2.5), we have

$$\int_0^{t_n} e^{-\int_s^{t_n} H(u)du}\omega(s)ds \leq \alpha.$$

This yields

$$\int_0^{t_n} e^{\int_0^{s} H(u)du}\omega(s)ds \leq \alpha e^{\int_0^{t_n} H(u)du} \leq J.$$

The sequence $\{\int_0^{t_n} e^{\int_0^s H(u)du}\omega(s)ds\}$ is bounded, so there exists a convergent subsequence. For brevity of notation, we may assume that

$$\lim_{n\to\infty} \int_0^{t_n} e^{\int_0^{s} H(u)du}\omega(s)ds = \gamma,$$

for some $\gamma \in \mathbb{R}^+$ and choose a positive integer m so large that

$$\int_{t_m}^{t_n} e^{\int_0^{s} H(u)du}\omega(s)ds < \delta_0/4K,$$

for all $n \geq m$, where $\delta_0 > 0$ satisfies $2\delta_0 K e^J + \alpha \leq 1$.

By (2.4), K in (2.7) is well defined. We now consider the solution $x(t) = x(t, t_m, \psi)$ of (1.1) with $\psi(t_m) = \delta_0$ and $|\psi(s)| \leq \delta_0$ for $s \leq t_m$. We may choose ψ so that $|x(t)| \leq 1$ for $t \geq t_m$ and

$$\begin{aligned}&\psi(t_m) - \sum_{j=1}^{N}[\frac{c_j(t_m)}{1-\tau_j'(t_m)}\psi(t_m - \tau_j(t_m)) \\ &+ \int_{t_m-\tau_j(t_m)}^{t_m} [h_j(s) + B_j(t_m, s)]\psi(s)ds] \\ &\geq \frac{1}{2}\delta_0.\end{aligned}$$

It follows from (2.9) with $x(t) = (Px)(t)$ that for $n \ge m$

$$\begin{aligned}
&\Big|x(t_n) - \sum_{j=1}^{N}\Big[\frac{c_j(t_n)}{1-\tau_j'(t_n)}x(t_n-\tau_j(t_n)) + \int_{t_n-\tau_j(t_n)}^{t_n}[h_j(s)+B_j(t_n,s)]x(s)ds\Big]\Big| \\
&\ge \frac{1}{2}\delta_0 e^{-\int_{t_m}^{t_n}H(u)du} - \int_{t_m}^{t_n} e^{-\int_s^{t_n}H(u)du}\omega(s)ds \\
&= \frac{1}{2}\delta_0 e^{-\int_{t_m}^{t_n}H(u)du} - e^{-\int_0^{t_n}H(u)du}\int_{t_m}^{t_n} e^{\int_0^s H(u)du}\omega(s)ds \\
&= e^{-\int_{t_m}^{t_n}H(u)du}\Big(\frac{1}{2}\delta_0 - e^{-\int_0^{t_m}H(u)du}\int_{t_m}^{t_n} e^{\int_0^s H(u)du}\omega(s)ds\Big) \\
&\ge e^{-\int_{t_m}^{t_n}H(u)du}\Big(\frac{1}{2}\delta_0 - K\int_{t_m}^{t_n} e^{\int_0^s H(u)du}\omega(s)ds\Big) \\
&\ge \frac{1}{4}\delta_0 e^{-\int_{t_m}^{t_n}H(u)du} \ge \frac{1}{4}\delta_0 e^{-2J} > 0.
\end{aligned} \tag{2.10}$$

On the other hand, if the zero solution of (1.1) is asymptotically stable, then $x(t) = x(t, t_m, \psi) \to 0$ as $t \to \infty$. Since $t_n - \tau_j(t_n) \to \infty$ as $n \to \infty$ and (2.5) holds, we have

$$x(t_n) - \sum_{j=1}^{N}\Big[\frac{c_j(t_n)}{1-\tau_j'(t_n)}x(t_n-\tau_j(t_n)) + \int_{t_n-\tau_j(t_n)}^{t_n}[h_j(s)+B_j(t_n,s)]x(s)ds\Big] \to 0$$

as $n \to \infty$, which contradicts (2.10). Hence condition (2.6) is necessary for the asymptotic stability of the zero solution of (1.1). The proof is complete. $\square$

Remark 2.3. It follows from the first part of the proof of Theorem 2.2 that the zero solution of (1.1) is stable under (2.4) and (2.5). Moreover, Theorem 2.2 still holds if (2.5) is satisfied for $t \ge t_\sigma$ for some $t_\sigma \in \mathbb{R}^+$.

For the special case $N = 1$ and $c_1 = 0$, we have the following result.

Corollary 2.4. *Suppose that τ_1 is differentiable and there exist continuous function $h_1 : [m_1(0), \infty) \to \mathbb{R}$ and a constant $\alpha \in (0, 1)$ such that for $t \ge 0$*

$$\liminf_{t\to\infty} \int_0^t h_1(s)ds > -\infty, \tag{2.11}$$

and

$$\begin{aligned}
&\int_{t-\tau_1(t)}^{t} |h_1(s) + B_1(t,s)|ds \\
&+ \int_0^t e^{-\int_s^t h_1(u)du}|h_1(s-\tau_1(s)) + B_1(s, s-\tau_1(s))||1-\tau_1'(s)|ds \\
&+ \int_0^t e^{-\int_s^t h_1(u)du}|h_1(s)|(\int_{s-\tau_1(s)}^{s}|h_1(u)+B_1(s,u)|du)ds \le \alpha.
\end{aligned} \tag{2.12}$$

Then the zero solution of (1.2) is asymptotically stable if and only if

$$\int_0^t h_1(s)ds \to \infty \quad \text{as } t \to \infty. \tag{2.13}$$

Obviously, Corollary 2.4 extends Theorem 1.3. Thus, Theorem 2.2 generalizes Theorem 1.3.

3. An Example

In this section, we give an example to illustrate the applications of Theorem 2.2.

Example 3.1. *Consider the linear neutral integro-differential equation with variable delays*

$$x'(t) = -\sum_{j=1}^{2} \int_{t-\tau_j(t)}^{t} a_j(t,s)x(s)ds + \sum_{j=1}^{2} c_j(t)x'(t-\tau_j(t)), \tag{3.1}$$

where $\tau_1(t) = 0.489t$, $\tau_2(t) = 0.478t$, $a_1(t,s) = 0.48/(s^2+1)$, $a_2(t,s) = 0.52/(s^2+1)$, $c_1(t) = 0.015$, $c_2(t) = 0.017$. *Then the zero solution of* (3.1) *is asymptotically stable.*

Proof. We have

$$B_1(t,s) = \int_t^s \frac{0.48}{s^2+1} du = \frac{0.48(s-t)}{s^2+1}, \quad B_2(t,s) = \int_t^s \frac{0.52}{s^2+1} du = \frac{0.52(s-t)}{s^2+1}.$$

Choosing $h_1(t) = 0.52t/(t^2+1)$ and $h_2(t) = 0.48t/(t^2+1)$ in Theorem 2.2, we have $H(t) = t/(t^2+1)$ and

$$\sum_{j=1}^{2} \left|\frac{c_j(t)}{1-\tau_j'(t)}\right| = \left|\frac{0.015}{1-0.489}\right| + \left|\frac{0.017}{1-0.478}\right| < 0.062,$$

$$\begin{aligned}
&\sum_{j=1}^{2} \int_{t-\tau_j(t)}^{t} |h_j(s) + B_j(t,s)|\, ds \\
&= \int_{0.511t}^{t} \left|\frac{s-0.48t}{s^2+1}\right| ds + \int_{0.522t}^{t} \left|\frac{s-0.52t}{s^2+1}\right| ds \\
&= \int_{0.511t}^{t} \frac{s-0.48t}{s^2+1} ds + \int_{0.522t}^{t} \frac{s-0.52t}{s^2+1} ds \\
&= t[0.48 \arctan 0.511t + 0.52 \arctan 0.522t - \arctan t] + \ln(t^2+1) \\
&\quad - \frac{1}{2}[\ln(0.511^2t^2+1) + \ln(0.522^2t^2+1)] \\
&= \omega(t).
\end{aligned}$$

Since the function ω is increasing in $[0,\infty)$ and

$$\lim_{t\to\infty} \omega(t) = 1 - 0.48/0.511 - 0.52/0.522 - \ln(0.511 \times 0.522) \simeq 0.386,$$

then

$$\sum_{j=1}^{2} \int_{t-\tau_j(t)}^{t} |h_j(s) + B_j(t,s)|\, ds < 0.386,$$

$$\sum_{j=1}^{2} \int_0^t e^{-\int_s^t H(u)du} |H(s)| \Big(\int_{s-\tau_j(s)}^{s} |h_j(u) + B_j(s,u)| du \Big) ds < 0.386,$$

and

$$
\begin{aligned}
&\sum_{j=1}^{2}\int_0^t e^{-\int_s^t H(u)du}|[h_j(s-\tau_j(s))+B(s,s-\tau_j(s))](1-\tau_j'(s))-r_j(s)|ds\\
&=\int_0^t e^{-\int_s^t \frac{u}{u^2+1}du}|0.511(\frac{0.52\times 0.511s}{0.511^2s^2+1}+\frac{0.48(0.511s-s)}{0.511^2s^2+1})-\frac{0.015s}{0.511(s^2+1)}|ds\\
&+\int_0^t e^{-\int_s^t \frac{u}{u^2+1}du}|0.522(\frac{0.48\times 0.522s}{0.522^2s^2+1}+\frac{0.52(0.522s-s)}{0.522^2s^2+1})-\frac{0.017s}{0.522(s^2+1)}|ds\\
&\le (1-\frac{0.48}{0.511})\int_0^t e^{-\int_s^t \frac{u}{u^2+1}du}\frac{s}{s^2+1/0.511^2}ds+\frac{0.015}{0.511}\int_0^t e^{-\int_s^t \frac{u}{u^2+1}du}\frac{s}{s^2+1}ds\\
&+(1-\frac{0.52}{0.522})\int_0^t e^{-\int_s^t \frac{u}{u^2+1}du}\frac{s}{s^2+1/0.522^2}ds+\frac{0.017}{0.522}\int_0^t e^{-\int_s^t \frac{u}{u^2+1}du}\frac{s}{s^2+1}ds\\
&<1-\frac{0.48}{0.511}+\frac{0.015}{0.511}+1-\frac{0.52}{0.522}+\frac{0.017}{0.522}<0.127.
\end{aligned}
$$

It is easy to see that all the conditions of Theorem 2.2 hold for $\alpha = 0.062+0.386+0.386+0.127 = 0.961 < 1$. Thus, Theorem 2.2 implies that the zero solution of (3.1) is asymptotically stable. □

Acknowledgements. The authors would like to express their sincere thanks to the anonymous referee for his/her careful reading of our manuscript and the helpful report.

References

[1] A. Ardjouni, A. Djoudi; Fixed points and stability in linear neutral differential equations with variable delays. *Nonlinear Analysis 74 (2011) 2062-2070.*

[2] A. Ardjouni, A. Djoudi; Stability in nonlinear neutral differential with variable delays using fixed point theory, *Electronic Journal of Qualitative Theory of Differential Equations 2011, No. 43, 1-11.*

[3] A. Ardjouni, A. Djoudi; Fixed point and stability in neutral nonlinear differential equations with variable delays, *Opuscula Mathematica, VoL. 32, No. 1, 2012, pp. 5-19.*

[4] L. C. Becker, T. A. Burton; Stability, fixed points and inverse of delays, *Proc. Roy. Soc. Edinburgh 136A (2006) 245-275.*

[5] T. A. Burton; Fixed points and stability of a nonconvolution equation, *Proceedings of the American Mathematical Society 132 (2004) 3679–3687.*

[6] T. A. Burton; Stability by Fixed Point Theory for Functional Differential Equations, *Dover Publications, New York, 2006.*

[7] T. A. Burton; Liapunov functionals, fixed points, and stability by Krasnoselskii's theorem, *Nonlinear Studies 9 (2001) 181–190.*

[8] T. A. Burton; Stability by fixed point theory or Liapunov's theory: A comparison, *Fixed Point Theory 4 (2003) 15–32.*

[9] T. A. Burton, T. Furumochi; A note on stability by Schauder's theorem, *Funkcialaj Ekvacioj 44 (2001) 73–82.*

[10] T. A. Burton, T. Furumochi; Fixed points and problems in stability theory, *Dynamical Systems and Applications 10 (2001) 89–116.*

[11] T. A. Burton, T. Furumochi; Asymptotic behavior of solutions of functional differential equations by fixed point theorems, *Dynamic Systems and Applications 11 (2002) 499–519.*

[12] T. A. Burton, T. Furumochi; Krasnoselskii's fixed point theorem and stability, *Nonlinear Analysis 49 (2002) 445–454.*

[13] Y. M. Dib, M. R. Maroun, Y. N. Raffoul; Periodicity and stability in neutral nonlinear differential equations with functional delay, *Electronic Journal of Differential Equations, Vol. 2005(2005), No. 142, pp. 1-11.*

[14] A. Djoudi, R. Khemis; Fixed point techniques and stability for neutral nonlinear differential equations with unbounded delays, *Georgian Mathematical Journal, Vol. 13 (2006), No. 1, 25-34.*
[15] C. H. Jin, J. W. Luo; Stability of an integro-differential equation, *Computers and Mathematics with Applications 57 (2009) 1080-1088.*
[16] C. H. Jin, J. W. Luo; Stability in functional differential equations established using fixed point theory, *Nonlinear Anal. 68 (2008) 3307-3315.*
[17] C. H. Jin, J. W. Luo; Fixed points and stability in neutral differential equations with variable delays, *Proceedings of the American Mathematical Society, Vol. 136, Nu. 3 (2008) 909-918.*
[18] Y. N. Raffoul; Stability in neutral nonlinear differential equations with functional delays using fixed-point theory, *Math. Comput. Modelling 40 (2004) 691–700.*
[19] D. R. Smart, Fixed point theorems; *Cambridge Tracts in Mathematics, No. 66. Cambridge University Press, London-New York, 1974.*
[20] B. Zhang; Fixed points and stability in differential equations with variable delays, *Nonlinear Anal. 63 (2005) e233–e242.*

Abdelouaheb Ardjouni
Applied Mathematics Lab., Facullty of Sciences, Department of Mathematics, Badji Mokhtar University of Annaba, P.O. Box 12, Annaba 23000, Algeria
E-mail address: abd_ardjouni@yahoo.fr

Ahcene Djoudi
Applied Mathematics Lab., Facullty of Sciences, Department of Mathematics, Badji Mokhtar University of Annaba, P.O. Box 12, Annaba 23000, Algeria
E-mail address: adjoudi@yahoo.com

Imene Soualhia
Applied Mathematics Lab., Facullty of Sciences, Department of Mathematics, Badji Mokhtar University of Annaba, P.O. Box 12, Annaba 23000, Algeria