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UNIFORM DECAY FOR A LOCAL DISSIPATIVE KLEIN-GORDON-SCHRÖDINGER TYPE SYSTEM

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Abstract. In this article, we consider a nonlinear Klein-Gordon-Schrödinger type system in $\mathbb{R}^n$, where the nonlinear term exists and the damping term is effective. We prove the existence and uniqueness of a global solution and its exponential decay. The result is achieved by using the multiplier technique.

1. Introduction

We consider the nonlinear system of Klein-Gordon-Schrödinger type with locally distributed damping

$$i\psi_t + \kappa\Delta\psi + i\alpha\psi = \phi\psi\chi_\omega, \quad x \in \Omega,\ t > 0, \tag{1.1}$$

$$\phi_{tt} - \Delta\phi + \phi + \lambda(x)\phi_t = -\operatorname{Re}(F(x)\cdot\nabla\psi), \quad x \in \Omega,\ t > 0, \tag{1.2}$$

$$\psi = \phi = 0 \quad \text{on } \Gamma \times (0,\infty), \tag{1.3}$$

$$\psi(0) = \psi_0, \quad \phi(0) = \phi_0, \quad \phi_t(0) = \phi_1 \tag{1.4}$$

where $\psi_0 \in H_0^1(\Omega) \cap H^2(\Omega)$, $\phi_0 \in H_0^1(\Omega) \cap H^2(\Omega)$, $\phi_1 \in H_0^1(\Omega)$, Ω is a bounded domain of $\mathbb{R}^n$, $n \le 2$, $\kappa > 0$, $\alpha > 0$, Γ is the smooth boundary of Ω, ω is an open subset of Ω such that $\operatorname{meas}(\omega) > 0$ and $\lambda \in W^{1,\infty}(\Omega)$ is a nonnegative function.

In what follows χ_ω represents the characteristic function of ω; that is, $\chi = 1$ in ω and $\chi = 0$ in $\Omega \setminus \omega$. So that the nonlinearity term $\phi\psi$ exists, where the damping $\lambda(x)\phi_t$ takes palce and reciprocally.

Systems of Klein-Gordon-Schrödinger type have been studied for many years. For example in [3, 4, 8, 5, 9, 10] the authors studied problems such as existence and uniqueness of solutions, exponential decay, the existence of a global attractor and its finite dimensionality in one or higher dimensions, in bounded or unbounded domains.

The majority of works in the literature deals with linear dissipative terms, acting on both equations. Very few is known about the polynomial decay of a Klein-Gordon-Schrödinger type system. In [2] the author proves the polynomial decay when dealing with a localized dissipation in the wave equation and in [1] the authors prove a similar result when dealing with a KGS system, idea that inspired this work.

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The rest of this article is divided into four sections. In Section 2, the basic notation is given and the main assumptions made are quoted. In Section 3, the existence and uniqueness of global solutions are proved. Finally in Section 4, integral inequalities for the energy of the system are proved using the multiplier method combined with integral inequalities that can be found in [6] (See also [7]).

2. Notation and assumptions

Let us introduce some notation that will be used throughout this work. Denote by $H^s(\Omega)$ both the standard real and complex Sobolev spaces on Ω. For simplicity reasons sometimes we use H^s, L^s for $H^s(\Omega)$, $L^s(\Omega)$. Let $\|\cdot\|$, $(\cdot,\cdot)$ denote the norm and inner product in $L^2(\Omega)$ respectively, as well as the symbol $\cdot$ denotes the inner product in $\mathbb{R}^n$. Finally, C is a general symbol for any positive constant.

Let $x^0 \in \mathbb{R}^n$, $n \leq 2$ and $n(x)$ be the unit exterior normal vector at $x \in \Gamma$, $m(x) = x - x^0$, $x \in \mathbb{R}^n$, $n \leq 2$ and

$$R(x^0) := \sup_{x \in \overline{\Omega}} m(x) = \sup_{x \in \overline{\Omega}} |x - x^0|. \tag{2.1}$$

We set the norms

$$\|u\|_p^p = \int_\Omega |u|^p dx, \quad \|u\|_{\Gamma,p}^p = \int_\Gamma |u(x)|^p d\Gamma, \quad \|u\|_\infty = \operatorname{ess\,sup}_{x\in\Omega} |u(x)|.$$

Some of the basic tools used are: the embedding inequality

$$\|u\|_4 \leq c_1 \|\nabla u\|_2, \quad \text{for all } u \in H_0^1(\Omega);$$

Gagliardo-Nirenberg inequality for $n = 1$,

$$\|u\|_4 \leq c_2 \|u\|_2^{3/4} \|\nabla u\|_2^{1/4}, \quad \text{for all } u \in H_0^1(\Omega);$$

Gagliardo-Nirenberg inequality for $n = 2$,

$$\|u\|_4 \leq c_3 \|u\|^{1/2} \|\nabla u\|^{1/2}, \quad \text{for all } u \in H_0^1(\Omega);$$

and Young's inequality

$$ab \leq \frac{1}{p} a^p + \frac{1}{p'} b^{p'}, \quad \text{for all } a, b \geq 0.$$

Assumption 2.1. Let $\lambda \in W^{1,\infty}(\Omega)$ be a nonnegative function such that $\lambda(x) \geq \lambda_0 > 0$, a.e. in ω. If $\lambda(x) \geq \lambda_0 > 0$ in Ω, then $\chi_\omega \equiv 1$ in Ω (See [9]).

Assumption 2.2. Let ω be a neighborhood of $\overline{\Gamma(x^0)}$, where $\Gamma(x^0) := \{x \in \Gamma : m(x) \cdot n(x) > 0\}$.

Assumption 2.3. We assume that $F \in C^1(\Omega)$ and $F \in L^\infty(\Omega)$ with $\|F\|_\infty = M < +\infty$.

Assumption 2.4. Let there be a neighborhood $\hat{\omega}$ of $\overline{\Gamma(x^0)}$ such that $\hat{\omega} \cap \Omega \subset \omega$ and a vector field $h \in (C^1(\overline{\Omega}))^n$, such that $h = n$ on $\Gamma(x^0)$ $h \cdot n \geq 0$ a.e. in Γ, $h = 0$ on $\Omega \setminus \hat{\omega}$.

We conclude this section with the following lemma which will play essential role when establishing the asymptotic behavior of solutions in Section 4.

Lemma 2.5 ([6, Lemma 9.1]). *Let $E : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be a non-increasing function and assume that there exist two constants $p > 0$ and $c > 0$ such that*

$$\int_s^{+\infty} E^{(p+1)/2}(t)dt \le cE(s), \quad 0 \le s < +\infty.$$

Then, for all $t \ge 0$,

$$E(t) \le \begin{cases} cE(0)(1+t)^{-2(p-1)} & \text{if } p > 1, \\ cE(0)e^{1-wt} & \text{if } p = 1, \end{cases}$$

where c and w are positive constants.

3. Existence and uniqueness of solutions

In this section we derive a priori estimates for the solutions of the Klein-Gordon-Schrödinger system (1.1) - (1.4). Let us represent by w_n a basis in $H_0^1(\Omega) \cap H^2(\Omega)$ formed by the eigenfunctions of $-\Delta$, by V_m the subspace of $H_0^1(\Omega) \cap H^2(\Omega)$ generated by the first m vectors and by

$$\psi_m(t) = \sum_{i=1}^m g_{im}(t)w_i, \quad \phi_m(t) = \sum_{i=1}^m h_{im}w_i,$$

where $(\psi_m(t), \phi_m(t))$ is the solution of the Cauchy problem

$$i(\psi_{t,m}, u) + \kappa(\Delta\psi, u) + i\alpha(\psi_m, u) = (\phi_m\psi_m\chi_\omega, u) \quad \forall u \in V_m, \tag{3.1}$$

$$(\phi_{tt,m}, v) - (\Delta\phi_m, v) + (\phi_m, v) + (\lambda(x)\phi_{t,m}, v) = -\operatorname{Re}(F(x)\cdot\nabla\psi_m, v), \quad \forall v \in V_m, \tag{3.2}$$

with

$$\begin{gathered} \psi_m(x,0) = \psi_{0m} \to \psi_0, \quad \phi(x,0) = \phi_{0m} \to \phi_0 \quad \text{in } H_0^1(\Omega) \cap H^2(\Omega), \\ \phi_{t,m}(0) = \phi_{1m} \to \phi_1 \quad \text{in } H_0^1(\Omega). \end{gathered} \tag{3.3}$$

Next we have the following existence and uniqueness result.

Theorem 3.1. *Let $(\psi_0, \phi_0, \phi_1) \in (H_0^1(\Omega) \cap H^2(\Omega))^2 \times H_0^1(\Omega)$ and Assumption 2.1 and 2.3 hold. Then, there exists a unique solution for the problem* (1.1)-(1.4) *such that*

$$\begin{gathered} \psi \in L^\infty(0,\infty; H_0^1(\Omega) \cap H^2(\Omega)), \quad \psi_t \in L^\infty(0,\infty; L^2(\Omega)), \\ \phi \in L^\infty(0,\infty; H_0^1(\Omega) \cap H^2(\Omega)), \quad \phi_t \in L^\infty(0,\infty; H_0^1(\Omega)), \\ \phi_{tt} \in L^\infty(0,\infty; L^2(\Omega)), \\ \psi(x,0) = \psi_0(x), \quad \phi(x,0) = \phi_0(x), \quad \phi_{t,0}(x,0) = \phi_1(x), \quad x \in \Omega. \end{gathered}$$

Proof. The main idea is to use the Galerkin Method. Setting as $u = \bar{\psi}_m(t)$ in (3.1) and by integrating and taking the imaginary part of the equation we obtain

$$\frac{1}{2}\frac{d}{dt}\|\psi_m(t)\|^2 + \alpha\|\psi_m\|^2 = 0. \tag{3.4}$$

Applying Gronwall's Lemma produces

$$\|\psi_m(t)\| \le \|\psi_m(0)\|e^{-2\alpha t}. \tag{3.5}$$

Therefore,

$$\|\psi_m\| \le R \quad \text{for all } t > 0. \tag{3.6}$$

Next, by setting $u = -\bar{\psi}_{t,m}$ in (3.1) and by integrating and taking the real part, (3.1) becomes

$$\frac{\kappa}{2}\frac{d}{dt}\int_\Omega |\nabla\psi_m|^2 + \alpha\,\mathrm{Im}\int_\Omega \psi_m\bar{\psi}_{t,m} = -\,\mathrm{Re}\int_\omega \phi_m\psi_m\bar{\psi}_{t,m}.$$

For the right hand side of the equation above we have

$$\frac{1}{2}\frac{d}{dt}\int_\omega \phi_m|\psi_m|^2 = \frac{1}{2}\int_\omega \phi_{t,m}|\psi_m|^2 + \mathrm{Re}\int_\omega \phi_m\psi\bar{\psi}_{t,m}.$$

But from equation (3.1) we also obtain

$$\alpha\,\mathrm{Im}\int_\Omega \psi_m\bar{\psi}_{t,m}dx = \kappa\alpha\int_\Omega |\nabla\psi_m|^2dx + \alpha\int_\omega \phi_m|\psi_m|^2dx.$$

Therefore,

$$\begin{aligned} &\frac{\kappa}{2}\frac{d}{dt}\int_\Omega |\nabla\psi_m|^2 + \kappa\alpha\int_\Omega |\nabla\psi_m|^2dx + \alpha\int_\omega \phi_m|\psi_m|^2dx \\ &= -\frac{1}{2}\frac{d}{dt}\int_\omega \phi_m|\psi_m|^2 + \frac{1}{2}\int_\omega \phi_{t,m}|\psi_m|^2. \end{aligned} \tag{3.7}$$

Next, letting $v = \phi_{t,m}$, equation (3.2) gives

$$\frac{1}{2}\frac{d}{dt}\Big[\|\phi_{t,m}\|^2 + \|\nabla\phi_m\|^2 + \|\phi_m\|^2\Big] + \lambda_0\|\phi_{t,m}\|^2 \le -\int_\Omega (F(x)\cdot\nabla\psi_m)\phi_{t,m}dx.$$

Now, by adding (3.4), (3.7) and the above inequality, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big[\|\psi_m\|^2 + \kappa\|\nabla\psi_m(t)\|^2 + \|\phi_{t,m}\|^2 + \|\nabla\phi_m\|^2 + \|\phi_m\|^2 + \int_\omega \phi_m|\psi_m|^2dx\Big] \\ &+ \alpha\|\psi_m\|^2 + \lambda_0\|\phi_{t,m}\|^2 + \kappa\alpha\|\nabla\psi_m\|^2 + \alpha\int_\omega \phi_m|\psi_m|^2dx \\ &- \frac{1}{2}\int_\omega \phi_{t,m}|\psi_m|^2 \\ &\le -\int_\Omega (F(x)\cdot\nabla\psi_m)\phi_{t,m}dx. \end{aligned}$$

Evaluating these integrals by using Assumption 2.3, Gagliardo-Nirenberg inequality and Young's inequality, we obtain

$$\begin{gathered} \Big|\frac{1}{2}\int_\omega \phi_{t,m}|\psi_m|^2dx\Big| \le \frac{\alpha}{4}\int_\Omega |\phi_{t,m}|^2dx + \frac{C^2}{4\alpha}\int_\Omega |\nabla\psi_m|^2dx, \\ \Big|\int_\Omega (F(x)\cdot\nabla\psi_m)\phi_{t,m}dx\Big| \le \frac{\alpha}{2}\int_\Omega |\phi_{t,m}|^2dx + \frac{M^2}{2\alpha}\int_\Omega |\nabla\psi_m|^2dx, \\ \Big|\alpha\int_\omega \phi_m|\psi_m|^2dx\Big| \le \alpha\int_\Omega |\phi_m|^2dx + \frac{\alpha C^2}{4}\int_\Omega |\nabla\psi_m|^2dx. \end{gathered}$$

Integrating the above expression over $(0,t)$ and applying Gronwall's Lemma we obtain the first estimate

$$\|\psi_m\|^2 + \kappa\|\nabla\psi_m\|^2 + \|\phi_{t,m}\|^2 + \|\nabla\phi_m\|^2 + \|\phi_m\|^2 + \int_\omega \phi_m|\psi_m|^2dx \le L_1, \tag{3.8}$$

where L_1 is a positive constant independent of $m \in \mathbb{N}$. Let

$$E(t) = \|\psi_m\|^2 + \kappa\|\nabla\psi_m\|^2 + \|\phi_{t,m}\|^2 + \|\nabla\phi_m\|^2 + \|\phi_m\|^2 + \int_\omega \phi_m|\psi_m|^2dx$$

evaluating the integral

$$\int_\omega |\phi_m||\psi_m|^2dx \le \|\phi_m\|\|\psi_m\|_4^2 \le C\|\phi_m\|\|\nabla\psi_m\|\|\psi_m\| \le \frac{1}{2}\|\nabla\phi_m\|^2+\frac{\kappa}{2}\|\nabla\psi_m\|^2+C$$

one can deduce that

$$E(t) \ge \|\psi_m\|^2 + \frac{\kappa}{2}\|\nabla\psi_m\|^2 + \|\phi_{t,m}\|^2 + \frac{1}{2}\|\nabla\phi_m\|^2 + \|\phi_m\|^2 + C,$$

and

$$E(t) \le \|\psi_m\|^2 + \frac{3\kappa}{2}\|\nabla\psi_m\|^2 + \|\phi_{t,m}\|^2 + \frac{3}{2}\|\nabla\phi_m\|^2 + \|\phi_m\|^2 + C.$$

so from (3.8) we have

$$\|\psi_m\|^2 + \|\nabla\psi_m\|^2 + \|\phi_{t,m}\|^2 + \|\nabla\phi_m\|^2 + \|\phi_m\|^2 \le L_1 + C.$$

Next, let $u = \Delta\bar{\psi}_{t,m} + \alpha\Delta\bar{\psi}_m$, in (3.1). Taking the real part and integrating over Ω produces

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\kappa\|\Delta\psi_m\|^2 + \kappa\alpha\|\Delta\psi_m\|^2\\ &= \operatorname{Re}\int_\omega \phi_m\psi_m\Delta\bar{\psi}_{t,m}dx + \alpha\operatorname{Re}\int_\omega \phi_m\psi_m\Delta\bar{\psi}_m dx.\end{aligned} \tag{3.9}$$

Furthermore, by letting $v = -\Delta\phi_{t,m}$ in (3.2) and integrating, we also obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\|\nabla\phi_{t,m}\|^2 + \|\Delta\phi_m\|^2 + \|\nabla\phi_m\|^2\Big) + \lambda_0\|\nabla\phi_{t,m}\|^2\\ &\le \operatorname{Re}\int_\Omega (F(x)\cdot\nabla\psi_m)\Delta\phi_{t,m}dx.\end{aligned} \tag{3.10}$$

Analyzing the right hand side of (3.9) produces the equation

$$\begin{aligned}&\operatorname{Re}\int_\omega \phi_m\psi_m\Delta\bar{\psi}_{t,m}dx\\ &= \frac{d}{dt}\operatorname{Re}\int_\omega \phi_m\psi_m\Delta\bar{\psi}_m dx - \operatorname{Re}\int_\omega \phi_{t,m}\psi_m\Delta\bar{\psi}_m dx - \operatorname{Re}\int_\omega \phi_m\psi_{t,m}\Delta\bar{\psi}_m dx,\end{aligned}$$

while by $\psi_{t,m} = -i(-\Delta\psi_m - i\alpha\psi_m - \phi_m\psi_m\chi_\omega)$,

$$\begin{aligned}-\operatorname{Re}\int_\omega \phi_m\psi_{t,m}\Delta\bar{\psi}_m dx &= \operatorname{Re}\int_\omega i\phi_m[-\Delta\psi_m - i\alpha\psi_m - \phi_m\psi_m]\Delta\bar{\psi}_m dx\\ &= \alpha\operatorname{Re}\int_\omega \phi_m\psi_m\Delta\bar{\psi}_m dx + \operatorname{Im}\int_\omega \phi_m^2\psi_m\Delta\bar{\psi}_m dx.\end{aligned}$$

Substituting the expressions above into (3.9) yields

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\kappa\|\Delta\psi_m\|^2 - 2\operatorname{Re}\int_\omega \phi_m\psi_m\Delta\bar{\psi}_m dx\Big) + \kappa\alpha\|\Delta\psi_m\|^2\\ &= 2\alpha\int_\omega \phi_m\psi_m\Delta\bar{\psi}_m dx + \operatorname{Im}\int_\omega \phi_m^2\psi_m\Delta\bar{\psi}_m dx - \operatorname{Re}\int_\omega \phi_{t,m}\psi_m\Delta\bar{\psi}_m dx.\end{aligned} \tag{3.11}$$

Next, adding (3.10) and (2.1) gives

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\kappa\|\Delta\psi_m\|^2 - 2\operatorname{Re}\int_\omega \phi_m\psi_m\Delta\bar{\psi}_m dx + \|\nabla\phi_{t,m}\|^2 + \|\Delta\phi_m\|^2 + \|\nabla\phi_m\|^2\Big)\\
&+ \kappa\alpha\|\Delta\psi_m\|^2 + \lambda_0\|\nabla\phi_{t,m}\|^2\\
&\le 2\alpha\int_\omega \phi_m\psi_m\Delta\bar{\psi}_m dx + \operatorname{Im}\int_\omega \phi_m^2\psi_m\Delta\bar{\psi}_m dx\\
&\quad - \operatorname{Re}\int_\omega \phi_{t,m}\psi_m\Delta\bar{\psi}_m dx + \operatorname{Re}\int_\Omega (F(x)\cdot\nabla\psi_m)\Delta\phi_{t,m} dx.
\end{aligned} \tag{3.12}
$$

Estimating the integrals on the right hand side of (3.12) using the Sobolev embedding theorem and Young's Inequality

$$
\begin{aligned}
\Big|\operatorname{Re}\int_\omega \phi_m\psi_m\Delta\bar{\psi}_m dx\Big| &\le \|\phi_m\|_4\|\psi_m\|_4\|\Delta\psi_m\| \le \frac{1}{4}\|\Delta\psi_m\|^2 + C\|\nabla\phi_m\|^2\|\nabla\psi_m\|^2,\\
\Big|\operatorname{Im}\int_\omega \phi_m^2\psi_m\Delta\bar{\psi}_m dx\Big| &\le \|\phi_m\|_6^2\|\psi_m\|_6\|\Delta\psi_m\| \le \frac{1}{4}\|\Delta\psi_m\|^2 + C\|\nabla\phi_m\|^4\|\nabla\psi_m\|^2,\\
\Big| - \operatorname{Re}\int_\omega \phi_m'\psi_m\Delta\bar{\psi}_m dx\Big| &\le \|\phi_m'\|_4\|\psi_m\|_4\|\Delta\psi_m\| \le \frac{1}{4}\|\Delta\psi_m\|^2 + C\|\nabla\phi_m'\|^2\|\nabla\psi_m\|^2.
\end{aligned}
$$

Now evaluating the last term of (3.10),

$$
\begin{aligned}
&\int_\Omega (F(x)\cdot\nabla\psi_m)\Delta\phi_{t,m} dx\\
&= -\int_\Omega \nabla(F(x)\cdot\nabla\psi_m)\nabla\phi_{t,m} dx\\
&= -\int_\Omega (F(x)\cdot\Delta\psi_m)\nabla\phi_{t,m} dx - \int_\Omega (\nabla F(x)\cdot\nabla\psi_m)\nabla\phi_{t,m} dx\\
&\quad - \int_\Omega (\nabla\psi_m\times(\nabla\times F(x)))\nabla\phi_{t,m} dx
\end{aligned}
$$

and taking into consideration Assumption 2.3 produces

$$
\begin{aligned}
\Big| -\int_\Omega (F(x)\cdot\Delta\psi_m)\nabla\phi_{t,m} dx\Big| &\le C\|\Delta\psi_m\|\|\nabla\phi_{t,m}\|,\\
\Big| -\int_\Omega (\nabla F(x)\cdot\nabla\psi_m)\nabla\phi_{t,m} dx\Big| &\le C\|\nabla\psi_m\|\|\nabla\phi_{t,m}\|,\\
\Big| -\int_\Omega (\nabla\psi_m\times(\nabla\times F(x)))\nabla\phi_{t,m} dx\Big| &\le C\|\nabla\psi_m\|\|\nabla\phi_{t,m}\|.
\end{aligned}
$$

Integrating over $(0,t)$ and applying Gronwall's Lemma we obtain the second estimate

$$
\|\Delta\psi_m\|^2 + \|\nabla\phi_{t,m}\|^2 + \|\Delta\phi_m\|^2 + \|\nabla\phi_m\|^2 \le L_2, \tag{3.13}
$$

where L_2 is a positive constant independent of $m \in \mathbb{N}$. The rest of the proof follows the same basic steps as the one of [5, Theorem 3.1]. $\square$

The energy associated to the problem is defined by

$$
E(t) := \frac{1}{2}\Big(\|\psi\|^2 + \kappa\|\nabla\psi\|^2 + \int_\omega \phi|\psi|^2 dx + \|\phi_t\|^2 + \|\nabla\phi\|^2 + \|\phi\|^2\Big). \tag{3.14}
$$

4. Exponential decay

Let $\{\psi(t), \phi(t), \phi_t(t)\}$ in $H_0^1(\Omega) \cap H^2 \times H_0^1(\Omega) \cap H^2(\Omega) \times H_0^1(\Omega)$ be a solution of the (1.1)- (1.4).

Lemma 4.1. *Let $\kappa, \alpha, \lambda_0$ be large and ϵ be small enough. Then for $\beta := \kappa\alpha - \frac{M^2}{2\lambda_0} - \frac{1}{2\epsilon} > 0$ the first order energy satisfies the inequality*

$$E'(t) \le -\alpha \int_\Omega |\psi|^2 dx - \beta \int_\Omega |\nabla\psi|^2 dx + c(\alpha, c_1, \epsilon) \int_\Omega |\phi|^2 dx - \frac{3}{8} \int_\Omega \lambda(x)|\phi_t|^2 dx.$$

Proof. Substituting into (3.1) $u = -\bar{\psi}_t$, taking the real part, next substituting also $v = \phi_t$, into (3.2) and integrating both over Ω, we have

$$\begin{gathered}\frac{\kappa}{2}\frac{d}{dt}\int_\Omega |\nabla\psi|^2 dx + \alpha \operatorname{Im} \int_\Omega \psi\bar{\psi}_t dx = -\operatorname{Re}\int_\omega \phi\psi\bar{\psi}_t dx,\\ \frac{1}{2}\frac{d}{dt}\int_\Omega (|\phi_t|^2 dx + |\nabla\phi|^2 + |\phi|^2) dx + \int_\Omega \lambda(x)|\phi_t|^2 dx = -\operatorname{Re}\int_\Omega (F(x)\cdot\nabla\psi)\phi_t dx.\end{gathered}$$

The right hand side of the first equation becomes

$$\frac{1}{2}\frac{d}{dt}\int_\omega \phi|\psi|^2 dx = \frac{1}{2}\int_\omega \phi_t|\psi|^2 dx + \operatorname{Re}\int_\omega \phi\psi\bar{\psi}_t dx.$$

Also from (3.1) we have

$$\alpha \operatorname{Im}\int_\Omega \psi\bar{\psi}_t dx = \kappa\alpha\int_\Omega |\nabla\psi|^2 dx + \alpha\int_\omega \phi|\psi|^2 dx.$$

Taking $u = \bar{\psi}$, in (3.1) integrating over Ω and taking the imaginary part yields

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\psi|^2 dx + \alpha\int_\Omega |\psi|^2 dx = 0.$$

Adding the above equations gives

$$\begin{aligned}&\frac{d}{dt}E(t) + \alpha\int_\Omega |\psi|^2 dx + \kappa\alpha\int_\Omega |\nabla\psi|^2 dx + \alpha\int_\omega \phi|\psi|^2 dx + \int_\Omega \lambda(x)|\phi_t|^2 dx\\ &= \frac{1}{2}\int_\omega \phi_t|\psi|^2 dx - \operatorname{Re}\int_\Omega (F(x)\cdot\nabla\psi)\phi_t dx.\end{aligned}$$

Evaluating the integrals by using Assumption 2.3, Gagliardo-Nirenberg inequality and Young's inequality produces

$$\begin{gathered}\Big|\frac{1}{2}\int_\omega \phi_t|\psi|^2 dx\Big| \le \frac{1}{8}\int_\Omega \lambda(x)|\phi_t|^2 dx + \frac{c_1^2}{2\lambda_0}\int_\Omega |\nabla\psi|^2 dx,\\ \Big|\int_\Omega (F(x)\cdot\nabla\psi)\phi_t dx\Big| \le \frac{1}{2}\int_\Omega \lambda(x)|\phi_t|^2 dx + \frac{M^2}{2\lambda_0}\int_\Omega |\nabla\psi|^2 dx,\\ \Big|\alpha\int_\omega \phi|\psi|^2 dx\Big| \le c(\alpha, c_1, \epsilon)\int_\Omega |\phi|^2 dx + \frac{1}{4\epsilon}\int_\Omega |\nabla\psi|^2 dx.\end{gathered}$$

Hence for $\kappa, \alpha, \lambda_0$ large, ϵ small enough and β as above, it holds that

$$E'(t) \le -\alpha\int_\Omega |\psi|^2 dx - \beta\int_\Omega |\nabla\psi|^2 dx + c(\alpha, c_1, \epsilon)\int_\Omega |\phi|^2 dx - \frac{3}{8}\int_\Omega \lambda(x)|\phi_t|^2 dx, \tag{4.1}$$

which concludes the proof of the Lemma. □

Lemma 4.2. *Let $\{\psi, \psi, \phi_t\}$ be solutions of (1.1) - (1.4), $\beta > 0$ and Assumptions 2.1 - 2.3 hold. Then there exists $T_0 > 0$ such that if $T > T_0$ we have*

$$\frac{2}{12}\int_s^T E^2(t)dt \le \frac{1}{2}\int_s^T E(t)\int_{\Gamma(x^0)}(m\cdot n)\Big(\frac{\partial\phi}{\partial n}\Big)^2 d\Gamma dt + \frac{1}{\alpha}|\chi| + C_0E(s),$$

where

$$\chi := \frac{1}{4}\Big[E(t)\Big(\int_\Omega\big(|\psi|^2 + \kappa|\nabla\psi|^2 + 4\alpha\phi_t(m\cdot\nabla\phi) + \alpha\delta\phi\Big(4\phi_t + 2\lambda(x)\phi\Big)\big)dx + \int_\omega \phi|\psi|^2 dx\Big)\Big]_s^T,$$

$0 < s < T < +\infty$ *and* C_0 *depends on* $\alpha, \beta, \lambda, R, M, \kappa, n, c_1$.

Proof. Multiplying (3.1) by $E(t)\bar{\psi}$, taking the imaginary part and integrating produces

$$\frac{1}{2}\Big[E(t)\int_\Omega|\psi|^2dx\Big]_s^T - \frac{1}{2}\int_s^T E'(t)\int_\Omega|\psi|^2dx + \alpha\int_s^T E(t)\int_\Omega|\psi|^2\,dx\,dt = 0.$$

Next, multiplying (3.1) by $-\frac{1}{2}E(t)\bar{\psi}_t$, taking the real part and adding the previous equation produces

$$\begin{aligned}
&\frac{1}{4}\Big[E(t)\Big(\int_\Omega(|\psi|^2 + \kappa|\nabla\psi|^2)dx + \int_\omega\phi|\psi|^2dx\Big)\Big]_s^T - \frac{\kappa}{4}\int_s^T E'(t)\int_\Omega|\nabla\psi|^2\,dx\,dt\\
&+ \frac{\alpha}{2}\int_s^T E(t)\int_\Omega|\psi|^2\,dx\,dt + \frac{\kappa\alpha}{2}\int_s^T E(t)\int_\Omega|\nabla\psi|^2\,dx\,dt\\
&+ \frac{\alpha}{2}\int_s^T E(t)\int_\omega\phi|\psi|^2\,dx\,dt - \frac{1}{4}\int_s^T E'(t)\int_\Omega|\psi|^2\,dx\,dt\\
&= \frac{1}{2}\int_s^T E'(t)\int_\omega\phi|\psi|^2\,dx\,dt + \frac{1}{2}\int_s^T E(t)\int_\omega\phi_t|\psi|^2\,dx\,dt
\end{aligned}\tag{4.2}$$

Multiplying the second equation by $\alpha E(t)(q\cdot\nabla\phi)$, where $q \in (W^{1,\infty}(\Omega))^n$, integrating by parts and using Green's identity we obtain

$$\begin{aligned}
&\Big[\alpha E(t)\int_\Omega\phi_t(q\cdot\nabla\phi)dx\Big]_s^T - \alpha\int_s^T E'(t)\int_\Omega\phi_t(q\cdot\nabla\phi)\,dx\,dt\\
&+ \frac{\alpha}{2}\int_s^T E(t)\int_\Omega \operatorname{div} q|\phi_t|^2\,dx\,dt + \alpha\int_s^T E(t)\int_\Omega\frac{\partial\phi}{\partial x_i}\frac{\partial q_k}{\partial x_i}\frac{\partial\phi}{\partial x_k}\,dx\,dt\\
&- \frac{\alpha}{2}\int_s^T E(t)\int_\Omega \operatorname{div} q|\nabla\phi|^2\,dx\,dt - \frac{\alpha}{2}\int_s^T E(t)\int_\Gamma(q\cdot n)\Big(\frac{\partial\phi}{\partial n}\Big)^2 d\Gamma dt\\
&+ \alpha\int_s^T E(t)\int_\Omega\phi(q\cdot\nabla\phi)\,dx\,dt + \alpha\int_s^T E(t)\int_\Omega\lambda(x)\phi_t(q\cdot\nabla\phi)\,dx\,dt\\
&= -\alpha\int_s^T E(t)\int_\Omega(F(x)\cdot\nabla\psi)(q\cdot\nabla\phi)\,dx\,dt.
\end{aligned}\tag{4.3}$$

Adding relations (4.2) and (4.3), we obtain

$$\frac{1}{4}\Big[E(t)\Big(\int_\Omega(|\psi|^2 + \kappa|\nabla\psi|^2 + 4\alpha\phi_t(q\cdot\nabla\phi))dx + \int_\omega\phi|\psi|^2dx)\Big)\Big]_s^T$$

$$
\begin{aligned}
&+\frac{\alpha}{2}\int_s^T E(t)\int_\Omega \operatorname{div} q[|\phi_t|^2-|\nabla\phi|^2]\,dx\,dt+\alpha\int_s^T E(t)\int_\Omega \frac{\partial\phi}{\partial x_i}\frac{\partial q_k}{\partial x_i}\frac{\partial\phi}{\partial x_k}\,dx\,dt\\
&-\frac{\kappa}{4}\int_s^T E'(t)\int_\Omega |\nabla\psi|^2\,dx\,dt+\alpha\int_s^T E(t)\int_\Omega \phi(q\cdot\nabla\phi)\,dx\,dt\\
&-\alpha\int_s^T E'(t)\int_\Omega \phi_t(q\cdot\nabla\phi)\,dx\,dt+\alpha\int_s^T E(t)\int_\Omega \lambda(x)\phi_t(q\cdot\nabla\phi)\,dx\,dt\\
&+\frac{\alpha}{2}\int_s^T E(t)\int_\Omega |\psi|^2\,dx\,dt+\frac{\kappa\alpha}{2}\int_s^T E(t)\int_\Omega |\nabla\psi|^2\,dx\,dt\\
&+\frac{\alpha}{2}\int_s^T E(t)\int_\omega \phi|\psi|^2\,dx\,dt-\frac{1}{2}\int_s^T E'(t)\int_\omega \phi|\psi|^2\,dx\,dt\\
&-\frac{1}{4}\int_s^T E'(t)\int_\Omega |\psi|^2\,dx\,dt\\
&=\frac{1}{2}\int_s^T E(t)\int_\omega \phi_t|\psi|^2\,dx\,dt-\alpha\int_s^T E(t)\int_\Omega (F(x)\cdot\nabla\psi)(q\cdot\nabla\phi)\,dx\,dt\\
&\quad+\frac{\alpha}{2}\int_s^T E(t)\int_\Gamma (q\cdot n)\Big(\frac{\partial\phi}{\partial n}\Big)^2 d\Gamma dt.
\end{aligned}\tag{4.4}
$$

Considering that $q(x)=m(x)=x-x_0$, relation (4.4) becomes

$$
\begin{aligned}
&\frac{1}{4}\Big[E(t)\Big(\int_\Omega(|\psi|^2+\kappa|\nabla\psi|^2+4\alpha\phi_t(m\cdot\nabla\phi))dx+\int_\omega \phi|\psi|^2dx)\Big)\Big]_s^T\\
&\quad+\alpha\int_s^T E(t)\int_\Omega |\nabla\phi|^2\,dx\,dt+\frac{\alpha n}{2}\int_s^T E(t)\int_\Omega [|\phi_t|^2-|\nabla\phi|^2]\,dx\,dt\\
&-\frac{\kappa}{4}\int_s^T E'(t)\int_\Omega |\nabla\psi|^2\,dx\,dt+\frac{\alpha}{2}\int_s^T E(t)\int_\Omega |\psi|^2\,dx\,dt\\
&+\alpha\int_s^T E(t)\int_\Omega \phi(m\cdot\nabla\phi)\,dx\,dt+\alpha\int_s^T E(t)\int_\Omega \lambda(x)\phi_t(m\cdot\nabla\phi)\,dx\,dt\\
&-\alpha\int_s^T E'(t)\int_\Omega \phi_t(q\cdot\nabla\phi)\,dx\,dt-\frac{1}{4}\int_s^T E'(t)\int_\Omega |\psi|^2\,dx\,dt\\
&+\frac{\kappa\alpha}{2}\int_s^T E(t)\int_\Omega |\nabla\psi|^2\,dx\,dt\\
&\le -\frac{\alpha}{2}\int_s^T E(t)\int_\omega \phi|\psi|^2\,dx\,dt+\frac{1}{2}\int_s^T E(t)\int_\omega \phi_t|\psi|^2\,dx\,dt\\
&\quad+\frac{1}{2}\int_s^T E'(t)\int_\omega \phi|\psi|^2\,dx\,dt+\frac{\alpha}{2}\int_s^T E(t)\int_{\Gamma(x^0)}(m\cdot n)\Big(\frac{\partial\phi}{\partial n}\Big)^2 d\Gamma dt\\
&\quad-\alpha\int_s^T E(t)\int_\Omega (F(x)\cdot\nabla\psi)(m\cdot\nabla\phi)\,dx\,dt.
\end{aligned}\tag{4.5}
$$

Now multiplying (3.2) by $\alpha E(t)\xi\phi$, where $\xi\in W^{1,\infty}(\Omega)$, and integrating we obtain

$$
\begin{aligned}
&\Big[\alpha E(t)\int_\Omega \phi\xi\Big(\phi_t+\frac{\lambda(x)\phi}{2}\Big)dx\Big]_s^T-\alpha\int_s^T E(t)\int_\Omega \xi|\phi_t|^2\,dx\,dt\\
&+\alpha\int_s^T E(t)\int_\Omega \phi(\nabla\xi\cdot\nabla\phi)\,dx\,dt+\alpha\int_s^T E(t)\int_\Omega \xi|\nabla\phi|^2\,dx\,dt
\end{aligned}
$$

$$
\begin{aligned}
&+\alpha\int_s^T E(t)\int_\Omega \xi|\phi|^2\,dx\,dt+\frac{\alpha}{2}\int_s^T E'(t)\int_\Omega \xi\lambda(x)|\phi|^2\,dx\,dt\\
&=-\alpha\int_s^T E(t)\int_\omega \xi\phi\nabla\psi\,dx\,dt+\alpha\int_s^T E'(t)\int_\Omega \xi\phi_t\phi\,dx\,dt.
\end{aligned}\tag{4.6}
$$

Taking $\xi = 2\delta \in \mathbb{R}$ and adding (4.5) and (4.6) produces the expression

$$
\begin{aligned}
&\chi+\alpha(\frac{n}{2}-2\delta)\int_s^T E(t)\int_\Omega |\phi_t|^2\,dx\,dt+\alpha(1+2\delta-\frac{n}{2})\int_s^T E(t)\int_\Omega |\nabla\phi|^2\,dx\,dt\\
&-\frac{\kappa}{4}\int_s^T E'(t)\int_\Omega |\nabla\psi|^2\,dx\,dt+\alpha\int_s^T E(t)\int_\Omega \phi(m\cdot\nabla\phi)\,dx\,dt\\
&-\frac{1}{4}\int_s^T E'(t)\int_\Omega |\psi|^2\,dx\,dt-\alpha\int_s^T E'(t)\int_\Omega \phi_t(m\cdot\nabla\phi)\,dx\,dt\\
&+2\alpha\delta\int_s^T E(t)\int_\Omega |\phi|^2\,dx\,dt+\frac{\alpha}{2}\int_s^T E(t)\int_\Omega |\psi|^2\,dx\,dt\\
&+\alpha\int_s^T E(t)\int_\Omega \lambda(x)\phi_t(m\cdot\nabla\phi)\,dx\,dt+\frac{\kappa\alpha}{2}\int_s^T E(t)\int_\Omega |\nabla\psi|^2\,dx\,dt\\
&+\alpha\delta\int_s^T E'(t)\int_\Omega \lambda(x)|\phi|^2\,dx\,dt+2\alpha\delta\int_s^T E(t)\int_\omega \phi\nabla\psi\,dx\,dt\\
&\le +\frac{1}{2}\int_s^T E'(t)\int_\omega \phi|\psi|^2\,dx\,dt-\frac{\alpha}{2}\int_s^T E(t)\int_\omega \phi|\psi|^2\,dx\,dt\\
&\quad+\frac{1}{2}\int_s^T E(t)\int_\omega \phi_t|\psi|^2\,dx\,dt+2\delta\int_s^T E'(t)\int_\Omega \phi_t\phi\,dx\,dt\\
&\quad-\alpha\int_s^T E(t)\int_\Omega (F(x)\cdot\nabla\psi)(m\cdot\nabla\phi)\,dx\,dt\\
&\quad+\frac{\alpha}{2}\int_s^T E(t)\int_{\Gamma(x^0)}(m\cdot n)\Big(\frac{\partial\phi}{\partial n}\Big)^2 d\Gamma dt.
\end{aligned}\tag{4.7}
$$

Choosing $\delta = \frac{n-1}{4}$, if $n = 2$ or $\delta \in (0, 1/2)$, if $n = 1$ relation (4.7) becomes

$$
\begin{aligned}
&\chi+\alpha\int_s^T E^2(t)dt-\frac{\kappa}{4}\int_s^T E'(t)\int_\Omega |\nabla\psi|^2\,dx\,dt+\alpha\int_s^T E(t)\int_\Omega \phi(m\cdot\nabla\phi)\,dx\,dt\\
&-\alpha\int_s^T E'(t)\int_\Omega \phi_t(m\cdot\nabla\phi)\,dx\,dt+\alpha\int_s^T E(t)\int_\Omega \lambda(x)\phi_t(m\cdot\nabla\phi)\,dx\,dt\\
&-\frac{1}{4}\int_s^T E'(t)\int_\Omega |\psi|^2\,dx\,dt+\frac{\alpha(n-1)}{4}\int_s^T E'(t)\int_\Omega \lambda(x)|\phi|^2\,dx\,dt\\
&-\frac{\alpha}{2}\int_s^T E(t)\int_\Omega |\phi|^2\,dx\,dt+\frac{\alpha(n-1)}{2}\int_s^T E(t)\int_\Omega |\phi|^2\,dx\,dt\\
&+\frac{\alpha(n-1)}{2}\int_s^T E(t)\int_\omega \phi\nabla\psi\,dx\,dt\\
&\le\frac{1}{2}\int_s^T E'(t)\int_\omega \phi|\psi|^2\,dx\,dt+\frac{1}{2}\int_s^T E(t)\int_\omega \phi_t|\psi|^2\,dx\,dt
\end{aligned}
$$

$$
\begin{aligned}
&- \alpha \int_s^T E(t) \int_\Omega (F(x)\cdot\nabla\psi)(m\cdot\nabla\phi)\,dx\,dt \\
&+ \frac{\alpha}{2}\int_s^T E(t)\int_{\Gamma(x^0)}(m\cdot n)\Big(\frac{\partial\phi}{\partial n}\Big)^2 d\Gamma dt + \frac{\alpha(n-1)}{2}\int_s^T E'(t)\int_\Omega \phi_t\phi\,dx\,dt.
\end{aligned} \tag{4.8}
$$

In order for the proof to be completed we need to estimate some of the terms appearing in (4.8). Let $I_1 = -\frac{1}{4}\int_s^T E'(t)\int_\Omega(\kappa|\nabla\psi|^2+|\psi|^2)\,dx\,dt$. Then taking into consideration (3.14),

$$
|I_1| \le \frac{1}{4}\int_s^T |E'(t)|E(t)dt \le -\frac{1}{8}\int_s^T (E^2(t))'dt \le \frac{1}{8}E(0)E(s).
$$

Let $I_2 = \frac{\alpha(n-2)}{2}\int_s^T E(t)\int_\Omega |\phi|^2\,dx\,dt$. Then taking into consideration (3.14),

$$
|I_2| \le -\frac{(n-2)}{\alpha\lambda_0}\int_s^T E(t)E'(t)dt \le \frac{(n-2)}{2\alpha\lambda_0}E(0)E(s).
$$

Let $I_3 = \alpha\int_s^T E(t)\int_\Omega \phi(m\cdot\nabla\phi)\,dx\,dt$. Then taking into consideration (3.14), (4.1) and Young's inequality gives

$$
\begin{aligned}
|I_3| &\le \frac{R^2\alpha^2}{4\epsilon}\int_s^T E(t)\int_\Omega |\phi|^2\,dx\,dt + \epsilon\int_s^T E(t)\int_\Omega |\nabla\phi|^2\,dx\,dt \\
&\le \frac{R^2}{4\epsilon\lambda_0}E(0)E(s) + 2\epsilon\int_s^T E^2(t)dt.
\end{aligned}
$$

Let $I_4 = -\alpha\int_s^T E'(t)\int_\Omega \phi_t(m\cdot\nabla\phi)\,dx\,dt$. Then taking into consideration (3.14),

$$
|I_4| \le \alpha R\int_s^T |E'(t)|E(t)dt \le \frac{\alpha R}{2}E(0)E(s).
$$

Let $I_5 = \alpha\int_s^T E(t)\int_\Omega \lambda(x)\phi_t(m\cdot\nabla\phi)\,dx\,dt$. Then taking into consideration (3.14) and (4.1) and Young's inequality produces

$$
|I_5| \le \frac{\alpha^2R^2\|\lambda\|_\infty}{3\epsilon}E(0)E(t) + 2\epsilon\int_s^T E^2(t)dt.
$$

Let $I_6 = \frac{\alpha(n-1)}{4}\int_s^T E'(t)\int_\Omega \lambda(x)|\phi|^2\,dx\,dt$. Then taking into consideration (3.14),

$$
|I_6| \le \frac{\alpha(n-1)\|\lambda\|_\infty}{2}\int_s^T |E'(t)|E(t)dt \le \frac{\alpha(n-1)\|\lambda\|_\infty}{4}E(0)E(s).
$$

Let $I_7 = \frac{\alpha(n-1)}{2}\int_s^T E(t)\int_\omega \phi\nabla\psi\,dx\,dt$. Then taking into consideration (3.14), (4.1) and Young's inequality produces

$$
\begin{aligned}
|I_7| &\le \frac{\alpha^2(n-1)^2}{16\epsilon}\int_s^T E(t)\int_\Omega |\phi|^2\,dx\,dt + 2\epsilon\int_s^T E^2(t)dt \\
&\le \frac{(n-1)^2}{8\epsilon\lambda_0}E(0)E(s) + 2\epsilon\int_s^T E^2(t)dt.
\end{aligned}
$$

Let $I_8 = \frac{1}{2}\int_s^T E'(t)\int_\omega \phi|\psi|^2\,dx\,dt$. Then from relation (3.14) and Young's inequality we obtain

$$
|I_8| \le \frac{c_1}{2}\int_s^T |E'(t)|\int_\Omega |\phi||\nabla\psi|^2\,dx\,dt \le \frac{c_1(\kappa+1)}{4}E(0)E(s).
$$

Let $I_9 = \frac{1}{2}\int_s^T E(t)\int_\omega \phi_t|\psi|^2\,dx\,dt$. Then taking into consideration (3.14) and Young's inequality we obtain

$$
\begin{aligned}
|I_9| &\le \frac{c_1^2}{16\epsilon}\int_s^T E(t)\int_\Omega |\nabla\psi|^2\,dx\,dt + 2\epsilon\int_s^T E^2(t)dt\\
&\le \frac{c_1^2}{32\epsilon\beta}E(0)E(s) + 2\epsilon\int_s^T E^2(t)dt.
\end{aligned}
$$

Let $I_{10} = -\alpha\int_s^T E(t)\int_\Omega (F(x)\cdot\nabla\psi)(m\cdot\nabla\phi)\,dx\,dt$. Then using (3.14), (4.1) and Young's inequality we obtain

$$
\begin{aligned}
|I_{10}| &\le \frac{\alpha^2R^2M^2}{4\epsilon}\int_s^T E(t)\int_\Omega |\nabla\psi|^2\,dx\,dt + 2\epsilon\int_s^T E^2(t)dt\\
&\le \frac{\alpha^2R^2M^2}{8\epsilon\beta}E(0)E(s) + 2\epsilon\int_s^T E^2(t)dt.
\end{aligned}
$$

Let $I_{11} = \frac{\alpha(n-1)}{2}\int_s^T E'(t)\int_\omega \phi_t\phi\,dx\,dt$. Then relation (3.14) implies

$$
|I_{11}| \le \frac{\alpha(n-1)}{2}\int_s^T |E'(t)|E(t)dt \le \frac{\alpha(n-1)}{4}E(0)E(s).
$$

Hence, the following inequality holds, with $\epsilon = 1/12$,

$$
\frac{2}{12}\int_s^T E^2(t)dt \le \frac{1}{2}\int_s^T E(t)\int_{\Gamma(x^0)}(m\cdot n)\Big(\frac{\partial\phi}{\partial n}\Big)^2 d\Gamma dt + \frac{1}{\alpha}|\chi| + C_0E(s), \tag{4.9}
$$

where

$$
\begin{aligned}
C_0 = \Big[&\frac{\alpha(n-1)^2 + 8\alpha^3R^2\|\lambda\|_\infty\lambda_0 + 6\alpha R^2 + (n-2)}{\lambda_0} + \frac{3c_1^2 + 12\alpha^2R^2M^2}{8\beta}\\
&+ \frac{2\alpha(n-1)(1+\|\lambda\|_\infty) + 2c_1(\kappa+1) + 1 + 4\alpha R}{8}\Big]E(0).
\end{aligned}
$$

□

Lemma 4.3. *Let the assumptions in Lemma 4.2 and Assumption 2.4 hold. Then there exists $T_0 > 0$ such that if $T > T_0$, we have*

$$
\begin{aligned}
&\frac{1}{24}\int_s^T E^2(t)dt\\
&\le \frac{1}{\alpha}|\chi| + \frac{R}{\alpha}|Y| + (C_0 + \frac{RC_1}{\alpha})E(s) + \frac{3}{2}\|h|_{W^{1,\infty}}R\int_s^T E(t)\int_{\hat\omega}|\nabla\phi|^2\,dx\,dt
\end{aligned}
$$

where

$$
Y := \frac{1}{4}\Big[E(t)\Big(\int_\Omega \big(|\psi|^2 + \kappa|\nabla\psi|^2 + 4\alpha\phi_t(h\cdot\nabla\phi)\big)dx + \int_{\hat\omega}\phi|\psi|^2dx\Big)\Big]_s^T
$$

and C_1 depends on $\alpha,\beta,\lambda,R,M,\kappa,h,c_1,c_2$.

Proof. For $q = h$ expression (4.4) becomes

$$
\begin{aligned}
&Y - \alpha \int_s^T E'(t) \int_{\hat\omega} \phi_t (h \cdot \nabla\phi)\,dx\,dt + \frac{\alpha}{2} \int_s^T E(t) \int_{\hat\omega} \operatorname{div} h[|\phi_t|^2 - |\nabla\phi|^2]\,dx\,dt \\
&+ \alpha \int_s^T E(t) \int_{\hat\omega} \frac{\partial\phi}{\partial x_i}\frac{\partial h_k}{\partial x_i}\frac{\partial\phi}{\partial x_k}\,dx\,dt - \frac{\kappa}{4}\int_s^T E'(t)\int_\Omega |\nabla\psi|^2\,dx\,dt \\
&+ \alpha \int_s^T E(t) \int_{\hat\omega} \phi (h \cdot \nabla\phi)\,dx\,dt + \alpha \int_s^T E(t) \int_{\hat\omega} \lambda(x)\phi_t (h \cdot \nabla\phi)\,dx\,dt \\
&+ \frac{\alpha}{2} \int_s^T E(t) \int_\Omega |\psi|^2\,dx\,dt + \frac{\kappa\alpha}{2} \int_s^T E(t) \int_\Omega |\nabla\psi|^2\,dx\,dt \\
&- \frac{1}{4} \int_s^T E'(t) \int_\Omega |\psi|^2\,dx\,dt + \frac{\alpha}{2} \int_s^T E(t) \int_\omega \phi|\psi|^2\,dx\,dt \\
&- \frac{1}{2} \int_s^T E'(t) \int_\omega \phi|\psi|^2\,dx\,dt - \frac{1}{2} \int_s^T E(t) \int_\omega \phi_t|\psi|^2\,dx\,dt \\
&+ \alpha \int_s^T E(t) \int_{\hat\omega} (F(x) \cdot \nabla\psi)(h \cdot \nabla\phi)\,dx\,dt \\
&\geq \frac{\alpha}{2} \int_s^T E(t) \int_{\Gamma(x^0)} (h \cdot n)\Big(\frac{\partial\phi}{\partial n}\Big)^2 d\Gamma dt.
\end{aligned} \tag{4.10}
$$

Evaluating some of the terms in the previous expression, we have

$$
\begin{gathered}
\begin{aligned}
&\Big|\frac{\alpha}{2} \int_s^T E(t) \int_{\hat\omega} \operatorname{div} h[|\phi_t|^2 - |\nabla\phi|^2]\,dx\,dt\Big| \\
&\leq \frac{\alpha\|h\|_{W^{1,\infty}}}{2\lambda_0} E(0)E(s) + \frac{\alpha\|h\|_{W^{1,\infty}}}{2} \int_s^T E(t) \int_{\hat\omega} |\nabla\phi|^2\,dx\,dt,
\end{aligned} \\
\Big|\alpha \int_s^T E(t) \int_{\hat\omega} \frac{\partial\phi}{\partial x_i}\frac{\partial h_k}{\partial x_i}\frac{\partial\phi}{\partial x_k}\,dx\,dt\Big| \leq \alpha\|h\|_{W^{1,\infty}} \int_s^T E(t) \int_{\hat\omega} |\nabla\phi|^2\,dx\,dt, \\
\Big|\frac{\kappa}{4} \int_s^T E'(t) \int_\Omega |\nabla\psi|^2 - |\psi|^2\,dx\,dt\Big| \leq \frac{\kappa+1}{4} E(0)E(s), \\
\Big|\alpha \int_s^T E'(t) \int_\Omega \phi_t (h \cdot \nabla\phi)\,dx\,dt\Big| \leq \frac{\alpha\|h\|_{W^{1,\infty}}}{2} E(0)E(s), \\
\Big|\alpha \int_s^T E(t) \int_{\hat\omega} \lambda(x)\phi_t (h \cdot \nabla\phi)\,dx\,dt\Big| \leq \frac{\alpha^2\|h\|^2_{W^{1,\infty}}\|\lambda\|_\infty}{6\epsilon} E(0)E(s) + 2\epsilon \int_s^T E^2(t)dt, \\
\Big|\frac{\kappa\alpha}{2} \int_s^T E(t) \int_\Omega |\nabla\psi|^2\,dx\,dt\Big| \leq \frac{\kappa\alpha}{4\beta} E(0)E(s), \\
\Big|\frac{\alpha}{2} \int_s^T E(t) \int_\omega \phi|\psi|^2\,dx\,dt\Big| \leq \frac{a^2c_1^2}{32\epsilon\beta} E(0)E(s) + 2\epsilon \int_s^T E^2(t)dt, \\
\Big|\frac{1}{4} \int_s^T E'(t) \int_\Omega |\psi|^2\,dx\,dt\Big| \leq \frac{1}{4} E(0)E(s), \\
\Big|\frac{1}{2} \int_s^T E(t) \int_\omega \phi_t|\psi|^2\,dx\,dt\Big| \leq \frac{c_1^4}{8\epsilon\beta} E(0)E(s) + 2\epsilon \int_s^T E^2(t)dt,
\end{gathered}
$$

and finally

$$\begin{aligned}&\Big|\alpha\int_s^T E(t)\int_\Omega (F(x)\cdot\nabla\psi)(h\cdot\nabla\phi)\,dx\,dt\Big|\\&\le \frac{a^2\|h\|^2_{W^{1,\infty}}M^2}{8\epsilon\beta}E(0)E(s)+2\epsilon\int_s^T E^2(t)dt,\end{aligned}$$

$$\Big|\alpha\int_s^T E(t)\int_\Omega \phi(h\cdot\nabla\phi)\,dx\,dt\Big|\le\frac{\|h\|^2_{W^{1,\infty}}}{4\epsilon\lambda_0}E(0)E(s)+2\epsilon\int_s^T E^2(t)dt,$$

$$\begin{aligned}&\Big|\frac12\int_s^T E'(t)\int_\omega \phi|\psi|^2\,dx\,dt\Big|\le\frac14\int_s^T E'(t)\int_\Omega|\phi|^2\,dx\,dt+\int_s^T E'(t)\int_\Omega|\psi|^4\,dx\,dt\\&\le\frac{1+c_2^2}{4}E(0)E(s).\end{aligned}$$

Taking into consideration the evaluations above and substituting them into (4.10), we obtain

$$\begin{aligned}\frac{\alpha R}{2}\int_s^T E(t)\int_{\Gamma(x^0)}\Big(\frac{\partial\phi}{\partial n}\Big)^2d\Gamma dt&\le R|Y|+10\epsilon R\int_s^T E^2(t)dt+RC_1E(s)\\&\quad+\frac{3\alpha\|h\|_{W^{1,\infty}}R}{2}\int_s^T E(t)\int_{\hat\omega}|\nabla\phi|^2\,dx\,dt,\end{aligned}\tag{4.11}$$

where

$$\begin{aligned}C_1:=\Big[&\frac{\kappa+3+c_2^2+2\alpha\|h\|_{W^{1,\infty}}}{4}+\frac{\alpha^2c_1^2+4\alpha^2\|h\|^2_{W^{1,\infty}}M^2}{32\epsilon\beta}+\frac{\alpha\|h|_{W^{1,\infty}}}{2\lambda_0}\\&+\frac{\|h\|^2_{W^{1,\infty}}}{4\epsilon\lambda_0}+\frac{\kappa\alpha}{4\beta}+\frac{\alpha^2\|h|^2_{W^{1,\infty}}\|\lambda\|_\infty}{6\epsilon}\Big]E(0).\end{aligned}$$

Combining (4.9) and (4.11) and choosing $\epsilon=\frac{\alpha}{80R}$, we deduce

$$\begin{aligned}\frac{1}{24}\int_s^T E^2(t)dt&\le\frac1\alpha|\chi|+\frac{R}{\alpha}|Y|+(C_0+\frac{RC_1}{\alpha})E(s)\\&\quad+\frac32\|h|_{W^{1,\infty}}R\int_s^T E(t)\int_{\hat\omega}|\nabla\phi|^2\,dx\,dt,\end{aligned}\tag{4.12}$$

which completes the proof of the Lemma. □

To evaluate $\int_s^T E(t)\int_{\hat\omega}|\nabla\phi|^2\,dx\,dt$, we construct a function $\eta\in W^{1,\infty}(\Omega)$ such that

$$0\le\eta\le1,\quad \text{a.e. in }\Omega,\text{ with }\eta=1,\text{ a.e. in }\hat\omega,\tag{4.13}$$

$$\eta=0,\quad\text{a.e. in }\Omega\backslash\omega,\tag{4.14}$$

$$\frac{|\nabla\eta|^2}{\eta}\in L^\infty(\omega).\tag{4.15}$$

Finally, we state and prove the main result of the present work.

Theorem 4.4. *Let Assumptions 2.1 and 2.2 hold and $\beta>0$. Then there exists some positive constant $C=C(E(0))$ such that the following decay rate holds for each solution (ψ,ϕ,ϕ_t) of (1.1)- (1.4),*

$$E(t)\le\frac{CE(0)}{1+t},\quad\text{for all }t\ge0.\tag{4.16}$$

Proof. To prove the Theorem it is sufficient to prove that
$$\int_s^T E^2(t)dt \le CE(S), \quad \text{for all } 0 \le s < T < +\infty,$$
for some positive constant C independent of T. Letting $\xi = \eta$ in (4.6) we obtain
$$\begin{aligned}
&\Big[\alpha E(t)\int_\omega \phi\eta\Big(\phi_t + \frac{\lambda(x)\phi}{2}\Big)dx\Big]_s^T - \alpha\int_s^T E(t)\int_\omega \eta|\phi_t|^2\,dx\,dt\\
&+\alpha\int_s^T E(t)\int_\omega \phi(\nabla\eta\cdot\nabla\phi)\,dx\,dt + \alpha\int_s^T E(t)\int_\omega \eta|\nabla\phi|^2\,dx\,dt\\
&+\alpha\int_s^T E(t)\int_\omega \eta|\phi|^2\,dx\,dt + \frac{\alpha}{2}\int_s^T E'(t)\int_\omega \eta\lambda(x)|\phi|^2\,dx\,dt\\
&= -\alpha\int_s^T E(t)\int_\omega \eta\phi\nabla\psi\,dx\,dt + \alpha\int_s^T E'(t)\int_\omega \eta\phi_t\phi\,dx\,dt.
\end{aligned} \tag{4.17}$$
Evaluating the integrals above produces
$$\begin{aligned}
&\alpha\int_s^T E(t)\int_\omega \eta|\nabla\phi|^2\,dx\,dt\\
&\le |Z| + C_2E(s) + 2\epsilon\int_s^T E^2(t)dt + \frac{\alpha}{2}\int_s^T E(t)\int_\omega \eta|\nabla\phi|^2\,dx\,dt,
\end{aligned} \tag{4.18}$$
where
$$Z := \Big[\alpha E(t)\int_\omega \phi\eta\Big(\phi_t + \frac{\lambda(x)\phi}{2}\Big)dx\Big]_s^T,$$
$$C_2 := \Big[\frac{\alpha\|\lambda\|_\infty}{2} + \|\frac{|\nabla\eta|^2}{\eta}\|_\infty + \frac{1}{2\alpha\lambda_0} + \frac{1}{\alpha\lambda_0} + \frac{\alpha^2}{8\epsilon\beta} + \frac{4\alpha}{3\lambda_0} + \frac{\alpha}{2}\Big]E(0).$$
Therefore combining (4.12) and (4.18), choosing $\epsilon = \frac{1}{288\|h\|_\infty R}$ and taking into consideration
$$\int_s^T E(t)\int_{\hat\omega} \eta|\nabla\phi|^2\,dx\,dt = \int_s^T E(t)\int_{\hat\omega} |\nabla\phi|^2\,dx\,dt,$$
we obtain
$$\frac{1}{48}\int_s^T E^2(t)dt \le \frac{1}{\alpha}|\chi| + \frac{R}{\alpha}|Y| + 3\|h\|_\infty R|Z| + (C_0 + \frac{RC_1}{\alpha} + \|h\|_\infty RC_2)E(s). \tag{4.19}$$
Note that the following estimate holds
$$\frac{1}{\alpha}|\chi| + \frac{R}{\alpha}|Y| + 3\|h\|_\infty R|Z| \le C_3E(0)E(s),$$
where $C_3 = C_3(R,\alpha,\kappa,\delta,\lambda_0,c_1,\|h\|_\infty,\|\lambda\|_\infty)$. Then
$$\int_s^T E^2(t)dt \le CE(0)E(s),$$
where $C = C(R,\alpha,\kappa,\delta,n,c_1,\beta,\|h\|_\infty,\|\lambda\|_\infty)$ is independent of T. Then employing Lemma 2.5 we deduce the desired decay rate. □

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