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ANALYTIC SOLUTIONS FOR ITERATIVE FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. Because of its technical difficulties the existence of analytic solutions to the iterative differential equation $x'(z) = x(az + bx(z) + cx'(z))$ is a source of open problems. In this article we obtain analytic solutions, using Schauder's fixed point theorem. Also we present a unique solution which is a nonconstant polynomial in the complex field.

1. Introduction

The study of differential equations with variable delays

$$x'(t) = f(t, x(t - \tau(t))), \tag{1.1}$$

began in 60's. Then both theory and applications have been considered; see the monographs [1, 5, 16]. In recent years equations with complicated deviating arguments $\tau(x(t))$ have attracted the attention of mathematicians, since these equations often appear in applied sciences. For example, Cooke established the following equation, related to a genetic phenomenon [2],

$$x'(t) + ax(t - h(t, x(t))) = F(t), \quad t \in \mathbb{R}.$$

Periodic solutions in periodic systems for the population model

$$x'(t) = f(t, x(t - \tau(t, x(t)))), \quad t \in \mathbb{R}$$

were investigated in [8], where f and τ have the same period. On the other hand, the analytic solutions of the equation $x'(z) = x(z-\tau(z))$, such as $x'(z) = x(az+bx(z))$ and $x'(z) = x(az + bx'(z))$, even the more complicated equation $\alpha z + \beta x'(z) = x(az+bx''(z))$, have been investigated in the complex field; see for example [4, 6, 7], [9]-[16], [18]-[19]. In these studies, the Schröder transformation is an important tool which can reduce the equation involving iteration of unknown functions to an auxiliary differential equation without the iterative operation. To the best of our knowledge, there are only a few results on analytic solutions of the iterative differential equation that includes state and state derivative simultaneously in $\tau(z)$. The main reason is that this equation being rewritten as another one which does

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not involve $\tau(z)$ is difficult to analyze; there the Schröder transformation, a very useful method in the eliminating iteration of unknown function, does not work.

In this article we study analytic solutions of the equation

$$x'(z) = x(az + bx(z) + cx'(z)), \quad z \in \mathbb{C} \tag{1.2}$$

by using Schauder's fixed point theorem, in which a, b, c are complex constants. Moreover, we find the unique nonconstant polynomial solution in complex field.

2. Analytic solution

In this section we aim to find analytic solutions of equation (1.2) in a neighborhood of the origin by using Schauder's fixed point theorem. In addition, a non-existence result on analytic solutions is also given, which is generalized to the equations involving higher order derivatives.

Theorem 2.1. *Suppose that $|a| < 1$ and $(1-|a|)|x_0| + |a| + |b| + |c| < 1$, then (1.2) has an analytic solution of the form*

$$x(z) = x_0 + \sum_{n=1}^{\infty} a_n z^n \tag{2.1}$$

in a neighborhood of the origin.

Proof. It suffices to find the solution of an integral equation

$$x(z) = x_0 + \int_0^z x(as + bx(s) + cx'(s))ds. \tag{2.2}$$

Since $(1-|a|)|x_0| + |a| + |b| + |c| < 1$, there exists positive value h satisfying

$$\frac{(|b| + |c|)|x_0|}{1 - |a| - |b| - |c|} < h < 1 - |x_0|. \tag{2.3}$$

For constants M and L satisfy

$$\frac{|x_0|}{1-h} \le M \le 1 \qquad \text{and} \qquad L > \frac{1 + M|b|}{1 - M|c|}, \tag{2.4}$$

we define functional subspace

$$\begin{aligned} X_h = \Big\{ & x \in C^1(\bar{B}_h(0), \mathbb{C}) : x(0) = x_0,\ |x(z_1) - x(z_2)| \le M|z_1 - z_2|, \\ & \|x'\|_{C^0} \le |x_0| + Mh, \|x_1{}' - x_2{}'\|_{C^0} \le L\|x_1 - x_2\|_{C^0}, \\ & \forall z_1, z_2 \in \bar{B}_h(0), \forall x_1, x_2 \in C^1(\bar{B}_h(0), \mathbb{C}) \Big\}, \end{aligned}$$

where $\bar{B}_h(0)$ is the closure of open ball $B_h(0)$ centered at 0 with radius h. Clearly, X_h is a convex and closed subset of $C^1(\bar{B}_h(0), \mathbb{C})$, a Banach space with the standard norm $\|x\|_C^0 = \|x\|_{C^0} + \|x'\|_{C^0}$, where $\|x\|_{C^0} = \max_{z\in\bar{B}_h(0)} |x(z)|$ and $\|x'\|_{C^0} = \max_{z\in\bar{B}_h(0)} |x'(z)|$. Moreover, we have $|x_n(z)| \le |x_0| + Mh$ for any $x_n \in X_h$. So $\{x_n(z)\}$ is compact in $\bar{B}_h(0)$ by Ascoli-Arzela Lemma. Therefore, X_h is a compact and convex subset of the Banach space $C^1(\bar{B}_h(0), \mathbb{C})$.

Now we define the operator $\mathcal{J}$ on X_h by

$$\mathcal{J}x(z) := x_0 + \int_0^z x(as + bx(s) + cx'(s))ds. \tag{2.5}$$

Then a fixed point of map $\mathcal{J}$ is a solution of the integral equation (2.2). From (2.3), we have

$$\begin{aligned}|az + bx(z) + cx'(z)| &\le |a|h + |b| \cdot (|x_0| + Mh) + |c| \cdot (|x_0| + Mh)\\ &\le |a|h + |b| \cdot (|x_0| + h) + |c| \cdot (|x_0| + h)\\ &\le h, \quad \forall z \in \bar{B}_h(0);\end{aligned}$$

i.e., $x(az + bx(z) + cx'(z))$ is well-defined for $x \in X_h$.

To verify $\mathcal{J}(X_h) \subset X_h$, we note that for arbitrary $x \in X_h$,

$$\mathcal{J}x(0) = x_0 + \int_0^0 x(as + bx(s) + cx'(s))ds = x_0,$$

and from (2.4),

$$\begin{aligned}|\mathcal{J}x(z_2) - \mathcal{J}x(z_1)| &\le |\int_{z_1}^{z_2} x(as + bx(s) + cx'(s))ds|\\ &\le (|x_0| + Mh)|z_2 - z_1|\\ &\le M|z_2 - z_1|.\end{aligned}$$

We also have $|(\mathcal{J}x)'(z)| \le |x_0| + Mh$; i.e., $\|(\mathcal{J}x)'\|_{C^0} \le |x_0| + Mh$. Moreover, for any $x_1, x_2 \in X_h$,

$$\begin{aligned}&|(\mathcal{J}x_1)'(z) - (\mathcal{J}x_2)'(z)|\\ &= |x_1(az + bx_1(z) + c{x_1}'(z)) - x_2(az + bx_2(z) + c{x_2}'(z))|\\ &\le |x_1(az + bx_1(z) + c{x_1}'(z)) - x_2(az + bx_1(z) + c{x_1}'(z))|\\ &\quad + |x_2(az + bx_1(z) + c{x_1}'(z)) - x_2(az + bx_2(z) + c{x_2}'(z))|\\ &\le \|x_1 - x_2\|_{C^0} + M(|b| \cdot \|x_1 - x_2\|_{C^0} + |c| \cdot \|{x_1}' - {x_2}'\|_{C^0})\\ &\le \|x_1 - x_2\|_{C^0} + M(|b| + |c|L) \cdot \|x_1 - x_2\|_{C^0}\\ &= (1 + M(|b| + |c|L))\|x_1 - x_2\|_{C^0}\\ &\le L\|x_1 - x_2\|_{C^0}.\end{aligned} \tag{2.6}$$

These relations imply that $\mathcal{J}x \in X_h$; i.e., $\mathcal{J}$ is a self-mapping.

Furthermore, from (2.6) we also see that

$$\begin{aligned}&|\mathcal{J}x_1(z) - \mathcal{J}x_2(z)|\\ &\le |\int_0^z |x_1(as + bx_1(s) + c{x_1}'(s)) - x_2(as + bx_2(s) + c{x_2}'(s))|ds\\ &\le |\int_0^z L\|x_1 - x_2\|_{C^0} ds\\ &\le hL\|x_1 - x_2\|_{C^0},\end{aligned}$$

implying that $\mathcal{J}$ is continuous.

Therefore we can conclude from Schauder's Fixed Point Theorem that $\mathcal{J}$ has a fixed point $x \in X_h$; i.e., $\mathcal{J}x = x$, which gives a solution for equation (1.2) associated with $x(0) = x_0$ on the domain $|z| \le h$, naturally, it is an analytic solution of equation (1.2) with the form $x(z) = x_0 + \sum_{n=1}^{\infty} a_n z^n$ in complex field $|z| \le h$. This completes the proof. $\square$

Example 2.2. Consider equation

$$x'(z) = x(az + bx(z) + cx'(z)), \tag{2.7}$$

where $a = \sqrt{2}/4 + i\sqrt{2}/4$, $b = c = \sqrt{2}/16 + i\sqrt{2}/16$.

Since $|a| = 1/2$, $|b| = |c| = 1/8$, then for arbitrary $|x_0| < 1/2$, equation (2.7) has an analytic solution with the form as $x(z) = x_0 + \sum_{n=1}^{\infty} a_n z^n$ in a neighborhood of the origin.

Next, we obtain a result on non-existence of analytic solution for equation (1.2), which can be generalized to the equations with high order derivatives.

Theorem 2.3. *Equation* (1.2) *has no nonzero analytic solution of the form*

$$x(z) = \sum_{n=2}^{\infty} a_n z^n. \tag{2.8}$$

Proof. Substituting (2.8) into (1.2), we have

$$\sum_{n=1}^{\infty} (n+1)a_{n+1} z^n = \sum_{n=2}^{\infty} a_n (q(z))^n, \tag{2.9}$$

where

$$q(z) = (a + 2a_2 c)z + \sum_{n=2}^{\infty} (a_n b + (n+1)a_{n+1} c) z^n.$$

Comparing coefficient of z in (2.9), we have $2a_2 = 0$; that is, $a_2 = 0$. In general we obtain $a_n = 0$, $n = 3, 4, \dots$, as coefficient of z^n in (2.9). Therefore, equation (1.2) has only the zero analytic solution of the form (2.8). □

To extend the above result, we have the non-existence of nonzero analytic solutions to the more general equations.

Theorem 2.4. *The equation*

$$x'(z) = x(az + bx(z) + cx^{(m)}(z)), \quad m = 2, 3, \dots, \tag{2.10}$$

has no nonzero analytic solution of the form

$$x(z) = \sum_{n=m+1}^{\infty} a_n z^n.$$

The proof of the above theorem is similar to Theorem 2.3 and is omitted.

3. Polynomial solution

In this section we find the unique nonconstant complex polynomial solution to (1.2). Furthermore, the iterative differential equations including high order derivatives have also unique nonconstant complex polynomial solution which is independent of the order of derivative.

Theorem 3.1. *Suppose that $a \neq 0, 1$ and $b \neq 0$, then* (1.2) *has a unique nonconstant complex polynomial solution*

$$x(z) = \frac{a^2 c + ab}{(a-1)b^2} - \frac{a}{b} z. \tag{3.1}$$

Proof. Let

$$x(z) = \sum_{i=0}^{n} a_i z^i. \tag{3.2}$$

Substituting (3.2) into (1.2), we have

$$\begin{aligned} &a_1 + 2a_2 z + 3a_3 z^2 + \dots + n a_n z^{n-1} \\ &= a_0 + a_1 p_n(z) + a_2 (p_n(z))^2 + \dots + a_n (p_n(z))^n, \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} p_n(z) = &(a_0 b + a_1 c) + (a + a_1 b + 2a_2 c)z + (a_2 b + 3a_3 c)z^2 \\ &+ (a_3 b + 4a_4 c)z^3 + \dots + (a_{n-1} b + n a_n c)z^{n-1} + a_n b z^n. \end{aligned} \tag{3.4}$$

Comparing coefficient of z^{n^2}, we have

$$a_n (a_n b)^n = 0;$$

that is, $a_n = 0$. Then (3.3) and (3.4) are simplified into

$$\begin{aligned} &a_1 + 2a_2 z + 3a_3 z^2 + \dots + (n-1) a_{n-1} z^{n-2} \\ &= a_0 + a_1 p_{n-1}(z) + a_2 (p_{n-1}(z))^2 + \dots + a_{n-1} (p_{n-1}(z))^{n-1}, \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} p_{n-1}(z) = &(a_0 b + a_1 c) + (a + a_1 b + 2a_2 c)z + (a_2 b + 3a_3 c)z^2 + (a_3 b + 4a_4 c)z^3 \\ &+ \dots + (a_{n-2} b + (n-1) a_{n-1} c)z^{n-2} + a_{n-1} b z^{n-1}, \end{aligned} \tag{3.6}$$

respectively. Comparing coefficient of $z^{(n-1)^2}$, we obtain

$$a_{n-1} (a_{n-1} b)^{n-1} = 0;$$

that is, $a_{n-1} = 0$. Thus, repeating the same procedure as before, we can obtain the simplest forms of (3.3) and (3.4) respectively as

$$\begin{gathered} a_1 = a_0 + a_1 p_1(z), \\ p_1(z) = (a_0 b + a_1 c) + (a + a_1 b)z. \end{gathered}$$

Namely,

$$a_1 = a_0 + a_1((a_0 b + a_1 c) + (a + a_1 b)z). \tag{3.7}$$

From which, we obtain a nonconstant complex polynomial solution (3.1), which also is unique since it is determined only by constants a, b, c in equation (1.2). $\square$

Example 3.2. Consider the equation

$$x'(z) = x(az + bx(z) + cx'(z)), \tag{3.8}$$

where $a = 1 + i$, $b = 1 + 2i$, $c = 3 + i$. Then (3.8) has a unique complex polynomial solution $x(z) = (3 - 9i)/(4 + 3i) - (1 + i)z/(1 + 2i)$.

For the iterative differential equations involving high-order derivatives, we have the following result.

Theorem 3.3. *Suppose that $a \neq 0, 1$ and $b \neq 0$, then the equation*
$$x'(z) = x(az + bx(z) + cx^{(m)}(z)), \quad m = 2, 3, \dots, \tag{3.9}$$
has a unique nonconstant complex polynomial solution
$$x(z) = \frac{a}{(a-1)b} - \frac{a}{b}z.$$

The proof of the above theorem is similar to Theorem 3.1, and is omitted.

Example 3.4. Consider the equation
$$x'(z) = x(az + bx(z) + cx^{(m)}(z)) \tag{3.10}$$
for arbitrary $m \in \{2, 3, \dots\}$, where $a = 2 + i$, $b = 1 + i$, $c = 3 + 3i$. Then (3.10) has a unique complex polynomial solution $x(z) = (1 - 2i)/2 - (2 + i)z/(1 + i)$.

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