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WEAK-STRONG UNIQUENESS OF HYDRODYNAMIC FLOW OF NEMATIC LIQUID CRYSTALS

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ABSTRACT. This article concerns a simplified model for a hydrodynamic system of incompressible nematic liquid crystal materials. It is shown that the weak-strong uniqueness holds for the class of weak solutions provided that either $(\mathbf{u},\nabla\mathbf{d})\in C([0,T),L^3(\mathbb{R}^3))$; or $(\mathbf{u},\nabla\mathbf{d})\in L^q(0,T;\dot{B}^{-1+3/p+2/q}_{p,q}(\mathbb{R}^3))$ with $2\le p<\infty$, $2<q<\infty$ and $\frac{3}{p}+\frac{2}{q}>1$.

1. Introduction

In this article, we study uniqueness criteria for solutions of a hydrodynamical system modeling the flow of nematic liquid crystals in the whole space $\mathbb{R}^3$, namely the Cauchy problem

$$\begin{gathered}\partial_t\mathbf{u}-\nu\Delta\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u}+\nabla\pi=-\lambda\operatorname{div}(\nabla\mathbf{d}\odot\nabla\mathbf{d}),\\ \partial_t\mathbf{d}+\mathbf{u}\cdot\nabla\mathbf{d}=\gamma(\Delta\mathbf{d}-g(\mathbf{d})),\\ \operatorname{div}\mathbf{u}=0,\\ (\mathbf{u},\mathbf{d})|_{t=0}=(\mathbf{u}_0,\mathbf{d}_0).\end{gathered}\tag{1.1}$$

This system describes the time evolution of nematic liquid crystal materials (cf. [18]), where $\mathbf{u}\in\mathbb{R}^3$ and $\pi\in\mathbb{R}$ denote, respectively, the velocity field and the pressure of the fluid, and $\mathbf{d}\in\mathbb{R}^3$ denotes the director field of the nematic liquid crystals; ν,λ,γ are positive constants, and $g(\mathbf{d})=\nabla G(\mathbf{d})$ with $G(\mathbf{d})=\frac{|\mathbf{d}|^4}{4}-\frac{|\mathbf{d}|^2}{2}$ is a Ginzburg-Landau approximation function; the unusual term $\nabla\mathbf{d}\odot\nabla\mathbf{d}=(\langle\partial_{x_i}\mathbf{d},\partial_{x_j}\mathbf{d}\rangle)_{1\le i,j\le 3}$ is the stress tensor induced by the director field $\mathbf{d}$, and the notation $\langle\cdot,\cdot\rangle$ denotes the inner product in $\mathbb{R}^3$. Since the sizes of the viscosity constants ν, λ and γ do not play important roles in the proof of our main result, we shall assume that $\nu=\lambda=\gamma=1$ throughout this paper.

As the authors pointed out in [20], although system (1.1) is a simplified version of the liquid crystal model proposed by Ericksen [3] and Leslie [16], but it still retains most of the interesting mathematical properties. We refer the reader to see [4, 10, 17, 18] and the references therein for more discussions of the physical background of this problem. In [20], using the modified Galerkin method and the

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compactness argument, Lin and Liu proved global existence of weak solutions of (1.1) with $g(\mathbf{d}) = \nabla G(\mathbf{d})$ for some smooth and bounded function $G : \mathbb{R}^3 \to \mathbb{R}$. Moreover, when $g(\mathbf{d}) = 0$, they established global existence of strong solutions if the initial data is sufficiently small (or if the viscosity ν is sufficiently large). The same as for the Navier-Stokes equations (which are equations obtained by putting $\mathbf{d} = \mathbf{0}$ in (1.1)), it is well known that weak solution of (1.1) is unique and regular in $\mathbb{R}^2$. However, the question of regularity and uniqueness of weak solution is an outstanding open problem in $\mathbb{R}^3$. Hence, it is meaningful to find sufficient conditions on a strong solution of (1.1) such that all weak solutions sharing the same initial data must coincide with the one which additionally satisfies these sufficient conditions, and we say then weak-strong uniqueness holds. For the three dimensional Navier-Stokes equations, Prodi [25] and Serrin [27] proved that weak-strong uniqueness holds in the class

$$\mathcal{P} = L^q(0,T; L^p(\mathbb{R}^3)) \quad \text{with } \frac{3}{p} + \frac{2}{q} = 1,\ 3 < p \le \infty.$$

Von Wahl [28] and Giga [9] improved this result in the class

$$\mathcal{P} = C([0,T], L^3(\mathbb{R}^3)).$$

Moreover, this last result was extended in the limit case by Kozono and Sohr [13], and Escauriaza, Seregin and Šverák [5], who proved that weak strong uniqueness holds for

$$\mathcal{P} = L^\infty(0,T; L^3(\mathbb{R}^3)).$$

For uniqueness criteria related to the Sobolev spaces, we refer the reader to [1, 26]. Recently, many researches have refined the above results. Kozono and Taniuchi [14] proved that weak-strong uniqueness holds in the class

$$\mathcal{P} = L^2(0,T; BMO).$$

Gallagher and Planchon [6] proved that weak-strong uniqueness holds for

$$\mathcal{P} = L^q(0,T; \dot{B}^{-1+3/p+2/q}_{p,q}(\mathbb{R}^3)) \quad \text{with } 2 \le p < \infty,\ 2 < q < \infty \text{ and } \frac{3}{p} + \frac{2}{q} > 1.$$

Lemarié-Rieusset [15] and Germain [8] proved that weak-strong uniqueness holds for

$$\mathcal{P} = C([0,T], X_1^{(0)}) \quad \text{or} \quad \mathcal{P} = L^{2/(1-r)}(0,T; X_r) \quad \text{with } r \in [-1,1).$$

Chen, Miao and Zhang [2] improved the above results by showing weak-strong uniqueness for

$$\mathcal{P} = L^q(0,T; \dot{B}^r_{p,\infty}(\mathbb{R}^3)) \quad \text{with } \frac{3}{1+r} < p \le \infty, r \in (0,1] \text{ and } (p,r) \ne (\infty,1).$$

We refer the reader to see [8] and [15] for definitions of these function spaces.

In this article, we are interested in finding uniqueness criteria for weak solutions of (1.1). For the two $n \times n$ matrixes $A = (a_{ij})_{i,j=1}^n$ and $B = (b_{ij})_{i,j=1}^n$, we define $A : B = \sum_{i,j=1}^n a_{ij} b_{ij}$, and denote by $\otimes$ the tensor product. Let us recall the definition of weak solutions.

Definition 1.1. The vector-valued function $(\mathbf{u}, \mathbf{d})$ is called a weak solution of (1.1) on $\mathbb{R}^3 \times (0,T)$ if it satisfies the following conditions:

(1) $(\mathbf{u},\nabla\mathbf{d})\in L^\infty(0,T;L^2(\mathbb{R}^3))\cap L^2(0,T;\dot H^1(\mathbb{R}^3)):=(\mathcal{LS})$, where $\dot H^1(\mathbb{R}^3)$ is the usual homogeneous Sobolev space; i.e., the space of functions whose gradient belongs to $L^2(\mathbb{R}^3)$.

(2) $(\mathbf{u},\mathbf{d})$ satisfies (1.1) in the sense of distributions; i.e., $\operatorname{div}\mathbf{u}=0$ in the distributional sense and for all $\mathbf{v}\in C_0^\infty(\mathbb{R}^3\times(0,T))$ and $\mathbf{e}\in C_0^\infty(\mathbb{R}^3\times(0,T))$ with $\operatorname{div}\mathbf{v}=0$, we have

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}^3}\mathbf{u}\cdot\partial_t\mathbf{v}\,dx\,dt-\int_0^T\int_{\mathbb{R}^3}\nabla\mathbf{u}:\nabla\mathbf{v}\,dx\,dt+\int_0^T\int_{\mathbb{R}^3}\mathbf{u}\otimes\mathbf{u}:\nabla\mathbf{v}\,dx\,dt\\&=-\int_0^T\int_{\mathbb{R}^3}\nabla\mathbf{d}\odot\nabla\mathbf{d}:\nabla\mathbf{v}\,dx\,dt\end{aligned}$$

and

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}^3}\mathbf{d}\cdot\partial_t\mathbf{e}\,dx\,dt-\int_0^T\int_{\mathbb{R}^3}\nabla\mathbf{d}:\nabla\mathbf{e}\,dx\,dt+\int_0^T\int_{\mathbb{R}^3}\mathbf{u}\otimes\mathbf{d}:\nabla\mathbf{e}\,dx\,dt\\&=\int_0^T\int_{\mathbb{R}^3}g(\mathbf{d})\cdot\mathbf{e}\,dx\,dt.\end{aligned}$$

(3) The following energy inequality holds (see (3.6) in the appendix):

$$\begin{aligned}&\|\mathbf{u}(t)\|_{L^2}^2+\|\nabla\mathbf{d}(t)\|_{L^2}^2+2\int_0^t(\|\nabla\mathbf{u}(\tau)\|_{L^2}^2+\|\Delta\mathbf{d}(\tau)\|_{L^2}^2)d\tau\\&+6\int_0^t\|\mathbf{d}\cdot\nabla\mathbf{d}\|_{L^2}^2(\tau)d\tau\\&\le\|\mathbf{u}_0\|_{L^2}^2+\|\nabla\mathbf{d}_0\|_{L^2}^2+\int_0^t\|\nabla\mathbf{d}(\tau)\|_{L^2}^2d\tau\quad\text{for all } t\ge0.\end{aligned}$$

Before presenting the exact statement of our result, let us first recall the definition of the homogeneous Besov spaces. Let $\mathcal{S}(\mathbb{R}^3)$ be the Schwartz space. We denote by $\{\Delta_j,S_j\}_{j\in\mathbb{Z}}$ the Littlewood-Paley decomposition. Let $\mathcal{Z}(\mathbb{R}^3)=\big\{f\in\mathcal{S}(\mathbb{R}^3):\ \partial^\alpha\widehat f(0)=0,\ \forall\alpha\in(\mathbb{N}\cup\{0\})^3\big\}$, and denote its dual by $\mathcal{Z}'(\mathbb{R}^3)$. Recall that for $s\in\mathbb{R}$ and $(p,q)\in[1,\infty]\times[1,\infty]$, the homogeneous Besov space $\dot B^s_{p,q}(\mathbb{R}^3)$ is defined by

$$\dot B^s_{p,q}(\mathbb{R}^3)=\big\{f\in\mathcal{Z}'(\mathbb{R}^3):\|f\|_{\dot B^s_{p,q}}<\infty\big\},$$

where

$$\|f\|_{\dot B^s_{p,q}}=\begin{cases}\big(\sum_{j\in\mathbb{Z}}2^{jsq}\|\Delta_jf\|_{L^p}^q\big)^{1/q} & \text{for } 1\le q<\infty,\\ \sup_{j\in\mathbb{Z}}2^{js}\|\Delta_jf\|_{L^p} & \text{for } q=\infty.\end{cases}$$

It is well-known that if either $s<\frac3p$ or $s=\frac3p$ and $q=1$, then $(\dot B^s_{p,q}(\mathbb{R}^3),\|\cdot\|_{\dot B^s_{p,q}})$ is a Banach space. For more details about the homogeneous Besov spaces, we refer the reader to see [15]. Next we introduce some notations. Given $0<T<\infty$ and a Banach space X, we denote by $C([0,T],X)$ the Banach space of all bounded and continuous mappings from $[0,T]$ to X, and for $p\ge1$, we denote by $L^p(0,T;X)$ the set of Bochner measurable X-valued time dependent functions f such that $t\to\|f\|_X$ belongs to $L^p(0,T)$. The product of Banach spaces $\mathcal{X}\times\mathcal{Y}$ will be equipped with the usual norm $\|(f,g)\|_{\mathcal{X}\times\mathcal{Y}}=\|f\|_{\mathcal{X}}+\|g\|_{\mathcal{Y}}$, and if $\mathcal{X}=\mathcal{Y}$, we use $\|(f,g)\|_{\mathcal{X}}$ to denote $\|(f,g)\|_{\mathcal{X}\times\mathcal{X}}$.

The main result of this paper is as follows.

Theorem 1.2. *Assume that* $(\mathbf{u}, \mathbf{d})$ *and* $(\tilde{\mathbf{u}}, \tilde{\mathbf{d}})$ *are two weak solutions of* (1.1) *for a given initial data* $(\mathbf{u}_0, \nabla \mathbf{d}_0) \in L^2(\mathbb{R}^3)$. *Assume furthermore that for some* $T > 0$, *either*

$$(\mathbf{u}, \nabla \mathbf{d}) \in C([0,T], L^3(\mathbb{R}^3)) \tag{1.2}$$

or

$$(\mathbf{u}, \nabla \mathbf{d}) \in L^q(0,T; \dot{B}_{p,q}^{-1+3/p+2/q}(\mathbb{R}^3)) \tag{1.3}$$

with $2 \le p < \infty$, $2 < q < \infty$ *and* $\frac{3}{p} + \frac{2}{q} > 1$. *Then* $\mathbf{u} = \tilde{\mathbf{u}}$ *and* $\mathbf{d} = \tilde{\mathbf{d}}$ *on the time interval* $[0,T]$.

Remark 1.3. Theorem 1.2 holds with $\frac{3}{p} + \frac{2}{q} = 1$ in (1.3) as well, with the space $L^q(0,T; \dot{B}_{p,q}^{-1+3/p+2/q}(\mathbb{R}^3))$ replaced by $L^q(0,T; L^p(\mathbb{R}^3))$, when $p > 3$, namely, if we assume that

$$(\mathbf{u}, \nabla \mathbf{d}) \in L^q(0,T;, L^p(\mathbb{R}^3)) \quad \text{with } 3 < p \le \infty,\ 2 \le q < \infty \text{ and } \frac{3}{p} + \frac{2}{q} = 1,$$

then $\mathbf{u} = \tilde{\mathbf{u}}$ and $\mathbf{d} = \tilde{\mathbf{d}}$ on the time interval $[0,T]$. This can be seen as a consequence of Prodi-Serrin's uniqueness criterion.

Remark 1.4. We extend, in Theorem 1.2, the uniqueness criteria of weak solutions of [28] and [6] for the system (1.1).

Let us sketch an idea leading to the proof of Theorem 1.2. We introduce the function

$$F = \nabla \mathbf{d}.$$

Let F^T be the transpose of F. Then, taking the gradient of second equation of (1.1), noticing the facts that $F \odot F = F^T F$ and

$$\frac{\partial}{\partial x_k}\Big(\sum_{j=1}^n \mathbf{u}_j \frac{\partial \mathbf{d}_i}{\partial x_j}\Big) = \sum_{j=1}^n \frac{\partial \mathbf{u}_j}{\partial x_k}\frac{\partial \mathbf{d}_i}{\partial x_j} + \sum_{j=1}^n \mathbf{u}_j \frac{\partial}{\partial x_j}\Big(\frac{\partial \mathbf{d}_i}{\partial x_k}\Big) = (F\nabla \mathbf{u} + \mathbf{u}\cdot\nabla F)_{ik}$$

for all $i, k = 1, 2, \dots, n$, system (1.1) reads

$$\begin{gathered}
\partial_t \mathbf{u} - \Delta \mathbf{u} = -\nabla \pi - \mathbf{u}\cdot\nabla \mathbf{u} - \operatorname{div}(F^T F), \\
\partial_t F - \Delta F = -\mathbf{u}\cdot\nabla F - F\nabla \mathbf{u} - (3|\mathbf{d}|^2 - 1)F, \\
\operatorname{div} \mathbf{u} = 0, \\
(\mathbf{u}, F)|_{t=0} = (\mathbf{u}_0, F_0),
\end{gathered} \tag{1.4}$$

where $F_0 = \nabla \mathbf{d}_0$. System (1.4) is more related to the viscoelastic fluids, which had attracted much attention recently; see for instance [21]. Using the technical matrixes analysis, the energy inequality and the similar argument in the studying of the incompressible Navier-Stokes equations in [28] and [6], we can obtain some important estimates which yield the proof of Theorem 1.2.

Before ending this section, we mention some well-posedness results of the system (1.1). Recently, when $g(\mathbf{d}) = 0$, by using the maximal regularity of Stokes equations and the parabolic equations, Hu and Wang [12] proved global existence of strong solutions to the system (1.1) for small initial data belonging to Besov spaces of positive-order. They also proved that when the strong solution exists, all global weak solutions constructed by [20] must be equal to the unique strong solution. In [19] and [22], the authors studied the system (1.1) with $g(\mathbf{d}) = |\nabla \mathbf{d}|^2 \mathbf{d}$ in two dimensions. They established the global existence, uniqueness and partial regularity of weak solutions and performed the blow-up analysis at each singular time. Hong [11] proved independently the global existence of weak solutions of the system (1.1) in two dimensions. In [29], Wang established global well-posedness of (1.1) with $g(\mathbf{d}) = |\nabla \mathbf{d}|^2\mathbf{d}$ for small initial data in $BMO^{-1} \times BMO$. Some regularity criteria for weak solutions of the system (1.1) were also established, see [7], [23] and [24].

The rest of this paper is organized as follows. In Section 2, we present the proof of Theorem 1.2. In appendix, we shall establish the basic energy inequality of the system (1.1), which gives global existence of weak solutions of (1.1).

2. The proof of Theorem 1.2

Throughout this section, we assume that $(\mathbf{u}_0, \nabla \mathbf{d}_0)$, $(\tilde{\mathbf{u}}_0, \nabla \tilde{\mathbf{d}}_0) \in L^2(\mathbb{R}^3)$, and denote by $(\mathbf{u}, \mathbf{d})$ and $(\tilde{\mathbf{u}}, \tilde{\mathbf{d}})$, respectively, be two weak solutions associated with initial conditions $(\mathbf{u}_0, \nabla \mathbf{d}_0)$ and $(\tilde{\mathbf{u}}_0, \nabla \tilde{\mathbf{d}}_0)$, respectively.

Let us define $F = \nabla \mathbf{d}$, $\tilde{F} = \nabla \tilde{\mathbf{d}}$, $F_0 = \nabla \mathbf{d}_0$ and $\tilde{F}_0 = \nabla \tilde{\mathbf{d}}_0$. Obviously, by Definition 1.1, $(\mathbf{u}, F)$ and $(\tilde{\mathbf{u}}, \tilde{F})$ verify equations (1.4) and satisfy

$$
\begin{aligned}
&\|\mathbf{u}(t)\|_{L^2}^2 + \|F(t)\|_{L^2}^2 + 2\int_0^t (\|\nabla \mathbf{u}(\tau)\|_{L^2}^2 + \|\nabla F(\tau)\|_{L^2}^2) d\tau \\
&+ 6\int_0^t \||\mathbf{d}|F\|_{L^2}^2(\tau) d\tau \\
&\le \|\mathbf{u}_0\|_{L^2}^2 + \|F_0\|_{L^2}^2 + 2\int_0^t \|F(\tau)\|_{L^2}^2 d\tau,
\end{aligned}
\tag{2.1}
$$

$$
\begin{aligned}
&\|\tilde{\mathbf{u}}(t)\|_{L^2}^2 + \|\tilde{F}(t)\|_{L^2}^2 + 2\int_0^t (\|\nabla \tilde{\mathbf{u}}(\tau)\|_{L^2}^2 + \|\nabla \tilde{F}(\tau)\|_{L^2}^2) d\tau \\
&+ 6\int_0^t \||\tilde{\mathbf{d}}|\tilde{F}\|_{L^2}^2(\tau) d\tau \\
&\le \|\tilde{\mathbf{u}}_0\|_{L^2}^2 + \|\tilde{F}_0\|_{L^2}^2 + 2\int_0^t \|\tilde{F}(\tau)\|_{L^2}^2 d\tau.
\end{aligned}
\tag{2.2}
$$

Setting $\mathbf{w} = \mathbf{u} - \tilde{\mathbf{u}}$, $E = F - \tilde{F}$, $\mathbf{w}_0 = \mathbf{u}_0 - \tilde{\mathbf{u}}_0$ and $E_0 = F_0 - \tilde{F}_0$, we divide the proof of Theorem 1.2 into the following two cases.

Case 1. $(\mathbf{u}, \nabla \mathbf{d}) \in C([0,T], L^3(\mathbb{R}^3))$. We shall prove the following stability result.

Proposition 2.1. *Assume that* $(\mathbf{u}, \nabla \mathbf{d}) \in C([0,T], L^3(\mathbb{R}^3))$. *Then*

$$
\begin{aligned}
&\|(\mathbf{w}(t), E(t))\|_{L^2}^2 + 2\int_0^t \|(\nabla \mathbf{w}(\tau), \nabla E(\tau))\|_{L^2}^2 d\tau \\
&\le \|(\mathbf{w}_0, E_0)\|_{L^2}^2 \exp\Big(Ct\big(\|(\mathbf{u}, F)\|_{C([0,T], L^3(\mathbb{R}^3))}^2 + 1\big)\Big),
\end{aligned}
\tag{2.3}
$$

where C is a constant depending on $\|(\mathbf{d}, \tilde{\mathbf{d}})\|_{L^\infty(0,T;\dot{H}^1)}$ *and* $\|(\mathbf{d}, \tilde{\mathbf{d}})\|_{L^2(0,T;\dot{H}^2)}$.

It is clear that, under the condition (1.2), Theorem 1.2 is an immediate consequence of Proposition 2.1. Note that, by (2.1) and (2.2), the left hand side of (2.3) satisfies

$$\begin{aligned}
&\|(\mathbf{w}(t),E(t))\|_{L^2}^2+2\int_0^t\|(\nabla\mathbf{w}(\tau),\nabla E(\tau))\|_{L^2}^2d\tau\\
&=\|(\mathbf{u}(t),F(t))\|_{L^2}^2+\|(\tilde{\mathbf{u}}(t),\tilde{F}(t))\|_{L^2}^2+2\int_0^t\|(\nabla\mathbf{u}(\tau),\nabla F(\tau))\|_{L^2}^2d\tau\\
&\quad+2\int_0^t\|(\nabla\tilde{\mathbf{u}}(\tau),\nabla\tilde{F}(\tau))\|_{L^2}^2d\tau-2\big(\mathbf{u}(t)|\tilde{\mathbf{u}}(t)\big)\\
&\quad-2\big(F(t)|\tilde{F}(t)\big)-4\int_0^t\big(\nabla\mathbf{u}(\tau)|\nabla\tilde{\mathbf{u}}(\tau)\big)d\tau-4\int_0^t\big(\nabla F(\tau)|\nabla\tilde{F}(\tau)\big)d\tau \qquad (2.4)\\
&\le\|(\mathbf{u}_0,F_0)\|_{L^2}^2+2\int_0^t\|F(\tau)\|_{L^2}^2d\tau-6\int_0^t\||\mathbf{d}|F\|_{L^2}^2(\tau)d\tau+\|(\tilde{\mathbf{u}}_0,\tilde{F}_0)\|_{L^2}^2\\
&\quad+2\int_0^t\|\tilde{F}(\tau)\|_{L^2}^2d\tau-6\int_0^t\||\tilde{\mathbf{d}}|\tilde{F}\|_{L^2}^2(\tau)d\tau-2\big(\mathbf{u}(t)|\tilde{\mathbf{u}}(t)\big)-2\big(F(t)|\tilde{F}(t)\big)\\
&\quad-4\int_0^t\big(\nabla\mathbf{u}(\tau)|\nabla\tilde{\mathbf{u}}(\tau)\big)d\tau-4\int_0^t\big(\nabla F(\tau)|\nabla\tilde{F}(\tau)\big)d\tau,
\end{aligned}$$

where we denote by $(\cdot|\cdot)$ the scalar product in $L^2(\mathbb{R}^3)$. Hence, we aim at proving the following lemma.

Lemma 2.2. *Under the assumptions of Proposition 2.1, the following equality holds for all $t\le T$,*

$$\begin{aligned}
&\big(\mathbf{u}(t)|\tilde{\mathbf{u}}(t)\big)+\big(F(t)|\tilde{F}(t)\big)+2\int_0^t\big(\nabla\mathbf{u}(\tau)|\nabla\tilde{\mathbf{u}}(\tau)\big)d\tau+2\int_0^t\big(\nabla F(\tau)|\nabla\tilde{F}(\tau)\big)d\tau\\
&=(\mathbf{u}_0|\tilde{\mathbf{u}}_0)+(F_0|\tilde{F}_0)-\int_0^t\int_{\mathbb{R}^3}\mathbf{w}\cdot\nabla\mathbf{w}\cdot\mathbf{u}\,dx\,d\tau+\int_0^t\int_{\mathbb{R}^3}\tilde{F}^T\tilde{F}:\nabla\mathbf{u}\,dx\,d\tau\\
&\quad+\int_0^t\int_{\mathbb{R}^3}F^TF:\nabla\mathbf{u}\,dx\,d\tau-\int_0^t\int_{\mathbb{R}^3}\nabla\mathbf{u}:(\tilde{F}^TF+F^T\tilde{F})\,dx\,d\tau\\
&\quad-\int_0^t\int_{\mathbb{R}^3}\mathbf{w}\cdot\nabla F:\tilde{F}\,dx\,d\tau+\int_0^t\int_{\mathbb{R}^3}F:\tilde{F}\nabla\mathbf{w}\,dx\,d\tau\\
&\quad-\int_0^t\int_{\mathbb{R}^3}(3|\mathbf{d}|^2-1)F:\tilde{F}\,dx\,d\tau-\int_0^t\int_{\mathbb{R}^3}(3|\tilde{\mathbf{d}}|^2-1)\tilde{F}:F\,dx\,d\tau.
\end{aligned} \qquad (2.5)$$

Proof. Let us choose two smooth sequences $\{(\tilde{\mathbf{u}}_n,\tilde{F}_n)\}$ ($\operatorname{div}\tilde{\mathbf{u}}_n=0$) and $\{(\mathbf{u}_n,F_n)\}$ ($\operatorname{div}\mathbf{u}_n=0$) such that

$$\begin{gathered}
\lim_{n\to\infty}(\tilde{\mathbf{u}}_n,\tilde{F}_n)=(\tilde{\mathbf{u}},\tilde{F})\quad\text{in } L^2(0,T;\dot{H}^1(\mathbb{R}^3)),\\
\lim_{n\to\infty}(\tilde{\mathbf{u}}_n,\tilde{F}_n)=(\tilde{\mathbf{u}},\tilde{F})\quad\text{weakly-star in } L^\infty(0,T;L^2(\mathbb{R}^3))
\end{gathered} \qquad (2.6)$$

and

$$\begin{gathered}
\lim_{n\to\infty}(\mathbf{u}_n,F_n)=(\mathbf{u},F)\quad\text{in } L^2(0,T;\dot{H}^1(\mathbb{R}^3))\cap C([0,T],L^3(\mathbb{R}^3)),\\
\lim_{n\to\infty}(\mathbf{u}_n,F_n)=(\mathbf{u},F)\quad\text{weakly-star in } L^\infty(0,T;L^2(\mathbb{R}^3)).
\end{gathered} \qquad (2.7)$$

We split the proof into the following two steps.

Step 1. Taking the scalar product with $\tilde{\mathbf{u}}_n$ and $\mathbf{u}_n$ of the equation (1.4) on $\mathbf{u}$ and $\tilde{\mathbf{u}}$ respectively, after integration in time and integration by parts in the space variables, we obtain

$$\int_0^t \Big((\partial_\tau \mathbf{u}|\tilde{\mathbf{u}}_n) + (\nabla\mathbf{u}|\nabla\tilde{\mathbf{u}}_n) + (\mathbf{u}\cdot\nabla\mathbf{u}|\tilde{\mathbf{u}}_n) + (\operatorname{div}(F^TF)|\tilde{\mathbf{u}}_n)\Big)d\tau = 0 \tag{2.8}$$

and

$$\int_0^t \Big((\partial_\tau \tilde{\mathbf{u}}|\mathbf{u}_n) + (\nabla\tilde{\mathbf{u}}|\nabla\mathbf{u}_n) + (\tilde{\mathbf{u}}\cdot\nabla\tilde{\mathbf{u}}|\mathbf{u}_n) + (\operatorname{div}(\tilde{F}^T\tilde{F})|\mathbf{u}_n)\Big)d\tau = 0. \tag{2.9}$$

By (2.6) and (2.7), it is obvious that

$$\lim_{n\to\infty}\Big(\int_0^t (\nabla\mathbf{u}|\nabla\tilde{\mathbf{u}}_n)d\tau + \int_0^t (\nabla\tilde{\mathbf{u}}|\nabla\mathbf{u}_n)d\tau\Big) = 2\int_0^t (\nabla\mathbf{u}|\nabla\tilde{\mathbf{u}})d\tau. \tag{2.10}$$

Applying the Hölder inequality and the Sobolev embedding inequality, it follows that

$$\begin{aligned}\int_0^t (\tilde{\mathbf{u}}\cdot\nabla\tilde{\mathbf{u}}|\mathbf{u}_n)d\tau &\le C\int_0^t \|\tilde{\mathbf{u}}\|_{L^6}\|\nabla\tilde{\mathbf{u}}\|_{L^2}\|\mathbf{u}_n\|_{L^3}d\tau \\ &\le C\|\tilde{\mathbf{u}}\|^2_{L^2(0,T;\dot{H}^1)}\|\mathbf{u}_n\|_{C([0,T],L^3)}.\end{aligned} \tag{2.11}$$

Since $\mathbf{u}_n$ converges to $\mathbf{u}$ in $C([0,T],L^3(\mathbb{R}^3))$, (2.11) implies that

$$\lim_{n\to\infty}\int_0^t (\tilde{\mathbf{u}}\cdot\nabla\tilde{\mathbf{u}}|\mathbf{u}_n)d\tau = \int_0^t (\tilde{\mathbf{u}}\cdot\nabla\tilde{\mathbf{u}}|\mathbf{u})d\tau. \tag{2.12}$$

Similarly, by applying (2.6), (2.7) and (2.11), we obtain the following three equalities:

$$\begin{aligned}&\lim_{n\to\infty}\int_0^t (\mathbf{u}\cdot\nabla\mathbf{u}|\tilde{\mathbf{u}}_n)d\tau = -\lim_{n\to\infty}\int_0^t (\mathbf{u}\cdot\nabla\tilde{\mathbf{u}}_n|\mathbf{u})d\tau \\ &= -\int_0^t (\mathbf{u}\cdot\nabla\tilde{\mathbf{u}}|\mathbf{u})d\tau = \int_0^t (\mathbf{u}\cdot\nabla\mathbf{u}|\tilde{\mathbf{u}})d\tau,\end{aligned} \tag{2.13}$$

$$\begin{aligned}&\lim_{n\to\infty}\int_0^t (\operatorname{div}(F^TF)|\tilde{\mathbf{u}}_n)d\tau = -\lim_{n\to\infty}\int_0^t (F^TF|\nabla\tilde{\mathbf{u}}_n)d\tau \\ &= -\int_0^t (F^TF|\nabla\tilde{\mathbf{u}})d\tau = \int_0^t (\operatorname{div}(F^TF)|\tilde{\mathbf{u}})d\tau,\end{aligned} \tag{2.14}$$

and

$$\begin{aligned}&\lim_{n\to\infty}\int_0^t (\operatorname{div}(\tilde{F}^T\tilde{F})\big|\mathbf{u}_n)d\tau = \lim_{n\to\infty}\int_0^t \Big(\sum_{i=1}^3(\partial_{x_i}\tilde{F}^T\tilde{F} + \tilde{F}^T\partial_{x_i}\tilde{F})|\mathbf{u}_n\Big)d\tau \\ &= \int_0^t \Big(\sum_{i=1}^3(\partial_{x_i}\tilde{F}^T\tilde{F} + \tilde{F}^T\partial_{x_i}\tilde{F})\big|\mathbf{u}\Big)d\tau = \int_0^t (\operatorname{div}(\tilde{F}^T\tilde{F})|\mathbf{u})d\tau.\end{aligned} \tag{2.15}$$

Since $\partial_t\tilde{\mathbf{u}} = \Delta\tilde{\mathbf{u}} - \tilde{\mathbf{u}}\cdot\nabla\tilde{\mathbf{u}} - \operatorname{div}(\tilde{F}^T\tilde{F}) - \nabla\pi$ holds in the sense of distribution, the estimates (2.10), (2.12)–(2.15) and $\operatorname{div}\mathbf{u}_n = 0$ imply in particular that

$$\begin{aligned}\lim_{n\to\infty}\int_0^t (\partial_\tau\tilde{\mathbf{u}}|\mathbf{u}_n)d\tau &= -\lim_{n\to\infty}\int_0^t \Big((\nabla\tilde{\mathbf{u}}|\nabla\mathbf{u}_n) + (\tilde{\mathbf{u}}\cdot\nabla\tilde{\mathbf{u}}|\mathbf{u}_n) + (\operatorname{div}(\tilde{F}^T\tilde{F})|\mathbf{u}_n)\Big)d\tau \\ &= -\int_0^t \Big((\nabla\tilde{\mathbf{u}}|\nabla\mathbf{u}) + (\tilde{\mathbf{u}}\cdot\nabla\tilde{\mathbf{u}}|\mathbf{u}) + (\operatorname{div}(\tilde{F}^T\tilde{F})|\mathbf{u})\Big)d\tau\end{aligned}$$

$$= \int_0^t (\partial_\tau \tilde{\mathbf{u}}|\mathbf{u}) d\tau.$$

It can be proved analogously that

$$\lim_{n\to\infty} \int_0^t (\partial_\tau \mathbf{u}|\tilde{\mathbf{u}}_n) d\tau = \int_0^t (\partial_\tau \mathbf{u}|\tilde{\mathbf{u}}) d\tau.$$

Putting these estimates together, and noticing that

$$\int_0^t (\partial_\tau \tilde{\mathbf{u}}|\mathbf{u}) + (\partial_\tau \mathbf{u}|\tilde{\mathbf{u}}) d\tau = (\mathbf{u}(t)|\tilde{\mathbf{u}}(t)), \tag{2.16}$$

$$\int_0^t \Big((\tilde{\mathbf{u}} \cdot \nabla \tilde{\mathbf{u}}|\mathbf{u}) - (\mathbf{u} \cdot \nabla \tilde{\mathbf{u}}|\mathbf{u}) \Big) d\tau = \int_0^t (\mathbf{w} \cdot \nabla \mathbf{w}|\mathbf{u}) d\tau, \tag{2.17}$$

we obtain

$$\begin{aligned} &(\mathbf{u}(t)|\tilde{\mathbf{u}}(t)) + 2\int_0^t (\nabla \mathbf{u}|\nabla \tilde{\mathbf{u}}) d\tau \\ &= (\mathbf{u}_0|\tilde{\mathbf{u}}_0) - \int_0^t (\mathbf{w} \cdot \nabla \mathbf{w}|\mathbf{u}) d\tau + \int_0^t \int_{\mathbb{R}^3} \tilde{F}^T \tilde{F} : \nabla \mathbf{u} \, dx \, d\tau \\ &\quad + \int_0^t \int_{\mathbb{R}^3} F^T F : \nabla \mathbf{u} \, dx \, d\tau. \end{aligned} \tag{2.18}$$

Step 2. Proceeding in the same way as (2.8) and (2.9), we obtain

$$\begin{aligned} &\int_0^t \Big((\partial_\tau F|\tilde{F}_n) + (\nabla F|\nabla \tilde{F}_n) + (\mathbf{u} \cdot \nabla F|\tilde{F}_n) \\ &+ (F\nabla \mathbf{u}|\tilde{F}_n) + ((3|\mathbf{d}|^2 - 1)F|\tilde{F}_n) \Big) d\tau = 0 \end{aligned} \tag{2.19}$$

and

$$\begin{aligned} &\int_0^t \Big((\partial_\tau \tilde{F}|F_n) + (\nabla \tilde{F}|\nabla F_n) + (\tilde{\mathbf{u}} \cdot \nabla \tilde{F}|F_n) \\ &+ (\tilde{F}\nabla \tilde{\mathbf{u}}|F_n) + ((3|\tilde{\mathbf{d}}|^2 - 1)\tilde{F}|F_n) \Big) d\tau = 0. \end{aligned} \tag{2.20}$$

By using assumptions (2.6)–(2.7) and similar argument in the proof of (2.11), we obtain

$$\lim_{n\to\infty} \Big(\int_0^t (\nabla F|\nabla \tilde{F}_n) d\tau + \int_0^t (\nabla \tilde{F}|\nabla F_n) d\tau \Big) = 2\int_0^t (\nabla F|\nabla \tilde{F}) d\tau, \tag{2.21}$$

$$\lim_{n\to\infty} \int_0^t (\tilde{\mathbf{u}} \cdot \nabla \tilde{F}|F_n) d\tau = \int_0^t (\tilde{\mathbf{u}} \cdot \nabla \tilde{F}|F) d\tau, \tag{2.22}$$

$$\lim_{n\to\infty} \int_0^t (\mathbf{u} \cdot \nabla F|\tilde{F}_n) d\tau = \int_0^t (\mathbf{u} \cdot \nabla F|\tilde{F}) d\tau, \tag{2.23}$$

$$\lim_{n\to\infty} \int_0^t (\tilde{F}\nabla \tilde{\mathbf{u}}|F_n) d\tau = \int_0^t (\tilde{F}\nabla \tilde{\mathbf{u}}|F) d\tau, \tag{2.24}$$

$$\lim_{n\to\infty} \int_0^t (F\nabla \mathbf{u}|\tilde{F}_n) d\tau = \int_0^t (F\nabla \mathbf{u}|\tilde{F}) d\tau. \tag{2.25}$$

To estimate the remaining terms, the Hölder inequality and the Sobolev embedding theorem yield

$$\int_0^t (F(\tau)|\tilde{F}_n(\tau))d\tau \le \|F\|_{L^2(0,T;L^2)}\|\tilde{F}_n\|_{L^2(0,T;L^2)}$$

and

$$\begin{aligned}\int_0^t (3|\mathbf{d}|^2F|\tilde{F}_n)(\tau)d\tau &\le C\int_0^t \|\mathbf{d}(\tau)\|_{L^6}\|F(\tau)\|_{L^2}\|\tilde{F}_n(\tau)\|_{L^6}d\tau\\ &\le C\|\mathbf{d}\|_{L^\infty(0,T;\dot{H}^1)}\|F\|_{L^2(0,T;L^2)}\|\tilde{F}_n\|_{L^2(0,T;\dot{H}^1)}.\end{aligned}$$

Hence, by (2.6)–(2.7), we can easily see that

$$\lim_{n\to\infty}\int_0^t ((3|\mathbf{d}|^2-1)F|\tilde{F}_n)d\tau = \int_0^t ((3|\mathbf{d}|^2-1)F|\tilde{F})d\tau. \tag{2.26}$$

Similarly,

$$\lim_{n\to\infty}\int_0^t ((3|\tilde{\mathbf{d}}|^2-1)\tilde{F}|F_n)d\tau = \int_0^t ((3|\tilde{\mathbf{d}}|^2-1)\tilde{F}|F)d\tau. \tag{2.27}$$

As in the derivations of estimates (2.16) and (2.17), the above estimates (2.21)–(2.27) imply

$$\begin{gathered}\lim_{n\to\infty}\int_0^t ((\partial_\tau F|\tilde{F}_n)d\tau = \int_0^t (\partial_\tau F|\tilde{F})d\tau,\\ \lim_{n\to\infty}\int_0^t (\partial_\tau \tilde{F}|F_n)d\tau = \int_0^t (\partial_\tau \tilde{F}|F)d\tau.\end{gathered}$$

Since

$$\begin{gathered}\int_0^t (\partial_\tau F|\tilde{F}) + (\partial_\tau \tilde{F}|F)d\tau = (F(t)|\tilde{F}(t)) - (F_0|\tilde{F}_0),\\ \int_0^t\int_{\mathbb{R}^3}\Big(\tilde{\mathbf{u}}\cdot\nabla\tilde{F}:F + \tilde{\mathbf{u}}\cdot\nabla F:\tilde{F}\Big)\,dx\,d\tau = \int_0^t\int_{\mathbb{R}^3}\tilde{\mathbf{u}}\cdot\nabla(\tilde{F}:F)\,dx\,d\tau = 0,\\ \tilde{F}\nabla\mathbf{u}:F + \tilde{F}:F\nabla\mathbf{u} = \nabla\mathbf{u}:(\tilde{F}^TF + F^T\tilde{F}),\end{gathered}$$

we have

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}^3}\Big(\mathbf{u}\cdot\nabla F:\tilde{F} + \tilde{\mathbf{u}}\cdot\nabla\tilde{F}:F + F\nabla\mathbf{u}:\tilde{F} + \tilde{F}\nabla\tilde{\mathbf{u}}:F\Big)\,dx\,d\tau\\ &= \int_0^t\int_{\mathbb{R}^3}\Big(\nabla\mathbf{u}:(\tilde{F}^TF + F^T\tilde{F}) + (\mathbf{u}-\tilde{\mathbf{u}})\cdot\nabla F:\tilde{F} - F:\tilde{F}\nabla(\mathbf{u}-\tilde{\mathbf{u}})\Big)\,dx\,d\tau.\end{aligned}$$

Here we have used the facts $\operatorname{div}\tilde{\mathbf{u}} = 0$ and $AB:C = A:CB^T = B:A^TC$ for any three $n\times n$ matrixes A, B and C. Finally, putting all above estimates together, we obtain

$$\begin{aligned}&(F(t)|\tilde{F}(t)) + 2\int_0^t(\nabla F|\nabla\tilde{F})d\tau\\ &= (F_0|\tilde{F}_0) - \int_0^t\int_{\mathbb{R}^3}\Big(\nabla\mathbf{u}:(\tilde{F}^TF + F^T\tilde{F}) + \mathbf{w}\cdot\nabla F:\tilde{F} - F:\tilde{F}\nabla\mathbf{w}\Big)\,dx\,d\tau\\ &\quad - \int_0^t\int_{\mathbb{R}^3}\Big(\big((3|\mathbf{d}|^2-1)F:\tilde{F}\big) + \big((3|\tilde{\mathbf{d}}|^2-1)\tilde{F}:F\big)\Big)\,dx\,d\tau.\end{aligned} \tag{2.28}$$

Now it is easy see that (2.5) follows from (2.18) and (2.28). This proves Lemma 2.3. □

The following result plays a very important role in the proof of Proposition 2.1.

Lemma 2.3 ([28]). *Let u be a measurable function in $(\mathcal{LS}) \cap C([0,T], L^3(\mathbb{R}^3))$. Then for each $\varepsilon > 0$ we can split u on $[0,T]$ in $u = m + l$ with $m \in L^\infty([0,T] \times \mathbb{R}^3)$ and $\|l\|_{L^\infty(0,T;L^3)} < \varepsilon$.*

Proof. The proof of this lemma is due to [28], but we give it for completeness. Since $u \in C([0,T], L^3(\mathbb{R}^3))$, by the uniform continuity, we can choose N large enough such that

$$\big\|u(x,t) - \sum_{k=0}^{N-1} \chi_{[\frac{k}{N}T, \frac{k+1}{N}T]}(t) u(x, \frac{k}{N}T)\big\|_{L^\infty(0,T;L^3)} < \frac{\varepsilon}{2},$$

where $\chi_{[a,b]}$ denotes the characteristic function on the interval $[a,b]$. Now we may approximate each $u(\cdot, \frac{k}{N}T)$ by a function $m_{k,N} \in L^\infty(\mathbb{R}^3)$ with an error controlled in L^3-norm by $\|u(\cdot, \frac{k}{N}T) - m_{k,N}(\cdot)\|_{L^3} < \varepsilon/2$. Now we define m as $m(x,t) = \sum_{0 \le k \le N-1} \chi_{[\frac{k}{N}T, \frac{k+1}{N}T]}(t) m_{k,N}(x)$, and $l = u - m$. This proves Lemma. □

Proof of Proposition 2.1. Since $\operatorname{div} \mathbf{w} = 0$, we obtain $\int_0^t \int_{\mathbb{R}^3} \mathbf{w} \cdot \nabla F : F \,dx\,d\tau = 0$. By (2.4) and Lemma 2.2, it follows immediately that

$$\begin{aligned}
&\|(\mathbf{w}(t), E(t))\|_{L^2}^2 + 2\int_0^t \|(\nabla \mathbf{w}(\tau), \nabla E(\tau))\|_{L^2}^2 d\tau \\
&\le \|(\mathbf{u}_0, F_0)\|_{L^2}^2 + \|(\tilde{\mathbf{u}}_0, \tilde{F}_0)\|_{L^2}^2 - 2(\mathbf{u}_0|\tilde{\mathbf{u}}_0) - 2(F_0|\tilde{F}_0) \\
&\quad + 2\int_0^t \int_{\mathbb{R}^3} \mathbf{w} \cdot \nabla \mathbf{w} \cdot \mathbf{u} \,dx\,d\tau - 2\int_0^t \int_{\mathbb{R}^3} \tilde{F}^T \tilde{F} : \nabla \mathbf{u} \,dx\,d\tau \\
&\quad - 2\int_0^t \int_{\mathbb{R}^3} F^T F : \nabla \tilde{\mathbf{u}} \,dx\,d\tau + 2\int_0^t \int_{\mathbb{R}^3} \nabla \mathbf{u} : (\tilde{F}^T F + F^T \tilde{F}) \,dx\,d\tau \\
&\quad + 2\int_0^t \int_{\mathbb{R}^3} \mathbf{w} \cdot \nabla F : \tilde{F} \,dx\,d\tau - 2\int_0^t \int_{\mathbb{R}^3} F : \tilde{F} \nabla \mathbf{w} \,dx\,d\tau - 2\int_0^t \|E\|_{L^2}^2 d\tau \\
&\quad - 6\int_0^t \int_{\mathbb{R}^3} \big(|\mathbf{d}|^2 E : F + |\tilde{\mathbf{d}}|^2 E : \tilde{F}\big) \,dx\,d\tau \\
&\le \|(\mathbf{w}_0, E_0)\|_{L^2}^2 + 2\int_0^t \int_{\mathbb{R}^3} \mathbf{w} \cdot \nabla \mathbf{w} \cdot \mathbf{u} \,dx\,d\tau - 2\int_0^t \int_{\mathbb{R}^3} E^T E : \nabla \mathbf{u} \,dx\,d\tau \\
&\quad - 2\int_0^t \int_{\mathbb{R}^3} \mathbf{w} \cdot \nabla F : E \,dx\,d\tau + 2\int_0^t \int_{\mathbb{R}^3} E^T F : \nabla \mathbf{w} \,dx\,d\tau - 2\int_0^t \|E\|_{L^2}^2 d\tau \\
&\quad - 6\int_0^t \int_{\mathbb{R}^3} \big(|\mathbf{d}|^2 E : F + |\tilde{\mathbf{d}}|^2 E : \tilde{F}\big) \,dx\,d\tau.
\end{aligned} \tag{2.29}$$

Since we have assumed that $(\mathbf{u}, F) \in C([0,T], L^3(\mathbb{R}^3))$, by Lemma 2.3, we can split $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$ and $F = F_1 + F_2$ such that $(\mathbf{u}_1, F_1) \in L^\infty([0,T] \times \mathbb{R}^3)$ and $\|(\mathbf{u}_2, F_2)\|_{L^\infty(0,T;L^3)} < \varepsilon$, respectively, where $\varepsilon > 0$ is a constant to be determined later. Then we see that

$$\begin{aligned}
&\Big|\int_0^t\int_{\mathbb{R}^3}\mathbf{w}\cdot\nabla\mathbf{w}\cdot\mathbf{u}\,dx\,d\tau\Big| \\
&\leq C\|\mathbf{u}_2\|_{C([0,T],L^3)}\int_0^t\|\nabla\mathbf{w}\|_{L^2}^2d\tau \\
&\quad+\|\mathbf{u}_1\|_{L^\infty([0,T]\times\mathbb{R}^3)}\Big(\int_0^t\|\nabla\mathbf{w}\|_{L^2}^2d\tau\Big)^{1/2}\Big(\int_0^t\|\mathbf{w}\|_{L^2}^2d\tau\Big)^{1/2} \\
&\leq 2C\varepsilon\int_0^t\|\nabla\mathbf{w}\|_{L^2}^2d\tau+\frac{4}{C\varepsilon}\|\mathbf{u}_1\|_{L^\infty([0,T]\times\mathbb{R}^3)}^2\int_0^t\|\mathbf{w}\|_{L^2}^2d\tau.
\end{aligned}\tag{2.30}$$

Similarly, we obtain

$$\begin{aligned}
&\Big|\int_0^t\int_{\mathbb{R}^3}E^TE:\nabla\mathbf{u}\,dx\,d\tau\Big| \\
&=\Big|-\int_0^t\int_{\mathbb{R}^3}\operatorname{div}(E^TE)\cdot\mathbf{u}\,dx\,d\tau\Big| \\
&\leq 2C\varepsilon\int_0^t\|\nabla E\|_{L^2}^2d\tau+\frac{4}{C\varepsilon}\|\mathbf{u}_1\|_{L^\infty([0,T]\times\mathbb{R}^3)}^2\int_0^t\|E\|_{L^2}^2d\tau;
\end{aligned}\tag{2.31}$$

$$\begin{aligned}
&\Big|\int_0^t\int_{\mathbb{R}^3}\mathbf{w}\cdot\nabla F:E\,dx\,d\tau\Big| \\
&=\Big|\int_0^t\int_{\mathbb{R}^3}(\mathbf{w}\otimes F)\cdot\nabla E\,dx\,d\tau\Big| \\
&\leq 2C\varepsilon\int_0^t\|(\nabla\mathbf{w},\nabla E)\|_{L^2}^2d\tau+\frac{4}{C\varepsilon}\|F_1\|_{L^\infty([0,T]\times\mathbb{R}^3)}^2\int_0^t\|\mathbf{w}\|_{L^2}^2d\tau;
\end{aligned}\tag{2.32}$$

$$\begin{aligned}
\Big|\int_0^t\int_{\mathbb{R}^3}E^TF:\nabla\mathbf{w}\,dx\,d\tau\Big| &\leq 2C\varepsilon\int_0^t\|(\nabla\mathbf{w},\nabla E)\|_{L^2}^2d\tau \\
&\quad+\frac{4}{C\varepsilon}\|F_1\|_{L^\infty([0,T]\times\mathbb{R}^3)}^2\int_0^t\|E\|_{L^2}^2d\tau;
\end{aligned}\tag{2.33}$$

$$\begin{aligned}
&\Big|\int_0^t\int_{\mathbb{R}^3}\big(|\mathbf{d}|^2E:F+|\tilde{\mathbf{d}}|^2E:\tilde{F}\big)\,dx\,d\tau\Big| \\
&\leq C\int_0^t(\|\mathbf{d}\|_{L^6}^2\|F\|_{L^6}+\|\tilde{\mathbf{d}}\|_{L^6}^2\|\tilde{F}\|_{L^6})\|E\|_{L^2}d\tau \\
&\leq C\int_0^t(\|\mathbf{d}\|_{\dot{H}^1}^2\|F\|_{\dot{H}^1}+\|\tilde{\mathbf{d}}\|_{\dot{H}^1}^2\|\tilde{F}\|_{\dot{H}^1})\|E\|_{L^2}d\tau \\
&\leq C\int_0^t\|E\|_{L^2}^2d\tau,
\end{aligned}\tag{2.34}$$

where C is a constant depending on $\|(\mathbf{d},\tilde{\mathbf{d}})\|_{L^\infty(0,T;\dot{H}^1)}$ and $\|(\mathbf{d},\tilde{\mathbf{d}})\|_{L^2(0,T;\dot{H}^2)}$. Returning back to the estimate (2.29), putting (2.30)–(2.34) together, and choosing ε sufficiently small such that $16C\varepsilon<1$, we obtain

$$\begin{aligned}
&\|(\mathbf{w}(t),E(t))\|_{L^2}^2+\int_0^t\|(\nabla\mathbf{w}(\tau),\nabla E(\tau))\|_{L^2}^2d\tau \\
&\leq\|(\mathbf{w}_0,E_0)\|_{L^2}^2+C\Big(\|(\mathbf{u}_2,F_2)\|_{L^\infty((0,T)\times\mathbb{R}^3)}^2+1\Big)\int_0^t(\|\mathbf{w}\|_{L^2}^2+\|E\|_{L^2}^2)d\tau.
\end{aligned}\tag{2.35}$$

The estimate above together with the Gronwall inequality yield the desired estimate (2.3) immediately. We complete the proof of Proposition 2.1. □

Case 2. $(\mathbf{u},\nabla\mathbf{d})\in L^q(0,T;\dot B_{p,q}^{-1+3/p+2/q}(\mathbb{R}^3))$. It suffices to establish the following stability result.

Proposition 2.4. *Assume that* $(\mathbf{u},\nabla\mathbf{d})\in L^q(0,T;\dot B_{p,q}^{-1+3/p+2/q}(\mathbb{R}^3))$ *with* $2\le p<\infty$, $2<q<\infty$ *and* $\frac{3}{p}+\frac{2}{q}>1$. *Then*

$$
\begin{aligned}
&\|(\mathbf{w}(t),E(t))\|_{L^2}^2+2\int_0^t\|(\nabla\mathbf{w}(\tau),\nabla E(\tau))\|_{L^2}^2d\tau\\
&\le\|(\mathbf{w}_0,E_0)\|_{L^2}^2\times\exp\Big(Ct+C\int_0^t\|(\mathbf{u}(\tau),F(\tau))\|^q_{\dot B_{p,q}^{-1+3/p+2/q}}d\tau\Big).
\end{aligned}\tag{2.36}
$$

where C is a constant depending on $\|(\mathbf{d},\tilde{\mathbf{d}})\|_{L^\infty(0,T;\dot H^1)}$ *and* $\|(\mathbf{d},\tilde{\mathbf{d}})\|_{L^2(0,T;\dot H^2)}$.

To prove Proposition 2.4, the key tool we shall use is the following Lemma whose proof can be found in [6].

Lemma 2.5 ([6]). *Let* $2\le p<\infty$ *and* $2<q<\infty$ *such that* $\frac{2}{q}+\frac{3}{p}>1$. *Then for every* $T>0$, *the trilinear form*

$$
(\mathbf{u},\mathbf{v},\mathbf{w})\in(\mathcal{LS})\times(\mathcal{LS})\times L^q(0,T;\dot B_{p,q}^{-1+3/p+2/q}(\mathbb{R}^3))\mapsto\int_0^T\int_{\mathbb{R}^3}\mathbf{u}\cdot\nabla\mathbf{v}\cdot\mathbf{w}\,dx\,dt
$$

is continuous. In particular, the following estimate holds:

$$
\begin{aligned}
&\Big|\int_0^T\int_{\mathbb{R}^3}\mathbf{u}\cdot\nabla\mathbf{v}\cdot\mathbf{w}\,dx\,dt\Big|\\
&\le C\|\mathbf{u}\|^{2/q}_{L^\infty(0,T;L^2)}\|\nabla\mathbf{u}\|^{1-2/q}_{L^2(0,T;L^2)}\|\nabla\mathbf{v}\|_{L^2(0,T;L^2)}\|\mathbf{w}\|_{L^q(0,T;\dot B_{p,q}^{-1+3/p+2/q})}\\
&\quad+\|\nabla\mathbf{u}\|_{L^2(0,T;L^2)}\|\mathbf{v}\|^{2/q}_{L^\infty(0,T;L^2)}\|\nabla\mathbf{v}\|^{1-2/q}_{L^2(0,T;L^2)}\|\mathbf{w}\|_{L^q(0,T;\dot B_{p,q}^{-1+3/p+2/q})}\\
&\quad+\|\mathbf{u}\|^{1/q}_{L^\infty(0,T;L^2)}\|\nabla\mathbf{u}\|^{1-1/q}_{L^2(0,T;L^2)}\|\mathbf{v}\|^{1/q}_{L^\infty(0,T;L^2)}\|\nabla\mathbf{v}\|^{1-1/q}_{L^2(0,T;L^2)}\\
&\quad\times\|\mathbf{w}\|_{L^q(0,T;\dot B_{p,q}^{-1+3/p+2/q})}.
\end{aligned}\tag{2.37}
$$

Note that (2.37) holds in both scalar and vector cases.

Note that for the Navier-Stokes equations, Gallagher and Planchon [6] proved that weak-strong uniqueness holds in the class

$$
\mathcal{P}=L^q(0,T;\dot B_{p,q}^{-1+3/p+2/q}(\mathbb{R}^3))\quad\text{with } 2\le p<\infty,\ 2<q<\infty\text{ and }\frac{3}{p}+\frac{2}{q}>1.
$$

Hence, we need only to deal with the remaining terms $\operatorname{div}(F^TF)$ and $F\nabla\mathbf{u}$ (the term $\mathbf{u}\cdot\nabla F$ can be treated as the term $\mathbf{u}\cdot\nabla\mathbf{u}$). Similarly as we have done before, we choose two smooth sequences of $\{(\tilde{\mathbf{u}}_n,\tilde F_n)\}$ ($\operatorname{div}\tilde{\mathbf{u}}_n=0$) and $\{(\mathbf{u}_n,F_n)\}$ ($\operatorname{div}\mathbf{u}_n=0$) such that

$$
\begin{gathered}
\lim_{n\to\infty}(\tilde{\mathbf{u}}_n,\tilde F_n)=(\tilde{\mathbf{u}},\tilde F)\quad\text{in } L^2(0,T;\dot H^1(\mathbb{R}^3)),\\
\lim_{n\to\infty}(\tilde{\mathbf{u}}_n,\tilde F_n)=(\tilde{\mathbf{u}},\tilde F)\quad\text{weakly-star in } L^\infty(0,T;L^2(\mathbb{R}^3))
\end{gathered}
$$

and

$$
\begin{gathered}
\lim_{n\to\infty}(\mathbf{u}_n,F_n)=(\mathbf{u},F)\quad\text{in } L^2(0,T;\dot H^1(\mathbb{R}^n))\cap L^q(0,T;\dot B_{p,q}^{-1+n/p+2/q}(\mathbb{R}^3)),\\
\lim_{n\to\infty}(\mathbf{u}_n,F_n)=(\mathbf{u},F)\quad\text{weakly-star in } L^\infty(0,T;L^2(\mathbb{R}^3)).
\end{gathered}
$$

Applying the above assumptions and Lemma 2.5, we obtain

$$\begin{aligned} &\lim_{n\to\infty}\int_0^t (\operatorname{div}(F^TF)|\tilde{\mathbf{u}}_n)d\tau = -\lim_{n\to\infty}\int_0^t (F^TF|\nabla\tilde{\mathbf{u}}_n)d\tau \\ &= -\int_0^t (F^TF|\nabla\tilde{\mathbf{u}})d\tau = \int_0^t (\operatorname{div}(F^TF)|\tilde{\mathbf{u}})d\tau \end{aligned} \tag{2.38}$$

and

$$\begin{aligned} &\lim_{n\to\infty}\int_0^t (\operatorname{div}(\tilde{F}^T\tilde{F})\big|\mathbf{u}_n)d\tau = \lim_{n\to\infty}\int_0^t \Big(\sum_{i=1}^n (\partial_{x_i}\tilde{F}^T\tilde{F} + \tilde{F}^T\partial_{x_i}\tilde{F})|\mathbf{u}_n\Big)d\tau \\ &= \int_0^t \Big(\sum_{i=1}^n (\partial_{x_i}\tilde{F}^T\tilde{F} + \tilde{F}^T\partial_{x_i}\tilde{F})\big|\mathbf{u}\Big)d\tau = \int_0^t (\operatorname{div}(\tilde{F}^T\tilde{F})|\mathbf{u})d\tau. \end{aligned} \tag{2.39}$$

Hence, (2.18) still holds under the assumption of Proposition 2.4.

It is clear that by Lemma 2.5,

$$\lim_{n\to\infty}\int_0^t (\tilde{F}\nabla\tilde{\mathbf{u}}|F_n)d\tau = \int_0^t (\tilde{F}\nabla\tilde{\mathbf{u}}|F)d\tau. \tag{2.40}$$

Since $\nabla\tilde{F}_n$ converges to $\nabla\tilde{F}$ in $L^2(0,T;L^2(\mathbb{R}^3))$, and $\{\tilde{F}_n\}$ is bounded in he space $L^\infty(0,T;L^2(\mathbb{R}^3))$ which was ensured by the Banach-Steinhaus theorem due to $\tilde{F}_n$ weakly-star converge to $\tilde{F}$ in $L^\infty(0,T;L^2(\mathbb{R}^3))$, by Lemma 2.5, we obtain

$$\lim_{n\to\infty}\int_0^t (F\nabla\mathbf{u}|\tilde{F}_n)d\tau = \int_0^t (F\nabla\mathbf{u}|\tilde{F})d\tau. \tag{2.41}$$

The two estimates (2.40)–(2.41) imply that the equality (2.28) still holds under the assumption of Proposition 2.4.

Now we finish the proof of Proposition 2.4. Using the similar argument as in the proof of Lemma 2.5 (see [6]), we obtain

$$\begin{aligned} &\Big|\int_0^t\int_{\mathbb{R}^3} \mathbf{w}\cdot\nabla\mathbf{w}\cdot\mathbf{u}\,dx\,d\tau\Big| \\ &\le C\int_0^t \|\mathbf{w}\|_{L^2}^{2/q}\|\nabla\mathbf{w}\|_{L^2}^{2-2/q}\|\mathbf{u}\|_{\dot{B}_{p,q}^{-1+3/p+2/q}}d\tau \\ &\le \frac{1}{2}\int_0^t \|\nabla\mathbf{w}\|_{L^2}^2 d\tau + C\int_0^t \|\mathbf{w}\|_{L^2}^2\|\mathbf{u}\|_{\dot{B}_{p,q}^{-1+3/p+2/q}}^q d\tau; \end{aligned} \tag{2.42}$$

$$\begin{aligned} &\Big|\int_0^t\int_{\mathbb{R}^3} E^TE:\nabla\mathbf{u}\,dx\,d\tau\Big| \\ &= \Big|-\int_0^t\int_{\mathbb{R}^3} \operatorname{div}(E^TE)\cdot\mathbf{u}\,dx\,d\tau\Big| \\ &\le C\int_0^t \|E\|_{L^2}^{2/q}\|\nabla E\|_{L^2}^{2-2/q}\|\mathbf{u}\|_{\dot{B}_{p,q}^{-1+3/p+2/q}}d\tau \\ &\le \frac{1}{2}\int_0^t \|\nabla E\|_{L^2}^2 d\tau + C\int_0^t \|E\|_{L^2}^2\|\mathbf{u}\|_{\dot{B}_{p,q}^{-1+3/p+2/q}}^q d\tau; \end{aligned} \tag{2.43}$$

$$\begin{aligned}
&\Big|\int_0^t\int_{\mathbb{R}^3}\mathbf{w}\cdot\nabla F:E\,dx\,d\tau\Big| \\
&=\Big|\int_0^t\int_{\mathbb{R}^3}(\mathbf{w}\otimes F)\cdot\nabla E\,dx\,d\tau\Big| \\
&\le\frac{1}{2}\int_0^t\|(\nabla\mathbf{w},\nabla E)\|_{L^2}^2d\tau+C\int_0^t\|(\mathbf{w},E)\|_{L^2}^2\|F\|_{\dot{B}_{p,q}^{-1+3/p+2/q}}^q d\tau
\end{aligned}\tag{2.44}$$

and

$$\begin{aligned}
\Big|\int_0^t\int_{\mathbb{R}^3}E^TF:\nabla\mathbf{w}\,dx\,d\tau\Big|\le\frac{1}{2}\int_0^t\|(\nabla\mathbf{w},\nabla E)\|_{L^2}^2d\tau \\
+C\int_0^t\|(\mathbf{w},E)\|_{L^2}^2\|F\|_{\dot{B}_{p,q}^{-1+3/p+2/q}}^q d\tau.
\end{aligned}\tag{2.45}$$

Returning back to the estimate (2.29) and putting the above estimates (2.42)–(2.45) and (2.34) together, we obtain

$$\begin{aligned}
&\|(\mathbf{w}(t),E(t))\|_{L^2}^2+2\int_0^t\|(\nabla\mathbf{w}(\tau),\nabla E(\tau))\|_{L^2}^2d\tau \\
&\le\|(\mathbf{w}_0,E_0)\|_{L^2}^2+C\int_0^t(\|\mathbf{w}\|_{L^2}^2+\|E\|_{L^2}^2)(1+\|\mathbf{u}\|_{\dot{B}_{p,q}^{-1+3/p+2/q}}^q \\
&\quad+\|F\|_{\dot{B}_{p,q}^{-1+3/p+2/q}}^q)d\tau.
\end{aligned}\tag{2.46}$$

Applying the Gronwall inequality, we obtain (2.36) immediately. The proof of Proposition 2.4 is complete. □

3. Appendix

In this section we shall establish the basic energy inequality (see Definition 1.1) governing the system (1.1). In order to do so, let us consider a classical solution $(\mathbf{u},\mathbf{d})$ of the problem (1.1). We first multiply the first equation of (1.1) by $\mathbf{u}$, integrate over $\mathbb{R}^3$, and use the fact $\nabla\cdot(\nabla\mathbf{d}\odot\nabla\mathbf{d})=\nabla(\frac{|\nabla\mathbf{d}|^2}{2})+\Delta\mathbf{d}\cdot\nabla\mathbf{d}$, we see that

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}\|_{L^2}^2+\|\nabla\mathbf{u}\|_{L^2}^2+(\Delta\mathbf{d}\cdot\nabla\mathbf{d},\mathbf{u})=0.\tag{3.1}$$

Next, we multiply the second equation of (1.1) by $-\Delta\mathbf{d}+g(\mathbf{d})$, integrate over $\mathbb{R}^3$, and use the fact that $(\mathbf{u}\cdot\nabla\mathbf{d},g(\mathbf{d}))=(\mathbf{u},\nabla G(\mathbf{d}))=0$, we see that

$$\frac{1}{2}\frac{d}{dt}\|\nabla\mathbf{d}\|_{L^2}^2+\frac{d}{dt}\int_{\mathbb{R}^3}G(\mathbf{d})dx+\|\Delta\mathbf{d}-g(\mathbf{d})\|_{L^2}^2-(\mathbf{u}\cdot\nabla\mathbf{d},\Delta\mathbf{d})=0.\tag{3.2}$$

Equations (3.1) and (3.2) together imply

$$\frac{1}{2}\frac{d}{dt}\Big(\|\mathbf{u}\|_{L^2}^2+\|\nabla\mathbf{d}\|_{L^2}^2+2\int_{\mathbb{R}^3}G(\mathbf{d})dx\Big)+\|\nabla\mathbf{u}\|_{L^2}^2+\|\Delta\mathbf{d}-g(\mathbf{d})\|_{L^2}^2=0.\tag{3.3}$$

Note that $\int_{\mathbb{R}^3}G(\mathbf{d})dx=\frac{1}{4}\|\mathbf{d}(t)\|_{L^4}^4-\frac{1}{2}\|\mathbf{d}(t)\|_{L^2}^2$. Hence, in order to calculate the term $\frac{d}{dt}\int_{\mathbb{R}^3}G(\mathbf{d})dx$, we multiply the second equation of (1.1) by $\mathbf{d}$ to yield that

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{d}\|_{L^2}^2+\|\nabla\mathbf{d}\|_{L^2}^2+\int_{\mathbb{R}^3}g(\mathbf{d})\cdot\mathbf{d}dx=0;$$

i.e.,

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{d}\|_{L^2}^2+\|\nabla\mathbf{d}\|_{L^2}^2+\|\mathbf{d}\|_{L^4}^4=\|\mathbf{d}\|_{L^2}^2.\tag{3.4}$$

Similarly, multiplying the second equation of (1.1) by $|\mathbf{d}|^2\mathbf{d}$, we obtain

$$\frac{1}{4}\frac{d}{dt}\|\mathbf{d}\|_{L^4}^4 + 3\|\mathbf{d}\cdot\nabla\mathbf{d}\|_{L^2}^2 + \|\mathbf{d}\|_{L^6}^6 = \|\mathbf{d}\|_{L^4}^4. \tag{3.5}$$

On the other hand, it is obvious that

$$\begin{aligned}
&\|\Delta\mathbf{d} - g(\mathbf{d})\|_{L^2}^2 = (\Delta\mathbf{d} - |\mathbf{d}|^2\mathbf{d} + \mathbf{d}, \Delta\mathbf{d} - |\mathbf{d}|^2\mathbf{d} + \mathbf{d})\\
&= \|\Delta\mathbf{d}\|_{L^2}^2 - 2(\Delta\mathbf{d}, |\mathbf{d}|^2\mathbf{d}) + 2(\Delta\mathbf{d}, \mathbf{d}) - 2(|\mathbf{d}|^2\mathbf{d}, \mathbf{d}) + (|\mathbf{d}|^2\mathbf{d}, |\mathbf{d}|^2\mathbf{d}) + (\mathbf{d}, \mathbf{d})\\
&= \|\Delta\mathbf{d}\|_{L^2}^2 + 6\||\mathbf{d}|\nabla\mathbf{d}\|_{L^2}^2 - 2\|\nabla\mathbf{d}\|_{L^2}^2 - 2\|\mathbf{d}\|_{L^4}^4 + \|\mathbf{d}\|_{L^6}^6 + \|\mathbf{d}\|_{L^2}^2.
\end{aligned}$$

Putting the estimates (3.3)–(3.5) together, we obtain

$$\frac{1}{2}\frac{d}{dt}\Big(\|\mathbf{u}\|_{L^2}^2 + \|\nabla\mathbf{d}\|_{L^2}^2\Big) + \|\nabla\mathbf{u}\|_{L^2}^2 + \|\Delta\mathbf{d}\|_{L^2}^2 + 3\||\mathbf{d}|\nabla\mathbf{d}\|_{L^2}^2 = \|\nabla\mathbf{d}\|_{L^2}^2. \tag{3.6}$$

This yields immediately the energy inequality in Definition 1.1. Finally, by applying the Gronwall inequality, we obtain the following basic energy inequality:

$$\begin{aligned}
&\|\mathbf{u}(t)\|_{L^2}^2 + \|\nabla\mathbf{d}(t)\|_{L^2}^2 + \int_0^t \big(\|\nabla\mathbf{u}(\tau)\|_{L^2}^2 + \|\Delta\mathbf{d}(\tau)\|_{L^2}^2\big)d\tau\\
&\le C(\|\mathbf{u}_0\|_{L^2}^2, \|\nabla\mathbf{d}_0\|_{L^2}^2)e^{2t}.
\end{aligned} \tag{3.7}$$

Combining the above energy estimate, the Galerkin approximate procedure and the compactness argument give global existence of weak solutions of (1.1).

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