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BASIC RESULTS ON NONLINEAR EIGENVALUE PROBLEMS OF FRACTIONAL ORDER

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Abstract. In this article, we discuss the basic theory of boundary-value problems of fractional order $1 < \delta < 2$ involving the Caputo derivative. By applying the maximum principle, we obtain necessary conditions for the existence of eigenfunctions, and show analytical lower and upper bounds estimates of the eigenvalues. Also we obtain a sufficient condition for the non existence of ordered solutions, by transforming the problem into equivalent integro-differential equation. By the method of lower and upper solution, we obtain a general existence and uniqueness result: We generate two well defined monotone sequences of lower and upper solutions which converge uniformly to the actual solution of the problem. While some fundamental results are obtained, we leave others as open problems stated in a conjecture.

1. Introduction

In this article, we study the eigenvalue problem of fractional order

$$D^{\delta}u(t) + g(t)u' + h(t)u = -\lambda k(t,u), \quad t \in (0,1),\ 1 < \delta < 2, \tag{1.1}$$

$$u(0) - \alpha u'(0) = 0, \quad u(1) + \beta u'(1) = 0, \quad \alpha, \beta \ge 0, \tag{1.2}$$

where $k \in C^1([0,1] \times \mathbb{R}), g$ and $h \in C[0,1]$, and D^{δ} is the Caputo fractional derivative of order δ.

In recent years a great interest was devoted to the study of boundary-value problems of fractional order. There are several definitions of fractional derivative. However, the most popular ones are the Riemann-Liouville and Caputo fractional derivatives. The two definitions differ only in the order of evaluation. In the Caputo definition, we first compute an ordinary derivative then a fractional integral, while in the Riemann-Liouville definition the operators are reversed. In this article, we use the Caputo's fractional derivative since mathematical modeling of many physical problems requires initial and boundary conditions. These demands are satisfied using the Caputo fractional derivative. For more details we refer the reader to [12, 18] and references therein.

The importance of fractional boundary-value problems stems from the fact that they model various applications in fluid mechanics, visco-elasticity, physics, biology

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and economics which can not be modeled by differential equations with integer derivatives [6, 9, 18].

The existence and uniqueness results of fractional boundary-value problems have been investigated by many authors using different techniques [2, 3, 4, 7, 8, 10, 11, 15, 21, 22]. The basic theory of fractional differential equations involving the Riemann-Liouville derivative of order $0 < \delta < 1$, has been investigated in [13, 14] by the classical approach of differential equations. Also, the ideas of comparison principle and the method of lower and upper solutions are applied to prove a general result of existence and uniqueness of solutions in [15]. By means of Schauder Fixed Point Theorem, Zhang [21] proved the existence of solutions for the following boundary-value problem of fractional order, involving Caputo's derivative

$$D_t^{\delta} u(t) = g(t,u), \quad 0 < t < 1,\ 1 < \delta < 2,$$
$$u(0) = a \neq 0, \quad u(1) = b \neq 0.$$

Also, Al-Refai and Hajji [4] established existence and uniqueness results by generating monotone iterative sequences of lower and upper solutions that converge uniformly to the actual solution of the above problem. In a recent work Qi and Chen [20] studied analytically an eigenvalue problem of order $0 < \delta < 1$, with left and right fractional derivatives, which models a bar of finite length with long range interactions. Using the spectral theory of self-adjoint compact operators in Hilbert spaces, they proved that the problem has a countable simple real eigenvalues and the corresponding eigenfunctions form a complete orthogonal system in the Hilbert space L^2.

The Sturm-liouville eigenvalue problem has played an important role in modeling many physical problems. The theory of the problem is well developed and many results have been obtained concerning the eigenvalues and corresponding eigenfunctions. However, up to our knowledge, there are no analytical studies for the eigenvalues and eigenfunctions of the fractional Sturm-Liouville eigenvalue problems of order $1 < \delta < 2$. Following the classical approach of the theory of differential equations of integer order and by means of the maximum principle and the method of lower and upper solutions [17, 19], this paper presents new results about the eigenvalues and eigenfunctions of the eigenvalue problem (1.1)-(1.2).

This article is organized as follows. In the next section, we present basic definitions and results of fractional derivative. In Section 3, we present a new positivity lemma, a uniqueness result and bounds for the eigenvalues of the problem. In Section 4, we present a sufficient condition for the non existence of ordered solutions. In Section 5, we obtain an existence and uniqueness result using the method of lower and upper solutions. We then discuss the linear eigenvalue problem and highlight future research directions in Section 6. We close up with some concluding remarks in Section 7.

2. Preliminaries

In this section, we present the definitions and some preliminary results of the Riemmann-Liouville fractional integral and the Caputo fractional derivative. We then give the definition of lower and upper solutions of the problem (1.1)-(1.2).

Definition 2.1. A real function $f(t), t > 0$, is said to be in the space $C_\mu, \mu \in \mathbb{R}$ if there exists a real number $p > \mu$, such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C[0,\infty)$, and it is said to be in the space C_μ^m if $f^{(m)} \in C_\mu$, $m \in \mathbb{N}$.

Definition 2.2. The left Riemann-Liouville fractional integral of order $\delta > 0$, of a function $f \in C_\mu, \mu \geq -1$, is defined by

$$I^\delta f(t) = \frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} f(s) ds, \quad t > 0. \tag{2.1}$$

The following result will be used throughout the text.

Lemma 2.3. *If $f(t) \in C[0,1]$, then $I^\delta f(t)$ exists and $\lim_{t\to 0} I^\delta f(t) = 0$.*

Proof. We have $r(s) = (t-s)^{\delta-1} \geq 0$ is integrable on $[0,t]$, since $\delta > 0$. Applying the mean value theorem for integrals we have

$$\int_0^t (t-s)^{\delta-1} f(s) ds = f(\xi) \int_0^t (t-s)^{\delta-1} ds = f(\xi) \frac{t^\delta}{\delta}, \quad \text{for some } 0 < \xi < t.$$

Thus,

$$\lim_{t\to 0} I^\delta f(t) = \lim_{t\to 0} f(\xi) \frac{t^\delta}{\Gamma(\delta+1)} = 0.$$

□

Definition 2.4. For $\delta > 0$, $m-1 < \delta < m$, $m \in \mathbb{N}$, $t > 0$, and $f \in C^m_{-1}$, the left Caputo fractional derivative is defined by

$$D^\delta f(t) = \frac{1}{\Gamma(m-\delta)} \int_0^t (t-s)^{m-1-\delta} f^{(m)}(s) ds, \tag{2.2}$$

where Γ is the well-known Gamma function.

The Caputo derivative defined in (2.2) is related to the Riemann-Liouville fractional integral, I^δ, of order $\delta \in \mathbb{R}^+$, by

$$D^\delta f(t) = I^{m-\delta} f^{(m)}(t).$$

It is known (see [12]) that

$$I^\delta (D^\delta f(t)) = f(t) - \sum_{k=0}^{m-1} c_k t^k, \tag{2.3}$$

$$D^\delta I^\delta f(t) = f(t), \tag{2.4}$$

where in (2.3), $c_k = \frac{f^{(k)}(0+)}{k!}$, $0 \leq k \leq m-1$.

Definition 2.5 ([4])**.** A function $v(t) \in C^2[0,1]$ is called a lower solution of the problem (1.1)-(1.2) if it satisfies

$$P(v) = D^\delta v(t) + g(t)v' + h(t)v + \lambda k(t,v) \geq 0, \quad t \in (0,1),\ 1 < \delta < 2, \tag{2.5}$$

and

$$v(0) - \alpha v'(0) \leq 0, \quad v(1) + \beta v'(1) \leq 0. \tag{2.6}$$

Analogously, a function $w(t) \in C^2[0,1]$ is called an upper solution if it satisfies (2.5)-(2.6) with reversed inequalities.

If $v(t) \leq w(t)$, for all $t \in [0,1]$, we say that v and w are ordered lower and upper solutions.

3. General results and estimates of the eigenvalues

In this section we present a positivity lemma which will be used throughout the text. We then prove general results concerning the lower and upper solutions of the eigenvalue problem (1.1)-(1.2). At the end, we present necessary conditions for the existence of eigenfunctions and give analytical estimates on the bounds of the eigenvalues. We have the following results, see [5].

Theorem 3.1. *Assume $f \in C^2[0,1]$ attains its minimum at $t_0 \in (0,1)$, then*

$$(D^\delta f)(t_0) \geq \frac{1}{\Gamma(2-\delta)}\Big[(\delta-1)t_0^{-\delta}(f(0)-f(t_0)) - t_0^{1-\delta}f'(0)\Big], \quad \textit{for all } 1<\delta<2.$$

Corollary 3.2. *Assume $f \in C^2[0,1]$ attains its minimum at $t_0 \in (0,1)$, and $f'(0) \leq 0$. Then $(D^\delta f)(t_0) \geq 0$, for all $1<\delta<2$.*

In the following we state and prove a positivity result which will be used throughout the text.

Lemma 3.3 (Positivity Result)**.** *Let $z(t) \in C^2[0,1], \mu(t,z) \in C\big([0,1]\times\mathbb{R}\big)$ and $\mu(t,z)<0, \forall\, t \in (0,1)$. If $z(t)$ satisfies the inequalities*

$$D^\delta z(t) + a(t)z'(t) + \mu(t,z)z \leq 0, \quad t \in (0,1), \tag{3.1}$$

$$z(0) - \alpha z'(0) \geq 0, \quad \textit{and} \quad z(1) + \beta z'(1) \geq 0, \tag{3.2}$$

where $a(t) \in C[0,1]$ and $\alpha, \beta \geq 0$, then $z(t) \geq 0$, for all $t \in [0,1]$ provided $\alpha \geq \frac{1}{\delta-1}$.

Proof. Assume that the conclusion is false, then $z(t)$ has absolute minimum at some t_0 with $z(t_0)<0$. Let $t_0 \in (0,1)$, then $z'(t_0)=0$. In the following we prove that $(D^\delta z)(t_0) \geq 0$. By Corollary 3.2 the result is true if $z'(0) \leq 0$. If $z'(0)>0$, by Theorem 3.1 there holds

$$\begin{aligned}\Gamma(2-\delta)(D^\delta z)(t_0) &\geq (\delta-1)t_0^{-\delta}(z(0)-z(t_0)) - t_0^{1-\delta}z'(0)\\ &= t_0^{-\delta}\Big((\delta-1)(z(0)-z(t_0)) - t_0 z'(0)\Big).\end{aligned}$$

Since $\alpha(\delta-1) \geq 1$ and from the boundary condition $z(0) \geq \alpha z'(0)$, we have

$$(\delta-1)(z(0)-z(t_0)) \geq (\delta-1)(\alpha z'(0)-z(t_0)) \geq z'(0) - (\delta-1)z(t_0).$$

Thus,

$$(\delta-1)(z(0)-z(t_0)) - t_0 z'(0) \geq z'(0) - (\delta-1)z(t_0) - t_0 z'(0) = z'(0)(1-t_0) - (\delta-1)z(t_0).$$

Since $z'(0)>0, 0<t_0<1$ and $z(t_0)<0$, we have $(D^\delta z)(t_0) \geq 0$. The above results together with $\mu(t_0,z(t_0))<0$, imply

$$(D^\delta z)(t_0) + a(t_0)z'(t_0) + \mu(t_0,z(t_0))z(t_0) = (D^\delta z)(t_0) + \mu(t_0,z(t_0))z(t_0) > 0,$$

which contradicts (3.1). If $t_0=0$, by simple maximum principle, $z'(0^+) \geq 0$. Applying the boundary condition $z(0)-\alpha z'(0) \geq 0$, we have $z(0) \geq 0$ and a contradiction is reached. Similarly, if $t_0=1$, then simple maximum principle implies $z'(1^-) \leq 0$. The boundary condition $z(1)+\beta z'(1^-) \geq 0$ yields $z(1) \geq 0$ and a contradiction is reached. ∎

Theorem 3.4. *Consider problem (1.1)–(1.2). If $h(t) + \lambda\frac{\partial k(t,u)}{\partial u} < 0$, for all $u \in C^2[0,1]$ and $t \in (0,1)$, then for $\alpha \geq \frac{1}{\delta-1}$ we have*

(1) *Any lower and upper solutions are ordered.*

(2) *The problem* (1.1)-(1.2) *possesses at most one solution.*

Proof. (1) Let v and w respectively, be any lower and upper solutions of the problem. We have

$$\begin{gathered} D^\delta v(t) + g(t)v'(t) + h(t)v + \lambda k(t,v) \geq 0, \quad t \in (0,1), \\ v(0) - \alpha v'(0) \leq 0, \quad v(1) + \beta v'(1) \leq 0, \end{gathered} \tag{3.3}$$

and

$$\begin{gathered} D^\delta w(t) + g(t)w'(t) + h(t)w + \lambda k(t,w) \leq 0, \quad t \in (0,1), \\ w(0) - \alpha w'(0) \geq 0, \quad w(1) + \beta w'(1) \geq 0. \end{gathered} \tag{3.4}$$

Subtracting (3.3) from (3.4) and applying the mean value theorem, we have

$$D^\delta (w-v) + g(t)(w'-v') + h(t)(w-v) + \lambda(k(t,w) - k(t,v)) \leq 0,$$

$$D^\delta (w-v) + g(t)(w'-v') + \Big(h(t) + \lambda \frac{\partial k}{\partial u}(\xi)\Big)(w-v) \leq 0,$$

where $\xi = \gamma w + (1-\gamma)v$, $0 \leq \gamma \leq 1$. Let $z = w - v$, then z satisfies

$$D^\delta z + g(t)z' + \Big(h(t) + \lambda \frac{\partial k}{\partial u}(\xi)\Big)z \leq 0,$$

with $z(0) - \alpha z'(0) \geq 0$ and $z(1) + \beta z'(1) \geq 0$. Since $h(t) + \lambda \frac{\partial k}{\partial u}(\xi) < 0$, by the positivity result, Lemma 3.3, $z \geq 0$, and thus $w \geq v$.

(2) Let u_1 and u_2 be two solutions of (1.1)-(1.2). We have

$$D^\delta u_1 + g(t)u_1' + h(t)u_1 + \lambda k(t,u_1) = 0, \tag{3.5}$$

$$D^\delta u_2 + g(t)u_2' + h(t)u_2 + \lambda k(t,u_2) = 0, \tag{3.6}$$

with $u_1(0) - \alpha u_1'(0) = u_2(0) - \alpha u_2'(0) = 0$, and $u_1(1) + \beta u_1'(1) = u_2(1) + \beta u_2'(1) = 0$. By subtracting (3.6) from (3.5), applying the mean value theorem, and substituting $z = u_1 - u_2$, we have

$$D^\delta z + g(t)z' + \Big(h(t) + \lambda \frac{\partial k}{\partial u}(\xi)\Big)z = 0, \tag{3.7}$$

for some ξ between u_1 and u_2, with

$$z(0) - \alpha z'(0) = 0 \quad \text{and} \quad z(1) + \beta z'(1) = 0. \tag{3.8}$$

Since $h(t) + \lambda \frac{\partial k}{\partial u}(\xi) < 0$, by the positivity result, Lemma 3.3, we have $z \geq 0$. On the other hand (3.7)-(3.8) are also satisfied by $-z$ and by Lemma 3.3, we have $-z \geq 0$. Thus, $z = 0$. This proves that $u_1 = u_2$ and problem (1.1)-(1.2) has at most one solution. □

The following corollary gives analytical lower and upper bounds estimates of the eigenvalues.

Corollary 3.5. *Consider the eigenvalue problem* (1.1)-(1.2), *with* $k(t,0) = 0$ *and* $\alpha \geq \frac{1}{\delta - 1}$. *We have the following necessary conditions for the existence of an eigenfunction.*

(1) *If there exists a negative constant* ξ *such that* $\frac{\partial k}{\partial u} \leq \xi < 0$, *then* $\lambda \leq \sup\{-h/\frac{\partial k}{\partial u}\}$.

(2) *If there exists a positive constant* μ *such that* $\frac{\partial k}{\partial u} \geq \mu > 0$, *then* $\lambda \geq \inf\{-h/\frac{\partial k}{\partial u}\}$.

Proof. (1) If $\lambda > \sup\{-h/\frac{\partial k}{\partial u}\}$, then $\lambda > -h/\frac{\partial k}{\partial u}$ for all $t \in [0,1]$. Since $\frac{\partial k}{\partial u} < 0$, we have $h(t) + \lambda \frac{\partial k}{\partial u} < 0$. Thus the eigenvalue problem possesses at most one solution. Since $k(t,0) = 0$, it can be easily seen that $u = 0$ is a solution. Thus the eigenvalue problem has only the trivial solution and hence no eigenfunction.

(2) If $\lambda < \inf\{-h/\frac{\partial k}{\partial u}\}$, then $\lambda < -h/\frac{\partial k}{\partial u}$ for all $t \in [0,1]$. Since $\frac{\partial k}{\partial u} > 0$, we have $h(t) + \lambda \frac{\partial k}{\partial u} < 0$. Thus the eigenvalue problem possesses only the trivial solution and hence no eigenfunction. $\square$

4. Existence of ordered solutions

In this section, we present a sufficient condition for the non existence of ordered solutions of the eigenvalue problem (1.1)-(1.2). This new result will be needed to prove a uniqueness result in the next section.

Definition 4.1. Let $u_1 \neq u_2$ be two solutions of (1.1)-(1.2). We say that u_1 and u_2 are ordered solutions, if either $u_1 \leq u_2$ or $u_2 \leq u_1$ for all $t \in [0,1]$.

Lemma 4.2. *Consider problem* (1.1)-(1.2) *with* $g, h \in C[0,1]$ *and* $k \in C^1([0,1] \times \mathbb{R})$. *A function* $u(t) \in C^2[0,1]$ *is a solution of the problem if and only if it is a solution of the integro-differential equation*

$$\begin{aligned} u(t) = \frac{\alpha + t}{\Gamma(\delta)(\alpha+\beta+1)} \Big(\int_0^1 (1-s)^{\delta-1} H(s,u) ds \\ + \beta(\delta-1) \int_0^1 (1-s)^{\delta-2} H(s,u) ds \Big) - \frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} H(s,u) ds, \end{aligned} \tag{4.1}$$

where $H(s,u) = g(s)u'(s) + h(s)u(s) + \lambda k(s,u)$.

Proof. Let $u(t)$ be a solution of (1.1). Applying the operator I^δ to both sides of (1.1) and using (2.3), we obtain

$$\begin{aligned} u(t) &= c_0 + c_1 t - I^\delta H(t,u(t)) = c_0 + c_1 t - \frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} H(s,u) ds \\ &= c_0 + c_1 t - \frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} (g(s)u'(s) + h(s)u(s) + \lambda k(s,u)) ds. \end{aligned} \tag{4.2}$$

Thus, $u(0) = c_0$ and $u(1) = c_0 + c_1 - \frac{1}{\Gamma(\delta)} \int_0^1 (1-s)^{\delta-1} H(s,u) ds$. Differentiating (4.2) yields

$$\begin{aligned} u'(t) &= c_1 - D^1 I^\delta H(t,u) = c_1 - D^1 I^1 I^{\delta-1} H(t,u) \\ &= c_1 - I^{\delta-1} H(t,u) = c_1 - \frac{1}{\Gamma(\delta-1)} \int_0^t (t-s)^{\delta-2} H(s,u) ds. \end{aligned} \tag{4.3}$$

Since $\delta - 1 > 0$ and $H(s,u) \in C[0,1]$ by Lemma 2.3, we have $u'(0) = c_1$, and

$$u'(1) = c_1 - \frac{1}{\Gamma(\delta-1)} \int_0^1 (1-s)^{\delta-2} H(s,u) ds.$$

Applying the boundary conditions (1.2) and using the fact that $\Gamma(\delta) = (\delta-1)\Gamma(\delta-1)$, we find that $c_0 - \alpha c_1 = 0$, and

$$c_0 + c_1 - \frac{1}{\Gamma(\delta)} \int_0^1 (1-s)^{\delta-1} H(s,u) ds + \beta\Big(c_1 - \frac{\delta-1}{\Gamma(\delta)} \int_0^1 (1-s)^{\delta-2} H(s,u) ds\Big) = 0. \tag{4.4}$$

Substituting $c_0 = \alpha c_1$ in (4.4) yields

$$c_1 = \frac{1}{\Gamma(\delta)(\alpha+\beta+1)}\Big(\int_0^1 (1-s)^{\delta-1}H(s,u)ds + \beta(\delta-1)\int_0^1 (1-s)^{\delta-2}H(s,u)ds\Big). \tag{4.5}$$

Substitution of c_0 and c_1 back into (4.2) gives

$$\begin{aligned} u(t) &= \frac{\alpha+t}{\Gamma(\delta)(\alpha+\beta+1)}\Big(\int_0^1 (1-s)^{\delta-1}H(s,u)ds \\ &\quad + \beta(\delta-1)\int_0^1 (1-s)^{\delta-2}H(s,u)ds\Big) - \frac{1}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-1}H(s,u)ds. \end{aligned}$$

Conversely, Let $u(t)$ be a solution of (4.1). Substituting

$$\nu = \frac{1}{(\alpha+\beta+1)}\Big(\int_0^1 (1-s)^{\delta-1}H(s,u)ds + \beta(\delta-1)\int_0^1 (1-s)^{\delta-2}H(s,u)ds\Big)$$

yields

$$u(t) = \frac{1}{\Gamma(\delta)}\Big(\nu(\alpha+t) - \int_0^t (t-s)^{\delta-1}H(s,u)ds\Big). \tag{4.6}$$

Applying the differential operator D^1, we have

$$\begin{aligned} u'(t) &= \frac{1}{\Gamma(\delta)}\nu - DI^\delta H(t,u) = \frac{1}{\Gamma(\delta)}\nu - I^{\delta-1}H(t,u) \\ &= \frac{1}{\Gamma(\delta)}\Big(\nu - (\delta-1)\int_0^t (t-s)^{\delta-2}H(s,u)ds\Big). \end{aligned} \tag{4.7}$$

Thus $u(0) = \frac{\alpha\nu}{\Gamma(\delta)}$ and by Lemma 2.3, $u'(0) = \frac{\nu}{\Gamma(\delta)}$ and hence; $u(0) - \alpha u'(0) = 0$. Also,

$$\begin{aligned} u(1) + \beta u'(1) &= \frac{1}{\Gamma(\delta)}\Big(\nu(\alpha+1) - \int_0^1 (1-s)^{\delta-1}H(s,u)ds \\ &\quad + \beta\nu - \beta(\delta-1)\int_0^1 (1-s)^{\delta-2}H(s,u)ds\Big) \\ &= \frac{1}{\Gamma(\delta)}\Big(\nu(\alpha+\beta+1) - [\int_0^1 (1-s)^{\delta-1}H(s,u)ds \\ &\quad + \beta(\delta-1)\int_0^1 (1-s)^{\delta-2}H(s,u)ds]\Big) \\ &= \frac{1}{\Gamma(\delta)}\Big(\nu(\alpha+\beta+1) - \nu(\alpha+\beta+1\Big) = 0. \end{aligned}$$

Next, since $D^\delta[\nu(\alpha+t)] = 0,\ 1<\delta<2$, application of D^δ to both sides of (4.6), gives

$$D^\delta u = -\frac{1}{\Gamma(\delta)}D^\delta\Big[\int_0^t (t-s)^{\delta-1}H(s,u(s))ds\Big] = -D^\delta I^\delta H(t,u) = -H(t,u).$$

Thus, u satisfies (1.1) and the proof is complete. □

We have the following important result.

Theorem 4.3. *Consider problem (1.1)-(1.2) with* $g, h \in C[0,1], k \in C^1([0,1] \times \mathbb{R}), u \in C^2[0,1]$, *and* $g(t) \geq 0,\ t \in [0,1]$. *If* $h(t) + \lambda \frac{\partial k}{\partial u} \geq \eta > 0$, *for some positive constant* η, *then there exists* $\alpha_0 > 0$ *such that the problem has no ordered solutions for* $\alpha \geq \alpha_0$.

Proof. Let $u_1 \leq u_2$ be two solutions of (1.1)-(1.2). We have

$$u_1(t) = c_0 + c_1 t - I^\delta H(t, u_1(t)) = c_0 + c_1 t - \frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} H(s, u_1) ds,$$
$$u_2(t) = d_0 + d_1 t - I^\delta H(t, u_2(t)) = d_0 + d_1 t - \frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} H(s, u_2) ds,$$

where $c_1 = u_1'(0)$, $d_1 = u_2'(0)$, and $H(s, u(s)) = g(s)u'(s) + h(s)u(s) + \lambda k(s, u(s))$. Let $z(t) = u_2(t) - u_1(t) \geq 0 \in [0,1]$. We have

$$\begin{gathered} z(t) = d_0 - c_0 + (d_1 - c_1)t - I^\delta \Big(H(t, u_2(t)) - H(t, u_1(t)) \Big) \\ z(0) = \alpha z'(0) \quad \text{and} \quad z(1) = -\beta z'(1). \end{gathered} \tag{4.8}$$

Thus,

$$\begin{aligned} z'(t) &= d_1 - c_1 - D^1 I^\delta \Big(H(t, u_2) - H(t, u_1) \Big) \\ &= u_2'(0) - u_1'(0) - D I^1 \Big[I^{\delta-1} \Big(H(t, u_2) - H(t, u_1) \Big) \Big] \\ &= z'(0) - \frac{1}{\Gamma(\delta-1)} \int_0^t (t-s)^{\delta-2} \Big(H(s, u_2) - H(s, u_1) \Big) ds. \end{aligned} \tag{4.9}$$

Substituting $z' = u_2' - u_1'$ in (4.9) and applying the mean value theorem yields

$$\begin{aligned} z'(t) = z'(0) &- \frac{1}{\Gamma(\delta-1)} \int_0^t (t-s)^{\delta-2} \Big(g(s)(u_2' - u_1') \\ &+ h(s)(u_2 - u_1) + \lambda[k(s, u_2) - k(s, u_1)] \Big) ds \\ = z'(0) &- \frac{1}{\Gamma(\delta-1)} \int_0^t (t-s)^{\delta-2} \Big(g(s) z'(s) + [h(s) + \lambda \frac{\partial k}{\partial u}(\xi)] z(s) \Big) ds, \end{aligned} \tag{4.10}$$

for some ξ between u_1 and u_2. Assume by contradiction $z \neq 0$, then $z(t)$ has a positive maximum in $[0,1]$. Let $t_0 \in [0,1]$ be the first point at which z has a positive maximum. If $t_0 = 0$, by simple maximum principle, $z'(0^+) \leq 0$. Applying the boundary condition in (4.8) we have $z(0) \leq 0$, and a contradiction is reached. Similarly, if $t_0 = 1$, then $z'(1^-) \geq 0$. Applying the boundary condition in (4.8) yields $z(1) \leq 0$ and a contradiction is reached. Thus $t_0 \in (0,1)$. We have $z(t_0) > 0$, $z'(t_0) = 0$ and since $z'(0) \geq 0$, $z'(t) \geq 0$, for all $t \in [0, t_0]$. We consider two cases for $z(0)$: case 1; $z(0) = 0$ and case 2; $z(0) = \zeta > 0$. If $z(0) = 0$, by the boundary condition (4.8), $z'(0) = 0$. Substituting the above results in (4.10) together with $g(s) \geq 0, s \in [0, t_0]$ and $h(s) + \lambda \frac{\partial k}{\partial u}(\xi) \geq \eta > 0$ yields $z'(t_0) < 0$, which contradicts $z'(t_0) = 0$. Now, let $z(0) = \zeta > 0$. We have

$$\begin{aligned} \kappa &= \frac{1}{\Gamma(\delta-1)} \int_0^{t_0} (t_0 - s)^{\delta-2} \Big(g(s) z'(s) + [h(s) + \lambda \frac{\partial k}{\partial u}(\xi)] z(s) \Big) ds \\ &> \frac{\eta\zeta}{\Gamma(\delta-1)} \int_0^{t_0} (t_0 - s)^{\delta-2} ds = \frac{\eta\zeta}{\Gamma(\delta-1)} \frac{t_0^{\delta-1}}{\delta-1} = \frac{\eta\zeta}{\Gamma(\delta)} t_0^{\delta-1}. \end{aligned} \tag{4.11}$$

Substituting (4.11) in (4.10), for $\alpha > \alpha_0 = \frac{\Gamma(\delta)}{\eta} t_0^{1-\delta}$, we have

$$z'(t_0) < \frac{\zeta}{\alpha} - \frac{\eta\zeta}{\Gamma(\delta)} t_0^{\delta-1} < 0, \tag{4.12}$$

and a contradiction is reached. Thus $z = 0$ and the proof is complete. □

5. Existence and uniqueness result

In this section we apply the method of lower and upper solutions to establish an existence and uniqueness result of the eigenvalue problem (1.1)-(1.2). Given ordered lower and upper solutions, $v^{(0)}(t)$ and $w^{(0)}(t)$ respectively, define the set

$$[v^{(0)}, w^{(0)}] = \{f \in C^2[0,1] : v^{(0)} \le f \le w^{(0)}\},$$

and let $Q_1(f) = f(0) - \alpha f'(0)$, $Q_2(f) = f(1) + \beta f'(1)$.

Theorem 5.1. *Consider the boundary-value problem* (1.1)-(1.2) *with* $\alpha \ge \frac{1}{\delta-1}$. *Let* $v^{(0)}$ *and* $w^{(0)}$ *be an initial ordered lower and upper solutions of* (1.1)-(1.2) *and assume that there exists a positive constant* c *such that*

$$-c < h(t) + \lambda \frac{\partial k}{\partial u} \quad \text{for all } u \in [v^{(0)}, w^{(0)}] \text{ and } t \in [0,1]. \tag{5.1}$$

Let $t \in (0,1)$, *let* $v^{(j)}, w^{(j)}$, $j \ge 1$, *be, respectively, the solutions of*

$$-D^\delta v^{(j)} - g(t) D v^{(j)} + c\, v^{(j)} = (h(t)+c) v^{(j-1)} + \lambda k(t, v^{(j-1)}), \tag{5.2}$$

$$Q_1(v^{(j)}) = Q_2(v^{(j)}) = 0, \tag{5.3}$$

and

$$-D^\delta w^{(j)} - g(t) D w^{(j)} + c\, w^{(j)} = (h(t)+c) w^{(j-1)} + \lambda k(t, w^{(j-1)}), \tag{5.4}$$

$$Q_1(w^{(j)}) = Q_2(w^{(j)}) = 0. \tag{5.5}$$

Then we have

1. *The sequence* $v^{(j)}$, $j \ge 1$, *is an increasing sequence of lower solutions of* (1.1)-(1.2).
2. *The sequence* $w^{(j)}$, $j \ge 1$, *is a decreasing sequence of upper solutions of* (1.1)-(1.2).
3. $v^{(j)} \le w^{(j)}$, *for all* $j \ge 1$.
4. *The sequences* $\{v^{(j)}\}$ *and* $\{w^{(j)}\}$, $j \ge 0$, *converge uniformly to* v^* *and* w^*, *respectively, with* $v^* \le w^*$.

The proof is a simple generalization to the one obtained in [4]. We shall skip it as not to impede the presentation of the main results and make the presentation easier to follow.

Theorem 5.2. *Consider problem* (1.1)-(1.2) *with* $k(t,0) \ne 0$. *Let* $v^{(j)}$ *and* $w^{(j)}$, $j \ge 0$, *be as defined in Theorem 5.1. Then problem* (1.1)-(1.2) *has a unique nontrivial solution* $u \in [v^{(0)}, w^{(0)}]$ *provided one of the following conditions holds*

- (H1) $h(t) + \lambda \frac{\partial k}{\partial u} < 0$, *for all* $u \in [v^{(0)}, w^{(0)}]$, $t \in (0,1)$ *and* $\alpha \ge \frac{1}{\delta-1}$.
- (H2) $h(t) + \lambda \frac{\partial k}{\partial u} \ge \eta > 0$, *for all* $u \in [v^{(0)}, w^{(0)}]$ *and* $t \in (0,1)$, $g(t) \ge 0$ *and* $\alpha \ge \alpha_0$ *is sufficiently large.*

Proof. First we prove that v^* and w^* are solutions to (1.1)-(1.2). Since $\{v^{(j)}\}$ converges uniformly and applying the result in [4], we have

$$-\lim_{j\to\infty} D^\delta v^{(j)} = -D^\delta \lim_{j\to\infty} v^{(j)} = -D^\delta v^*.$$

Also, the uniform convergence of the sequence $\{v^{(j)}\}$ together with the continuity of $k(t,u)$ yields

$$\lim_{j\to\infty} k(t,v^{(j)}) = k(t,v^*).$$

Applying the above results in (5.2), we have

$$\lim_{j\to\infty}\Big(-D^\delta v^{(j)} - g(t)Dv^{(j)} + c\, v^{(j)}\Big) = \lim_{k\to\infty}\Big((h(t)+c)v^{(j-1)} + \lambda k(t,v^{(j-1)})\Big),$$
$$-D^\delta v^* - g(t)Dv^* + c\, v^* = (h(t)+c)v^* + \lambda k(t,v^*).$$

Thus,

$$D^\delta v^* + g(t)Dv^* + h(t)v^* = -\lambda k(t,v^*)$$

which together with $Q_1(v^*) = Q_2(v^*) = 0$ proves that v^* is a solution to (1.1)-(1.2). Similar arguments can be applied to show that w^* is a solution to (1.1)-(1.2). If (H1) holds, by Theorem 3.4, the solution is unique and hence $v^* = w^* = u$ is the unique solution of (1.1)-(1.2) in $[v^{(0)}, w^{(0)}]$. If (H2) holds then by Theorem 4.3 there is no ordered solutions. Since v^* and w^* are solutions of the problem with $v^* \le w^*$ then $v^* = w^* = u$. Lastly, the condition $k(t,0) \neq 0$ guarantees that $u \neq 0$. □

Remark 5.3. The condition (H1) in Theorem 5.2 guarantees the uniqueness of solutions for the eigenvalue problem (1.1)-(1.2) with integer order $\delta = 2$, (see [17, p. 104]), while the condition (H2) is new.

6. The linear eigenvalue problem

As a special case of problem (1.1)-(1.2), we consider the linear eigenvalue problem

$$D^\delta u(t) + g(t)u' + h(t)u = -\lambda r(t)u, \quad t\in(0,1),\ 1<\delta<2, \tag{6.1}$$
$$u(0) - \alpha u'(0) = 0, \quad u(1) + \beta u'(1) = 0, \tag{6.2}$$

where $k(t,u) = r(t)u$. Here we discuss two cases of $r(t)$. First $r(t) > 0$ and there exists a positive constant μ such that $r(t) \ge \mu > 0$, and second $r(t) < 0$ and there exists a negative constant ξ such that $r(t) \le \xi < 0$. Since $k(t,0) = 0$ by Corollary 3.5 we have for $\alpha \ge 1/(\delta-1)$,

$$\begin{aligned}\lambda &\ge \inf_{t\in[0,1]}\Big(-\frac{h(t)}{r(t)}\Big), \quad \text{if } r(t) > 0,\\ \lambda &\le \sup_{t\in[0,1]}\Big(-\frac{h(t)}{r(t)}\Big), \quad \text{if } r(t) < 0.\end{aligned} \tag{6.3}$$

Thus for $h(t) = 0$, we have $\lambda \ge 0$, if $r(t) > 0$ and $\lambda \le 0$, if $r(t) < 0$. These results are well-known for the eigenvalue problem (6.1)-(6.2) with integer order $\delta = 2$, see [19]. And here we proved that they are valid for any $1 < \delta < 2$. As a simple illustration consider the eigenvalue problem

$$D^\delta u(t) + g(t)u' - u = -\lambda e^t u, \quad t\in(0,1),\ 1<\delta<2, \tag{6.4}$$

with boundary conditions (6.2). For $\alpha \geq \frac{1}{\delta-1}$ the eigenvalues of the problem satisfy

$$\lambda \geq \inf_{t\in[0,1]} \Big(- \frac{-1}{e^t} \Big) = \inf_{t\in[0,1]} \Big(e^{-t} \Big) = e^{-1}.$$

We now consider another special case of the linear eigenvalue problem (6.1)-(6.2) where $g(t) = 0, r(t) > 0$ and $h(t) < 0$. We have

$$D^\delta u + h(t)u = -\lambda r(t)u, \quad t \in (0,1),\ 1 < \delta < 2. \tag{6.5}$$

For $\alpha \geq 1/(\delta - 1)$ the eigenvalues of the problem satisfy

$$\lambda \geq \inf_{t\in[0,1]} \Big(- \frac{h(t)}{r(t)} \Big) > 0.$$

From the above discussion and numerical results in [1, 16, 20] we believe that the eigenvalues and eigenfunctions of the eigenvalue problem (6.5)-(6.2) satisfy the following claim:

Conjecture 6.1. (1) *The first eigenfunction ϕ_1 has one sign on $[0,1]$. That is, either $\phi_1 \geq 0$ or $\phi_1 \leq 0$ on $[0,1]$.*
(2) *The first eigenvalue λ_1 increases with δ.*

We leave these open problems for future work and hopefully they will open new areas of research for interested researchers. It is well-known that the first eigenfunction of the Sturm-Liouville eigenvalue problem is important and has various applications. We expect the first eigenfunction of the problem (6.5)-(6.2) will play the same role. Also, if (2) is true, then we can use the eigenvalues of the Sturm-Liouville problem as upper bounds for the ones of fractional order.

7. Concluding Remarks

We have studied analytically a class of eigenvalue problems of fractional order $1 < \delta < 2$. We have obtained a new positivity result and used it to derive a sufficient condition (H1):$h(t) + \lambda \frac{\partial k}{\partial u} < 0$, which guarantees the uniqueness of solutions and ordering of any lower and upper solutions for any $\alpha \geq \frac{1}{\delta-1}$. For the case where the nonlinear term $k(t,u)$ in the problem satisfies $k(t,0) = 0$, analytical upper and lower bounds estimates of the eigenvalues are obtained. These bounds have closed forms and can be easily computed. A sufficient condition (H2) for the non existence of ordered solution is obtained by transforming the problem into equivalent integro-differential equation. The method of lower and upper solutions is used to obtain two well-defined sequences of lower and upper solutions which converge uniformly to a solution of the problem. Under the condition (H1) or (H2) we proved that these lower and upper solutions converge to the same limit, which is the unique solution of the problem. We illustrated these result by considering some linear eigenvalue problems. We believe that these analytical results are useful in the applications and numerical treatments and leads to better understanding of the problem. While some results have been established, many open problems are still not verified and we leave them for future work.

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