SOLUTIONS FOR LINEAR DIFFERENTIAL EQUATIONS WITH MEROMORPHIC COEFFICIENTS OF (P,Q)-ORDER IN THE PLANE

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Abstract. In this article we study the growth of meromorphic solutions of high order linear differential equations with meromorphic coefficients of (p,q)-order. We extend some previous results due to Belaïdi, Cao-Xu-Chen, Kin-nunen, Liu- Tu -Shi, and others.

1. Introduction and main results

For $k \geq 2$, consider the linear differential equations

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = 0,$$  \hspace{1cm} (1.1)

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = F(z),$$  \hspace{1cm} (1.2)

where $A_0 \not\equiv 0$ and $F \not\equiv 0$. When the coefficients $A_0, A_1, \ldots, A_{k-1}$ and $F$ are entire functions, it is well known that all solutions of (1.1) and (1.2) are entire functions, and that if some coefficients of (1.1) are transcendental then (1.1) has at least one solution with infinite order. We refer to [16] for the literature on the growth of entire solutions of (1.1) and (1.2).

As far as we known, Bernal [4] firstly introduced the idea of iterated order to express the fast growth of solutions of complex linear differential equations. Since then, many authors obtained further results on iterated order of solutions of (1.1) and (1.2), see e.g. [1, 2, 4, 5, 6, 15, 19]. Recently, Liu, Tu and Shi [17] firstly introduced the concept of (p, q)-order for the case $p \geq q \geq 1$ to investigate the entire solutions of (1.1) and (1.2), and obtained some results which improve and generalize some previous results.

Theorem 1.1 ([17, Theorems 2.2-2.3]). Let $p \geq q \geq 1$, and let $A_0, A_1, \ldots, A_{k-1}$ be entire functions such that either

$$\max\{\sigma_{(p,q)}(A_j) : j \neq 0\} < \sigma_{(p,q)}(A_0) < +\infty,$$

or

$$\max\{\sigma_{(p,q)}(A_j) : j \neq 0\} \leq \sigma_{(p,q)}(A_0) < +\infty,$$

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Let \( A_0, A_1, \ldots, A_{k-1} \) be meromorphic functions in the plane, and let \( i(A_0) = p \) \((0 < p < \infty)\). Assume that either \( i_\lambda(\frac{1}{A_0}) < p \) or \( \lambda_p(\frac{1}{A_0}) < \sigma_p(A_0) \), and that either

\[
\max \{i(A_j) : j = 1, 2, \ldots, k-1\} < p
\]

or

\[
\max \{\sigma_p(A_j) : j = 1, 2, \ldots, k-1\} \leq \sigma_p(A_0) := \sigma \quad (0 < \sigma < \infty),
\]

\[
\max \{\tau_p(A_j) : \sigma_p(A_j) = \sigma_p(A_0)\} < \tau_p(A_0) := \tau \quad (0 < \tau < \infty).
\]

Then every meromorphic solution \( f \neq 0 \) whose poles are of uniformly bounded multiplicities, of equation (1.1) satisfies \( i(f) = p+1 \) and \( \sigma_{p+1}(f) = \sigma_p(A_0) \).

There exists a natural question: How about the growth of meromorphic solutions of equations (1.1) and (1.2) with meromorphic coefficients of finite \((p, q)\)-order in the plane?

The main purpose of this paper is to consider the above question. Now we show our main results. For homogeneous linear differential equation (1.1), we obtain the following results.

**Theorem 1.3.** Let \( A_0, A_1, \ldots, A_{k-1} \) be meromorphic functions in the plane. Suppose that there exists one coefficient \( A_s \) \((s \in \{0, 1, \ldots, k-1\})\) such that

\[
\max \{\sigma_{(p, q)}(A_j), \lambda_{(p, q)}(\frac{1}{A_s}) : j \neq s\} < \sigma_{(p, q)}(A_s) < +\infty,
\]

then every transcendental meromorphic solution \( f \) whose poles are of uniformly bounded multiplicities of (1.1) satisfies

\[
\sigma_{(p+1, q)}(f) \leq \sigma_{(p, q)}(A_s) \leq \sigma_{(p, q)}(f).
\]

Furthermore, if all solutions of (1.1) are meromorphic solutions, then there is at least one meromorphic solution, say \( f_1 \), satisfies

\[
\sigma_{(p+1, q)}(f_1) = \sigma_{(p, q)}(A_s).
\]

Now replacing the arbitrary coefficient \( A_s \) by the dominant fixed coefficient \( A_0 \), then we obtain the following result.

**Theorem 1.4.** Let \( A_0, A_1, \ldots, A_{k-1} \) be meromorphic functions in the plane satisfying

\[
\max \{\sigma_{(p, q)}(A_j), \lambda_{(p, q)}(\frac{1}{A_0}) : j = 1, 2, \ldots, k-1\} < \sigma_{(p, q)}(A_0) < +\infty,
\]

then every meromorphic solution \( f \) whose poles are of uniformly bounded multiplicities of (1.1) satisfies

\[
\sigma_{(p+1, q)}(f) = \sigma_{(p, q)}(A_0).
\]
If there exist some other coefficients $A_j$ ($j \in \{1, 2, \ldots, k-1\}$) having the same (p,q)-order as $A_0$, then we have the following result by making use of the concept of (p,q)-type.

**Theorem 1.5.** Let $A_0, A_1, \ldots, A_{k-1}$ be meromorphic functions in the plane, assume that

$$\lambda_{(p,q)}\left(\frac{1}{A_0}\right) < \sigma_{(p,q)}(A_0)$$

and

$$\max\{\sigma_{(p,q)}(A_j) : j = 1, 2, \ldots, k-1\} = \sigma_{(p,q)}(A_0) < +\infty,$$

$$\max\{\tau_{(p,q)}(A_j) : \sigma_{(p,q)}(A_j) = \sigma_{(p,q)}(A_0) > 0\} < \tau_{(p,q)}(A_0).$$

Then any nonzero meromorphic solution $f$ whose poles are of uniformly bounded multiplicities of (1.1) satisfies

$$\sigma_{(p+1,q)}(f) = \sigma_{(p,q)}(A_0).$$

Obviously, Theorems 1.4 and 1.5 are a generalization of Theorems 1.1 and 1.2. Considering nonhomogeneous linear differential equation (1.2), we obtain the following three results.

**Theorem 1.6.** Assume that $A_0, A_1, \ldots, A_{k-1}, F \neq 0$ be meromorphic functions in the plane satisfying

$$\max\{\sigma_{(p,q)}(A_j), \lambda_{(p,q)}\left(\frac{1}{A_0}\right), \sigma_{(p+1,q)}(F) : j = 1, 2, \ldots, k-1\} < \sigma_{(p,q)}(A_0),$$

then all meromorphic solutions $f$ whose poles are of uniformly bounded multiplicities of (1.2) satisfy

$$\lambda_{(p+1,q)}(f) = \lambda_{(p+1,q)}(f) = \sigma_{(p+1,q)}(f) = \sigma_{(p,q)}(A_0)$$

with at most one exceptional solution $f_0$ satisfying $\sigma_{(p+1,q)}(f_0) < \sigma_{(p,q)}(A_0)$.

**Theorem 1.7.** Let $A_0, A_1, \ldots, A_{k-1}, F \neq 0$ be meromorphic functions in the plane satisfying

$$\max\{\sigma_{(p,q)}(A_j) : j = 0, 1, \ldots, k-1\} < \sigma_{(p+1,q)}(F).$$

Suppose that all solutions of (1.2) are meromorphic functions whose poles are of uniformly bounded multiplicities, then $\sigma_{(p+1,q)}(f) = \sigma_{(p+1,q)}(F)$ holds for all solutions of (1.2).

**Theorem 1.8.** Let $H \subset (1, \infty)$ be a set satisfying $\log \text{dens} \{|z| : |z| \in H\} > 0$ and let $A_0, A_1, \ldots, A_{k-1}, F \neq 0$ be meromorphic functions in the plane satisfying

$$\max\{\sigma_{(p,q)}(A_j) : j = 0, 1, \ldots, k-1\} < \alpha_1,$$

where $\alpha_1$ is a constant, and there exists another constant $\alpha_2$ ($\alpha_2 < \alpha_1$) such that for any given $\epsilon$ ($0 < \epsilon < \alpha_1 - \alpha_2$), we have

$$|A_0(z)| \geq \exp_{p+1}\{(\alpha_1 - \epsilon) \log q r\}, |A_j(z)| \leq \exp_{p+1}\{\alpha_2 \log q r\}$$

for $|z| \in H, j = 1, 2, \ldots, k-1$. Then we have:

(i) If $\sigma_{(p+1,q)}(F) \geq \alpha_1$, then all meromorphic solutions whose poles are of uniformly bounded multiplicities of (1.2) satisfy

$$\sigma_{(p+1,q)}(f) = \sigma_{(p+1,q)}(F).$$
(ii) If \( \sigma_{(p+1,q)}(F) < \alpha_1 \), then all meromorphic solutions whose poles are of uniformly bounded multiplicities of (1.2) satisfy

\[ \lambda_{(p+1,q)}(f) = \lambda_{(p+1,q)}(f) = \sigma_{(p+1,q)}(f) = \alpha_1 \]

with at most one exceptional solution \( f_2 \) satisfying

\[ \sigma_{(p+1,q)}(f_2) < \alpha_1. \]

Recently, B. Belaïdi [3] investigated the growth of solutions of differential equations (1.1) and (1.2) with analytic coefficients of \((p, q)\)-order in the unit disc. So, it is also interesting to consider the growth of meromorphic solutions of differential equations with coefficients of \((p, q)\)-order in the unit disc?

2. Preliminaries and some lemmas

We shall introduce some notation. Let us define inductively, for \( r \in [0, +\infty) \),

\[ \exp_1 r = e^r \quad \text{and} \quad \exp_{n+1} r = \exp(\exp_n r), \quad n \in \mathbb{N}. \]

For all \( r \) sufficiently large, we define \( \log_1 r = \log^+ r = \max\{\log r, 0\} \) and \( \log_{n+1} r = \log(\log_n r), \quad n \in \mathbb{N}. \)

We also denote \( \exp_n r = r = \log_0 r, \quad \log_{-1} r = \exp_1 r \) and \( \exp_{-1} r = \log_1 r. \) Moreover, we denote the linear measure and the logarithmic measure of a set \( E \subset (1, \infty) \) by

\[ m_E = \int_E dt \quad \text{and} \quad m_l E = \int_E \frac{dt}{t}. \]

The upper logarithmic density of \( E \subset (1, \infty) \) is defined by

\[ \log dens E = \limsup_{r \to \infty} \frac{m_l(E \cap [1, r])}{\log r}. \]

We assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna’s value distribution theory of meromorphic functions (e.g. see [11, 20]), such as \( T(r, f), m(r, f), \) and \( N(r, f) \). In this section, a meromorphic function \( f \) means meromorphic in the complex plane \( \mathbb{C} \). To express the rate of fast growth of meromorphic functions, we recall the following definitions (e.g. see [4, 5, 15, 16, 18]).

**Definition 2.1.** The iterated \( p \)-order \( \sigma_p(f) \) of a meromorphic function \( f \) is defined by

\[ \sigma_p(f) = \limsup_{r \to \infty} \frac{\log_p T(r, f)}{\log r} \quad (p \in \mathbb{N}). \]

If \( f \) is an entire function, then

\[ \sigma_{p,M}(f) = \limsup_{r \to \infty} \frac{\log_{p+1} M(r, f)}{\log r} \quad (p \in \mathbb{N}). \]

**Definition 2.2.** The growth index of the iterated order of a meromorphic function \( f \) is defined by

\[ i(f) = \begin{cases} 0 & \text{if } f \text{ is rational}, \\ \min\{n \in \mathbb{N} : \sigma_n(f) < \infty\} & \text{if } f \text{ is transcendental and } \sigma_n(f) < \infty \\ \infty & \text{for some } n \in \mathbb{N}, \\ \infty & \text{if } \sigma_n(f) = \infty \text{ for all } n \in \mathbb{N}. \end{cases} \]

**Definition 2.3.** The iterated \( p \)-type of a meromorphic function \( f \) with iterated order \( p \)-order \( 0 < \sigma_p(f) < \infty \) is defined by

\[ \tau_p(f) = \limsup_{r \to \infty} \frac{\log_{p-1} T(r, f)}{\log^{\sigma_p(f)} r} \quad (p \in \mathbb{N}). \]
If $f$ is an entire function, then
\[ \tau_{p,M}(f) = \limsup_{r \to \infty} \frac{\log_p M(r, f)}{r^{\sigma(f)}} \quad (p \in \mathbb{N}). \]

**Definition 2.4.** The iterated convergence exponent of the sequence of zeros of a meromorphic function $f$ is defined by
\[ \lambda_p(f) = \limsup_{r \to \infty} \frac{\log_p N(r, \frac{1}{f})}{\log r} \quad (p \in \mathbb{N}). \]

**Definition 2.5.** The growth index of the iterated convergence exponent of the sequence of zeros of a meromorphic function $f$ with iterated order is defined by
\[ i_{\lambda}(f) = \begin{cases} 
0 & \text{if } n(r, \frac{1}{f}) = O(\log r), \\
\min\{n \in \mathbb{N} : \lambda_n(f) < \infty\} & \text{if } \lambda_n(f) < \infty \text{ for some } n \in \mathbb{N}, \\
\infty & \text{if } \lambda_n(f) = \infty \text{ for all } n \in \mathbb{N}.
\end{cases} \]

Similarly, we can use the notation $\bar{\lambda}_p(f)$ to denote the iterated convergence exponent of the sequence of distinct zeros, and use the notation $i_{\bar{\lambda}}(f)$ to denote the growth index of $\bar{\lambda}_p(f)$.

Now, we shall introduce the definition of meromorphic functions of $(p, q)$-order, where $p, q$ are positive integers satisfying $p \geq q \geq 1$. In order to keep accordance with Definition 2.1, we will give a minor modification to the original definition of $(p, q)$-order (e.g. see [13]).

**Definition 2.6.** The $(p, q)$-order of a transcendental meromorphic function $f$ is defined by
\[ \sigma_{(p,q)}(f) = \limsup_{r \to \infty} \frac{\log_p T(r, f)}{\log_q r}. \]

If $f$ is a transcendental entire function, then
\[ \sigma_{(p,q)}(f) = \limsup_{r \to \infty} \frac{\log_{p+1} M(r, f)}{\log_q r}. \]

It is easy to show that $0 \leq \sigma_{(p,q)} \leq \infty$. By Definition 2.6 we note that $\sigma_{(1,1)}(f) = \sigma_1(f) = \sigma(f)$, $\sigma_{(2,1)}(f) = \sigma_2(f)$ and $\sigma_{(p,1)}(f) = \sigma_p(f)$.

**Remark 2.7.** If $f$ is a meromorphic function satisfying $0 \leq \sigma_{(p,q)} \leq \infty$, then
(i) $\sigma_{(p-n,q)} = \infty \ (n < p)$, $\sigma_{(p,q-n)} = 0 \ (n < q)$, and $\sigma_{(p+n,q+n)} = 1 \ (n < p)$ for $n = 1$ to $\infty$.
(ii) If $(p_1, q_1)$ is another pair of integers satisfying $p_1 - q_1 = p - q$ and $p_1 < p$, then we have $\sigma_{(p_1,q_1)} = 0$ if $0 < \sigma_{(p,q)} < 1$ and $\sigma_{(p_1,q_1)} = \infty$ if $1 < \sigma_{(p,q)} < \infty$.
(iii) $\sigma_{(p_1,q_1)} = \infty$ for $p_1 - q_1 > p - q$ and $\sigma_{(p_1,q_1)} = 0$ for $p_1 - q_1 > p - q$.

**Remark 2.8.** Suppose that $f_1$ is a meromorphic function of $(p, q)$-order $\sigma_1$ and $f_2$ is a meromorphic function of $(p_1, q_1)$-order $\sigma_2$, let $p \leq p_1$. We can easily deduce the result about their comparative growth:
(i) If $p_1 - q_1 > p - q$, then the growth of $f_1$ is slower than the growth of $f_2$.
(ii) If $p_1 - q_1 < p - q$, then $f_1$ grows faster than $f_2$.
(iii) If $p_1 - q_1 = p - q > 0$, then the growth of $f_1$ is slower than the growth of $f_2$ if $\sigma_2 > 1$, and the growth of $f_1$ is faster than the growth of $f_2$ if $\sigma_2 < 1$.
(iv) Especially, when $p_1 = p$ and $q_1 = q$ then $f_1$ and $f_2$ are of the same index-pair $(p, q)$. If $\sigma_1 > \sigma_2$, then $f_1$ grows faster than $f_2$; and if $\sigma_1 < \sigma_2$, then $f_1$ grows slower.
Lemma 2.12. The \((p, q)\)-type of a meromorphic function \(f\) with \((p, q)\)-order \(\sigma_{(p, q)}(f) \in (0, \infty)\) is defined by
\[
\tau_{(p, q)}(f) = \limsup_{r \to \infty} \frac{\log_{p-1} T(r, f)}{(\log_{q-1} r)^{\sigma_{(p, q)}(f)}}.
\]

Definition 2.10. The \((p, q)\) convergence exponent of the sequence of zeros of a meromorphic function \(f\) is defined by
\[
\lambda_{(p, q)}(f) = \limsup_{r \to \infty} \frac{\log_{p} N(r, \frac{1}{f})}{\log_{q} r}.
\]

Similarly, we can use the notation \(\overline{\lambda}_{(p, q)}(f)\) to denote the \((p, q)\) convergence exponent of the sequence of distinct zeros of \(f\). To prove our results, we need the following lemmas.

Lemma 2.11 \((\text{[8]}\)) Let \(f_1, f_2, \ldots, f_k\) be linearly independent meromorphic solutions of the differential equation \([1.1]\) with meromorphic functions \(A_0, A_1, \ldots, A_{k-1}\) as the coefficients, then
\[
m(r, A_j) = O\{\max_{1 \leq n \leq k} T(r, f_n)\} \quad (j = 0, 1, \ldots, k - 1).
\]

Lemma 2.12 \((\text{[7]}\)) Let \(f\) be a meromorphic solution of equation \([1.1]\), assuming that not all coefficients \(A_j\) are constants. Given a real constant \(\gamma > 1\), and denoting \(T(r) = \sum_{j=0}^{k-1} T(r, A_j)\), we have
\[
\log m(r, f) < T(r)\{(\log r) \log T(r)\}^\gamma, \quad \text{if} \ s = 0,
\]
\[
\log m(r, f) < r^{2s+\gamma-1} T(r)\{(\log T(r))\}^\gamma, \quad \text{if} \ s > 0
\]
outside of an exceptional set \(E_s\) with \(\int_{E_s} e^{st-1} dt < \infty\).

By inequalities in \([12]\) Chapter 6 and in \([10]\) Corollary 2.3.5, we obtain the following lemma.

Lemma 2.13. If \(f\) is a meromorphic function, then
\[
\sigma_{(p, q)}(f) = \sigma_{(p, q)}(f').
\]

Lemma 2.14 \((\text{[9]}\)) Let \(f\) be a transcendental meromorphic function, and let \(\alpha\) be a given constant. Then there exist a set \(E_1 \subset (1, \infty)\) that has finite logarithmic measure and a constant \(B > 0\) depending only on \(\alpha\) and \((m, n) (m, n \in \{0, 1, \ldots, k\})\), \(m < n\) such that for all \(z\) with \(|z| = r \notin [0, 1] \cup E_1\), we have
\[
\left| \frac{f^{(m)}(z)}{f^{(n)}(z)} \right| \leq B \left( \frac{T(\alpha r, f)}{r} \right) \left( \log^a r \right)^{n-m}.
\]

Lemma 2.15. Let \(f\) be a meromorphic function of \((p, q)\)-order satisfying \(\sigma_{(p, q)}(f) < \infty\). Then there exists a set \(E_2 \subset (1, \infty)\) having infinite logarithmic measure such that for all \(r \in E_2\), we have
\[
\lim_{r \to \infty} \frac{\log_{p} T(r, f)}{\log_{q} r} = \sigma_{(p, q)}(f).
\]
Proof. By Definition 2.6 there exists a sequence \( \{r_n\}_{n=1}^\infty \) tending to \( \infty \), satisfying \((1 + \frac{1}{n})r_n < r_{n+1}\), and

\[
\lim_{n \to \infty} \frac{\log_p T(r_n, f)}{\log_q r_n} = \sigma_{(p,q)}(f).
\]

There exists a \( n_1 \in \mathbb{N} \), such that for \( n \geq n_1 \), and for any \( r \in [r_n, (1 + \frac{1}{n})r_n] \), we have

\[
\frac{\log_p T(r_n, f)}{\log_q (1 + \frac{1}{n})r_n} \leq \frac{\log_p T(r, f)}{\log_q r} \leq \frac{\log_p T((1 + \frac{1}{n})r_n, f)}{\log_q r_n}.
\]

Set \( E_2 = \cup_{n=n_1}^{\infty} [r_n, (1 + \frac{1}{n})r_n] \), then for any \( r \in E_2 \), we have

\[
\lim_{r \to \infty} \frac{\log_p T(r, f)}{\log_q r} = \lim_{n \to \infty} \frac{\log_p T(r_n, f)}{\log_q r_n} = \sigma_{(p,q)}(f),
\]

where

\[
m_1E_2 = \sum_{n=n_1}^{\infty} \int_{r_n}^{(1 + \frac{1}{n})r_n} \frac{dt}{t} = \sum_{n=n_1}^{\infty} \log(1 + \frac{1}{n}) = \infty.
\]

Lemma 2.16. Let \( \varphi(r) \) be a continuous and positive increasing function, defined for \( r \in [0, \infty) \) with \( \sigma_{(p,q)}(\varphi) = \limsup_{r \to \infty} \frac{\log_p \varphi(r)}{\log_q r} \), then for any subset \( E_3 \subset (0, \infty) \) that has a finite linear measure, there exists a sequence \( \{r_n\}, r_n \notin E_3 \) such that

\[
\sigma_{(p,q)}(\varphi) = \lim_{r_n \to \infty} \frac{\log_p \varphi(r_n)}{\log_q r_n}.
\]

Proof. Since \( \sigma_{(p,q)}(\varphi) = \limsup_{r \to \infty} \frac{\log_p \varphi(r)}{\log_q r} \), then there exists a sequence \( \{r'_n\} \) tending to \( \infty \), such that

\[
\lim_{r'_n \to \infty} \frac{\log_p \varphi(r'_n)}{\log_q r'_n} = \sigma_{(p,q)}(\varphi).
\]

Set \( mE_3 = \delta < \infty \), then for \( r_n \in [r'_n, r'_n + \delta + 1] \), we have

\[
\frac{\log_p \varphi(r_n)}{\log_q r_n} \geq \frac{\log_p \varphi(r'_n)}{\log_q (r'_n + \delta + 1)} = \frac{\log_p \varphi(r'_n)}{\log_q (r'_n + \log(1 + \frac{\delta + 1}{r'_n}))}.
\]

Hence

\[
\lim_{r_n \to \infty} \frac{\log_p \varphi(r_n)}{\log_q r_n} \geq \lim_{r'_n \to \infty} \frac{\log_p \varphi(r'_n)}{\log_q (r'_n + \log(1 + \frac{\delta + 1}{r'_n}))} = \lim_{r'_n \to \infty} \frac{\log_p \varphi(r'_n)}{\log_q r'_n} = \sigma_{(p,q)}(\varphi),
\]

this gives

\[
\sigma_{(p,q)}(\varphi) = \lim_{r_n \to \infty} \frac{\log_p \varphi(r_n)}{\log_q r_n}.
\]

Lemma 2.17 (\cite{H3}). Let \( f \) be an entire function of \( (p,q) \)-order, and let \( \nu_f(r) \) be the central index of \( f \), then

\[
\limsup_{r \to \infty} \frac{\log_p \nu_f(r)}{\log_q r} = \sigma_{(p,q)}(f).
\]
Lemma 2.18. Let $f$ be a meromorphic function of $(p,q)$-order satisfying $0 < \sigma_{(p,q)}(f) < \infty$, let $\tau_{(p,q)}(f) > 0$, then for any given $\tau_{(p,q)}(f) > \beta$, there exists a set $E_4 \subset (1, \infty)$ that has infinite logarithmic measure such that for all $r \in E_4$, we have
\[ \log_{p-1} T(r, f) > \beta (\log_{q-1} r)^{\sigma_{(p,q)}(f)}. \]

Proof. (i) (see [3]) when $q = 1$, it holds absolutely. (ii) when $q \geq 2$, by Definition 2.9 there exists an increasing sequence $\{r_m\}$ satisfying $(1 + \frac{1}{m})r_m < r_{m+1}$, and
\[ \lim_{m \to \infty} \frac{\log_{p-1} T(r_m, f)}{(\log_{q-1} r_m)^{\sigma_{(p,q)}(f)}} = \tau_{(p,q)}(f). \]
Then there exists a positive constant $m_0$ such that for all $m > m_0$ and for any given $\epsilon$ ($0 < \epsilon < \tau_{(p,q)}(f) - \beta$) we have
\[ \log_{p-1} T(r_m, f) > (\tau_{(p,q)}(f) - \epsilon)(\log_{q-1} r_m)^{\sigma_{(p,q)}(f)}. \]
For any $r \in [r_m, (1 + \frac{1}{m})r_m]$, we have
\[ \lim_{r_m \to \infty} \frac{\log_{q-1} r_m}{\log_{q-1} r} = 1. \]
Since $\beta < \tau_{(p,q)}(f) - \epsilon$, there exists a positive constant $m_1$ such that for all $m > m_1$, we have
\[ (\log_{q-1} r_m)^{\sigma_{(p,q)}(f)} > \frac{\beta}{\tau_{(p,q)}(f) - \epsilon}, \]
i.e.,
\[ (\tau_{(p,q)}(f) - \epsilon)(\log_{q-1} r_m)^{\sigma_{(p,q)}(f)} > \beta (\log_{q-1} r)^{\sigma_{(p,q)}(f)}. \]
Now we take $m_2 = \max\{m_0, m_1\}$ and $E_4 = \bigcup_{m=m_2}^{\infty} [r_m, (1 + \frac{1}{m})r_m]$, then by (2.1)-(2.2), for any $r \in E_4$, we have
\[ \log_{p-1} T(r, f) \geq \log_{p-1} T(r_m, f) \]
\[ > (\tau_{(p,q)}(f) - \epsilon)(\log_{q-1} r_m)^{\sigma_{(p,q)}(f)} \]
\[ > \beta (\log_{q-1} r)^{\sigma_{(p,q)}(f)}, \]
where
\[ m_1 E_4 = \sum_{m=m_2}^{\infty} \int_{r_m}^{(1 + \frac{1}{m})r_m} \frac{dt}{t} = \sum_{m=m_2}^{\infty} \log(1 + \frac{1}{m}) = \infty. \]

Lemma 2.19 ([10]). Let $g(r)$ and $h(r)$ be monotone nondecreasing functions on $[0, \infty)$ such that $g(r) \leq h(r)$ for all $r \not\in [0, 1] \cup E_5$, where $E_5 \subset (1, \infty)$ is a set of finite logarithmic measure. Then for any constant $\alpha > 1$, there exists $r_0 = r_0(\alpha) > 0$ such that $g(r) \leq h(\alpha r)$ for all $r \geq r_0$.

Lemma 2.20. Let $A_0, A_1, \ldots, A_{k-1}$, $F \not\equiv 0$ be meromorphic functions and let $f$ be a meromorphic solution of equation $\{1, 2\}$. If
\[ \max\{\sigma_{(p+1,q)}(A_j), \sigma_{(p+1,q)}(F) : j = 0, 1, \ldots, k - 1\} < \sigma_{(p+1,q)}(f), \]
then we have
\[ \overline{\lambda}_{(p+1,q)}(f) = \lambda_{(p+1,q)}(f) = \sigma_{(p+1,q)}(f). \]
Hence, by (2.7) and (2.10), we obtain that for sufficiently large \( r \) holds for any given \( \lambda \).

By the lemma of the logarithmic derivative and (2.3), we have

\[
m(r, \frac{1}{f}) \leq m(r, \frac{1}{F}) + \sum_{j=0}^{k-1} m(r, A_j) + O(\log T(r, f) + \log r) \tag{2.6}
\]
holds for all \( |z| = r \notin E_6 \), where \( E_6 \) is a set of finite linear measure. By (2.5), (2.6) and the first main theorem, we have

\[
T(r, f) = T(r, \frac{1}{f}) + O(1) \leq kN(r, \frac{1}{f}) + T(r, F) + \sum_{j=0}^{k-1} T(r, A_j) + O(\log(rT(r, f)))
\tag{2.7}
\]
holds for all sufficiently \( r \notin E_6 \).

Assume that \( \max \\{ \sigma_{(p+1,q)}(A_j), \sigma_{(p+1,q)}(F) : j = 0, 1, \ldots, k-1 \} < \sigma_{(p+1,q)}(f) \). By Lemma 2.16 there exists a sequence \( \{ r_n \} \), \( r_n \notin E_6 \) such that

\[
\lim_{r_n \to \infty} \frac{\log_{p+1} T(r_n, f)}{\log_{q} r_n} = \sigma_{(p+1,q)}(f) =: \sigma_1.
\]

Hence, if \( r_n \notin E_6 \) is sufficiently large, since \( \sigma_1 > 0 \), then we have

\[
T(r_n, f) \geq \exp_{p+1} \{ (\sigma_1 - \epsilon) \log_q r_n \} \tag{2.8}
\]
holds for any given \( \epsilon (0 < 2\epsilon < \sigma_1 - \sigma_2) \), where \( \sigma_2 = \max \{ \sigma_{(p+1,q)}(A_j), \sigma_{(p+1,q)}(F) : j = 0, 1, \ldots, k-1 \} \). We have

\[
\max \{ T(r_n, F), T(r_n, A_j) : j = 0, 1, \ldots, k-1 \} \leq \exp_{p+1} \{ (\sigma_2 + \epsilon) \log_q r_n \}. \tag{2.9}
\]

Since \( \epsilon (0 < 2\epsilon < \sigma_1 - \sigma_2) \), then from (2.8) and (2.9) we obtain

\[
\max \{ \frac{T(r_n, F)}{T(r_n, f)}, \frac{T(r_n, A_j)}{T(r_n, f)} : j = 0, 1, \ldots, k-1 \} \to 0 \quad (r_n \to \infty). \tag{2.10}
\]

For sufficiently large \( r_n \), we have

\[
O(\log(r_n T(r_n, f))) = O(T(r_n, f)).
\]

Hence, by (2.7) and (2.10), we obtain that for sufficiently large \( r_n \notin E_6 \), there holds

\[
(1 - o(1)) T(r_n, f) \leq kN(r_n, \frac{1}{f}).
\]

Then we have \( \lambda_{(p+1,q)}(f) \geq \sigma_{(p+1,q)}(f) \), and by definitions we have \( \lambda_{(p+1,q)}(f) \leq \lambda_{(p+1,q)}(f) \leq \sigma_{(p+1,q)}(f) \). Therefore

\[
\lambda_{(p+1,q)}(f) = \sigma_{(p+1,q)}(f) = \sigma_{(p+1,q)}(f).
\]

\]
3. Proofs of Theorems 1.3–1.8

**Proof of Theorem 1.3.** We shall divide the proof into two parts.

- Firstly, we prove that \( \sigma_{(p+1,q)}(f) \leq \sigma_{(p,q)}(A_s) \leq \sigma_{(p,q)}(f) \) holds for every transcendental meromorphic function \( f \) of (1.1). By (1.1), we know that the poles of \( f \) can only occur at the poles of \( A_0, A_1, \ldots, A_{k-1} \), note that the multiplicities of poles of \( f \) are uniformly bounded, then we have

\[
N(r, f) \leq C_1 \overline{N}(r, f) \\
\leq C_1 \sum_{j=0}^{k-1} \overline{N}(r, A_j) \\
\leq C_2 \max \{N(r, A_j) : j = 0, 1, \ldots, k-1\} \leq O(T(r, A_s)),
\]

where \( C_1 \) and \( C_2 \) are suitable positive constants. Then we have

\[
\log T(r, f) \leq \log m(r, f) + \log N(r, f) + \log 2 \leq \log m(r, f) + O\{\log T(r, A_s)\}. \tag{3.1}
\]

By (3.1) and Lemma 2.12 we obtain

\[
\log T(r, f) \leq \log m(r, f) + O\{\log T(r, A_s)\} \\
= O\{(r, A_s)\{\log r\log T(r, A_s)\}\}
\]

outside of an exceptional set \( E_0 \) with \( \int_{E_0} \frac{dt}{r} < \infty \), this implies \( \sigma_{(p+1,q)}(f) \leq \sigma_{(p,q)}(A_s) \). On the other hand, by (1.1), we obtain

\[
-A_s = f^{(k)}(r, f) + f^{(k-1)}(r, f) + \cdots + f^{(s)}(r, f) + f^{(s-1)}(r, f) + \cdots + f^{(1)}(r, f) + f^{(0)}(r, f).
\]

Since

\[
m(r, f^{(s)}) \leq T(r, f) + T(r, f^{(s)}) = T(r, f) + T(r, f^{(s)}) + O(1) = O(T(r, f))
\]

then by the lemma of logarithmic derivative we have

\[
T(r, A_s) \leq N(r, A_s) + \sum_{j \neq s} m(r, A_j) + O(\log rT(r, f)) + O(T(r, f)) \tag{3.2}
\]

hold for all \( |z| = r \notin E_7 \), where \( E_7 \) is a set of finite linear measure. By Lemma 2.16 and similar discussion as in the proof of Lemma 2.20 we see that there exists a sequence \( \{r_n\} \) \( (r_n \to \infty) \) such that

\[
\sigma_1 := \sigma_{(p,q)}(A_s) = \lim_{r_n \to \infty} \frac{\log_p T(r_n, A_s)}{\log_q r_n}
\]

and

\[
T(r_n, A_s) \geq \exp_p\{\sigma_1 - \epsilon \log_q r_n\}, \tag{3.3}
\]

\[
N(r_n, A_s) \leq \exp_p\{\sigma_2 + \epsilon \log_q r_n\}, \tag{3.4}
\]

\[
m(r_n, A_j) \leq \exp_p\{\sigma_2 + \epsilon \log_q r_n\} \quad (j \neq s), \tag{3.5}
\]

where \( \sigma_2 := \max\{\sigma_{(p,q)}(A_j), \lambda_{(p,q)}(\frac{1}{A_j}) : j \neq s\} \) and \( 0 < 2\epsilon < \sigma_1 - \sigma_2 \).

By (3.2)–(3.5), we obtain

\[
(1 - o(1)) \exp_p\{\sigma_1 - \epsilon \log_q r_n\} \leq O\{\log r_n T(r_n, f)\} + O(T(r_n, f)).
\]

Hence we have \( \sigma_{(p,q)}(A_s) = \sigma_1 \leq \sigma_{(p,q)}(f) \).
Secondly, we prove that there exists at least one meromorphic solution that satisfies
\[ \sigma_{(p+1,q)}(f) = \sigma_{(p,q)}(A_s). \]
Now we can assume that \{f_1, f_2, \ldots, f_k\} is a meromorphic solution base of (1.1).
By Lemma 2.11,
\[ m(r, A_s) \leq O\left( \log(\max_{1 \leq n \leq k} T(r, f_n)) \right). \]
Now we assert that \( m(r, A_s) > N(r, A_s) \) holds for sufficiently large \( r \). Indeed, if \( m(r, A_s) \leq N(r, A_s) \), then
\[ T(r, A_s) = m(r, A_s) + N(r, A_s) \leq 2N(r, A_s), \]
so
\[ \limsup_{r \to \infty} \frac{\log T(r, A_s)}{\log r} \leq \limsup_{r \to \infty} \frac{\log 2N(r, A_s)}{\log r}, \]
then we have \( \sigma_{(p,q)}(A_s) \leq \lambda_{(p,q)}\left(\frac{1}{A_s}\right) \), which contradicts the condition \( \lambda_{(p,q)}\left(\frac{1}{A_s}\right) \neq \sigma_{(p,q)}(A_s) \). Hence,
\[ T(r, A_s) = O(m(r, A_s)) \leq O\left( \log(\max_{1 \leq n \leq k} T(r, f_n)) \right). \]
By Lemma 2.16, there exists a set \( E_9 \subset (0, \infty) \) has finite linear measure , and a sequence \( \{r_n\}, r_n \notin E_9 \), such that
\[ \lim_{r_n \to \infty} \frac{\log T(r_n, A_s)}{\log r_n} = \sigma_{(p,q)}(A_s). \]
Set
\[ T_n = \{ r : r \in (0, \infty) \setminus E_9, \quad T(r, A_s) \leq O(\log(T(r, f_n))) \} \quad (n = 1, 2, \ldots, k) \]
By Lemma 2.11, we have \( \bigcup_{n=1}^k T_n = (0, \infty) \setminus E_9 \). It is easy to see that there exists at least one \( T_n \), say \( T_1 \subset (0, \infty) \setminus E_9 \), that has infinite linear measure and satisfies
\[ T(r, A_s) \leq O(\log T(r, f_1)). \] (3.6)
From (3.6), we have \( \sigma_{(p+1,q)}(f_1) \geq \sigma_{(p,q)}(A_s). \)
In the first part we have proved that \( \sigma_{(p+1,q)}(f_1) \leq \sigma_{(p,q)}(A_s) \). Therefore, we have that there is at least one meromorphic solution \( f_1 \) satisfies
\[ \sigma_{(p+1,q)}(f_1) = \sigma_{(p,q)}(A_s). \]

**Proof of Theorem 1.4** Suppose that \( f \) is a nonzero meromorphic solution whose poles are of uniformly bounded multiplicities of (1.1), then (1.1) can be written
\[ -A_0 = \frac{f^{(k)}}{f} + A_{k-1} \frac{f^{(k-1)}}{f} + \cdots + A_1 \frac{f'}{f}. \] (3.7)
By the lemma of the logarithmic derivative and (3.7), we have
\[ m(r, A_0) \leq \sum_{j=1}^{k-1} m(r, A_j) + \sum_{j=1}^{k} m(r, \frac{f^{(j)}}{f}) + O(1) \]
\[ = \sum_{j=1}^{k-1} m(r, A_j) + O\{\log(rT(r, f))\}. \] (3.8)
holds for all sufficiently large \( r \not\in E_{10} \), where \( E_{10} \subset (0, \infty) \) has finite linear measure. Hence
\[
T(r, A_0) = m(r, A_0) + N(r, A_0) \leq N(r, A_0) + \sum_{j=1}^{k-1} m(r, A_j) + O(\log(rT(r, f))) \tag{3.9}
\]
holds for all sufficiently large \( |r| = r \not\in E_{10} \).

Since \( \max \{ \sigma_{(p,q)}(A_j) : j \neq 0 \} < \sigma_{(p,q)}(A_0) < \infty \), by Lemma 2.15 there exist a set \( E_{11} \subset (1, \infty) \) having infinite logarithmic measure such that for all \( z \) satisfying \( |z| = r \in E_{11} \), we have
\[
\lim_{r \to \infty} \frac{\log p T(r, A_0)}{\log q r} = \sigma_{(p,q)}(A_0), \quad \frac{m(r, A_j)}{m(r, A_0)} = o(1) \quad (r \in E_{10}, j = 1, 2, \ldots, k - 1).
\tag{3.10}
\]
By (3.8) and (3.10), for all sufficiently large \( r \in E_{11} \setminus E_{10} \), we have
\[
\frac{1}{2} m(r, A_0) \leq O(\log(rT(r, f))). \tag{3.11}
\]
Using a similar discussion as in second part of proof of Theorem 1.3, we can get that
\[
m(r, A_0) > N(r, A_0), \tag{3.12}
\]
and, hence,
\[
T(r, A_0) = m(r, A_0) + N(r, A_0) = O(m(r, A_0)) = O(\log rT(r, f))
\]
for all sufficiently large \( r \in E_{11} \setminus E_{10} \), this means
\[
\sigma_{(p+1,q)}(f) \geq \sigma_{(p,q)}(A_0).
\]

On the other hand, by Theorem 1.3 we have
\[
\sigma_{(p+1,q)}(f) \leq \sigma_{(p,q)}(A_0).
\]
Therefore, every meromorphic solution \( f \) whose poles are of uniformly bounded multiplicities of (1.1) satisfies
\[
\sigma_{(p+1,q)}(f) = \sigma_{(p,q)}(A_0).
\]

**Proof of Theorem 1.5.** When \( A_0, A_1, \ldots, A_{k-1} \) satisfy
\[
\max \{ \sigma_{(p,q)}(A_j) : j \neq 0 \} < \sigma_{(p,q)}(A_0),
\]
then by Theorem 1.4 it is easy to see that Theorem 1.5 holds. Now we assume that there exists at least one of \( A_j \) \( (j = 1, 2, \ldots, k - 1) \) satisfies \( \sigma_{(p,q)}(A_j) = \sigma_{(p,q)}(A_0) \).

Suppose that \( f \) is a nonzero meromorphic solution of (1.1), we have
\[
|A_0| \leq \left| \frac{f^{(k)}(z)}{f(z)} \right| + |A_{k-1}| \left| \frac{f^{(k-1)}(z)}{f(z)} \right| + \cdots + |A_1| \left| \frac{f'(z)}{f(z)} \right|. \tag{3.13}
\]
Using a similar discussion as in the proof of Theorem 1.4, we can get that (3.8) and (3.9) hold for all sufficiently large \( r \not\in E_{12} \), where \( E_{12} \subset (0, \infty) \) has finite linear measure. Since
\[
\max \{ \sigma_{(p,q)}(A_j) : j = 1, 2, \ldots, k - 1 \} = \sigma_{(p,q)}(A_0)
\]
and
\[
\max \{ \tau_{(p,q)}(A_j) : \sigma_{(p,q)}(A_j) = \sigma_{(p,q)}(A_0) > 0 \} < \tau_{(p,q)}(A_0),
\]
then there exists a set \( J \subset \{1, 2, \ldots, k-1\} \) such that for \( j \in J \), we have \( \sigma_{(p,q)}(A_j) = \sigma_{(p,q)}(A_0) \) and \( \tau_{(p,q)}(A_j) < \tau_{(p,q)}(A_0) \).
Hence, there exist two constants $\beta_1$ and $\beta_2$ satisfying $\max\{\tau_r : r \in J\} < \beta_1 < \beta_2 \leq \tau_0$. By Definitions 2.6 and 2.9, we obtain that

$$m(r, A_j) \leq T(r, A_j) < \exp\{\beta_1(\log_{q-1} r)^{\sigma_q(A_0)}\}. \quad (3.14)$$

Since $\lambda_0(\frac{1}{A_0}) < \sigma_q(A_0)$, we have

$$N(r, A_0) \leq \exp\{\lambda_0(\frac{1}{A_0}) + \epsilon \log_q r\} \leq \exp\{\beta_1(\log_{q-1} r)^{\sigma_q(A_0)}\}. \quad (3.15)$$

By Lemma 2.18, there exists a set of $E_{13}$ having infinite logarithmic measure such that for all $r \in E_{13}$, we have

$$T(r, A_0) \geq \exp\{\beta_2(\log_{q-1} r)^{\sigma_q(A_0)}\}. \quad (3.16)$$

Now, substituting (3.14) - (3.16) into (3.9), we have

$$(1 - o(1)) \exp\{\beta_2(\log_{q-1} r)^{\sigma_q(A_0)}\} \leq O(\log(rT(r, f)))$$

for all $r \in E_{13} \setminus E_{12}$, this implies

$$\sigma_{p+1,q}(f) \geq \sigma_q(A_0).$$

On the other hand, by Theorem 1.3, we have

$$\sigma_{p+1,q}(f) \leq \sigma_q(A_0).$$

Then we have that

$$\sigma_{p+1,q}(f) = \sigma_q(A_0)$$

holds for any nonzero meromorphic solution $f$ whose poles are of uniformly bounded multiplicities of (1.1).

**Proof of Theorem 1.6** Since all solutions of equation (1.2) are meromorphic functions, all solutions of the homogeneous differential equation (1.1) corresponding to equation (1.2) are still meromorphic functions.

Now we assume that $\{f_1, f_2, \ldots, f_k\}$ is a meromorphic solution base of (1.1), then by the elementary theory of differential equations (see, e.g. [16]), any solution of (1.2) has the form

$$f = c_1(z)f_1 + c_2(z)f_2 + \cdots + c_k(z)f_k, \quad (3.17)$$

where $c_1, c_2, \ldots, c_k$ are suitable meromorphic functions satisfying

$$c_j' = FG_j(f_1, f_2, \ldots, f_k)W(f_1, f_2, \ldots, f_k)^{-1} \quad (j = 1, 2, \ldots, k), \quad (3.18)$$

where $G_j(f_1, f_2, \ldots, f_k)$ are differential polynomials in $\{f_1, f_2, \ldots, f_k\}$ and their derivatives, and $W(f_1, f_2, \ldots, f_k)^{-1}$ is the Wronskian of $\{f_1, f_2, \ldots, f_k\}$. By Theorem 1.4, we have

$$\sigma_{p+1,q}(f_j) = \sigma_q(A_0) \quad (j = 1, 2, \ldots, k).$$

By Lemma 2.13, (3.17) and (3.18), we obtain

$$\sigma_{p+1,q}(f) \leq \max\{\sigma_{p+1,q}(f_j), \sigma_{p+1,q}(F) : j = 1, 2, \ldots, k\} = \sigma_q(A_0).$$

Now we assert that all solutions $f$ of (1.2) satisfy $\sigma_{p+1,q}(f) = \sigma_q(A_0)$ with at most one exceptional solution, say $f_0$, satisfying $\sigma_{p+1,q}(f_0) < \sigma_q(A_0)$. In fact, if there exists two distinct meromorphic functions $f_0$ and $f_1$ of (1.2) satisfying

$$\sigma_{p+1,q}(f_j) < \sigma_q(A_0) \quad (j = 0, 1),$$

then $f = f_0 - f_1$ is a nonzero meromorphic solution of (1.1), and satisfying $\sigma_{p+1,q}(f) < \sigma_q(A_0)$, this contradicts Theorem 1.4.
For all the solutions \( f \) of (1.2) satisfying \( \sigma_{(p+1,q)}(f) = \sigma_{(p,q)}(A_0) \), we have
\[
\max\{\sigma_{(p+1,q)}(A_j), \sigma_{(p+1,q)}(F) : j = 0, 1, \ldots, k-1\} < \sigma_{(p+1,q)}(f).
\]
Thus by Lemma 2.20 we obtain
\[
\overline{\lambda}_{(p+1,q)}(f) = \lambda_{(p+1,q)}(f) = \sigma_{(p+1,q)}(f).
\]
Therefore, Theorem 1.8 is proved.

### 3.1. Proof of Theorem 1.7

Suppose that \( \{g_1, g_2, \ldots, g_k\} \) is a meromorphic solution base of (1.1) corresponding to (1.2). By a similar discussion as in the proof of Theorem 1.6, we obtain
\[
\sigma_{(p+1,q)}(f) \leq \max\{\sigma_{(p+1,q)}(g_j), \sigma_{(p+1,q)}(F) : j = 1, 2, \ldots, k\}
\]
By the first part of the proof of Theorem 1.3, we can get that
\[
\sigma_{(p+1,q)}(g_j) \leq \max\{\sigma_{(p,q)}(A_j) : j = 0, 1, \ldots, k-1\} \leq \sigma_{(p+1,q)}(F),
\]
then we can get
\[
\sigma_{(p+1,q)}(f) \leq \sigma_{(p+1,q)}(F).
\]
On the other hand, by the simple order comparison from (1.2), we have
\[
\sigma_{(p+1,q)}(F) \leq \max\{\sigma_{(p+1,q)}(A_j), \sigma_{(p+1,q)}(f) : j = 0, 1, \ldots, k-1\}.
\]
Since \( \sigma_{(p+1,q)}(A_j) < \sigma_{(p+1,q)}(F) \), we have
\[
\sigma_{(p+1,q)}(F) \leq \sigma_{(p+1,q)}(f).
\]
By (3.19)-(3.20), we obtain
\[
\sigma_{(p+1,q)}(F) = \sigma_{(p+1,q)}(f).
\]
Therefore, the proof of Theorem 1.7 is complete.

### Proof of Theorem 1.8

(i) By the simple order comparison from (1.2) it is easy to see that all meromorphic solutions of (1.2) satisfy
\[
\sigma_{(p+1,q)}(f) \geq \sigma_{(p+1,q)}(F).
\]
On the other hand, by the similar proof in (3.17)-(3.18), we obtain that all meromorphic solutions of (1.2) satisfy
\[
\sigma_{(p+1,q)}(f) \leq \sigma_{(p+1,q)}(F)
\]
if \( \sigma_{(p+1,q)}(F) \geq \alpha_1 \). Therefore, all meromorphic solutions whose poles are of uniformly bounded multiplicities of (1.2) satisfy
\[
\sigma_{(p+1,q)}(f) = \sigma_{(p+1,q)}(F).
\]
(ii) By the hypotheses that
\[
|A_0(z)| \geq \exp_{p+1}\{(\alpha_1 - c) \log r\},
\]
and \( |A_j(z)| \leq \exp_{p+1}\{\alpha_2 \log r\} \), we can easily obtain that \( \sigma_{(p+1,q)}(A_0) = \alpha_1 \). Since \( \sigma_{(p+1,q)}(F) < \alpha_1 = \sigma_{(p+1,q)}(A_0) \), by the similar proof in Theorem 1.6, we obtain that all meromorphic solutions whose poles are of uniformly bounded multiplicities of (1.2) satisfy
\[
\overline{\lambda}_{(p+1,q)}(f) = \lambda_{(p+1,q)}(f) = \sigma_{(p+1,q)}(f) = \alpha_1
\]
with at most one exceptional solution \( f_2 \) satisfying \( \sigma_{(p+1,q)}(f_2) < \alpha_1 \). Therefore, we completely prove Theorem 1.8.
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