Electronic Journal of Differential Equations, Vol. 2012 (2012), No. 195, pp. 1–15.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

SOLUTIONS FOR LINEAR DIFFERENTIAL EQUATIONS WITH MEROMORPHIC COEFFICIENTS OF (P,Q)-ORDER IN THE PLANE

LEI-MIN LI, TING-BIN CAO

Abstract. In this article we study the growth of meromorphic solutions of high order linear differential equations with meromorphic coefficients of (p,q)-order. We extend some previous results due to Belaïdi, Cao-Xu-Chen, Kinnunen, Liu- Tu -Shi, and others.

1. Introduction and main results

For $k \geq 2$, consider the linear differential equations

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = 0, \tag{1.1}$$

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = F(z), \tag{1.2}$$

where $A_0 \not\equiv 0$ and $F \not\equiv 0$. When the coefficients $A_0, A_1, \dots, A_{k-1}$ and F are entire functions, it is well known that all solutions of (1.1) and (1.2) are entire functions, and that if some coefficients of (1.1) are transcendental then (1.1) has at least one solution with infinite order. We refer to [16] for the literature on the growth of entire solutions of (1.1) and (1.2).

As far as we known, Bernal [4] firstly introduced the idea of iterated order to express the fast growth of solutions of complex linear differential equations. Since then, many authors obtained further results on iterated order of solutions of (1.1) and (1.2), see e.g. [1, 2, 4, 5, 6, 15, 19]. Recently, Liu, Tu and Shi [17] firstly introduced the concept of (p, q)-order for the case $p \geq q \geq 1$ to investigate the entire solutions of (1.1) and (1.2), and obtained some results which improve and generalize some previous results.

Theorem 1.1 ([17, Theorems 2.2-2.3])**.** *Let $p \geq q \geq 1$, and let $A_0, A_1, \dots, A_{k-1}$ be entire functions such that either*

$$\max\{\sigma_{(p,q)}(A_j) : j \neq 0\} < \sigma_{(p,q)}(A_0) < +\infty,$$

or

$$\max\{\sigma_{(p,q)}(A_j) : j \neq 0\} \leq \sigma_{(p,q)}(A_0) < +\infty,$$

2000 *Mathematics Subject Classification.* 34M10, 30D35, 34M05.
Key words and phrases. Linear differential equation; meromorphic function; (p,q)-order; Nevanlinna theory.

Submitted January 6, 2012. Published November 8, 2012.

$$\max\{\tau_{(p,q)}(A_j) : \sigma_{(p,q)}(A_j) = \sigma_{(p,q)}(A_0) > 0\} < \tau_{(p,q)}(A_0),$$

then every nontrivial solution f of (1.1) *satisfies* $\sigma_{(p+1,q)}(f) = \sigma_{(p,q)}(A_0)$.

Recently, Cao, Xu and Chen [5] considered the growth of meromorphic solutions of equations (1.1) and (1.2) with meromorphic coefficients of finite iterated order, and obtained some results which improve and generalize some previous results.

Theorem 1.2 ([5, Theorem 2.1])**.** *Let $A_0, A_1, \dots, A_{k-1}$ be meromorphic functions in the plane, and let $i(A_0) = p$ $(0 < p < \infty)$. Assume that either $i_\lambda(\frac{1}{A_0}) < p$ or $\lambda_p(\frac{1}{A_0}) < \sigma_p(A_0)$, and that either*

$$\max\{i(A_j) : j = 1, 2, \dots, k-1\} < p$$

or

$$\max\{\sigma_p(A_j) : j = 1, 2, \dots, k-1\} \le \sigma_p(A_0) := \sigma \quad (0 < \sigma < \infty),$$
$$\max\{\tau_p(A_j) : \sigma_p(A_j) = \sigma_p(A_0)\} < \tau_p(A_0) := \tau \quad (0 < \tau < \infty).$$

Then every meromorphic solution $f \not\equiv 0$ whose poles are of uniformly bounded multiplicities, of equation (1.1) *satisfies $i(f) = p + 1$ and $\sigma_{p+1}(f) = \sigma_p(A_0)$.*

There exists a natural question: *How about the growth of meromorphic solutions of equations* (1.1) *and* (1.2) *with meromorphic coefficients of finite (p, q)-order in the plane?*

The main purpose of this paper is to consider the above question. Now we show our main results. For homogeneous linear differential equation (1.1), we obtain the following results.

Theorem 1.3. *Let $A_0, A_1, \dots, A_{k-1}$ be meromorphic functions in the plane. Suppose that there exists one coefficient A_s $(s \in \{0, 1, \dots, k-1\})$ such that*

$$\max\{\sigma_{(p,q)}(A_j), \lambda_{(p,q)}(\frac{1}{A_s}) : j \ne s\} < \sigma_{(p,q)}(A_s) < +\infty,$$

then every transcendental meromorphic solution f whose poles are of uniformly bounded multiplicities of (1.1) *satisfies*

$$\sigma_{(p+1,q)}(f) \le \sigma_{(p,q)}(A_s) \le \sigma_{(p,q)}(f).$$

Furthermore, if all solutions of (1.1) *are meromorphic solutions, then there is at least one meromorphic solution, say f_1, satisfies*

$$\sigma_{(p+1,q)}(f_1) = \sigma_{(p,q)}(A_s).$$

Now replacing the arbitrary coefficient A_s by the dominant fixed coefficient A_0, then we obtain the following result.

Theorem 1.4. *Let $A_0, A_1, \dots, A_{k-1}$ be meromorphic functions in the plane satisfying*

$$\max\{\sigma_{(p,q)}(A_j), \lambda_{(p,q)}(\frac{1}{A_0}) : j = 1, 2, \dots, k-1\} < \sigma_{(p,q)}(A_0) < +\infty,$$

then every meromorphic solution f whose poles are of uniformly bounded multiplicities of (1.1) *satisfies*

$$\sigma_{(p+1,q)}(f) = \sigma_{(p,q)}(A_0).$$

If there exist some other coefficients A_j ($j \in \{1, 2, \dots, k-1\}$) having the same (p,q)-order as A_0, then we have the following result by making use of the concept of (p,q)-type.

Theorem 1.5. *Let $A_0, A_1, \dots, A_{k-1}$ be meromorphic functions in the plane, assume that*

$$\lambda_{(p,q)}(\frac{1}{A_0}) < \sigma_{(p,q)}(A_0)$$

and

$$\max\{\sigma_{(p,q)}(A_j) : j = 1, 2, \dots, k-1\} = \sigma_{(p,q)}(A_0) < +\infty,$$
$$\max\{\tau_{(p,q)}(A_j) : \sigma_{(p,q)}(A_j) = \sigma_{(p,q)}(A_0) > 0\} < \tau_{(p,q)}(A_0).$$

Then any nonzero meromorphic solution f whose poles are of uniformly bounded multiplicities of (1.1) satisfies

$$\sigma_{(p+1,q)}(f) = \sigma_{(p,q)}(A_0).$$

Obviously, Theorems 1.4 and 1.5 are a generalization of Theorems 1.1 and 1.2. Considering nonhomogeneous linear differential equation (1.2), we obtain the following three results.

Theorem 1.6. *Assume that $A_0, A_1, \dots, A_{k-1}, F \not\equiv 0$ be meromorphic functions in the plane satisfying*

$$\max\{\sigma_{(p,q)}(A_j), \lambda_{(p,q)}(\frac{1}{A_0}), \sigma_{(p+1,q)}(F) : j = 1, 2, \dots, k-1\} < \sigma_{(p,q)}(A_0),$$

then all meromorphic solutions f whose poles are of uniformly bounded multiplicities of (1.2) satisfy

$$\overline{\lambda}_{(p+1,q)}(f) = \lambda_{(p+1,q)}(f) = \sigma_{(p+1,q)}(f) = \sigma_{(p,q)}(A_0)$$

with at most one exceptional solution f_0 satisfying $\sigma_{(p+1,q)}(f_0) < \sigma_{(p,q)}(A_0)$.

Theorem 1.7. *Let $A_0, A_1, \dots, A_{k-1}, F \not\equiv 0$ be meromorphic functions in the plane satisfying*

$$\max\{\sigma_{(p,q)}(A_j) : j = 0, 1, \dots, k-1\} < \sigma_{(p+1,q)}(F).$$

Suppose that all solutions of (1.2) are meromorphic functions whose poles are of uniformly bounded multiplicities, then $\sigma_{(p+1,q)}(f) = \sigma_{(p+1,q)}(F)$ holds for all solutions of (1.2).

Theorem 1.8. *Let $H \subset (1, \infty)$ be a set satisfying $\overline{\log dens}\{|z| : |z| \in H\} > 0$ and let $A_0, A_1, \dots, A_{k-1}$, $F \not\equiv 0$ be meromorphic functions in the plane satisfying*

$$\max\{\sigma_{(p,q)}(A_j) : j = 1, 2, \dots, k-1\} < \alpha_1,$$

where α_1 is a constant, and there exists another constant α_2 ($\alpha_2 < \alpha_1$) such than for any given ϵ ($0 < \epsilon < \alpha_1 - \alpha_2$), we have

$$|A_0(z)| \geq \exp_{p+1}\{(\alpha_1 - \epsilon) \log_q r\}, |A_j(z)| \leq \exp_{p+1}\{\alpha_2 \log_q r\}$$

for $|z| \in H$, $j = 1, 2, \dots, k-1$. Then we have:

(i) If $\sigma_{(p+1,q)}(F) \geq \alpha_1$, then all meromorphic solutions whose poles are of uniformly bounded multiplicities of (1.2) satisfy

$$\sigma_{(p+1,q)}(f) = \sigma_{(p+1,q)}(F).$$

(ii) If $\sigma_{(p+1,q)}(F) < \alpha_1$, then all meromorphic solutions whose poles are of uniformly bounded multiplicities of (1.2) *satisfy*

$$\overline{\lambda}_{(p+1,q)}(f) = \lambda_{(p+1,q)}(f) = \sigma_{(p+1,q)}(f) = \alpha_1$$

with at most one exceptional solution f_2 satisfying

$$\sigma_{(p+1,q)}(f_2) < \alpha_1.$$

Recently, B. Belaïdi [3] investigated the growth of solutions of differential equations (1.1) and (1.2) with analytic coefficients of (p,q)-order in the unit disc. So, it is also interesting to consider the growth of meromorphic solutions of differential equations with coefficients of (p,q)-order in the unit disc?

2. Preliminaries and some lemmas

We shall introduce some notation. Let us define inductively, for $r \in [0, +\infty)$, $\exp_1 r = e^r$ and $\exp_{n+1} r = \exp(\exp_n r)$, $n \in \mathbb{N}$. For all r sufficiently large, we define $\log_1 r = \log^+ r = \max\{\log r, 0\}$ and $\log_{n+1} r = \log(\log_n r)$, $n \in \mathbb{N}$. We also denote $\exp_0 r = r = \log_0 r$, $\log_{-1} r = \exp_1 r$ and $\exp_{-1} r = \log_1 r$. Moreover, we denote the linear measure and the logarithmic measure of a set $E \subset (1,\infty)$ by $mE = \int_E dt$ and $m_l E = \int_E \frac{dt}{t}$. The upper logarithmic density of $E \subset (1,\infty)$ is defined by

$$\overline{\log\text{dens}}E = \limsup_{r\to\infty} \frac{m_l(E \cap [1,r])}{\log r}.$$

We assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna's value distribution theory of meromorphic functions (e.g. see [11, 20]), such as $T(r,f)$, $m(r,f)$, and $N(r,f)$. In this section, a meromorphic function f means meromorphic in the complex plane $\mathbb{C}$. To express the rate of fast growth of meromorphic functions, we recall the following definitions (e.g. see [4, 5, 15, 16, 18]).

Definition 2.1. The iterated p-order $\sigma_p(f)$ of a meromorphic function f is defined by

$$\sigma_p(f) = \limsup_{r\to\infty} \frac{\log_p T(r,f)}{\log r} \quad (p \in \mathbb{N}).$$

If f is an entire function, then

$$\sigma_{p,M}(f) = \limsup_{r\to\infty} \frac{\log_{p+1} M(r,f)}{\log r} \quad (p \in \mathbb{N}).$$

Definition 2.2. The growth index of the iterated order of a meromorphic function f is defined by

$$i(f) = \begin{cases} 0 & \text{if } f \text{ is rational,} \\ \min\{n \in \mathbb{N} : \sigma_n(f) < \infty\} & \text{if } f \text{ is transendental and } \sigma_n(f) < \infty \\ & \text{for some } n \in \mathbb{N}, \\ \infty & \text{if } \sigma_n(f) = \infty \text{ for all } n \in \mathbb{N}. \end{cases}$$

Definition 2.3. The iterated p-type of a meromorphic function f with iterated order p-order $0 < \sigma_p(f) < \infty$ is defined by

$$\tau_p(f) = \limsup_{r\to\infty} \frac{\log_{p-1} T(r,f)}{r^{\sigma_p(f)}} \quad (p \in \mathbb{N}).$$

If f is an entire function, then

$$\tau_{p,M}(f) = \limsup_{r\to\infty} \frac{\log_p M(r,f)}{r^{\sigma_p(f)}} \quad (p \in \mathbb{N}).$$

Definition 2.4. The iterated convergence exponent of the sequence of zeros of a meromorphic function f is defined by

$$\lambda_p(f) = \limsup_{r\to\infty} \frac{\log_p N(r,\frac{1}{f})}{\log r} \quad (p \in \mathbb{N}).$$

Definition 2.5. The growth index of the iterated convergence exponent of the sequence of zeros of a meromorphic function f with iterated order is defined by

$$i_\lambda(f) = \begin{cases} 0 & \text{if } n(r,\frac{1}{f}) = O(\log r), \\ \min\{n \in \mathbb{N} : \lambda_n(f) < \infty\} & \text{if } \lambda_n(f) < \infty \text{ for some } n \in \mathbb{N}, \\ \infty & \text{if } \lambda_n(f) = \infty \text{ for all } n \in \mathbb{N}. \end{cases}$$

Similarly, we can use the notation $\overline{\lambda}_p(f)$ to denote the iterated convergence exponent of the sequence of distinct zeros, and use the notation $i_{\overline{\lambda}}(f)$ to denote the growth index of $\overline{\lambda}_p(f)$.

Now, we shall introduce the definition of meromorphic functions of (p,q)-order, where p, q are positive integers satisfying $p \geq q \geq 1$. In order to keep accordance with Definition 2.1, we will give a minor modification to the original definition of (p,q)-order (e.g. see [13, 14]).

Definition 2.6. The (p,q)-order of a transcendental meromorphic function f is defined by

$$\sigma_{(p,q)}(f) = \limsup_{r\to\infty} \frac{\log_p T(r,f)}{\log_q r}.$$

If f is a transcendental entire function, then

$$\sigma_{(p,q)}(f) = \limsup_{r\to\infty} \frac{\log_{p+1} M(r,f)}{\log_q r}.$$

It is easy to show that $0 \leq \sigma_{(p,q)} \leq \infty$. By Definition 2.6 we note that $\sigma_{(1,1)}(f) = \sigma_1(f) = \sigma(f)$, $\sigma_{(2,1)}(f) = \sigma_2(f)$ and $\sigma_{(p,1)}(f) = \sigma_p(f)$.

Remark 2.7. If f is a meromorphic function satisfying $0 \leq \sigma_{(p,q)} \leq \infty$, then

(i) $\sigma_{(p-n,q)} = \infty$ $(n < p)$, $\sigma_{(p,q-n)} = 0$ $(n < q)$, and $\sigma_{(p+n,q+n)} = 1$ $(n < p)$ for $n = 1$ to ∞.

(ii) If (p_1,q_1) is another pair of integers satisfying $p_1 - q_1 = p - q$ and $p_1 < p$, then we have $\sigma_{(p_1,q_1)} = 0$ if $0 < \sigma_{(p,q)} < 1$ and $\sigma_{(p_1,q_1)} = \infty$ if $1 < \sigma_{(p,q)} < \infty$.

(iii) $\sigma_{(p_1,q_1)} = \infty$ for $p_1 - q_1 > p - q$ and $\sigma_{(p_1,q_1)} = 0$ for $p_1 - q_1 > p - q$.

Remark 2.8. Suppose that f_1 is a meromorphic function of (p,q)-order σ_1 and f_2 is a meromorphic function of (p_1,q_1)-order σ_2, let $p \leq p_1$. We can easily deduce the result about their comparative growth:

(i) If $p_1 - q_1 > p - q$, then the growth of f_1 is slower than the growth of f_2.

(ii) If $p_1 - q_1 < p - q$, then f_1 grows faster than f_2.

(iii) If $p_1 - q_1 = p - q > 0$, then the growth of f_1 is slower than the growth of f_2 if $\sigma_2 \geq 1$,and the growth of f_1 is faster than the growth of f_2 if $\sigma_2 < 1$.

(iv) Especially, when $p_1 = p$ and $q_1 = q$ then f_1 and f_2 are of the same index-pair (p,q). If $\sigma_1 > \sigma_2$, then f_1 grows faster than f_2; and if $\sigma_1 < \sigma_2$, then f_1 grows slower than f_2. If $\sigma_1 = \sigma_2$, Definition 1.6 does not show any precise estimate about the relative growth of f_1 and f_2.

Definition 2.9. The (p,q)-type of a meromorphic function f with (p,q)-order $\sigma_{(p,q)}(f) \in (0,\infty)$ is defined by

$$\tau_{(p,q)}(f) = \limsup_{r\to\infty} \frac{\log_{p-1} T(r,f)}{(\log_{q-1} r)^{\sigma_{(p,q)}(f)}}.$$

Definition 2.10. The (p,q) convergence exponent of the sequence of zeros of a meromorphic function f is defined by

$$\lambda_{(p,q)}(f) = \limsup_{r\to\infty} \frac{\log_p N(r,\frac{1}{f})}{\log_q r}.$$

Similarly, we can use the notation $\overline{\lambda}_{(p,q)}(f)$ to denote the (p,q) convergence exponent of the sequence of distinct zeros of f. To prove our results, we need the following lemmas.

Lemma 2.11 ([8])**.** *Let $f_1, f_2, \dots, f_k$ be linearly independent meromorphic solutions of the differential equation* (1.1) *with meromorphic functions $A_0, A_1, \dots, A_{k-1}$ as the coefficients, then*

$$m(r, A_j) = O\{\log(\max_{1\le n\le k} T(r,f_n))\} \quad (j = 0, 1, \dots, k-1).$$

Lemma 2.12 ([7])**.** *Let f be a meromorphic solution of equation* (1.1)*, assuming that not all coefficients A_j are constants. Given a real constant $\gamma > 1$, and denoting $T(r) = \Sigma_{j=0}^{k-1} T(r, A_j)$, we have*

$$\log m(r,f) < T(r)\{(\log r)\log T(r)\}^{\gamma}, \quad \text{if } s = 0,$$
$$\log m(r,f) < r^{2s+\gamma-1} T(r)\{\log T(r)\}^{\gamma}, \quad \text{if } s > 0$$

outside of an exceptional set E_s with $\int_{E_s} t^{s-1}dt < \infty$.

By inequalities in [12, Chapter 6] and in [16, Corollary 2.3.5], we obtain the following lemma.

Lemma 2.13. *If f is a meromorphic function, then*

$$\sigma_{(p.q)}(f) = \sigma_{(p.q)}(f').$$

Lemma 2.14 ([9])**.** *Let f be a transcendental meromorphic function, and let α be a given constant. Then there exist a set $E_1 \subset (1,\infty)$ that has finite logarithmic measure and a constant $B > 0$ depending only on α and $(m,n)(m,n \in \{0,1,\dots,k\})$, $m < n$ such that for all z with $|z| = r \notin [0,1] \cup E_1$, we have*

$$\Big|\frac{f^{(n)}(z)}{f^{(m)}(z)}\Big| \le B\Big(\frac{T(\alpha r,f)}{r}(\log^{\alpha} r)T(\alpha r,f)\Big)^{n-m}.$$

Lemma 2.15. *Let f be a meromorphic function of $(p.q)$-order satisfying $\sigma_{(p.q)}(f) < \infty$. Then there exists a set $E_2 \subset (1,\infty)$ having infinite logarithmic measure such that for all $r \in E_2$, we have*

$$\lim_{r\to\infty} \frac{\log_p T(r,f)}{\log_q r} = \sigma_{(p.q)}(f).$$

Proof. By Definition 2.6, there exists a sequence $\{r_n\}_{n=1}^{\infty}$ tending to ∞, satisfying $(1+\frac{1}{n})r_n < r_{n+1}$, and

$$\lim_{n\to\infty} \frac{\log_p T(r_n, f)}{\log_q r_n} = \sigma_{(p.q)}(f).$$

There exists a $n_1 \in \mathbb{N}$, such that for $n \geq n_1$, and for any $r \in [r_n, (1+\frac{1}{n})r_n]$, we have

$$\frac{\log_p T(r_n, f)}{\log_q (1+\frac{1}{n})r_n} \leq \frac{\log_p T(r, f)}{\log_q r} \leq \frac{\log_p T((1+\frac{1}{n})r_n, f)}{\log_q r_n}.$$

Set $E_2 = \cup_{n=n_1}^{\infty}[r_n, (1+\frac{1}{n})r_n]$, then for any $r \in E_2$, we have

$$\lim_{r\to\infty} \frac{\log_p T(r, f)}{\log_q r} = \lim_{n\to\infty} \frac{\log_p T(r_n, f)}{\log_q r_n} = \sigma_{(p.q)}(f),$$

where

$$m_l E_2 = \Sigma_{n=n_1}^{\infty} \int_{r_n}^{(1+\frac{1}{n})r_n} \frac{dt}{t} = \Sigma_{n=n_1}^{\infty} \log(1+\frac{1}{n}) = \infty. \qquad \square$$

Lemma 2.16. *Let $\varphi(r)$ be a continuous and positive increasing function, defined for $r \in [0, \infty]$ with $\sigma_{(p.q)}(\varphi) = \limsup_{r\to\infty} \frac{\log_p \varphi(r)}{\log_q r}$, then for any subset $E_3 \subset (0, \infty)$ that has a finite linear measure, there exists a sequence $\{r_n\}, r_n \notin E_3$ such that*

$$\sigma_{(p.q)}(\varphi) = \lim_{r_n\to\infty} \frac{\log_p \varphi(r_n)}{\log_q r_n}.$$

Proof. Since $\sigma_{(p.q)}(\varphi) = \limsup_{r\to\infty} \frac{\log_p \varphi(r)}{\log_q r}$, then there exists a sequence $\{r'_n\}$ tending to ∞,such that

$$\lim_{r'_n\to\infty} \frac{\log_p \varphi(r'_n)}{\log_q r'_n} = \sigma_{(p.q)}(\varphi).$$

Set $mE_3 = \delta < \infty$,then for $r_n \in [r'_n, r'_n + \delta + 1]$, we have

$$\frac{\log_p \varphi(r_n)}{\log_q r_n} \geq \frac{\log_p \varphi(r'_n)}{\log_q (r'_n + \delta + 1)} = \frac{\log_p \varphi(r'_n)}{\log_{q-1}(\log r'_n + \log(1 + \frac{\delta+1}{r'_n}))}.$$

Hence

$$\begin{aligned} \lim_{r_n\to\infty} \frac{\log_p \varphi(r_n)}{\log_q r_n} &\geq \lim_{r'_n\to\infty} \frac{\log_p \varphi(r'_n)}{\log_{q-1}(\log r'_n + \log(1 + \frac{\delta+1}{r'_n}))} \\ &= \lim_{r'_n\to\infty} \frac{\log_p \varphi(r'_n)}{\log_q r'_n} = \sigma_{(p.q)}(\varphi), \end{aligned}$$

this gives

$$\sigma_{(p.q)}(\varphi) = \lim_{r_n\to\infty} \frac{\log_p \varphi(r_n)}{\log_q r_n}. \qquad \square$$

Lemma 2.17 ([13])**.** *Let f be an entire function of $(p.q)$-order,and let $\nu_f(r)$ be the central index of f, then*

$$\limsup_{r\to\infty} \frac{\log_p \nu_f(r)}{\log_q r} = \sigma_{(p.q)}(f).$$

Lemma 2.18. *Let f be a meromorphic function of $(p.q)$-order satisfying $0 < \sigma_{(p.q)}(f) < \infty$, let $\tau_{(p.q)}(f) > 0$, then for any given $\tau_{(p.q)}(f) > \beta$, there exists a set $E_4 \subset (1, \infty)$ that has infinite logarithmic measure such that for all $r \in E_4$, we have*

$$\log_{p-1} T(r, f) > \beta(\log_{q-1} r)^{\sigma_{(p.q)}(f)}.$$

.

Proof. (i) (see [5]) when $q = 1$, it holds absolutely. (ii) when $q \geq 2$, by Definition 2.9, there exists an increasing sequence $\{r_m\}(r_m \to \infty)$ satisfying $(1 + \frac{1}{m})r_m < r_{m+1}$, and

$$\lim_{m\to\infty} \frac{\log_{p-1} T(r_m, f)}{(\log_{q-1} r_m)^{\sigma_{(p.q)}(f)}} = \tau_{(p.q)}(f).$$

Then there exists a positive constant m_0 such that for all $m > m_0$ and for any given ϵ $(0 < \epsilon < \tau_{(p.q)}(f) - \beta)$ we have

$$\log_{p-1} T(r_m, f) > (\tau_{(p.q)}(f) - \epsilon)(\log_{q-1} r_m)^{\sigma_{(p.q)}(f)}. \tag{2.1}$$

For any $r \in [r_m, (1 + \frac{1}{m})r_m]$, we have

$$\lim_{r_m\to+\infty} \frac{\log_{q-1} r_m}{\log_{q-1} r} = 1.$$

Since $\beta < \tau_{(p.q)}(f) - \epsilon$, there exists a positive constant m_1 such that for all $m > m_1$, we have

$$(\frac{\log_{q-1} r_m}{\log_{q-1} r})^{\sigma_{(p.q)}(f)} > \frac{\beta}{\tau_{(p.q)}(f) - \epsilon};$$

i.e.,

$$(\tau_{(p.q)}(f) - \epsilon)(\log_{q-1} r_m)^{\sigma_{(p.q)}(f)} > \beta(\log_{q-1} r)^{\sigma_{(p.q)}(f)}. \tag{2.2}$$

Now we take $m_2 = \max\{m_0, m_1\}$ and $E_4 = \cup_{m=m_2}^{\infty}[r_m, (1 + \frac{1}{m})r_m]$, then by (2.1)-(2.2), for any $r \in E_4$, we have

$$\begin{aligned}
\log_{p-1} T(r, f) &\geq \log_{p-1} T(r_m, f)\\
&> (\tau_{(p.q)}(f) - \epsilon)(\log_{q-1} r_m)^{\sigma_{(p.q)}(f)}\\
&> \beta(\log_{q-1} r)^{\sigma_{(p.q)}(f)},
\end{aligned}$$

where

$$m_l E_4 = \Sigma_{m=m_2}^{\infty} \int_{r_m}^{(1+\frac{1}{m})r_m} \frac{dt}{t} = \Sigma_{m=m_2}^{\infty} \log(1 + \frac{1}{m}) = \infty.$$ □

Lemma 2.19 ([10])**.** *Let $g(r)$ and $h(r)$ be monotone nondecreasing functions on $[0, \infty)$ such that $g(r) \leq h(r)$ for all $r \notin [0, 1] \cup E_5$, where $E_5 \in (1, \infty)$ is a set of finite logarithmic measure. Then for any constant $\alpha > 1$, there exists $r_0 = r_0(\alpha) > 0$ such that $g(r) \leq h(\alpha r)$ for all $r \geq r_0$.*

Lemma 2.20. *Let $A_0, A_1, \dots, A_{k-1}$, $F \not\equiv 0$ be meromorphic functions and let f be a meromorphic solution of equation* (1.2)*. If*

$$\max\{\sigma_{(p+1,q)}(A_j), \sigma_{(p+1,q)}(F) : j = 0, 1, \dots, k-1\} < \sigma_{(p+1,q)}(f),$$

then we have

$$\overline{\lambda}_{(p+1,q)}(f) = \lambda_{(p+1,q)}(f) = \sigma_{(p+1,q)}(f).$$

Proof. By (1.1), we have

$$\frac{1}{f}=\frac{1}{F}(\frac{f^{(k)}}{f}+A_{k-1}\frac{f^{(k-1)}}{f}+\dots+A_0). \tag{2.3}$$

It is easy to see that if f has a zero at z_0 of order β ($\beta>k$) and if $A_0,A_1,\dots,A_{k-1}$ are all analytic at z_0, then F has a zero at z_0 of order at least $\beta-k$. Hence

$$n(r,\frac{1}{f})\le k\overline{n}(r,\frac{1}{f})+n(r,\frac{1}{F})+\Sigma_{j=0}^{k-1}n(r,A_j), \tag{2.4}$$

$$N(r,\frac{1}{f})\le k\overline{N}(r,\frac{1}{f})+N(r,\frac{1}{F})+\Sigma_{j=0}^{k-1}N(r,A_j). \tag{2.5}$$

By the lemma of the logarithmic derivative and (2.3), we have

$$m(r,\frac{1}{f})\le m(r,\frac{1}{F})+\Sigma_{j=0}^{k-1}m(r,A_j)+O(\log T(r,f)+\log r) \tag{2.6}$$

holds for all $|z|=r\notin E_6$, where E_6 is a set of finite linear measure. By (2.5),(2.6) and the first main theorem, we have

$$T(r,f)=T(r,\frac{1}{f})+O(1)\le k\overline{N}(r,\frac{1}{f})+T(r,F)+\Sigma_{j=0}^{k-1}T(r,A_j)+O(\log(rT(r,f))) \tag{2.7}$$

holds for all sufficiently $r\notin E_6$.

Assume that $\max\{\sigma_{(p+1,q)}(A_j),\sigma_{(p+1,q)}(F):j=0,1,\dots,k-1\}<\sigma_{(p+1,q)}(f)$. By Lemma 2.16, there exists a sequence $\{r_n\}$, $r_n\notin E_6$ such that

$$\lim_{r_n\to\infty}\frac{\log_{p+1}T(r_n,f)}{\log_q r_n}=\sigma_{(p+1,q)}(f)=:\sigma_1.$$

Hence , if $r_n\notin E_6$ is sufficiently large, since $\sigma_1>0$, then we have

$$T(r_n,f)\ge\exp_{p+1}\{(\sigma_1-\epsilon)\log_q r_n\} \tag{2.8}$$

holds for any given ϵ ($0<2\epsilon<\sigma_1-\sigma_2$), where $\sigma_2=\max\{\sigma_{(p+1,q)}(A_j),\sigma_{(p+1,q)}(F):j=0,1,\dots,k-1\}$. We have

$$\max\{T(r_n,F),T(r_n,A_j):j=0,1,\dots,k-1\}\le\exp_{p+1}\{(\sigma_2+\epsilon)\log_q r_n\}. \tag{2.9}$$

Since ϵ ($0<2\epsilon<\sigma_1-\sigma_2$), then from (2.8) and (2.9) we obtain

$$\max\{\frac{T(r_n,F)}{T(r_n,f)},\frac{T(r_n,A_j)}{T(r_n,f)}:j=0,1,\dots,k-1\}\to 0\quad(r_n\to\infty). \tag{2.10}$$

For sufficiently large r_n, we have

$$O(\log(r_nT(r_n,f)))=O(T(r_n,f)).$$

Hence, by (2.7) and (2.10), we obtain that for sufficiently large $r_n\notin E_6$, there holds

$$(1-o(1))T(r_n,f)\le k\overline{N}(r_n,\frac{1}{f}).$$

Then we have $\overline{\lambda}_{(p+1,q)}(f)\ge\sigma_{(p+1,q)}(f)$, and by definitions we have $\overline{\lambda}_{(p+1,q)}(f)\le\lambda_{(p+1,q)}(f)\le\sigma_{(p+1,q)}(f)$. Therefore

$$\overline{\lambda}_{(p+1,q)}(f)=\lambda_{(p+1,q)}(f)=\sigma_{(p+1,q)}(f).$$

□

3. Proofs of Theorems 1.3-1.8

Proof of Theorem 1.3. We shall divide the proof into two parts.

• Firstly, we prove that $\sigma_{(p+1,q)}(f) \leq \sigma_{(p,q)}(A_s) \leq \sigma_{(p,q)}(f)$ holds for every transcendental meromorphic function f of (1.1). By (1.1), we know that the poles of f can only occur at the poles of $A_0, A_1, \dots, A_{k-1}$, note that the multiplicities of poles of f are uniformly bounded, then we have

$$\begin{aligned} N(r,f) &\leq C_1 \overline{N}(r,f) \\ &\leq C_1 \Sigma_{j=0}^{k-1} \overline{N}(r,A_j) \\ &\leq C_2 \max\{N(r,A_j) : j=0,1,\dots,k-1\} \leq O(T(r,A_s)), \end{aligned}$$

where C_1 and C_2 are suitable positive constants. Then we have

$$\log T(r,f) \leq \log m(r,f) + \log N(r,f) + \log 2 \leq \log m(r,f) + O\{\log T(r,A_s)\}. \quad (3.1)$$

By (3.1) and Lemma 2.12, we obtain

$$\begin{aligned} \log T(r,f) &\leq \log m(r,f) + O\{\log T(r,A_s)\} \\ &= O\left(T(r,A_s)\{(\log r)\log T(r,A_s)\}^{\lambda}\right) \end{aligned}$$

outside of an exceptional set E_0 with $\int_{E_0} \frac{dt}{t} < \infty$, this implies $\sigma_{(p+1,q)}(f) \leq \sigma_{(p,q)}(A_s)$. On the other hand, by (1.1), we obtain

$$\begin{aligned} -A_s &= \frac{f^{(k)}}{f^{(s)}} + A_{k-1}\frac{f^{(k-1)}}{f^{(s)}} + \dots + A_{s+1}\frac{f^{(s+1)}}{f^{(s)}} + A_{s-1}\frac{f^{(s-1)}}{f^{(s)}} + \dots + A_0 \frac{f}{f^{(s)}} \\ &= \frac{f}{f^{(s)}}\{\frac{f^{(k)}}{f} + A_{k-1}\frac{f^{(k-1)}}{f} + \dots + A_{s+1}\frac{f^{(s+1)}}{f} + A_{s-1}\frac{f^{(s-1)}}{f} + \dots + A_0\}. \end{aligned}$$

Since

$$m(r,\frac{f}{f^{(s)}}) \leq T(r,f) + T(r,\frac{1}{f^{(s)}}) = T(r,f) + T(r,f^{(s)}) + O(1) = O(T(r,f)),$$

then by the lemma of logarithmic derivative we have

$$T(r,A_s) \leq N(r,A_s) + \Sigma_{j\neq s} m(r,A_j) + O(\log rT(r,f)) + O(T(r,f)) \quad (3.2)$$

hold for all $|z| = r \notin E_7$, where E_7 is a set of finite linear measure. By Lemma 2.16 and similar discussion as in the proof of Lemma 2.20, we see that there exists a sequence $\{r_n\}$ ($r_n \to \infty$) such that

$$\sigma_1 := \sigma_{(p,q)}(A_s) = \lim_{r_n\to\infty} \frac{\log_p T(r_n,A_s)}{\log_q r_n}$$

and

$$T(r_n,A_s) \geq \exp_p\{(\sigma_1 - \epsilon)\log_q r_n\}, \quad (3.3)$$

$$N(r_n,A_s) \leq \exp_p\{(\sigma_2 + \epsilon)\log_q r_n\}, \quad (3.4)$$

$$m(r_n,A_j) \leq \exp_p\{(\sigma_2 + \epsilon)\log_q r_n\} \quad (j \neq s), \quad (3.5)$$

where $\sigma_2 := \max\{\sigma_{(p,q)}(A_j), \lambda_{(p,q)}(\frac{1}{A_s}) : j \neq s\}$ and $0 < 2\epsilon < \sigma_1 - \sigma_2$.

By (3.2)-(3.5), we obtain

$$(1 - o(1))\exp_p\{(\sigma_1 - \epsilon)\log_q r_n\} \leq O\{\log r_n T(r_n,f)\} + O(T(r_n,f)).$$

Hence we have $\sigma_{(p,q)}(A_s) = \sigma_1 \leq \sigma_{(p,q)}(f)$.

• Secondly, we prove that there exists at least one meromorphic solution that satisfies

$$\sigma_{(p+1,q)}(f) = \sigma_{(p,q)}(A_s).$$

Now we can assume that $\{f_1, f_2, \dots, f_k\}$ is a meromorphic solution base of (1.1). By Lemma 2.11,

$$m(r, A_s) \le O\left(\log(\max_{1\le n\le k} T(r, f_n))\right).$$

Now we assert that $m(r, A_s) > N(r, A_s)$ holds for sufficiently large r. Indeed, if $m(r, A_s) \le N(r, A_s)$, then

$$T(r, A_s) = m(r, A_s) + N(r, A_s) \le 2N(r, A_s),$$

so

$$\limsup_{r\to\infty} \frac{\log_p T(r, A_s)}{\log_q r} \le \limsup_{r\to\infty} \frac{\log_p 2N(r, A_s)}{\log_q r},$$

then we have $\sigma_{(p,q)}(A_s) \le \lambda_{(p,q)}(\frac{1}{A_s})$, which contradicts the condition $\lambda_{(p,q)}(\frac{1}{A_s}) < \sigma_{(p,q)}(A_s)$. Hence,

$$T(r, A_s) = O(m(r, A_s)) \le O\Big(\log(\max_{1\le n\le k} T(r, f_n))\Big).$$

By Lemma 2.16, there exists a set $E_9 \subset (0, \infty)$ has finite linear measure , and a sequence $\{r_n\}$, $r_n \notin E_9$, such that

$$\lim_{r_n\to\infty} \frac{\log_p T(r_n, A_s)}{\log_q r_n} = \sigma_{(p,q)}(A_s).$$

Set

$$T_n = \{r : r \in (0, \infty) \setminus E_9, \quad T(r, A_s) \le O(\log(T(r, f_n))) \quad (n = 1, 2, \dots, k)$$

By Lemma 2.11, we have $\cup_{n=1}^{k} T_n = (0, \infty) \setminus E_9$. It is easy to see that there exists at least one T_n, say $T_1 \subset (0, \infty) \setminus E_9$, that has infinite linear measure and satisfies

$$T(r, A_s) \le O(\log T(r, f_1)). \tag{3.6}$$

From (3.6), we have $\sigma_{(p+1,q)}(f_1) \ge \sigma_{(p,q)}(A_s)$.

In the first part we have proved that $\sigma_{(p+1,q)}(f_1) \le \sigma_{(p,q)}(A_s)$. Therefore, we have that there is at least one meromorphic solution f_1 satisfies

$$\sigma_{(p+1,q)}(f_1) = \sigma_{(p,q)}(A_s).$$

Proof of Theorem 1.4. Suppose that f is a nonzero meromorphic solution whose poles are of uniformly bounded multiplicities of (1.1), then (1.1) can be written

$$-A_0 = \frac{f^{(k)}}{f} + A_{k-1}\frac{f^{(k-1)}}{f} + \dots + A_1\frac{f'}{f}. \tag{3.7}$$

By the lemma of the logarithmic derivative and (3.7), we have

$$\begin{aligned} m(r, A_0) &\le \sum_{j=1}^{k-1} m(r, A_j) + \sum_{j=1}^{k} m(r, \frac{f^{(j)}}{f}) + O(1) \\ &= \sum_{j=1}^{k-1} m(r, A_j) + O\{\log(rT(r, f))\} \end{aligned} \tag{3.8}$$

holds for all sufficiently large $r \notin E_{10}$, where $E_{10} \subset (0, \infty)$ has finite linear measure. Hence

$$T(r, A_0) = m(r, A_0)+N(r, A_0) \leq N(r, A_0)+\sum_{j=1}^{k-1} m(r, A_j)+O\{\log(rT(r, f))\} \quad (3.9)$$

holds for all sufficiently large $|r| = r \notin E_{10}$.

Since $\max\{\sigma_{(p,q)}(A_j) : j \neq 0\} < \sigma_{(p,q)}(A_0) < \infty$, by Lemma 2.15, there exist a set $E_{11} \subset (1, \infty)$ having infinite logarithmic measure such that for all z satisfying $|z| = r \in E_{11}$, we have

$$\lim_{r\to\infty} \frac{\log_p T(r, A_0)}{\log_q r} = \sigma_{(p,q)}(A_0), \ \frac{m(r, A_j)}{m(r, A_0)} = o(1) \quad (r \in E_{10}, j = 1, 2, \dots, k-1). \quad (3.10)$$

By (3.8) and (3.10), for all sufficiently large $r \in E_{11} \setminus E_{10}$, we have

$$\frac{1}{2} m(r, A_0) \leq O\{\log(rT(r, f))\}. \quad (3.11)$$

Using a similar discussion as in second part of proof of Theorem 1.3, we can get that

$$m(r, A_0) > N(r, A_0), \quad (3.12)$$

hence,

$$T(r, A_0) = m(r, A_0) + N(r, A_0) = O(m(r, A_0)) = O(\log rT(r, f))$$

for all sufficiently large $r \in E_{11} \setminus E_{10}$, this means

$$\sigma_{(p+1,q)}(f) \geq \sigma_{(p,q)}(A_0).$$

On the other hand, by Theorem 1.3, we have

$$\sigma_{(p+1,q)}(f) \leq \sigma_{(p,q)}(A_0).$$

Therefore, every meromorphic solution f whose poles are of uniformly bounded multiplicities of (1.1) satisfies

$$\sigma_{(p+1,q)}(f) = \sigma_{(p,q)}(A_0).$$

Proof of Theorem 1.5. When $A_0, A_1, \dots, A_{k-1}$ satisfy

$$\max\{\sigma_{(p,q)}(A_j) : j \neq 0\} < \sigma_{(p,q)}(A_0),$$

then by Theorem 1.4, it is easy to see that Theorem 1.5 holds. Now we assume that there exists at least one of A_j $(j = 1, 2, \dots, k-1)$ satisfies $\sigma_{(p,q)}(A_j) = \sigma_{(p,q)}(A_0)$.

Suppose that f is a nonzero meromorphic solution of (1.1), we have

$$|A_0| \leq |\frac{f^{(k)}(z)}{f(z)}| + |A_{k-1}||\frac{f^{(k-1)}(z)}{f(z)}| + \dots + |A_1||\frac{f'(z)}{f(z)}|. \quad (3.13)$$

Using a similar discussion as in the proof of Theorem 1.4, we can get that (3.8) and (3.9) hold for all sufficiently large $r \notin E_{12}$, where $E_{12} \subset (0, \infty)$ has finite linear measure. Since

$$\max\{\sigma_{(p,q)}(A_j) : j = 1, 2, \dots, k-1\} = \sigma_{(p,q)}(A_0)$$

and

$$\max\{\tau_{(p,q)}(A_j) : \sigma_{(p,q)}(A_j) = \sigma_{(p,q)}(A_0) > 0\} < \tau_{(p,q)}(A_0),$$

then there exists a set $J \subset \{1, 2, \dots, k-1\}$ such that for $j \in J$, we have $\sigma_{(p,q)}(A_j) = \sigma_{(p,q)}(A_0)$ and $\tau_{(p,q)}(A_j) < \tau_{(p,q)}(A_0)$.

Hence, there exist two constants β_1 and β_2 satisfying $\max\{\tau_{(p,q)} : j \in J\} < \beta_1 < \beta_2 \le \tau_{(p,q)}(A_0)$. By Definitions 2.6 and 2.9, we obtain that

$$m(r, A_j) \le T(r, A_j) < \exp_{p-1}\{\beta_1(\log_{q-1} r)^{\sigma_{(p,q)}(A_0)}\}. \tag{3.14}$$

Since $\lambda_{(p,q)}(\frac{1}{A_0}) < \sigma_{(p,q)}(A_0)$, we have

$$N(r, A_0) \le \exp_p\{(\lambda_(\frac{1}{A_0}) + \epsilon)\log_q r\} \le \exp_{p-1}\{\beta_1(\log_{q-1} r)^{\sigma_{(p,q)}(A_0)}\}. \tag{3.15}$$

By Lemma 2.18, there exists a set of E_{13} having infinite logarithmic measure such that for all $r \in E_{13}$, we have

$$T(r, A_0) \ge \exp_{p-1}\{\beta_2(\log_{q-1} r)^{\sigma_{(p,q)}(A_0)}\}. \tag{3.16}$$

Now, substituting (3.14)-(3.16)into (3.9), we have

$$(1 - o(1))\exp_{p-1}\{\beta_2(\log_{q-1} r)^{\sigma_{(p,q)(A_0)}}\} \le O(\log(rT(r, f)))$$

for all $r \in E_{13} \setminus E_{12}$, this implies

$$\sigma_{(p+1,q)}(f) \ge \sigma_{(p,q)}(A_0).$$

On the other hand ,by Theorem 1.3, we have

$$\sigma_{(p+1,q)}(f) \le \sigma_{(p,q)}(A_0).$$

Then we have that

$$\sigma_{(p+1,q)}(f) = \sigma_{(p,q)}(A_0)$$

holds for any nonzero meromorphic solution f whose poles are of uniformly bounded multiplicities of (1.1).

Proof of Theorem 1.6. Since all solutions of equation (1.2) are meromorphic functions, all solutions of the homogeneous differential equation (1.1) corresponding to equation (1.2) are still meromorphic functions.

Now we assume that $\{f_1, f_2, \dots, f_k\}$ is a meromorphic solution base of (1.1), then by the elementary theory of differential equations (see, e.g. [16]), any solution of (1.2) has the form

$$f = c_1(z)f_1 + c_2(z)f_2 + \dots + c_k(z)f_k, \tag{3.17}$$

where $c_1, c_2, \dots, c_k$ are suitable meromorphic functions satisfying

$$c_j' = FG_j(f_1, f_1, \dots, f_k)W(f_1, f_1, \dots, f_k)^{-1} \quad (j = 1, 2, \dots, k), \tag{3.18}$$

where $G_j(f_1, f_1, \dots, f_k)$ are differential polynomials in $\{f_1, f_2, \dots, f_k\}$ and their derivatives, and $W(f_1, f_1, \dots, f_k)^{-1}$ is the Wronskian of $\{f_1, f_2, \dots, f_k\}$. By Theorem 1.4, we have

$$\sigma_{(p+1,q)}(f_j) = \sigma_{(p,q)}(A_0) \quad (j = 1, 2, \dots, k).$$

By Lemma 2.13, (3.17) and (3.18), we obtain

$$\sigma_{(p+1,q)}(f) \le \max\{\sigma_{(p+1,q)}(f_j), \sigma_{(p+1,q)}(F) : j = 1, 2, \dots, k\} = \sigma_{(p,q)}(A_0).$$

Now we assert that all solutions f of (1.2) satisfy $\sigma_{(p+1,q)}(f) = \sigma_{(p,q)}(A_0)$ with at most one exceptional solution, say f_0, satisfying $\sigma_{(p+1,q)}(f_0) < \sigma_{(p,q)}(A_0)$. In fact, if there exists two distinct meromorphic functions f_0 and f_1 of (1.2) satisfying

$$\sigma_{(p+1,q)}(f_j) < \sigma_{(p,q)}(A_0) \quad (j = 0, 1),$$

then $f = f_0 - f_1$ is a nonzero meromorphic solution of (1.1), and satisfying $\sigma_{(p+1,q)}(f) < \sigma_{(p,q)}(A_0)$, this contradicts Theorem 1.4.

For all the solutions f of (1.2) satisfying $\sigma_{(p+1,q)}(f) = \sigma_{(p,q)}(A_0)$, we have

$$\max\{\sigma_{(p+1,q)}(A_j), \sigma_{(p+1,q)}(F) : j = 0, 1, \dots, k-1\} < \sigma_{(p+1,q)}(f).$$

Thus by Lemma 2.20, we obtain

$$\overline{\lambda}_{(p+1,q)}(f) = \lambda_{(p+1,q)}(f) = \sigma_{(p+1,q)}(f).$$

Therefore, Theorem 1.6 is proved.

3.1. **Proof of Theorem 1.7.** Suppose that $\{g_1, g_2, \dots, g_k\}$ is a meromorphic solution base of (1.1) corresponding to (1.2). By a similar discussion as in the proof of Theorem 1.6, we obtain

$$\sigma_{(p+1,q)}(f) \le \max\{\sigma_{(p+1,q)}(g_j), \sigma_{(p+1,q)}(F) : j = 1, 2, \dots, k\}$$

By the first part of the proof of Theorem 1.3, we can get that

$$\sigma_{(p+1,q)}(g_j) \le \max\{\sigma_{(p,q)}(A_j) : j = 0, 1, \dots, k-1\} \le \sigma_{(p+1,q)}(F),$$

then we can get

$$\sigma_{(p+1,q)}(f) \le \sigma_{(p+1,q)}(F). \tag{3.19}$$

On the other hand, by the simple order comparison from (1.2), we have

$$\sigma_{(p+1,q)}(F) \le \max\{\sigma_{(p+1,q)}(A_j), \sigma_{(p+1,q)}(f) : j = 0, 1, \dots, k-1\}.$$

Since $\sigma_{(p+1,q)}(A_j) < \sigma_{(p+1,q)}(F)$, we have

$$\sigma_{(p+1,q)}(F) \le \sigma_{(p+1,q)}(f). \tag{3.20}$$

By (3.19)-(3.20), we obtain

$$\sigma_{(p+1,q)}(F) = \sigma_{(p+1,q)}(f).$$

Therefore, the proof of Theorem 1.7 is complete.

Proof of Theorem 1.8. (i) By the simple order comparison from (1.2) it is easy to see that all meromorphic solutions of (1.2) satisfy

$$\sigma_{(p+1,q)}(f) \ge \sigma_{(p+1,q)}(F).$$

On the other hand, by the similar proof in (3.17)-(3.18), we obtain that all meromorphic solutions of (1.2) satisfy

$$\sigma_{(p+1,q)}(f) \le \sigma_{(p+1,q)}(F)$$

if $\sigma_{(p+1,q)}(F) \ge \alpha_1$. Therefore, all meromorphic solutions whose poles are of uniformly bounded multiplicities of (1.2) satisfy

$$\sigma_{(p+1,q)}(f) = \sigma_{(p+1,q)}(F).$$

(ii) By the hypotheses that

$$|A_0(z)| \ge \exp_{p+1}\{(\alpha_1 - \epsilon)\log_q r\},$$

and $|A_j(z)| \le \exp_{p+1}\{\alpha_2 \log_q r\}$, we can easily obtain that $\sigma_{(p+1,q)}(A_0) = \alpha_1$. Since $\sigma_{(p+1,q)}(F) < \alpha_1 = \sigma_{(p+1,q)}(A_0)$, by the similar proof in Theorem 1.6, we obtain that all meromorphic solutions whose poles are of uniformly bounded multiplicities of (1.2) satisfy

$$\overline{\lambda}_{(p+1,q)}(f) = \lambda_{(p+1,q)}(f) = \sigma_{(p+1,q)}(f) = \alpha_1$$

with at most one exceptional solution f_2 satisfying $\sigma_{(p+1,q)}(f_2) < \alpha_1$. Therefore, we completely prove Theorem 1.8.

Acknowledgements. This research was supported by grants 11101201 from the NSFC, and 2010GQS0139 from the NSF of Jiangxi of China. The authors would like to thank the anonymous referee for making valuable suggestions and comments to improve the present paper.

References

[1] B. Belaïdi; *On the iterated order and the fixed points of entire solutions of some complex linear differential equations*, Electron. J. Qual. Theory Differ. Equ. 2006, No. 9, 1-11.

[2] B. Belaïdi; *Growth and oscillation of solutions to linear differential equations with entire coefficients having the same order*, Electron. J. Diff. Equ. 2009(2009), No. 70, 1-10.

[3] B. Belaïdi; *Growth of solutions to linear differential equations with analytic coefficients of $[p,q]$-order in the unit disc*, Electron. J. Diff. Equ. 2011(2011), No. 156, 1-11.

[4] L. G. Bernal; *On growth k-order of solutions of a complex homogeneous linear differential equations*, Proc. Amer. Math. Soc. 101 (1987) 317-322.

[5] T. B. Cao, J. F. Xu, Z. X. Chen; *On the meromorphic solutions of linear differential equations on the complex plane*, J. Math. Anal. Appl. 364 (2010) 130-142.

[6] Z. X. Chen, C. C. Yang; *Quantitative estimations on the zeros and growths of entire solutions of linear differential equations*, Complex Variables 42 (2000) 119-133.

[7] Y. M. Chiang, W. K. Hayman; *Estimates on the growth of meomorphic solutions of linear differertial equations*, Comment. Math. Helv. 79 (2004) 451-470.

[8] G. Frank, S. Hellerstein; *On the meromorphic solutions of non-homogeneous linear differential equations with polynomial coefficients*, Proc. London Math. Soc. 53 (3) (1986) 407-428.

[9] G. G. Gundersen; *Estimates for the logarithmic derivate of a meromorphic function, plus similar estimates*, J. London Math. Soc. 37 (2) (1988) 88-104.

[10] G. G. Gundersen; *Finite order solutions of second order linear differential equations*, Trans. Amer. Math. Soc. 305 (1988), No.1, 415-429.

[11] W. K. Hayman; *Meromorphic Functions, Clarendon Press*, Oxford, 1964.

[12] G. Jank, L. Volkman; *Meromorphe Funktionen und Differentialgleichungen*, Birkäuser, 1985.

[13] O. P. Juneja, G. P. Kapoor, S. K. Bajpai; *On the (p,q)-order and lower (p,q)-order of an entire function*, J. Reine Angew. Math. 282 (1976), 53-57.

[14] O. P. Juneja, G. P. Kapoor, S. K. Bajpai; *On the (p,q)-type and lower (p,q)-type of an entire function*, J. Reine Angew. Math. 290 (1977), 180-190.

[15] L. Kinnunen; *Linear differential equations with solutions of finite iterated order*, Southeast Asian Bull. Math. 22 (1998), 385-405.

[16] I. Laine; *Nevanlinna Theory and Complex Differential Equations*, W. de Gruyter, Berlin, 1993.

[17] J. Liu, J. Tu, L. Z. Shi; *Linear differential equations with entire coefficients of (p,q)-order in the complex plane*, J. Math. Anal. Appl. 372(2010), No. 1, 55-67.

[18] D. Sato; *On the rate of growth of entire functions of fast growth*, Bull. Amer. Math. Soc. 69 (1963) 411-414.

[19] J. Tu, C. F. Yi; *On the growth of solutions of a class of higher order linear differential equations with coefficients having the same order*, J. Math. Anal. Appl. 340 (2008) 487-497.

[20] L. Yang; *Value Distribution Theory*, Springer-Verlag, Berlin, 1993, and Science Press, Beijing, 1982.

Lei-Min Li

Department of Mathematics, Nanchang University, Nanchang, Jiangxi 330031, China

E-mail address: leiminli@hotmail.com

Ting-Bin Cao

Department of Mathematics, Nanchang University, Nanchang, Jiangxi 330031, China

E-mail address: tbcao@ncu.edu.cn, tingbincao@hotmail.com (corresponding author)