Electronic Journal of Differential Equations, Vol. 2012 (2012), No. 197, pp. 1–13.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

TRAVELING WAVES IN A DIFFUSIVE PREDATOR-PREY MODEL WITH GENERAL FUNCTIONAL RESPONSE

ZHAOQUAN XU, PEIXUAN WENG

Abstract. This article concerns the existence of traveling waves in a diffusive predator-prey model with general functional response. By applying the Schauder fixed theorem, we establish existence results of traveling wave solutions. The results are then applied to the predator-prey model with Holling type-II response. Our results indicate that there is a transition zone moving from the state with no species to the coexistence state of both species.

1. Introduction

Dynamical relations among species can be very complicated. Due to their presence in natural environments, various types of predator-prey models have been widely studied, for example, see [1]–[18]. Nonlinear reaction-diffusion equations describe the dynamical relationship between predator and prey. In many situations, traveling waves determine the long term behavior of predator and prey.

Fundamental and important predator-prey models with diffusion are given by:

$$\begin{gathered} u_t = D_1 u_{xx} + Au(1-\frac{u}{K}) - UW, \\ w_t = D_2 w_{xx} - Cw + Duw; \end{gathered} \tag{1.1}$$

$$\begin{gathered} u_t = D_1 u_{xx} + Au(1-\frac{u}{K}) - B\frac{uw}{1+Eu}, \\ w_t = D_2 w_{xx} - Cw + D\frac{uw}{1+Eu}; \end{gathered} \tag{1.2}$$

and

$$\begin{gathered} u_t = D_1 u_{xx} + Au(1-\frac{u}{K}) - B\frac{u^2 w}{1+Eu^2}, \\ w_t = D_2 w_{xx} - Cw + D\frac{u^2 w}{1+Eu^2}; \end{gathered} \tag{1.3}$$

where $u(t,x)$, $w(t,x)$ are the density functions of prey and predator, respectively; $D_1 > 0$ and $D_2 > 0$ represent the diffusive rates; A is the growth factor for the prey species, $K > 0$ is the carrying capacity of prey species, $C > 0$ is the death rate for the predator in the absence of prey. For more details about the biological meaning of the parameters, we refer the readers to [4, 7, 9, 17].

2000 *Mathematics Subject Classification.* 34C12, 34C37, 35K57, 92D25.
Key words and phrases. Traveling waves; general functional response;
Schauder's fixed theorem.

Submitted February 20, 2012. Published November 10, 2012.

System (1.1) is the familiar Lotka-Volterra model (with Holling type-I functional response) and the systems (1.2), (1.3) have the Holling type-II and Holling type-III functional response, respectively. In [1, 2, 3], Dunbar obtained the existence of several kinds of traveling wave solutions for diffusive predator-prey systems with type I and type II functional responses ($D_1 = 0$ [1] and $D_1 \neq 0$ [2] for (1.1), $D_1 = 0$ [3] for (1.2)). He considered the existence of small amplitude periodic traveling waves, and "heteroclinic traveling waves" that correspond to heteroclinic orbits connecting two equilibria (point-to-point) or an equilibrium and a periodic orbit (point-to-periodic). The methods used by Dunbar include the invariant manifold theory, the shooting method, Hopf bifurcation analysis, and LaSalle's invariance principle. Huang, Lu & Ruan [9] extended the work in [2] to $\mathbb{R}^4$ ($D_1 \neq 0$ for (1.2)) using Dunbar's method in [2]. An interesting question is whether those results can be extended to a system with type III functional response. Recently, Li & Wu [12] proved the existence of traveling waves in a diffusive predator-prey system (1.3) with $D_1 = 0$ by employing a method similar to that used in [1, 2]. We emphasize that in [9] and [12] only heteroclinic orbits connecting equilibrium-to-equilibrium (point-to-point) are considered.

The shooting method used by Dunbar is based on a variant of Wazewski's theorem [1, 2, 3]. In Dunbar and Wazewski set $\mathbb{W}$, there is an orbit starting at the unstable manifold of an equilibrium that stays in $\mathbb{W}$ in the future. However, the Wazewski set $\mathbb{W}$ constructed in [1, 2, 3] is unbounded. To ensure the boundedness of the orbit, several additional lemmas were proved to rule out the possibility that the constructed orbit may escape to infinity. The use of unbounded sets $\mathbb{W}$ in $\mathbb{R}^3$ or $\mathbb{R}^4$ makes the argument long and hard to read. In a recent work, Lin, Weng & Wu[14] constructed a simple bounded Wazewski set $\mathbb{W}$ and use the original Wazewski's theorem to simplify the proof of the existence of heteroclinic traveling waves connecting two equilibria related to the following predator-prey system with Sigmoidal response function:

$$\begin{aligned} \frac{\partial u}{\partial t} &= ru(1-\frac{u}{K}) - \frac{u^2}{a_1+b_1u+u^2}v \\ \frac{\partial v}{\partial t} &= D_2\frac{\partial^2 v}{\partial x^2} + v\Big(\frac{\alpha u^2}{a_1+b_1u+u^2} - e\Big). \end{aligned} \tag{1.4}$$

Liang, Weng and Wu [13] considered the delayed diffusive predator-prey system

$$\begin{aligned} \frac{\partial u(t,x)}{\partial t} &= Au(t,x)\Big(1-\frac{u(t,x)}{K}\Big) - B\frac{u(t-\tau,x)w(t,x)}{1+Eu(t-\tau,x)}, \\ \frac{\partial w(t,x)}{\partial t} &= D_2\frac{\partial^2 w(t,x)}{\partial x^2} - Cw(t,x) + D\frac{u(t-\tau,x)w(t,x)}{1+Eu(t-\tau,x)}, \end{aligned} \tag{1.5}$$

where $\tau \geq 0$ measures the retarded response of growth for the prey species or the time for the prey species taken from birth to maturity. They proved the existence of small amplitude periodic traveling wave solutions of (1.5) for small $\tau > 0$. Furthermore, they developed a new method for combining the singular limit argument and the singular perturbation technique to establish the existence of the point-to-periodic traveling wave solutions for (1.5) with small delay $\tau > 0$, and also proved the existence of point-to-point traveling wave solutions for the any given $\tau > 0$.

It is very interesting to develop simpler methods to treat the problem of traveling waves for diffusive predator-prey systems. Recently, Lin et al [11] studied the existence of point-to-point traveling wave solutions of the following Lotka-Volterra system:

$$\begin{aligned} \frac{\partial u_1(t,x)}{\partial t} &= d_1 \Delta u_1(x,t) + r_1 u_1 [1 - a_{11} u_1(t-\tau_1, x) - a_{12} u_2(t-\tau_2, x)], \\ \frac{\partial u_2(t,x)}{\partial t} &= d_2 \Delta u_2(x,t) + r_2 u_2 [1 + a_{21} u_1(t-\tau_3, x) - a_{22} u_2(t-\tau_4, x)] \end{aligned} \tag{1.6}$$

by introducing the mixed quasi-monotone condition(MQM) and the exponentially mixed quasi-monotone condition(EMQM).

Motivated by the work in [11], in the present article, we consider the existence of traveling wave solutions of the following predator-prey system with general functional response:

$$\begin{aligned} \frac{\partial u}{\partial t} &= D_1 \frac{\partial^2 u}{\partial x^2} + h_1(u) - f(u)w, \\ \frac{\partial w}{\partial t} &= D_2 \frac{\partial^2 w}{\partial x^2} + h_2(w) + \mu f(u)w, \end{aligned} \tag{1.7}$$

where $D_1 > 0$, $D_2 > 0$ are the diffusive rates of the prey and predator, respectively. Also $h_1(u)$ denotes the growth function of prey which is a positive function within the maximal carrying capacity of the prey, and $h_2(v)$ denotes the growth function of predator. If the predator only depends on the prey given in (1.6), then $h_2(v)$ is a negative function. The function f denotes the predator response function.

For the functions h_1, h_2 and f, we make assumptions as follows.

- (H1) There exist two positive numbers u_0,w_0 such that $h_1(u_0) - f(u_0)w_0 = 0$, $h_2(w_0) + \mu f(u_0)w_0 = 0$, and $f(0) = h_1(0) = h_2(0) = 0$;
- (H2) f, h_1 and h_2 are Lipschitz continuous functions on any compact interval;
- (H3) f is nondecreasing on $[0, +\infty)$.

Remark 1.1. The hypothesis (H1) guarantees that $(0,0)$ is a steady state for the system (1.6) and it has another positive steady state (u_0, w_0). Moreover all the response functions in (1.1)-(1.3) satisfy the conditions (H2) and (H3). On the other hand, (H1) and (H3) imply that $f(u) \geq 0$ for $u \in \mathbb{R}$.

The rest of the paper is organized as follows. In section 2, some preliminaries are given. In section 3, we show the main results on the existence of traveling wave solutions for (1.6). In the last section, as an application of our main results, we shall establish the existence results of traveling wave solutions for system

$$\begin{aligned} u_t &= D_1 u_{xx} + \alpha u(\beta - u) - w f(u), \\ w_t &= D_2 w_{xx} + \gamma w(\delta - w) + \mu w f(u), \end{aligned} \tag{1.8}$$

with $f(u) = \frac{u}{1+u}$.

2. Preliminaries

In this article, we adopt the usual notation for the standard partial ordering in $\mathbb{R}^2$; i.e., if $a_1 \leq a_2$ and $b_1 \leq b_2$, we say that $(a_1, b_1) \leq (a_2, b_2)$. Let $|\cdot|$ denote the Euclidean norm in $\mathbb{R}^2$ and $\|\cdot\|$ denote the supermum norm in space $C(\mathbb{R}, \mathbb{R}^2)$.

A traveling wave solution of (1.6) is a solution with the form $(u(t,x), w(t,x)) = (\varphi(x+ct), \psi(x+ct))$, where $(\phi, \psi) \in C^2(\mathbb{R}, \mathbb{R}^2)$ is the wave profile which propagates at a constant velocity $c > 0$.

We study traveling wave solutions of (1.6) that connect $(0,0)$ and (u_0, v_0). By substituting such (φ,ψ) into (1.6) and replacing $x+ct$ by t, we know that (φ,ψ) satisfy the wave profile system

$$\begin{aligned} c\varphi'(t) &= D_1\varphi''(t) + h_1(\varphi(t)) - f(\varphi(t))\psi(t), \\ c\psi'(t) &= D_2\psi''(t) + h_2(\psi(t)) + \mu f(\varphi(t))\psi(t) \end{aligned} \tag{2.1}$$

accompanied with asymptotic boundary conditions

$$\lim_{t\to-\infty}(\varphi(t),\psi(t)) = (0,0), \quad \lim_{t\to+\infty}(\varphi(t),\psi(t)) = (u_0,w_0). \tag{2.2}$$

If, for some $c>0$, system (2.1) has a solution $(\varphi(t),\psi(t))$ satisfying the asymptotic boundary conditions (2.2), then $(u(t,x),v(t,x)) = (\varphi(x+ct),\psi(x+ct))$ is the traveling wave solution of system (1.6).

Let

$$C_{[0,K]}(\mathbb{R},\mathbb{R}^2) = \{(\varphi,\psi)\in C(\mathbb{R},\mathbb{R}^2) : 0 \le (\varphi,\psi)(t) \le K \text{ for } t\in\mathbb{R}\},$$

where $K=(k_1,k_2)$ is some constant vector such that $(u_0,w_0)\le(k_1,k_2)$.

For $(\varphi,\psi)\in C_{[0,K]}(\mathbb{R},\mathbb{R}^2)$, define the operator $Q=(Q_1,Q_2): C_{[0,K]}(\mathbb{R},\mathbb{R}^2)\to C(\mathbb{R},\mathbb{R}^2)$ by

$$\begin{aligned} Q_1(\varphi,\psi)(t) &= d_1\varphi(t) + h_1(\varphi(t)) - f(\varphi(t))\psi(t), \\ Q_2(\varphi,\psi)(t) &= d_2\psi(t) + h_2(\psi(t)) + \mu f(\varphi(t))\psi(t), \end{aligned} \tag{2.3}$$

where $d_1 = L_{h_1} + k_2L_f$, $d_2 = L_{h_2}$, L_{h_2} is the Lipschitz constant of h_2 on $[0,k_2]$ and L_f, L_{h_1} are the Lipschitz constants of f, h_1 on $[0,k_1]$, respectively. Hence, (2.1) is equivalent to

$$\begin{aligned} c\varphi'(t) &= D_1\varphi''(t) - d_1\varphi(t) + Q_1(\varphi,\psi)(t), \\ c\psi'(t) &= D_2\psi''(t) - d_2\psi(t) + Q_2(\varphi,\psi)(t)\,. \end{aligned} \tag{2.4}$$

Let

$$r_{i1} = \frac{c-\sqrt{c^2+4D_id_i}}{2D_i}, \quad r_{i2} = \frac{c+\sqrt{c^2+4D_id_i}}{2D_i}, \quad i=1,2.$$

Clearly, we have $r_{i1} < 0 < r_{i2}$ and

$$D_ir_{ij}^2 - cr_{ij} - d_i = 0, \quad i,j=1,2.$$

For $(\varphi,\psi)\in C_{[0,K]}(\mathbb{R},\mathbb{R}^2)$, define an operator $P=(P_1,P_2): C_{[0,K]}(\mathbb{R},\mathbb{R}^2)\to C(\mathbb{R},\mathbb{R}^2)$ by

$$\begin{aligned} P_1(\varphi,\psi)(t) &= \frac{1}{D_1(r_{12}-r_{11})}\Big[\int_{-\infty}^t e^{r_{11}(t-s)} + \int_t^{+\infty} e^{r_{12}(t-s)}\Big]Q_1(\varphi,\psi)(s)ds, \\ P_2(\varphi,\psi)(t) &= \frac{1}{D_2(r_{22}-r_{21})}\Big[\int_{-\infty}^t e^{r_{21}(t-s)} + \int_t^{+\infty} e^{r_{22}(t-s)}\Big]Q_2(\varphi,\psi)(s)ds. \end{aligned} \tag{2.5}$$

Note that fixed points of P are solutions to (2.1). Therefore, to prove the existence of traveling wave solutions of (1.3) connecting $(0,0)$ and (u_0,w_0), it is sufficient to consider fixed points of P that satisfy the asymptotic boundary conditions (2.2).

3. Main results

We first give the definition of upper-lower solutions of (2.1) which is crucial in proving our main results.

Definition 3.1. A pair of continuous functions $\overline{\Phi} = (\overline{\varphi}, \overline{\psi})$ and $\underline{\Phi} = (\underline{\varphi}, \underline{\psi}) \in C_{[0,K]}(\mathbb{R}, \mathbb{R}^2)$ is called an upper solution and a lower solution of (2.1), respectively, if $(\overline{\varphi}'(t), \overline{\psi}'(t))$, $(\underline{\varphi}''(t), \underline{\psi}''(t))$ exist and bounded on $R \setminus \Upsilon$ and satisfy

$$D_1\overline{\varphi}''(t) - c\overline{\varphi}'(t) + h_1(\overline{\varphi}(t)) - f(\overline{\varphi}(t))\underline{\psi}(t) \le 0,$$
$$D_2\overline{\psi}''(t) - c\overline{\psi}'(t) + h_2(\overline{\psi}(t)) + \mu f(\overline{\varphi}(t))\overline{\psi}(t) \le 0,$$

and

$$D_1\underline{\varphi}''(t) - c\underline{\varphi}'(t) + h_1(\underline{\varphi}(t)) - f(\underline{\varphi}(t))\overline{\psi}(t) \ge 0,$$
$$D_2\underline{\psi}''(t) - c\underline{\psi}'(t) + h_2(\underline{\psi}(t)) + \mu f(\underline{\varphi}(t))\underline{\psi}(t) \ge 0$$

for $R \setminus \Upsilon$, where $\Upsilon = \{t_1, t_2, \dots, t_n\}$, with $t_1 < t_2 < \dots < t_n$, is a finite set of points.

In what follows, we assume that (2.1) admits an upper solution $\overline{\Phi} = (\overline{\varphi}, \overline{\psi})$ and a lower solution $\underline{\Phi} = (\underline{\varphi}, \underline{\psi})$ such that

- (G1) $(0,0) \le (\underline{\varphi}, \underline{\psi})(t) \le (\overline{\varphi}, \overline{\psi})(t) \le (k_1, k_2)$, $t \in \mathbb{R}$;
- (G2) $\lim_{t\to-\infty}(\overline{\varphi}, \overline{\psi})(t) = (0,0)$,
  $\lim_{t\to+\infty}(\underline{\varphi}, \underline{\psi})(t) = \lim_{t\to+\infty}(\overline{\varphi}, \overline{\psi})(t) = (u_0, w_0)$;
- (G3) $(\overline{\varphi}', \overline{\psi}')(t_i^+) \le (\overline{\varphi}', \overline{\psi}')(t_i^-)$, $(\underline{\varphi}', \underline{\psi}')(t_i^+) \ge (\underline{\varphi}', \underline{\psi}')(t_i^-)$.

Lemma 3.2. *If $\Phi_1 = (\varphi_1, \psi_1)$, $\Phi_2 = (\varphi_2, \psi_2) \in C(\mathbb{R}, \mathbb{R}^2)$ with $0 \le \Phi_2(t) \le \Phi_1(t) \le K$, $t \in \mathbb{R}$, then*

1. $Q_1(\varphi_2, \psi_1)(t) \le Q_1(\varphi_1, \psi_2)(t)$, $P_1(\varphi_2, \psi_1)(t) \le P_1(\varphi_1, \psi_2)(t)$,
2. $Q_2(\varphi_2, \psi_2)(t) \le Q_2(\varphi_1, \psi_1)(t)$, $P_2(\varphi_2, \psi_2)(t) \le P_2(\varphi_1, \psi_1)(t)$.

Proof. From the definition of Q, we have

$$\begin{aligned}
&Q_1(\varphi_1, \psi_2)(t) - Q_1(\varphi_2, \psi_1)(t) \\
&= d_1(\varphi_1(t) - \varphi_2(t)) + [h_1(\varphi_1(t)) - h_1(\varphi_2(t))] - f(\varphi_1(t))\psi_2(t) + f(\varphi_2(t))\psi_1(t) \\
&= d_1(\varphi_1(t) - \varphi_2(t)) + [h_1(\varphi_1(t)) - h_1(\varphi_2(t))] - f(\varphi_1(t))[\psi_2(t) - \psi_1(t)] \\
&\quad - \psi_1(t)[f(\varphi_1(t)) - f(\varphi_2(t))] \\
&\ge (d_1 - L_{h_1} - k_2 L_f)(\varphi_1(t) - \varphi_2(t)) - f(\varphi_1(t))[\psi_2(t) - \psi_1(t)] \ge 0,
\end{aligned}$$

$$\begin{aligned}
&Q_2(\varphi_1, \psi_1)(t) - Q_2(\varphi_2, \psi_2)(t) \\
&= d_2(\psi_1(t) - \psi_2(t)) + [h_2(\psi_1(t)) - h_2(\psi_2(t))] + \mu f(\varphi_1(t))\psi_1(t) - \mu f(\varphi_2(t))\psi_2(t) \\
&= d_2(\psi_1(t) - \psi_2(t)) + [h_2(\psi_1(t)) - h_2(\psi_2(t))] + \mu f(\varphi_1(t))[\psi_1(t) - \psi_2(t)] \\
&\quad + \mu\psi_2(t)[f(\varphi_1(t)) - f(\varphi_2(t))] \\
&\ge (d_2 - L_{h_2})(\psi_1(t) - \psi_2(t)) + \mu\psi_2(t)[f(\varphi_1(t)) - f(\varphi_2(t))] \ge 0.
\end{aligned}$$

A similar argument leads to the inequalities about P. We omit the details. □

Define a set

$$\Omega = \{\Phi = (\varphi, \psi) \in C_{[0,K]}(\mathbb{R}, \mathbb{R}^2) : (\underline{\varphi}, \underline{\psi}) \le (\varphi, \psi) \le (\overline{\varphi}, \overline{\psi})\}.$$

Clearly, Ω is nonempty, bounded, closed and convex subset of $C(\mathbb{R},\mathbb{R}^2)$ with respect to the norm $\|\cdot\|$.

Lemma 3.3. *The operator $P=(P_1,P_2): C_{[0,K]}(\mathbb{R},\mathbb{R}^2)\to C(\mathbb{R},\mathbb{R}^2)$ is continuous with respect to the norm $\|\cdot\|$.*

Proof. For any $\Phi_1=(\varphi_1,\psi_1)$, $\Phi_2=(\varphi_2,\psi_2)\in C_{[0,K]}(\mathbb{R},\mathbb{R}^2)$, we have

$$\begin{aligned}
&|Q_1(\varphi_1,\psi_1)(t)-Q_1(\varphi_2,\psi_2)(t)|\\
&=|d_1(\varphi_1(t)-\varphi_2(t))+[h_1(\varphi_1(t))-h_1(\varphi_2(t))]-f(\varphi_1(t))\psi_1(t)+f(\varphi_2(t))\psi_2(t)|\\
&=|d_1(\varphi_1(t)-\varphi_2(t))+[h_1(\varphi_1(t))-h_1(\varphi_2(t))]-f(\varphi_1(t))[\psi_1(t)-\psi_2(t)]\\
&\quad-\psi_2(t)[f(\varphi_1(t))-f(\varphi_2(t))]|\\
&\le(d_1+L_{h_1}+k_2L_f)|\varphi_1(t)-\varphi_2(t)|+f(k_1)|\psi_1(t)-\psi_2(t)|
\end{aligned}$$

which implies

$$\sup_{t\in\mathbb{R}}|Q_1(\varphi_1,\psi_1)(t)-Q_1(\varphi_2,\psi_2)(t)|\ \to 0\quad \text{as } \|\Phi_1-\Phi_2\|\to 0.$$

By the definition of P, we have

$$\begin{aligned}
&|P_1(\varphi_1,\psi_1)(t)-P_1(\varphi_2,\psi_2)(t)|\\
&=\frac{1}{D_1(r_{12}-r_{11})}\Big[\int_{-\infty}^{t}e^{r_{11}(t-s)}+\int_t^{+\infty}e^{r_{12}(t-s)}\Big]\\
&\quad\times\big|Q_1(\varphi_1,\psi_1)(s)-Q_1(\varphi_2,\psi_2)(s)\big|ds\\
&\le\frac{1}{D_1(r_{12}-r_{11})}\sup_{s\in\mathbb{R}}\big|Q_1(\varphi_1,\psi_1)(s)-Q_1(\varphi_2,\psi_2)(s)\big|\\
&\quad\times\Big[\int_{-\infty}^{t}e^{r_{11}(t-s)}ds+\int_t^{+\infty}e^{r_{12}(t-s)}ds\Big]\\
&=\frac{-1}{D_1r_{11}r_{12}}\sup_{s\in\mathbb{R}}|Q_1(\varphi_1,\psi_1)(s)-Q_1(\varphi_2,\psi_2)(s)|\\
&=\frac{1}{d_1}\|Q_1(\varphi_1,\psi_1)-Q_1(\varphi_2,\psi_2)\|.
\end{aligned}$$

Therefore,

$$\sup_{t\in\mathbb{R}}|P_1(\varphi_1,\psi_1)(t)-P_1(\varphi_2,\psi_2)(t)|\ \to 0 \text{ if } \|\Phi_1-\Phi_2\|\to 0.$$

which implies P_1 is continuous. In a similar way, we can get that P_2 is also continuous. □

Lemma 3.4. *For P and Ω as above, $P(\Omega)\subset\Omega$.*

Proof. For any $\Phi=(\varphi,\psi)\in\Omega$, we have from Lemma 3.2 that

$$\begin{gathered}
Q_1(\underline{\varphi},\overline{\psi})(t)\le Q_1(\varphi,\psi)(t)\le Q_1(\overline{\varphi},\underline{\psi})(t),\\
Q_2(\underline{\varphi},\underline{\psi})(t)\le Q_2(\varphi,\psi)(t)\le Q_2(\overline{\varphi},\overline{\psi})(t),\\
P_1(\underline{\varphi},\overline{\psi})(t)\le P_1(\varphi,\psi)(t)\le P_1(\overline{\varphi},\underline{\psi})(t),\\
P_2(\underline{\varphi},\underline{\psi})(t)\le P_2(\varphi,\psi)(t)\le P_2(\overline{\varphi},\overline{\psi})(t).
\end{gathered}\tag{3.1}$$

Now, it is sufficient to show that

$$\begin{aligned} &\underline{\varphi}(t) \le P_1(\underline{\varphi}, \overline{\psi})(t) \le P_1(\overline{\varphi}, \underline{\psi})(t) \le \overline{\varphi}(t)\,, \\ &\underline{\psi}(t) \le P_2(\underline{\varphi}, \underline{\psi})(t) \le P_2(\overline{\varphi}, \overline{\psi})(t) \le \overline{\psi}(t)\,. \end{aligned} \tag{3.2}$$

According to the definitions of upper-lower solutions and the operator P, we have that

$$Q_1(\underline{\varphi}, \overline{\psi})(t) \ge d_1\underline{\varphi}(t) + c\underline{\varphi}'(t) - D_1\underline{\varphi}''(t), \quad t \in \mathbb{R} \setminus \Upsilon.$$

Let $t_0 = -\infty$ and $t_{n+1} = +\infty$, then for $t_{k-1} < t < t_k$ with $k = 1, 2, \dots, n+1$, we have from (G3) that

$$\begin{aligned} &P_1(\underline{\varphi}, \overline{\psi})(t) \\ &= \frac{1}{D_1(r_{12} - r_{11})}\Big[\int_{-\infty}^{t} e^{r_{11}(t-s)} + \int_{t}^{+\infty} e^{r_{12}(t-s)}\Big] Q_1(\underline{\varphi}, \overline{\psi})(s)ds \\ &\ge \frac{1}{D_1(r_{12} - r_{11})}\Big[\int_{-\infty}^{t} e^{r_{11}(t-s)} + \int_{t}^{+\infty} e^{r_{12}(t-s)}\Big] (d_1\underline{\varphi}(s) + c\underline{\varphi}'(s) - D_1\underline{\varphi}''(s))ds \\ &= \underline{\varphi}(t) + \frac{1}{r_{12} - r_{11}}\Big[\sum_{i=1}^{k} e^{r_{11}(t-t_i)}(\underline{\varphi}'(t_i^+) - \underline{\varphi}'(t_i^-)) \\ &\quad + \sum_{i=k+1}^{n} e^{r_{12}(t-t_i)}(\underline{\varphi}'(t_i^+) - \underline{\varphi}'(t_i^-))\Big] \\ &\ge \underline{\varphi}(t) \quad \text{for } t \in \mathbb{R} \setminus \Upsilon. \end{aligned}$$

By the continuity of $P_1(\underline{\varphi}, \overline{\psi})(t)$ and $\underline{\varphi}(t)$, we obtain

$$\underline{\varphi}(t) \le P_1(\underline{\varphi}, \overline{\psi})(t) \quad \text{for } t \in \mathbb{R}.$$

In a similar way, we can show that (3.2) holds for $t \in \mathbb{R}$. □

Lemma 3.5. *The operator $P : \Omega \to \Omega$ is compact with respect to the norm $\|\cdot\|$.*

The proof of of the above lemma is similar to that of [11, Lemma 3.5]; since it is independent of the monotone condition, so we omit it here. Now, we are in a position to state and prove our main results.

Theorem 3.6. *Assume* (H1)–(H3) *hold. If* (2.1) *has a pair of upper-lower solutions $\overline{\Psi} = (\overline{\varphi}, \overline{\psi})$ and $\underline{\Psi} = (\underline{\varphi}, \underline{\psi})$ satisfying (G1)-(G3). Then* (1.6) *admits a traveling wave solution connecting* $(0, 0)$ *and* (u_0, w_0).

Proof. By Lemma 3.3-3.5 and the Schauder's fixed point theorem, we know that the operator P admits a fixed point $(\varphi*, \psi^*) \in \Omega$ which is a solution of (2.1). Noting the fact that

$$(\underline{\varphi}, \underline{\psi}) \le (\varphi^*, \psi^*) \le (\overline{\varphi}, \overline{\psi}),$$

then we have from (G2) that

$$\lim_{t\to-\infty}(\varphi^*, \psi^*) = (0, 0) \quad \text{and} \quad \lim_{t\to+\infty}(\varphi^*, \psi^*) = (u_0, w_0).$$

Therefore, the fixed point (φ^*, ψ^*) satisfies the asymptotic boundary condition (2.2), and thus it is a traveling wave solution of (1.6) connecting $(0, 0)$ and (u_0, w_0). □

4. Applications

In this section we apply our results in section 3 to establish the existence of traveling wave solution for (1.8) with $f(u) = \frac{u}{1+u}$. In view of Theorem 3.6, the key point is to construct a pair of upper-lower solutions satisfying (G1)–(G3).

Example 4.1. Consider the existence of traveling wave solution for the system

$$\begin{aligned} u_t &= D_1 u_{xx} + \alpha u(\beta - u) - \frac{uw}{1+u}, \\ w_t &= D_2 w_{xx} + \gamma w(\delta - w) + \frac{\mu uw}{1+u}. \end{aligned} \tag{4.1}$$

We are interested in the co-existence of species, so we assume that (4.1) has a unique positive equilibrium (u_0, w_0) satisfying

$$\alpha\beta - \alpha u_0 - \frac{w_0}{1+u_0} = 0, \quad \gamma\delta - \gamma w_0 + \frac{\mu u_0}{1+u_0} = 0. \tag{4.2}$$

It is clear that $\gamma w_0 > \frac{\mu u_0}{1+u_0}$. Moreover, for the technique reason, we assume that

$$\alpha u_0 > 2w_0. \tag{4.3}$$

Clearly, the wave system corresponding to (4.1) is

$$\begin{aligned} c\varphi' &= D_1\varphi'' + \alpha\varphi(\beta - \varphi) - \frac{\varphi\psi}{1+\varphi}, \\ c\psi' &= D_2\psi' + \gamma\psi(\delta - \psi) + \frac{\mu\varphi\psi}{1+\varphi}. \end{aligned} \tag{4.4}$$

As mentioned above, we are interested in the solution of (4.4) with asymptotic boundary conditions

$$\lim_{t\to-\infty}(\varphi(t), \psi(t)) = (0, 0), \quad \lim_{t\to+\infty}(\varphi(t), \psi(t)) = (u_0, w_0). \tag{4.5}$$

In this example, we choose $k_1 = \beta$, $k_2 = \delta + \frac{\mu\beta}{\gamma(1+\beta)}$, then we have $k_1 > u_0$ and $k_2 > w_0$. Let $c > c^* := \max\{2\sqrt{D_1\alpha k_1}, 2\sqrt{D_2\gamma k_2}\}$, then there exist

$$0 < \lambda_{11} < \lambda_{12}, \quad 0 < \lambda_{21} < \lambda_{22}$$

such that

$$D_1\lambda_{1i} - c\lambda_{1i} + \alpha k_1 = 0, \quad D_2\lambda_{2i} - c\lambda_{2i} + \gamma k_2 = 0, \quad i = 1, 2.$$

Since $\gamma w_0 > \frac{\mu u_0}{1+u_0}$, $\alpha u_0 > 2w_0$, there exist $\varepsilon_1 \in (0, u_0)$, $\varepsilon_2 \in (0, w_0)$ such that

$$\gamma\varepsilon_2 > \frac{\mu u_0}{1+u_0}, \quad \alpha\varepsilon_1 > 2w_0. \tag{4.6}$$

For a small $\lambda > 0$, let $f(t) := \min\{e^{\lambda_{11}t}, u_0 + u_0e^{-\lambda t}\}$, $g(t) := \min\{e^{\lambda_{21}t}, w_0 + w_0e^{-\lambda t}\}$ and denote

$$m_1 = \max_{t\in\mathbb{R}}\{f(t)\}, \qquad m_2 = \max_{t\in\mathbb{R}}\{g(t)\}.$$

If $m_1 > k_1$, $m_2 > k_2$, define the following continuous functions:

$$\overline{\varphi}(t) = \begin{cases} e^{\lambda_{11}t}, & t \le t_1, \\ k_1, & t_1 < t < t_2, \\ u_0 + u_0e^{-\lambda t}, & t \ge t_2, \end{cases} \qquad \underline{\varphi}(t) = \begin{cases} 0, & t \le t_3, \\ u_0 - \varepsilon_1 e^{-\lambda t}, & t > t_3, \end{cases}$$

$$\overline{\psi}(t)=\begin{cases}e^{\lambda_{21}t}, & t\le t_4,\\ k_2, & t_4<t<t_5,\\ w_0+w_0e^{-\lambda t}, & t\ge t_5,\end{cases}\qquad \underline{\psi}(t)=\begin{cases}0, & t\le t_6,\\ w_0-\varepsilon_2e^{-\lambda t}, & t>t_6.\end{cases}$$

If $m_1\le k_1$, $m_2\le k_2$, then redefine the above $\overline{\varphi}(t),\overline{\psi}(t)$ as

$$\overline{\varphi}(t)=\begin{cases}e^{\lambda_{11}t}, & t\le t_1,\\ u_0+u_0e^{-\lambda t}, & t\ge t_1,\end{cases}\qquad \overline{\psi}(t)=\begin{cases}e^{\lambda_{21}t}, & t\le t_4,\\ w_0+w_0e^{-\lambda t}, & t\ge t_4.\end{cases}$$

The other two cases: either $m_1>k_1$, $m_2\le k_2$, or $m_1\le k_1$, $m_2>k_2$ can be considered similarly.

In what follows, we consider only the situation: $m_1>k_1$, $m_2\ge k_2$. The discussions of other cases will be omitted. It is easily seen that $(\overline{\varphi}(t),\overline{\psi}(t))$, $(\underline{\varphi}(t),\underline{\psi}(t))$ satisfy (G1)-(G3). Furthermore, we have from (4.2) and (4.3) that

$$k_1-u_0=\beta-u_0=\frac{\frac{w_0}{\alpha}}{1+u_0}<\frac{u_0}{2+2u_0}<u_0,$$

which leads to $\frac{u_0}{k_1-u_0}>1$. Note that $\frac{\varepsilon_2}{w_0}<1$, and thus we have

$$t_2=\frac{1}{\lambda}\ln\frac{u_0}{k_1-u_0}>0>t_6=\frac{1}{\lambda}\ln\frac{\varepsilon_2}{w_0}.$$

Lemma 4.2. *If $\lambda>0$ is small enough, then $\overline{\Phi}(t)=(\overline{\varphi},\overline{\psi})(t)$ and $\underline{\Phi}(t)=(\overline{\varphi},\underline{\psi})(t)$ is a pair of upper-lower solutions of* (4.4).

Proof. For $t<t_1$, we have $\overline{\varphi}(t)=e^{\lambda_{11}t}$ and

$$\begin{aligned}&D_1\overline{\varphi}''(t)-c\overline{\varphi}'(t)+\alpha\overline{\varphi}(t)(\beta-\overline{\varphi}(t))-\frac{\overline{\varphi}(t)}{1+\overline{\varphi}(t)}\underline{\psi}(t)\\&\le D_1\lambda_{11}^2e^{\lambda_{11}t}-c\lambda_{11}e^{\lambda_{11}t}+\alpha\beta e^{\lambda_{11}t}\\&=e^{\lambda_{11}t}[D_1\lambda_{11}^2-c\lambda_{11}+\alpha k_1]=0.\end{aligned}$$

For $t_1<t<t_2$, then we have $\overline{\varphi}(t)=k_1=\beta$ and

$$\begin{aligned}&D_1\overline{\varphi}''(t)-c\overline{\varphi}'(t)+a\overline{\varphi}(t)(\beta-\overline{\varphi}(t))-\frac{\overline{\varphi}(t)}{1+\overline{\varphi}(t)}\underline{\psi}(t)\\&\le D_1\overline{\varphi}''(t)-c\overline{\varphi}'(t)-\alpha\overline{\varphi}(t)(\beta-\overline{\varphi}(t))=0.\end{aligned}$$

For $t>t_2$, by (4.2), we have $\overline{\varphi}(t)=u_0+u_0e^{-\lambda t}$, $\underline{\psi}(t)=w_0-\varepsilon_2e^{-\lambda t}$ and

$$\begin{aligned}&D_1\overline{\varphi}''(t)-c\overline{\varphi}'(t)+\alpha\overline{\varphi}(t)(\beta-\overline{\varphi}(t))-\frac{\overline{\varphi}(t)}{1+\overline{\varphi}(t)}\underline{\psi}(t)\\&=D_1u_0\lambda^2e^{-\lambda t}+cu_0\lambda e^{-\lambda t}+(u_0+u_0e^{-\lambda t})\\&\quad\times\big[\alpha\beta-\alpha(u_0+u_0e^{-\lambda t})-\frac{w_0-\varepsilon_2e^{-\lambda t}}{1+(u_0+u_0e^{-\lambda t})}\big]\\&=D_1u_0\lambda^2e^{-\lambda t}+cu_0\lambda e^{-\lambda t}+(u_0+u_0e^{-\lambda t})\\&\quad\times\big[-\alpha u_0e^{-\lambda t}+\frac{w_0}{1+u_0}-\frac{w_0-\varepsilon_2e^{-\lambda t}}{1+(u_0+u_0e^{-\lambda t})}\big]\\&=:I_1(\lambda,t)=p_1(\lambda,t)+q_1(\lambda,t),\end{aligned}$$

where

$$p_1(\lambda,t)=D_1u_0\lambda^2e^{-\lambda t}+cu_0\lambda e^{-\lambda t},\quad q_1(\lambda,t)=\overline{q}_1(\lambda,t)\cdot\underline{q}_1(\lambda,t),$$

$$\overline{q}_1(\lambda,t)=u_0+u_0e^{-\lambda t},\quad \underline{q}_1(\lambda,t)=-\alpha u_0e^{-\lambda t}+\frac{w_0}{1+u_0}-\frac{w_0-\varepsilon_2e^{-\lambda t}}{1+(u_0+u_0e^{-\lambda t})}.$$

Since $\alpha u_0>2w_0$, then for $t>t_2$ uniformly, we have

$$I_1(0,t)=2u_0(-\alpha u_0+\frac{w_0}{1+u_0}-\frac{w_0-\varepsilon_2}{1+2u_0})<2u_0(-\alpha u_0+w_0)<0.$$

Furthermore, $\underline{q}_1(\lambda,0)=-\alpha u_0+\frac{w_0}{1+u_0}-\frac{w_0-\varepsilon_2}{1+2u_0}<0$, and for any fixed $\lambda>0$, $I(\lambda,\infty)=0$. Note that for any fixed $\lambda>0$, $\overline{q}_1(\lambda,t)=u_0+u_0e^{-\lambda t}>0$ and is decreasing on $t>0$, and $\underline{q}_1(\lambda,t)<0$ and is increasing on $t>0$. We know that $q_1(\lambda,t)<0$ and is increasing on $t>0$. On the other hand, $p_1(\lambda,t)>0$ and is decreasing on $t>0$. For all $\lambda_1>0$, $t>t_2$, we have

$$p_1(\lambda_1,t)=D_1u_0\lambda^2e^{-\lambda t}+cu_0\lambda e^{-\lambda t}<D_1u_0\lambda_1^2+cu_0\lambda_1\quad\text{for}\quad \lambda\in(0,\lambda_1).$$

From the monotone property of $p_1(\lambda,t)$ and $q_1(\lambda,t)$, one can choose $\lambda_1>0$ small such that $D_1u_0\lambda_1^2+cu_0\lambda_1$ is small and $I_1(\lambda,t)=p_1(\lambda,t)+q_1(\lambda,t)<0$ for $t>t_2$ and $\lambda\in(0,\lambda_1)$. That is,

$$D_1\overline{\varphi}''(t)-c\overline{\varphi}'(t)+\alpha\overline{\varphi}(t)(\beta-\overline{\varphi}(t))-\frac{\overline{\varphi}(t)}{1+\overline{\varphi}(t)}\underline{\psi}(t)<0$$

for $\lambda\in(0,\lambda_1),\ t>t_2$.

Note that $f(u)=\frac{u}{1+u}$ is nondecreasing on $[0,+\infty)$, then for $t<t_4$, we have $\overline{\psi}(t)=e^{\lambda_{21}t}$ and

$$\begin{aligned}
&D_2\overline{\psi}''(t)-c\overline{\psi}'(t)+\gamma\overline{\psi}(t)(\delta-\overline{\psi}(t))+\frac{\mu\overline{\varphi}(t)}{1+\overline{\varphi}(t)}\overline{\psi}(t)\\
&\le D_2\lambda_{21}^2e^{\lambda_{21}t}-c\lambda_{21}e^{\lambda_{21}t}+\gamma\delta e^{\lambda_{21}t}+\frac{\mu k_1}{1+k_1}e^{\lambda_{21}t}\\
&=e^{\lambda_{21}t}[D_2\lambda_{21}^2-c\lambda_{21}+\gamma\delta+\frac{\mu k_1}{1+k_1}]\\
&=e^{\lambda_{21}t}[D_2\lambda_{21}^2-c\lambda_{21}+\gamma k_2]=0.
\end{aligned}$$

For $t_4<t<t_5$, we have $\overline{\psi}(t)=k_2$ and

$$\begin{aligned}
&D_2\overline{\psi}''(t)-c\overline{\psi}'(t)+\gamma\overline{\psi}(t)(\delta-\overline{\psi}(t))+\frac{\mu\overline{\varphi}(t)}{1+\overline{\varphi}(t)}\overline{\psi}(t)\\
&\le D_2\overline{\psi}''(t)-c\overline{\psi}'(t)+\overline{\psi}(t)(\gamma\delta-\gamma\overline{\psi}(t)+\frac{\mu k_1}{1+k_1})=0.
\end{aligned}$$

For $t>t_5$, by (4.2), we have $\overline{\psi}(t)=w_0+w_0e^{-\lambda t}$, $\overline{\varphi}(t)\le u_0+u_0e^{-\lambda t}$, and

$$\begin{aligned}
&D_2\overline{\psi}''(t)-c\overline{\psi}'(t)+\gamma\overline{\psi}(t)(\delta-\overline{\psi}(t))+\frac{\mu\overline{\varphi}(t)}{1+\overline{\varphi}(t)}\overline{\psi}(t)\\
&\le D_2w_0\lambda^2e^{-\lambda t}+cw_0\lambda e^{-\lambda t}+(w_0+w_0e^{-\lambda t})\\
&\quad\times\big[\gamma\delta-\gamma(w_0+w_0e^{-\lambda t})+\frac{\mu(u_0+u_0e^{-\lambda t})}{1+(u_0+u_0e^{-\lambda t})}\big]\\
&=D_2w_0\lambda^2e^{-\lambda t}+cw_0\lambda e^{-\lambda t}+(w_0+w_0e^{-\lambda t})\\
&\quad\times\big[-\gamma w_0e^{-\lambda t}-\frac{\mu u_0}{1+u_0}+\frac{\mu(u_0+u_0e^{-\lambda t})}{1+(u_0+u_0e^{-\lambda t})}\big]=:I_2(\lambda,t).
\end{aligned}$$

Since $\gamma w_0 > \frac{\mu u_0}{1+u_0}$, then for $t > t_5$ uniformly, we have

$$I_2(0,t) = 2w_0(-\gamma w_0 - \frac{\mu u_0}{1+u_0} + \frac{2\mu u_0}{1+2u_0}) < 2w_0(-\gamma w_0 + \frac{\mu u_0}{1+u_0}) < 0.$$

Similar to the discussion of $I_1(\lambda,t)$, there exists $\lambda_2 > 0$ such that

$$D_2\overline{\psi}''(t) - c\overline{\psi}'(t) + \gamma\overline{\psi}(t)(\delta - \overline{\psi}(t)) + \frac{\mu\overline{\varphi}(t)}{1+\overline{\varphi}(t)}\overline{\psi}(t) < 0$$

for $\lambda \in (0,\lambda_2)$ and $t > t_5$.

For $t < t_3$, we have $\underline{\varphi}(t) = 0$, and

$$D_1\underline{\varphi}''(t) - c\underline{\varphi}'(t) + \alpha\underline{\varphi}(t)(\beta - \underline{\varphi}(t)) - \frac{\underline{\varphi}(t)}{1+\underline{\varphi}(t)}\overline{\psi}(t) = 0.$$

For $t > t_3$, by (4.2), we have $\underline{\varphi}(t) = u_0 - \varepsilon_1 e^{-\lambda t}$, $\overline{\psi}(t) \le w_0 + w_0 e^{-\lambda t}$ and

$$\begin{aligned}
&D_1\underline{\varphi}''(t) - c\underline{\varphi}'(t) + \alpha\underline{\varphi}(t)(\beta - \underline{\varphi}(t)) - \frac{\underline{\varphi}(t)}{1+\underline{\varphi}(t)}\overline{\psi}(t)\\
&\ge -D_1\varepsilon_1\lambda^2 e^{-\lambda t} - c\varepsilon_1\lambda e^{-\lambda t} + (u_0 - \varepsilon_1 e^{-\lambda t})\\
&\quad \times \big[\alpha\beta - \alpha(u_0 - \varepsilon_1 e^{-\lambda t}) - \frac{w_0 + w_0 e^{-\lambda t}}{1+(u_0 - \varepsilon_1 e^{-\lambda t})} big]\\
&= -D_1\varepsilon_1\lambda^2 e^{-\lambda t} - c\varepsilon_1\lambda e^{-\lambda t} + (u_0 - \varepsilon_1 e^{-\lambda t})\\
&\quad \times \big[\alpha\varepsilon_1 e^{-\lambda t} + \frac{w_0}{1+u_0} - \frac{w_0 + w_0 e^{-\lambda t}}{1+(u_0 - \varepsilon_1 e^{-\lambda t})}\big] =: I_3(\lambda,t).
\end{aligned}$$

It follows from (4.6) that for $t > t_3$ uniformly, we have

$$I_3(0,t) = (u_0 - \varepsilon_1)\Big[\alpha\varepsilon_1 + \frac{w_0}{1+u_0} - \frac{2w_0}{1+(u_0-\varepsilon_1)}\Big] > (u_0 - \varepsilon_1)(\alpha\varepsilon_1 - 2w_0) > 0.$$

Therefore, there exists $\lambda_3 > 0$ such that

$$D_1\underline{\varphi}''(t) - c\underline{\varphi}'(t) + \alpha\underline{\varphi}(t)(\beta - \underline{\varphi}(t)) - \frac{\underline{\varphi}(t)}{1+\underline{\varphi}(t)}\overline{\psi}(t) > 0$$

for $\lambda \in (0,\lambda_3)$ and $t > t_3$.

For $t < t_6$, we have $\underline{\psi}(t) = 0$ and

$$D_2\underline{\psi}''(t) - c\underline{\psi}'(t) + \gamma\underline{\psi}(t)(\delta - \underline{\psi}(t)) + \frac{\mu\underline{\varphi}(t)}{1+\underline{\varphi}(t)}\underline{\psi}(t) = 0$$

For $t > t_6$, by (4.2), we have $\underline{\psi}(t) = w_0 - \varepsilon_2 e^{-\lambda t}$, and

$$\begin{aligned}
&D_2\underline{\psi}''(t) - c\underline{\psi}'(t) + \gamma\underline{\psi}(t)(\delta - \underline{\psi}(t)) + \frac{\mu\underline{\varphi}(t)}{1+\underline{\varphi}(t)}\underline{\psi}(t)\\
&\ge -D_2\varepsilon_2\lambda^2 e^{-\lambda t} - c\varepsilon_2\lambda e^{-\lambda t} + (w_0 - \varepsilon_2 e^{-\lambda t})[\gamma\delta - \gamma(w_0 - \varepsilon_2 e^{-\lambda t})]\\
&= -D_2\varepsilon_2\lambda^2 e^{-\lambda t} - c\varepsilon_2\lambda e^{-\lambda t} + (w_0 - \varepsilon_2 e^{-\lambda t})(\gamma\varepsilon_2 e^{-\lambda t} - \frac{\mu u_0}{1+u_0}) =: I_4(\lambda,t).
\end{aligned}$$

It follows from (4.6) that for $t > t_6$ uniformly, we have

$$I_4(0,t) = (w_0 - \varepsilon_2)(\gamma\varepsilon_2 - \frac{\mu u_0}{1+u_0}) > 0.$$

Therefore, there exists $\lambda_4 > 0$ such that

$$D_2\underline{\psi}''(t) - c\underline{\psi}'(t) + \gamma\underline{\psi}(t)(\delta - \underline{\psi}(t)) + \frac{\mu\underline{\varphi}(t)}{1+\underline{\varphi}(t)}\underline{\psi}(t) < 0$$

for $\lambda \in (0, \lambda_4)$ and $t > t_6$. By the above argument, we know that $\overline{\Phi} = (\overline{\varphi}, \overline{\psi})(t)$ and $\underline{\Phi} = (\overline{\varphi}, \underline{\psi})$ is a pair of upper-lower solutions of (4.4) for $\lambda > 0$ small enough. $\square$

As a direct consequence of Theorem 3.6, we have the following result.

Theorem 4.3. *Assume that* (4.3) *holds and* $c > c^* := \max\{2\sqrt{D_1\alpha k_1}, 2\sqrt{D_2\gamma k_2}\}$, *where* $k_1 = \beta$, $k_2 = \delta + \frac{\mu\beta}{\gamma(1+\beta)}$. *For* $c > c^*$, *system* (4.1) *has a traveling wave* $\Psi(t) = (\varphi(t), \psi(t))$ *satisfying* $\Psi(-\infty) = (0, 0)$, $\Psi(\infty) = (u_0, w_0)$.

5. Concluding discussion

In this article we have dealt with the existence of traveling wave solutions for a reaction-diffusion system based on a predator-prey model with a general functional response. By constructing an admissible pair of upper and lower solutions and using Schauder's fixed point theorem, we show that there is a traveling wave solution connecting the trivial equilibrium $(0, 0)$ and the positive equilibrium (u_0, w_0). That is, there is a zone of transition from the steady state with no species to the steady state with the coexistence of both species. In comparison, the technique used here is simpler than those of the works mentioned in the introduction.

Predator-prey systems admit multiple equilibria. Our work here considered only one case. Traveling waves connecting other pairs of equilibrium are also possible. It would be interesting to use the techniques in the present paper to investigate the existence of traveling waves connecting $(\beta, 0)$ and (u_0, w_0) which would explain the situation where the habitat is first saturated with prey to its carrying capacity, then the invasion of predator may result in co-existence of both species in the long term.

Acknowledgments. Research is supported by grants 11171120 from the Natural Science Foundation of China, 20094407110001 from the Doctoral Program of Higher Education of China, and 10151063101000003 from the Natural Science Foundation of Guangdong Province. We want to thank the anonymous referee for his/her helpful suggestions which led to improvements of this article.

References

[1] S. Dunbar; *Traveling wave solutions of diffusive Lotka-Volterra equations*, J. Math. Biol. **17** (1983), 11-32.
[2] S. Dunbar; *Traveling wave solutions of diffusive Lotka-Volterra equations: A heteroclinic connection in R^4*, Trans. Amer. Math. Soc. 286, **286** (1984), 557-594.
[3] S. Dunbar; *Traveling waves in diffusive predator-prey equations: periodic orbits and point-to periodic heteroclinic orbits*, SIAM J. Appl. Math. **46** (1986), 1057-1078.
[4] Y. H. Fan, W. T. Li; *Global asymptotic stability of a ratio-dependent predator-prey system with diffusion*, J Comput. Appl. Math. 188 (2005), 205-227.
[5] Y. H. Fan, W. T. Li, L. L. Wang; *Periodic solutions of delayed ratio-dependent predator-prey models with monotonic or nonmonotonic functional responses*, Nonlinear Anal RWA. **5** (2004), 247-263.
[6] S. A. Gourley, S. A. Britton; *A predator-prey reaction-diffusion system with nonlocal effects*, J. Math. Biol. **34** (1996), 297–333.
[7] R. Gardner; *Existence of traveling wave solutions of predator-prey systems via the connection index*, SIAM J. Appl. Math **44** (1984), 56-79.

[8] H. F. Huo, W. T. Li; *Periodic solutions of a delayed predator-prey system with Michaelis-Menten type functional response*, J. Comput. Appl. Math. **166** (2004), 453-463.
[9] J. H. Huang, G. Lu, S. G. Ruan; *Existence of traveling wave solutions in a diffusive predator-prey model*, J. Math. Biol. **46** (2003), 132-152.
[10] G. R. Jiang, Q. S. Lu, L. N. Qian; *Complex dynamics of a Holling type II prey-preadator system with state feedback control*, Chaos, Solitons, Fractals **31** (2007), 448-461.
[11] G. Lin, W. T. Li, M. G. Ma; *Traveling wave solutions in delayed reaction diffusion systems with applications to multi-species models*, Discrete Cont. Dyn.-B **13** (2010), 393-414.
[12] W. T. Li, S. L. Wu; *Traveling waves in a diffusive predator-prey model with holling type-III functional response*, Chaos, Solitons and Fractals **37** (2008), 476-486.
[13] D. Liang, P. X. Weng, J. H. Wu; *Traveling Wave Solutions in A Delayed Predator-Prey Diffusion PDE System: Point-to-Periodic and Point-to-Point Waves*, IMA J. Appl. Math, **77** (2012), 516-545.
[14] X. B. Lin, P. X. Weng, C. F. Wu; *Traveling wave solutions for a predator-prey system with Sigmoidal response function*, J. Dynm. Diff. Eqs. 23 (2011), 903-921.
[15] B. Liu, Y. J. Zhang, L. S. Chen; *The dynamical behaviors of a Lotka-Volterra predator-prey model with Holling type functional response*, J. Comput. Appl. Math. **162** (2004), 341-357.
[16] K. Mischaikow, J. F. Reineck; *Traveling waves in predator-prey systems*, SIAM J. Math. Anal. **24** (1993), 1179-1214.
[17] J. D. Murray; *Mahtematical Biology*, Springer-Verlag, Berlin, Heidelberg, New York, 1993.
[18] L. L. Wang, W. T. Li; *Periodic solutions and permanence for a delayed nonautonomous ratio-dependent predator-prey model with Holling type functional response*, J. Comput. Appl. Math. **162** (2004), 341-357.

Zhaoquan Xu
School of Mathematics, South China Normal University, Guangzhou 510631, China
E-mail address: xiaozhao20042008@163.com

Peixuan Weng
School of Mathematics, South China Normal University, Guangzhou 510631, China
E-mail address: wengpx@scnu.edu.cn, Tel: 0086-20-85213533