NECESSARY AND SUFFICIENT CONDITIONS FOR THE
EXISTENCE OF PERIODIC SOLUTIONS IN A
PREDATOR-PREY MODEL ON TIME SCALES

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Abstract. This article explores the existence of periodic solutions for non-
autonomous impulsive semi-ratio-dependent predator-prey systems on time
scales. Based on a continuous theorem in coincidence degree theory, sharp suf-
ficient and necessary conditions are derived in which most popular monotonic,
non-monotonic and predator functional responses are applicable. This article
extends the work in [6, 10, 12, 13, 14, 18, 25].

1. Introduction

It is well known that time scales were introduced by Hilger [17] in his Doctoral de-
gree thesis to unify the continuous and discrete analysis. Today, it has become a new
important branch for its tremendous potential applications in many mathematical
models of real process and phenomena such as population dynamics, biotechnology,
economics, neural networks and social science; see, e.g., Agarwal [1, 2], Aulbach [3],
Bohner [7, 8], Erbe [11], Lakshmikantham [19] and the reference therein. In the way
of time scales, not only are the results related to the set of real numbers or to
the set of integers, but also pertaining to more general time scales are obtained.
On the other hand, impulsive effects always occur in the simulation of process and
phenomena observed in control theory, chemistry, population dynamics, industrial
robotics etc. To incorporate it into those models, impulsive differential/difference
equations are an adequate mathematical apparatus. The interesting in impulsive
systems has grown because of the importance of both theoretical and practice need,
and more richer dynamics are observed, see, e.g. [9, 14, 20].

Let \( \mathbb{T} \) be a time scale; i.e., \( \mathbb{T} \) is a nonempty closed subset of \( \mathbb{R} \) (see Definition
2.1–2.5 in Section 2), \( \{ t_k \}_{k \in \mathbb{N} \subset \mathbb{T} \} \) (\( \mathbb{N} \) is the set of positive integers) is the impulsive
moment sequence with \( t_0 = \min \{ [0, \infty) \cap \mathbb{T} \} < t_1 < \cdots < t_k < \cdots, \lim_{k \to \infty} t_k = \infty, k \in \mathbb{N} \). In present paper, we consider following impulsive semi-ratio-dependent
The key term $\varphi$ or its equivalent form $\varphi$ dependent responses are presented. For example, dependence. After the classical work of Lotka\cite{21} and Volterra\cite{22}, various prey-consumption by an average predator and can be classified as prey-dependence and predator-prey model on time scale $\mathbb{T}$

$$
x^1_\Delta(t) = a(t) - b(t)e^{x_1(t)} - \varphi(t, e^{x_1(t)}, e^{x_2(t)})e^{x^2(t)-x_1(t)}, \quad t \in \mathbb{T}\setminus t_k,
$$

$$
x^2_\Delta(t) = d(t) - \beta(t)e^{x_2(t)} - x_1(t), \quad t \in \mathbb{T}\setminus t_k,
$$

$$
\Delta x_1(t) = \ln(1 + c_{1k}), \quad t = t_k, \quad k \in \mathbb{N},
$$

$$
\Delta x_2(t) = \ln(1 + c_{2k}), \quad t = t_k, \quad k \in \mathbb{N},
$$

where $x_1(t)$ and $x_2(t)$ stand for the population (or density) of the prey and the predator, respectively. The same symbol $\Delta$ in (1.1) in differential positions has different meanings, we think, which are easily distinguished by readers, that is, $x^1_\Delta(t)$ is the delta-derivative of $x_i$ at $t$, and $\Delta x_1(t) = x_i(t^+) - x_i(t^-) = \lim_{s \to t^-}, x_i(s) - \lim_{s \to t^-} x_i(s), \quad i = 1, 2$ are impulsive perturbations. A natural constraint is $1 + c_{ik} > 0, \quad k \in \mathbb{N}, \quad i = 1, 2$. In (1.1), it has been assumed that the prey grows logistically with growth rate $a$ and carrying capacity $a/b$ in the absence of predation. The predator consumes the prey according to the function response $\varphi(t, x, y)$ and grow logistically with growth rate $d$ and carrying capacity $x/\beta$ proportional to the population size of prey (or prey abundance). The parameter $\beta$ is a measure of the food quality that the prey provides for conversion into predator birth.

As mentioned above, time scales can unify continuous and discrete analysis. If $\mathbb{T} = \mathbb{R}$, (1.1) reduces the following impulsive differential equations

$$
x'_1(t) = a(t) - b(t)e^{x_1(t)} - \varphi(t, e^{x_1(t)}, e^{x_2(t)})e^{x^2(t)-x_1(t)}, \quad t \in \mathbb{R}\setminus t_k,
$$

$$
x'_2(t) = d(t) - \beta(t)e^{x_2(t)} - x_1(t), \quad t \in \mathbb{R}\setminus t_k,
$$

$$
\Delta x_1(t) = \ln(1 + c_{1k}), \quad t = t_k, \quad k \in \mathbb{N},
$$

$$
\Delta x_2(t) = \ln(1 + c_{2k}), \quad t = t_k, \quad k \in \mathbb{N},
$$

or its equivalent form

$$
x'(t) = x(t)[a(t) - b(t)x(t)] - \varphi(t, x(t), y(t))y(t), \quad t \in \mathbb{R}\setminus t_k,
$$

$$
y'(t) = y(t)[d(t) - \beta(t)y(t)x(t)], \quad t \in \mathbb{R}\setminus t_k,
$$

$$
\Delta x(t) = c_{1k}x(t), \quad t = t_k, \quad k \in \mathbb{N},
$$

$$
\Delta y(t) = c_{2k}y(t), \quad t = t_k, \quad k \in \mathbb{N}.
$$

If $\mathbb{T} = \mathbb{Z}$, then $\{t_k\} \subset \mathbb{Z}$ and system (1.1) may turn into the following impulsive difference equations

$$
x(t + 1) = x(t)\exp(a(t) - b(t)x(t)) - \varphi(t, x(t), y(t))\frac{y(t)}{x(t)}), \quad t \in \mathbb{Z}\setminus t_k,
$$

$$
y(t + 1) = y(t)\exp(d(t) - \beta(t)\frac{y(t)}{x(t)}), \quad t \in \mathbb{Z}\setminus t_k,
$$

$$
\Delta x(t_k + 1) = (1 + c_{1k})x(t_k), \quad k \in \mathbb{N},
$$

$$
\Delta y(t_k + 1) = (1 + c_{2k})y(t_k), \quad k \in \mathbb{N}.
$$

The key term $\varphi(t, x, y)$ in (1.1) is called functional response, which is the rate of prey consumption by an average predator and can be classified as prey-dependence and predator-dependence. The response is a function of prey alone in prey-dependence while both predator and prey density have an effect on the response in predator-dependence. After the classical work of Lotka\cite{21} and Volterra\cite{22}, various prey-dependent responses are presented. For example, $\varphi_1(t, x) = r(t)x, \varphi_2(t, x) =$
$r(t)x/(A(t)+x), \varphi_3(t,x) = r(t)x^2/(A(t)+x^2)$ and $\varphi_4(t,x) = r(t)x/(A(t)+B(t)x+C(t)x^2)$ are well known as Holling type I, II, III and IV respectively. Particularly, $\varphi_4$ is non-monotone and declines at high prey densities, while $\varphi_1-\varphi_3$ are monotone, and in more general case is $\varphi_5(t,x) = r(t)x^7/(A(t)+x^9)$, $\theta > 2$ which is known as the sigmoidal response. Similar monotone responses as $\varphi_6(t,x) = r(t)x^2/((A(t)+x)(D(t)+x))$ and $\varphi_7(t,x) = r(t)(1-e^{-A(t)x})$ can be found in Freedman\cite{15}.

On the other hand, there are evidences to show predator density also has an effect on functional response. A typical predator-dependent response is proposed by Beddington and DeAngelis, now, popular referred to as Beddington-DeAngelis functional response taking the form $\varphi_8(t,x,y) = r(t)x/(A(t)+B(t)x+C(t)y)$. Recently, D. Miller et al\cite{24} proposed the following modified Holling type II and III response $\varphi_9(t,x,y) = r(t)x/((A(t)+x)(D(t)+y))$, $\varphi_{10}(t,x,y) = r(t)x^2/((A(t)+x^2)(D(t)+y))$. This dynamical relationship between predators and their prey has long been and will continue to be one of the dominant themes in both ecology and mathematical ecology due to its universal existence and importance\cite{5}.

It is an interesting topic to explore the existence of periodic solutions in nonautonomous semi-ratio-dependent predator-prey dynamical systems; see, e.g., \cite{6, 12, 13, 14, 18, 25} and the reference therein. For the case without impulses (i.e., $c_{ik} = 0$, $i = 1,2, k \in \mathbb{N}$), the existence of periodic solutions for system (1.1)–(1.4) has been studied by many authors. For example, Huo and Li\cite{13} considered system (1.3) with $\varphi(t,x,y) = \varphi_1$, which is called Leslie-Gower system. For more general monotone functional responses in (1.3), some criteria of existence are presented by Wang et al\cite{25}. Ding et al\cite{10} also establish a criterion for (1.3) with non-monotone functional response $\varphi_4$. The discrete analogue (1.4) was then explored by Fazly and Hesaaraki\cite{13}, Fan and Wang\cite{12}. Recently, Bohner et al.\cite{6}, Fazly and Hesaaraki\cite{14} investigate the dynamical system (1.1) on time scales with monotone functional responses $\varphi_1 - \varphi_3$ and $\varphi_5 - \varphi_7$. Especially, for some widely recognized functional responses which are not monotone such as $\varphi_4$, some sufficient conditions are derived in \cite{14}.

In this paper, our approach is based on continuation theorem developed by Gaines and Mawhin\cite{16} and also used by many authors. However, by the invariance property of homotopy and analysis technique, we establish some new sufficient and necessary results where the exponential or monotone conditions are not necessary, which improves and extends many previous work in the literature \cite{6, 10, 12, 13, 14, 18, 25}, see, Remark 3.5, Remark 4.2, Remark 4.4, and Proposition 4.5.

The rest of the paper is arranged as follows. In Section 2, we introduce some notation and concepts for time scales and continuous theorem of coincidence, at the same time we give some necessary lemmas. In Section 3, we establish new sharp conditions for the existence of periodic solutions for system (1.1). Its applications then are illustrated in Section 4.

2. Preliminaries

Denote $\mathbb{R}, \mathbb{R}_+, \mathbb{Z}, \mathbb{Z}_+$ and $\mathbb{N}$ are real numbers set, non-negative real numbers set, integer numbers set, non-negative integer numbers set and positive integer numbers set respectively. For the convenience of the reader, we list some definitions and notations on the time scale calculus as follows. These definitions and notations are common in the related literature.
Definition 2.1. A time scale is an arbitrary nonempty closed subset $\mathbb{T}$ of $\mathbb{R}$. The set $\mathbb{T}$ inherits the standard topology of $\mathbb{R}$.

Let $\omega > 0$, throughout this paper, the time scale $\mathbb{T}$, impulsive sequence and impulsive functions are assumed to be $\omega$-periodic; i.e., $t \in \mathbb{T}$ implies $t + \omega \in \mathbb{T}$ and there exists an integer $p \geq 1$ such that $\mathbb{I}_\omega \cap \{tk\} = \{t_1, t_2, \ldots, t_p\}$, $tk + \omega = tk + \omega$, $c_{ik} = c_{i(k+p)}$, $i = 1, 2, k \in \mathbb{N}$. Some examples of such time scales are $\mathbb{R}$, $\mathbb{Z}$, $\bigcup_{k \in \mathbb{Z}} [2k, 2k + 1]$.

Definition 2.2. For $t \in \mathbb{T}$, the forward jump operator $\sigma : \mathbb{T} \rightarrow \mathbb{T}$ and the backward jump operator $\rho : \mathbb{T} \rightarrow \mathbb{T}$ are defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) = \sup\{s \in \mathbb{T} : s < t\},$$

respectively.

In this definition, a point $t \in \mathbb{T}$ is said to be left-dense if $\rho(t) = t$, left-scattered if $\rho(t) < t$, left-dense if $\sigma(t) = t$ and right-scattered if $\sigma(t) > t$. The graininess $\mu$ of the time scale is defined by $\mu(t) = \sigma(t) - t$.

Definition 2.3. A function $f : \mathbb{T} \rightarrow \mathbb{R}$ is said to be rd-continuous if it is continuous at right-dense points in $\mathbb{T}$ and its left-sides limits exist (finite) at left-dense points in $\mathbb{T}$. The set of rd-continuous functions is shown by $C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R})$.

Definition 2.4. For $f : \mathbb{T} \rightarrow \mathbb{R}$ and $t \in \mathbb{T}$ we define $f^\Delta(t)$, the delta-derivative of $f$ at $t$, to be the number (provided it exists) with the property that, given any $\epsilon > 0$, there is a neighborhood $U$ of $t$ in $\mathbb{T}$ such that for any $s \in U$ it holds that

$$||f(\sigma(t)) - f(s) - f^\Delta(t)[\sigma(t) - s]|| \leq \epsilon|\sigma(t) - s|.$$ 

Thus, $f$ is said to be delta-differential if its delta-derivative exists. The set of function $f : \mathbb{T} \rightarrow \mathbb{R}$ that are delta-differentiable and whose delta-derivative are rd-continuous functions is denoted by $C^1_{rd} = C^1_{rd}(\mathbb{T}) = C^1_{rd}(\mathbb{T}, \mathbb{R})$.

Definition 2.5. A function $F : \mathbb{T} \rightarrow \mathbb{R}$ is called a delta-antiderivative of $f : \mathbb{T} \rightarrow \mathbb{R}$ provided $F^\Delta(t) = f(t)$, for all $t \in \mathbb{T}$. Then for all $a, b \in \mathbb{T}$ we write

$$\int_a^b f(s) \Delta s = F(b) - F(a).$$

Lemma 2.6 (Existence of Antiderivatives). Every rd-continuous function has an antiderivative. In particular if $t_0 \in \mathbb{T}$, then $F$ defined by

$$F(t) = \int_{t_0}^{t} f(\tau) \Delta \tau, \quad \text{for } t \in \mathbb{T}$$

is an antiderivative of $f$.

In fact, for the usual time scale $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$ we have

$$\sigma(t) = \rho(t) = t, \quad \mu(t) = 0, \quad f^\Delta(t) = f'(t),$$

$$\int_a^b f(t) \Delta t = \int_a^b f(t) \, dt, \quad \sigma(t) = t + 1, \quad \rho(t) = t - 1,$$

$$\mu(t) = 1, \quad f^\Delta(t) = f(t + 1) - f(t), \quad \int_a^b f(t) \Delta t = \sum_{t=a}^{b-1} f(t),$$
Let $X, Z$ be two Banach spaces, $L : X \cap \text{Dom} L \to Z$ be a linear mapping, $N : X \to Z$ be a continuous mapping. The mapping $L$ will be called a Fredholm mapping of index zero if $\dim \ker L = \text{codim} \text{Im} L < +\infty$ and $\text{Im} L$ is closed in $Z$. If $L$ is a Fredholm mapping of index zero and there exist continuous projections $P : X \to X$ and $Q : Z \to Z$ such that $\text{Im} P = \ker L$, $\text{Im} L = \ker Q = \text{Im} (I - Q)$, then it follows that $L|_{\text{Dom} L \cap \ker P} : (I - P)X \to \text{Im} L$ is invertible. We denote the inverse of that map by $K_P$. If $\Omega$ is an open bounded subset of $X$, the mapping $N$ will be called $L$-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_P(I - Q)N : \overline{\Omega} \to X$ is compact. Since $\text{Im} Q$ is isomorphic to $\ker L$, there exists an isomorphism $J : \text{Im} Q \to \ker L$.

**Lemma 2.7.** Let $L$ be a Fredholm mapping of index zero and $N$ be $L$-compact on $\Omega$. Suppose that

1. $Lx \neq \lambda Nx$, for any $x \in \partial \Omega$ and $\lambda \in (0, 1)$;
2. $QN x \neq 0$, for any $x \in \partial \Omega \cap \ker L$;
3. $\deg \{JQN, \Omega \cap \ker L, 0\} \neq 0$.

Then the operator equation $Lx = Nx$ has at least one solution in $\text{Dom} L \cap \Omega$.

In what follows in this section, we will translate (1.1) into its equivalent operator equations. Firstly of all, the following notation are introduced

$$Lx = [t_0, t_0 + \omega] \cap \mathbb{T}, \quad \tilde{f} = \frac{1}{\omega} \int_{I_\omega} f(t) \Delta t,$$

where $f \in C_{rd}(\mathbb{T})$ is an $\omega$-periodic real function. Moreover, we denote

$$PC_{\omega} = \left\{ \phi : \mathbb{T} \to \mathbb{R} : (i) \phi(t) \text{ is rd-continuous for } t \in \mathbb{T} \setminus t_k \text{ and } \omega \text{-periodic;} \right. \quad (ii) \lim_{s \to t_k^- 0} \phi(s) = \phi(t_k^-) \text{ and } \lim_{s \to t_k^+ 0} \phi(s) = \phi(t_k^+) \text{ exist.} \right\},$$

and

$$PC_{\omega}^1 = \{ \phi \in PC_{\omega} : \phi^\Delta \in PC_{\omega} \}.$$ Let

$$X = \{ x = (x_1, x_2)^T : x_1 \in PC_{\omega}, i = 1, 2 \}, \quad Z = X \times \mathbb{R}^{2p}$$

with the norms

$$\| x \|_X = \sum_{i=1}^2 \sup_{t \in I_\omega} | x_i(t) |, \quad x = (x_1, x_2)^T \in X,$$

$$\| z \|_Z = \| x \|_X + \| y \|, \quad z = (x, y) \in Z,$$

where $\| \cdot \|$ is the Euclidean norm of $\mathbb{R}^{2p}$. Then $X$ and $Z$ are Banach spaces when they are endowed with above norms. Set

$$L : \text{Dom} L \cap X \to Z, \quad Lx = (x^\Delta, \Delta x(t_1), \ldots, \Delta x(t_p)), \quad x = (x_1, x_2)^T \in X$$

with $\text{Dom} L = \{ x = (x_1, x_2)^T : x_1 \in PC_{\omega}^1, i = 1, 2 \}$ and $N : X \to Z$ as

$$Nx = \begin{bmatrix} x_1^\Delta(t_1), & \ln(1 + c_{11}), & \ldots, & \ln(1 + c_{1p}) \\ x_2^\Delta(t_1), & \ln(1 + c_{21}), & \ldots, & \ln(1 + c_{2p}) \end{bmatrix} x \in X.$$
Using this notation we may rewrite (1.1) in the equivalent form $Lx = Nx, x \in X$. So, $\ker L = \mathbb{R}^2$ and $\text{Im } L = \{z = (\phi, \gamma_1, \ldots, \gamma_p) \in Z : \int_1^0 \phi(t) \Delta t + \sum_{k=1}^p \gamma_k = 0\}$ is closed in $Z$, and $\dim \ker L = \text{codim } \text{Im } L = 2$. Thus, $L$ is a Fredholm mapping of index zero. Define two projections $P : X \to X$ and $Q : Z \to Z$ as

$$Px = \bar{x}, \quad x \in X,$$

$$Qz = Q(x, \gamma_1, \ldots, \gamma_p) = (\bar{x} + \frac{1}{\omega} \sum_{k=1}^p \gamma_k, 0, \ldots, 0).$$

It is trivial to show that $P, Q$ are continuous projections such that $\text{Im } P = \ker L, \quad \text{Im } L = \ker Q = \text{Im } (I - Q)$, and hence, the generalized inverse $K_p$ exists. For $x \in \text{Dom } L \subset X$, it is not difficult to get

$$QN x = \frac{1}{\omega} \left( \int_1^t x_1^1(t) \Delta t + \sum_{k=1}^p \ln(1 + c_k), 0, \ldots, 0 \right),$$

and

$$K_p(I - Q) N x = \left( \int_0^t x_1^1(s) \Delta s + \sum_{k=1}^p \ln(1 + c_k) \right) \left( \int_0^t x_2^1(s) \Delta s + \sum_{k=1}^p \ln(1 + c_k) \right)$$

$$- \left( \frac{t}{\omega} - \frac{1}{2} \left( \int_1^t x_1^2(t) \Delta t + \sum_{k=1}^p \ln(1 + c_k) \right) \right)$$

$$- \frac{1}{\omega} \left( \int_1^t \int_0^t x_1^1(s) \Delta s \Delta t + \omega \sum_{k=1}^p \ln(1 + c_k) - \sum_{k=1}^p \ln(1 + c_k) t_k \right).$$

Clearly, $QN$ and $K_p(I - Q)$ are continuous. It is not difficult to show that $K_p(I - Q) N(\overline{\Omega})$ is compact for any open-bounded set $\Omega \subset X$. In addition, $QN(\overline{\Omega})$ is bounded. Therefore, $N$ is $L$ compact on $\overline{\Omega}$ with any open-bounded set $\Omega \subset X$.

3. Main Results

**Theorem 3.1.** Assume that the following conditions hold.

(H1) $a(t), b(t), d(t)$ and $\beta(t)$ are non-negative $\omega$-periodic rd-continuous real functions and $\bar{a} > 0, \bar{d} > 0$;

(H2) The functional response $\varphi : \mathbb{T} \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is rd-continuous and $\omega$-periodic with respect to $t, \varphi(t, 0, y) = 0$ for any $t \in \mathbb{T}, \quad y \geq 0$. In addition, there exist $m \in \mathbb{N}$ and $\omega$-periodic rd-continuous functions $\alpha_i : \mathbb{T} \to \mathbb{R}_+, \quad i = 0, \ldots, m - 1$ such that

$$\varphi(t, x, y) \leq \alpha_0(t)x^m + \cdots + \alpha_{m-1}(t)x \quad (3.1)$$

for $t \in \mathbb{T}, \quad x \geq 0, \quad y \geq 0$.

Then, system (1.1) has at least one $\omega$-periodic solution if and only if

$$\bar{a} \omega + \sum_{k=1}^p \ln(1 + c_k) > 0, \quad \bar{d} \omega + \sum_{k=1}^p \ln(1 + c_k) > 0. \quad (3.2)$$
Proof. First, suppose \((x(t), y(t))\) is a periodic solution of (1.1), integrating both sides of the first two equations in (1.1) on \(I\), we have

\[
\tilde{a} \omega + \sum_{k=1}^{p} \ln(1 + c_{1k}) = \int_{I_{\omega}} b(t) \exp(x(t)) \Delta t
\]

\[
+ \int_{I_{\omega}} \varphi(t, \exp(x(t)), \exp(x(t))) \exp(x(t) - x(t)) \Delta t > 0,
\]

\[
\tilde{d} \omega + \sum_{k=1}^{p} \ln(1 + c_{2k}) = \int_{I_{\omega}} \beta(t) \exp(x(t) - x(t)) \Delta t > 0,
\]

which shows the condition is necessary.

Next, we show it is sufficient. By Lemma 2.7, it suffices to search for an appropriate open bounded subset \(\Omega \subset X\). For some \(\lambda \in (0, 1)\), suppose that \((x_{1}, x_{2})^{T} \in X\) is a solution of

\[
x_{1}(t) = \lambda \left[ a(t) - b(t) e^{x_{1}(t)} - \varphi(t, x_{1}(t), x_{2}(t)) e^{x_{2}(t) - x_{1}(t)} \right], \quad t \in T \setminus t_{k},
\]

\[
x_{2}(t) = \lambda \left[ d(t) - \beta(t) e^{x_{2}(t) - x_{1}(t)} \right], \quad t \in T \setminus t_{k},
\]

\[
\Delta x_{1}(t) = \lambda \ln(1 + c_{1k}), \quad t = t_{k}, \quad k \in \mathbb{N},
\]

\[
\Delta x_{2}(t) = \lambda \ln(1 + c_{2k}), \quad t = t_{k}, \quad k \in \mathbb{N}.
\]

Integrating both sides of the first and second equation of (3.3) over \(I_{\omega}\), one has

\[
\tilde{a} \omega + \sum_{k=1}^{p} \ln(1 + c_{1k})
\]

\[
= \int_{I_{\omega}} b(t) \exp(x(t)) \Delta t \tag{3.4}
\]

\[
+ \int_{I_{\omega}} \varphi(t, \exp(x(t)), \exp(x(t))) \exp(x(t) - x(t)) \Delta t,
\]

\[
\tilde{d} \omega + \sum_{k=1}^{p} \ln(1 + c_{2k}) = \int_{I_{\omega}} \beta(t) \exp(x(t) - x(t)) \Delta t. \tag{3.5}
\]

Thus, it follows that

\[
\int_{I_{\omega}} |x_{1}(t)| \Delta t \leq 2\tilde{a} \omega + \sum_{k=1}^{p} \ln(1 + c_{1k}),
\]

\[
\int_{I_{\omega}} |x_{2}(t)| \Delta t \leq 2\tilde{d} \omega + \sum_{k=1}^{p} \ln(1 + c_{2k}). \tag{3.6}
\]

For any \((x_{1}, x_{2})^{T} \in X\), clearly there exist \(\zeta_{i}, \eta_{i} \in I_{\omega}, i = 1, 2\), such that

\[
x_{i}(\zeta_{i}) (\text{or } x_{i}(\zeta_{i}^{+})) = \inf_{t \in I_{\omega}} x_{i}(t), \quad x_{i}(\eta_{i}) (\text{or } x_{i}(\eta_{i}^{+})) = \sup_{t \in I_{\omega}} x_{i}(t), \quad i = 1, 2.
\]

We only consider the following case (other cases are proved only by replacing \(\zeta_{i}(\eta_{i})\) by \(\zeta_{i}^{+}(\eta_{i}^{+})\))

\[
x_{i}(\zeta_{i}) = \inf_{t \in I_{\omega}} x_{i}(t), \quad x_{i}(\eta_{i}) = \sup_{t \in I_{\omega}} x_{i}(t), \quad i = 1, 2. \tag{3.7}
\]
For any $\xi \in I_\omega$, it holds that
\[
x_i(t) \geq x_i(\xi) - \int_{I_\omega} |x_i^\Delta(s)| \Delta s - \sum_{k=1}^{p} |\ln(1 + c_{ik})|, \quad i = 1, 2,
\]
\[
x_i(t) \leq x_i(\xi) + \int_{I_\omega} |x_i^\Delta(s)| \Delta s + \sum_{k=1}^{p} |\ln(1 + c_{ik})|, \quad i = 1, 2.
\]

Now we find the bound from above for solutions of (3.3). By (3.4) and the first equation in (3.6), we have
\[
x_1(\zeta_1) \leq \ln \hat{\alpha}_\omega + \sum_{k=1}^{p} \ln(1 + c_{1k}) \hat{\beta}_\omega \exp(x_2(\zeta_2) - x_1(\eta_1)).
\]
Thus,
\[
x_1(t) \leq \ln \hat{\alpha}_\omega + \sum_{k=1}^{p} \ln(1 + c_{1k}) \hat{\beta}_\omega \exp(\hat{d}_\omega + \sum_{k=1}^{p} \ln(1 + c_{2k})) := M_1.
\]

By (3.5) and (3.7), one has
\[
\hat{d}_\omega + \sum_{k=1}^{p} \ln(1 + c_{2k}) \geq \hat{\beta}_\omega \exp(x_2(\zeta_2) - x_1(\eta_1)),
\]
\[
\hat{d}_\omega + \sum_{k=1}^{p} \ln(1 + c_{2k}) \leq \hat{\beta}_\omega \exp(x_2(\eta_2) - x_1(\zeta_1)).
\]
It follows from (3.8) and (3.9) that
\[
x_2(\zeta_2) \leq M_1 + \ln \frac{\hat{d}_\omega + \sum_{k=1}^{p} \ln(1 + c_{2k})}{\hat{\beta}_\omega} := A
\]
which leads to
\[
x_2(t) \leq A + 2\hat{d}_\omega + 2 \sum_{k=1}^{p} |\ln(1 + c_{2k})| := M_2.
\]

Next, we find a bound from below for solutions of (3.3). This technique is similar to that in [13]. By (3.4) and the condition (3.1) of Theorem 3.1 we have
\[
\hat{a} + \frac{1}{\omega} \sum_{k=1}^{p} \ln(1 + c_{1k}) \leq \hat{b} \exp(x_1(\eta_1)) + \left[\hat{\alpha}_0(\exp(x_1(\eta_1)))^{m-1}
\right.
\]
\[
+ \hat{\alpha}_1(\exp(x_1(\eta_1)))^{m-2} + \cdots + \hat{\alpha}_{m-1}\exp(x_1(\eta_1)).
\]
There are two cases: $x_1(\eta_1) \leq x_2(\eta_2)$ and $x_1(\eta_1) \geq x_2(\eta_2)$.

Case (1): $x_1(\eta_1) \leq x_2(\eta_2)$. By (3.11), we obtain
\[
x_2(\eta_2) \geq \ln \left[\frac{\hat{a} + \frac{1}{\omega} \sum_{k=1}^{p} \ln(1 + c_{1k})}{(\hat{b} + \hat{\alpha}_{m-1}) + \hat{\alpha}_{m-2}\exp(W_{M1}) + \cdots + \hat{\alpha}_0(\exp(W_{M1}))^{m-1}}\right] := B,
\]

hence,
\[
x_2(t) \geq B - 2\hat{d}_\omega - 2 \sum_{k=1}^{p} |\ln(1 + c_{2k})| := l^{(1)}_2.
\]
Therefore, by (3.9), one has \( x_1(\eta_1) \geq l_2^{(1)} + \ln \left[ \frac{\beta \omega}{d \omega + \sum_{k=1}^{p \omega} \ln(1 + c_{2k})} \right] \) and then

\[
x_1(t) \geq l_2^{(1)} + \ln \left[ \frac{\beta \omega}{d \omega + \sum_{k=1}^{p \omega} \ln(1 + c_{2k})} \right] - 2 \hat{a} \omega - 2 \sum_{k=1}^{p \omega} \ln(1 + c_{1k}) : = l_1^{(1)}.
\]

Case (2): \( x_1(\eta_1) \geq x_2(\eta_2) \). In this case, by (3.11) we may have

\[
\hat{a} + \frac{1}{\omega} \sum_{k=1}^{p \omega} \ln(1 + c_{1k}) \leq \hat{a}_0 (\exp(x_1(\eta_1)))^m + \cdots + \hat{\alpha}_{m-2} (\exp(x_1(\eta_1)))^m + (\hat{b} + \hat{\alpha}_{m-1}) \exp(x_1(\eta_1)).
\]

Consider the function

\[
\varpi(t) = \hat{a}_0 t^m + \cdots + (\hat{b} + \hat{\alpha}_{m-1}) t,
\]

which is increasing for \( t \geq 0 \) with \( \lim_{t \to -\infty} \varpi(t) = \infty \) and \( \varpi(0) = 0 \), so there exists \( t^* > 0 \) such that \( \varpi(t^*) = \hat{a} + \frac{1}{\omega} \sum_{k=1}^{p \omega} \ln(1 + c_{1k}) \). The inequality (3.12) implies \( \exp(x_1(\eta_1)) \geq t^* \), namely, \( x_1(\eta_1) \geq \ln t^* \), which yields

\[
x_1(t) \geq \ln t^* - 2 \hat{a} \omega - 2 \sum_{k=1}^{p \omega} \ln(1 + c_{1k}) : = l_1^{(2)}.
\]

Thus, by (3.10) and (3.13), one has

\[
x_2(\eta_2) \geq l_1^{(2)} + \ln \left[ \frac{\hat{d} \omega + \sum_{k=1}^{p \omega} \ln(1 + c_{2k})}{\beta \omega} \right]
\]

and hence

\[
x_2(t) \geq l_1^{(2)} + \ln \left[ \frac{\hat{d} \omega + \sum_{k=1}^{p \omega} \ln(1 + c_{2k})}{\beta \omega} \right] - 2 \hat{d} \omega - 2 \sum_{k=1}^{p \omega} \ln(1 + c_{2k}) : = l_2^{(2)}.
\]

Choose \( l_i = \min \{ l_1^{(1)}, l_2^{(2)} \} \), \( i = 1, 2 \) such that any solution of (3.3) satisfies

\[
| x_i(t) | \leq \max \{ |M_i|, |l_i| \} : = W_i, \quad i = 1, 2.
\]

Clearly, \( W_1, W_2 \) are independent of \( \lambda \).

Consider the algebraic equations

\[
\hat{a} - \hat{b} \exp(x_1) + \frac{1}{\omega} \sum_{k=1}^{p \omega} \ln(1 + c_{1k})
\]

\[
- \frac{\exp(x_2 - x_1)}{\omega} \int_{t_0}^{t} \mu \varphi(t, \exp(x_1), \exp(x_2)) \, dt = 0,
\]

\[
\hat{a} + \frac{1}{\omega} \sum_{k=1}^{p \omega} \ln(1 + c_{2k}) - \hat{\beta} \exp(x_2 - x_1) = 0
\]

for \( (x_1, x_2)^T \in \mathbb{R}^2 \), where \( \mu \in [0, 1] \). Replace \( \varphi \) with \( \mu \varphi \) in (3.3), then the key inequality (3.11) still holds. So carry out similar arguments as above, any solution \( (x_1^*, x_2^*) \) of (3.15) with \( \mu \in [0, 1] \) also satisfies

\[
| x_i^* | \leq \max \{ |M_i|, |l_i| \} : = W_i, \quad i = 1, 2.
\]

Define \( \Omega = \{ x \in X ||x||_X < W \} \) with \( W > W_1 + W_2 \), it is easy to see \( \Omega \) satisfies the condition (1) of Lemma 2.7. Let \( x \in \partial \Omega \cap \ker L = \partial \Omega \cap \mathbb{R}^2 \), then \( x \) is a constant
vector in $\mathbb{R}^2$ with $\|x\|_X = W$. Hence from (3.15) and the definition of $W$, we see that $QNx \neq 0$.

Define a homotopy

$$H_\mu((x_1, x_2)^T) = \mu QN((x_1, x_2)^T) + (1 - \mu)U((x_1, x_2)^T), \quad \mu \in [0, 1],$$

(3.17)

where

$$U((x_1, x_2)^T) = \begin{pmatrix}
\hat{a} - \hat{b} \exp(x_1) + \frac{1}{2} \sum_{k=1}^{p} \ln(1 + c_{1k}) \\
\hat{d} + \frac{1}{2} \sum_{k=1}^{p} \ln(1 + c_{2k}) - \beta \exp(x_2 - x_1)
\end{pmatrix}.$$  

(3.18)

then it follows from (3.14) and (3.16) that $H_\mu(x) \neq 0$ for $x \in \partial \Omega \cap \ker L$ and $\mu \in [0, 1]$. In addition, it is clear that the algebraic equation $U((x_1, x_2)^T) = 0$ has a unique solution in $\mathbb{R}^2$. Choose the isomorphism $J$ to be the identity mapping, by a direct computation and the invariance property of homotopy, so

$$\deg\{JQN, \Omega \cap \ker L, 0\} = \deg\{QN, \Omega \cap \ker L, 0\} = \deg\{U, \Omega \cap \ker L, 0\} \neq 0,$$

where $\deg\{\cdot, \cdot, \cdot\}$ is the Brouwer degree.

Therefore, $\Omega$ defined above satisfies all conditions of Lemma 2.7, then, the system (1.1) has at least one $\omega$-periodic solution in $\text{Dom} L \cap \Omega$. This completes the proof. \hfill \Box

For the time scale of non-negative real numbers; i.e., $\mathbb{T} = \mathbb{R}_+ = [0, \infty)$, Equation (1.1) reduces to (1.2) or its equivalent form (1.3), and $\hat{f} = \frac{1}{\omega} \int_0^\omega f(s) \, ds$ for $\omega$-periodic function $f$. Correspondingly, we have following result.

**Theorem 3.2.** Suppose (1.3) satisfies the following assumptions:

\begin{enumerate}[(R1)]
  \item $a(t), b(t), d(t)$ and $\beta(t)$ are non-negative $\omega$-periodic continuous real functions and $\hat{a} > 0$, $\hat{d} > 0$;
  \item The functional response $\varphi : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and $\omega$-periodic with respect to $t$, $\varphi(t, 0, y) = 0$ for any $t \in \mathbb{R}_+$, $y \geq 0$. In addition, there exists positive integer $m$ and $\omega$-periodic continuous functions $\alpha_i : \mathbb{R} \to \mathbb{R}_+$, $i = 0, \ldots, m - 1$ such that

$$\varphi(t, x, y) \leq \alpha_0(t)x^m + \cdots + \alpha_{m-1}(t)x,$$

(3.19)

for $t \in \mathbb{R}_+, x \geq 0, y \geq 0$.
\end{enumerate}

Then, system (1.3) has at least one $\omega$-periodic solution if and only if

$$\int_0^\omega a(s) \, ds + \sum_{k=1}^{p} \ln(1 + c_{1k}) > 0, \quad \int_0^\omega d(s) \, ds + \sum_{k=1}^{p} \ln(1 + c_{2k}) > 0.$$

If $\mathbb{T}$ is another usual time scale $\mathbb{Z}_+ = \{0, 1, \ldots, n, \ldots\}$, Equation (1.1) reduces to impulsive difference system (1.4) and $\hat{f} = \frac{1}{\omega} \sum_{k=1}^{\omega-1} f(k)$. Similarly, we have following theorem.

**Theorem 3.3.** Assume that in system (1.4) the following conditions hold.

\begin{enumerate}[(Z1)]
  \item $a(j), b(j), d(j)$ and $\beta(j)$ are non-negative $\omega$-periodic real sequences and $\hat{a} > 0$, $\hat{d} > 0$;
  \item The functional response $\varphi : \mathbb{Z}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is $\omega$-periodic with respect to $t$, $\varphi(j, 0, y) = 0$ for any $j \in \mathbb{Z}_+$, $y \geq 0$. In addition, there exists positive integer $m$ and $\omega$-periodic sequences $\alpha_i : \mathbb{Z}_+ \to \mathbb{R}_+$, $i = 0, \ldots, m - 1$ such that

$$\varphi(j, x, y) \leq \alpha_0(j)x^m + \cdots + \alpha_{m-1}(j)x,$$

(3.20)
\end{enumerate}
\[ \phi \]

of dynamical system (1.1) on time scales with monotonic functional responses \( \phi \) are satisfied in system notation used here are same as that in corresponding papers cited

when \( c_{ik} = 0, i = 1, 2, k \in \mathbb{N} \) the impulses in (1.1) disappear. In this case, (3.2) holds naturally. By Theorem 3.1, we have the following corollary.

Corollary 3.4. It must have at least one \( \omega \)-periodic positive solution if (H1)–(H2) are satisfied in system (1.1) with \( c_{ik} = 0, i = 1, 2, k \in \mathbb{N} \).

Remark 3.5. When \( c_{ik} = 0, k \in \mathbb{N}, i = 1, 2 \), the existence of periodic solutions of dynamical system (1.1) on time scales with monotonic functional responses \( \varphi_1 \)- \( \varphi_3 \) and \( \varphi_5 \)-\( \varphi_7 \) is investigated by Bohner et al [6]. The following conditions (the notation used here are same as that in corresponding papers cited)

(i) \( c(t, x) \leq C_0(t)x \) and \( \bar{b}e > \bar{C}_0\bar{d}\exp\{\bar{a}|a| + \bar{d} + |\bar{d}|\} \) in [6] Theorem 3.4;

(ii) \( c(t, x) \leq C_1(t) \) and \( e^\bar{a} < C^\dagger_1 \dagger \bar{d} \) in [6] Theorem 3.5

can be eliminated.

Moreover, our results are applicable for both monotone and nonmonotone functional responses. By Corollary 3.4, the conditions

(iii) global monotonicity of response function \( f(t, x) \) with respect to \( x \) in [14] Theorem 1 ;

(iv) \( f(\cdot, x) \) is monotone function for \( 0 < x < \bar{a}/\bar{b} \) in [14] Theorem 2 ;

(v) \( A^\dagger > (\bar{a}/\bar{b})^2 \) in [14] Corollary 3

are not necessary. Thus, our results improve and extend their related work.

4. Applications

In this section, we apply Theorem 3.2 Theorem 3.3 and Corollary 3.4 to impulsive differential/difference systems (1.3)–(1.4) with all kinds of functional responses mentioned in Section 1. The functions \( r, A, B, C \) and \( D \), appearing in the functional responses \( \varphi_1 \)-\( \varphi_{10} \), are all \( \omega \)-periodic and \( r(t) > 0, A(t) > 0, B(t) \geq 0, C(t) \geq 0 \) and \( D(t) > 0 \). Then it holds that

\[ \begin{align*}
\varphi_1(t, x) &\leq r(t)x, \quad \varphi_2(t, x) \leq r(t)x/A(t), \quad \varphi_3(t, x) \leq r(t)x^2/A(t), \\
\varphi_4(t, x) &\leq r(t)x/A(t), \quad \varphi_5(t, x) \leq r(t)x^\theta/A(t)(\theta > 2), \\
\varphi_6(t, x) &\leq r(t)x^2/(A(t)D(t)), \quad \varphi_7(t, x) \leq r(t)A(t)x, \quad \varphi_8(t, x, y) \leq r(t)x/A(t), \\
\varphi_9(t, x, y) &\leq r(t)x/(A(t)D(t)), \quad \varphi_{10}(t, x, y) \leq r(t)x^2/(A(t)D(t))
\end{align*} \]

for \( t \in T, x \in \mathbb{R}^+ \) and \( y \in \mathbb{R}^+ \). Consequently, we have the following results.

Proposition 4.1. Impulsive differential / difference system (1.3) or (1.4) with monotonic functional response \( \varphi_1 \)-\( \varphi_3 \) and \( \varphi_5 \)-\( \varphi_7 \) has at least one positive periodic solution if (3.2) holds. Moreover, it must have at least one positive periodic solution if \( c_{ik} = 0, k \in \mathbb{N}, i = 1, 2 \).

Remark 4.2. The case of \( c_{ik} = 0, k \in \mathbb{N}, i = 1, 2 \) in (1.3) or (1.4) with monotonic functional responses \( \varphi_1 \)-\( \varphi_3 \) and \( \varphi_5 \)-\( \varphi_7 \) is studied by Bohner et al [6], Huo and Li [18], Fan and Wang [12], Wang et al [25]. By Proposition 4.1, the conditions

(vi) \( r_2(t) \geq a_2(t) \) in [18] Theorem 2.1;
(vii) (A7): $C_0 \bar{d} \exp\{2(\bar{a} + \bar{d})\omega\}/(\bar{b} \bar{e}) < 1$ in [25, Theorem 3.3];
(viii) $\bar{b} \bar{e} > \bar{p}_0 \bar{d} \exp\{A + \bar{a} + \bar{D} + \bar{d})\omega\}$ in [12, Theorem 2.1];
(ix) $e^\bar{a} > p_1^* \bar{d}$ in [12, Theorem 2.2]

may be redundant.

**Proposition 4.3.** Impulsive differential/difference system (1.3) or (1.4) with non-monotonic functional response $\varphi_4$ has at least one positive periodic solution if \( (3.2) \) holds. Moreover, it must have at least one positive periodic solution if $c_{ik} = 0$, $k \in \mathbb{N}$, $i = 1, 2$.

**Remark 4.4.** By Proposition 4.3 the following conditions are not necessary:

(x) $\bar{r}_1 - a_{12} \exp\{H_2\}/m^2 > 0$ in [10, Theorem 2.1];

(xi) the monotonicity of response function with respect to prey in [12, condition (H2), P55] and in Fazly and Hesaaraki [13, Theorem 1].

For predator-dependent response functions we also have a result.

**Proposition 4.5.** Impulsive differential / difference system (1.3) or (1.4) with predator-dependent function responses $\varphi_8 - \varphi_{10}$ has at least one positive periodic solution if \( (3.2) \) holds. Moreover, it must have at least one positive periodic solution if $c_{ik} = 0$, $k \in \mathbb{N}$, $i = 1, 2$.

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