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EXISTENCE AND UNIQUENESS FOR BOUNDARY-VALUE PROBLEM WITH ADDITIONAL SINGLE POINT CONDITIONS OF THE STOKES-BITSADZE SYSTEM

MUHAMMAD TAHIR

ABSTRACT. This article shows the uniqueness of a solution to a Bitsadze system of equations, with a boundary-value problem that has four additional single point conditions. It also shows how to construct the solution.

1. Introduction

The planar Stokes flow based on stream function $\psi(x,y)$ and stress function $\phi(x,y)$, is expressed as

$$\begin{aligned} \phi_{xx} - \phi_{yy} &= -4\eta\psi_{xy}, \\ -\phi_{xy} &= \eta(\psi_{yy} - \psi_{xx}), \end{aligned} \tag{1.1}$$

where η is a material constant, see for the details [4, 5, 9]. The re-scaling $(2\eta\psi \to \psi)$ reduces the system (1.1) to

$$\begin{aligned} \phi_{xx} - \phi_{yy} + 2\psi_{xy} &= 0, \\ \psi_{xx} - \psi_{yy} - 2\phi_{xy} &= 0, \end{aligned} \tag{1.2}$$

which is the famous second order elliptic system called the Bitsadze system of equations and is identified as Stokes-Bitsadze system [10]. In the literature Bitsadze appears to have been the first to question the uniqueness and existence or even the well-posedness of (1.2) subject to certain boundary conditions, see for reference [2, 3, 7]. Oshorov [8] finds well-posed problems for the Cauchy-Riemann system and extends those to the Bitsadze system (1.2). Vaitekhovich [12] discusses Dirichlet and Schwarz problems for the inhomogeneous Bitsadze equation for a circular ring domain. In the interior of unit disc a boundary value problem for the Bitsadze equation is considered by Babayan [1] and is proved to be Noetherian. In his paper Babayan also proposes solvability conditions for the inhomogeneous Bitsadze equation. The unique solvability in a unit disc for the inhomogeneous Bitsadze system is discussed in [6].

The Stokes-Bitsadze system (1.2) can be expressed in the matrix form as

$$A\mathbf{U}_{xx} + 2B\mathbf{U}_{xy} + C\mathbf{U}_{yy} = \mathbf{0}, \tag{1.3}$$

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where

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad C = -A, \quad \mathbf{U}(x,y) = \begin{pmatrix} \phi \\ \psi \end{pmatrix}.$$

In a domain $\Omega \subset \mathbb{R}^2$ with boundary Γ a linear boundary value problem of Poincaré for the system (1.3) can be formulated as

$$p_1 \mathbf{U}_x + p_2 \mathbf{U}_y + q\mathbf{U} = \boldsymbol{\alpha}(x,y), \quad (x,y) \in \Gamma \tag{1.4}$$

where p_1, p_2, q are real 2×2 matrices and $\boldsymbol{\alpha}(x,y)$ a real vector given on the boundary Γ. The boundary-value problems of Poincaré for the Stokes-Bitsadze system will be discussed elsewhere. In this paper we are interested in a boundary value problem with four additional single point conditions.

2. A boundary value problem with additional single point conditions

We consider the Stokes-Bitsadze system (1.2) in domain $\Omega \subset \mathbb{R}^2$ with boundary Γ subject to the following boundary conditions.

$$\psi = f, \quad \psi_n = g \quad \text{on } \Gamma, \tag{2.1}$$

and

$$\phi = \phi^P, \quad \nabla\phi = (\nabla\phi)^P, \quad \Delta\phi = (\Delta\phi)^P, \quad \text{at a single point } P \in \bar{\Omega}. \tag{2.2}$$

Theorem 2.1. *For $f, g \in C(\Gamma)$, the boundary value problem (2.1)–(2.2) for the Stokes-Bitsadze system (1.2) has a unique solution $(\phi,\psi) \in C^4(\Omega) \times C^4(\Omega)$.*

Proof. Suppose $\phi, \psi \in C^4(\Omega)$. If (ϕ,ψ) satisfies (1.2), then ϕ and ψ are biharmonic in Ω, and for $f, g \in C(\Gamma)$ the problem

$$\begin{gathered} \Delta^2\psi = 0 \quad \text{in } \Omega \\ \psi = f \quad \text{on } \Gamma \\ \psi_n = g \quad \text{on } \Gamma \end{gathered} \tag{2.3}$$

has a unique solution $\psi \in C^4(\Omega)$, [11], that satisfies (1.2) and (2.1). Let the unique solution be denoted by $\widetilde{\psi}$. Now we show that for the unique $\widetilde{\psi}$ if there exists ϕ satisfying (1.2) and (2.1)–(2.2) then that ϕ is unique. Assume that the pairs $(\phi_1, \widetilde{\psi})$ and $(\phi_2, \widetilde{\psi})$ with $\phi_1 \neq \phi_2$ satisfy (1.2) and (2.1)–(2.2) and that $\delta = \phi_1 - \phi_2$. Then from (1.2) it immediately follows that

$$\delta_{xx} - \delta_{yy} = 0, \quad \delta_{xy} = 0 \quad \text{on } \Omega. \tag{2.4}$$

But (2.2) then yields

$$\delta = 0, \quad \nabla\delta = 0, \quad \Delta\delta = 0 \quad \text{at } P, \tag{2.5}$$

and the general solution of the system (2.4) becomes,

$$\delta = ax + by + c(x^2 + y^2) + d, \tag{2.6}$$

which on imposing the conditions (2.5) gives $\delta \equiv 0$ in $\bar{\Omega}$ and uniqueness of ϕ thus follows. Hence there exists at most one pair $(\phi,\psi) \in C^4(\Omega) \times C^4(\Omega)$ that can satisfy (1.2) and (2.1)–(2.2). We are now in a position to assume (without proof) that $(\widetilde{\phi}, \widetilde{\psi})$ is a solution of (1.2) and (2.1)–(2.2).

Next, we suppose that $P(x_P, y_P)$ and $Q(x, y_P)$ are the points in $\bar{\Omega}$, refer to the Figure 1.

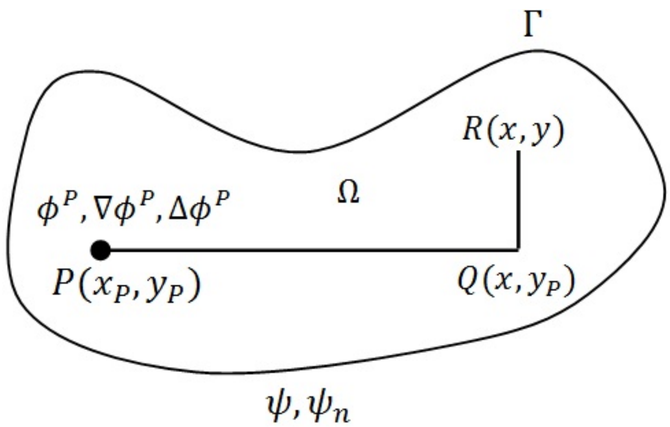

FIGURE 1. Boundary conditions and additional single point conditions

At point P the expressions (1.2)(a) and (2.2)(c) respectively take the form

$$\begin{gathered}\phi^P_{xx} - \phi^P_{yy} = -2\psi^P_{xy},\\ \phi^P_{xx} + \phi^P_{yy} = \Delta\phi^P,\end{gathered} \tag{2.7}$$

from which it is obvious that ϕ^P_{xx} and ϕ^P_{yy} are known at P. Since $(\widetilde{\phi}, \widetilde{\psi})$ satisfies (1.2)(b), therefore

$$\widetilde{\phi}_{xyy} = \frac{1}{2}[\widetilde{\psi}_{xxy} - \widetilde{\psi}_{yyy}], \tag{2.8}$$

and on integration along PQ we have

$$\widetilde{\phi}_{yy}(x, y_P) = \phi^P_{yy} + \frac{1}{2}\int_{x_P}^{x} [\widetilde{\psi}_{xxy}(\lambda, y_P) - \widetilde{\psi}_{yyy}(\lambda, y_P)]d\lambda, \tag{2.9}$$

$$\widetilde{\phi}_{y}(x, y_P) = \phi^P_{y} + \frac{1}{2}\int_{x_P}^{x} [\widetilde{\psi}_{xx}(\lambda, y_P) - \widetilde{\psi}_{yy}(\lambda, y_P)]d\lambda. \tag{2.10}$$

Since all the terms on right hand sides of (2.9) and (2.10) are known therefore $\widetilde{\phi}_{yy}$ and $\widetilde{\phi}_y$ are known along PQ. Since $(\widetilde{\phi}, \widetilde{\psi})$ satisfies (1.2)(a), we have

$$\widetilde{\phi}_{xx} = \widetilde{\phi}_{yy} - 2\widetilde{\psi}_{xy}, \tag{2.11}$$

and using (2.9), can further be expressed as

$$\widetilde{\phi}_{xx}(x, y_P) = \phi^P_{yy} + \frac{1}{2}\int_{x_P}^{x} [\widetilde{\psi}_{xxy}(\lambda, y_P) - \widetilde{\psi}_{yyy}(\lambda, y_P)]d\lambda - 2\widetilde{\psi}_{xy}(\lambda, y_P). \tag{2.12}$$

Further on integration along PQ, we have

$$\begin{aligned}\widetilde{\phi}_{x}(x, y_P) = \phi^P_{x} &+ \int_{x_P}^{x} \Big[\phi^P_{yy} + \frac{1}{2}\int_{x_P}^{\mu} [\widetilde{\psi}_{xxy}(\lambda, y_P) - \widetilde{\psi}_{yyy}(\lambda, y_P)]\Big]\, d\lambda\, d\mu\\ &- 2\int_{x_P}^{x} \widetilde{\psi}_{xy}(\lambda, y_P)\, d\lambda,\end{aligned} \tag{2.13}$$

whence

$$\begin{aligned} &\widetilde{\phi}(x, y_P) \\ &= \phi^P + (x - x_P)\phi_x^P + \frac{1}{2}(x - x_P)^2\phi_{yy}^P - 2\int_{x_P}^{x}\int_{x_P}^{\mu} \widetilde{\psi}_{xy}(\lambda, y_P) d\lambda\, d\mu \\ &\quad + \frac{1}{2}\int_{x_P}^{x}\int_{x_P}^{\nu}\int_{x_P}^{\mu} \big[\widetilde{\psi}_{xxy}(\lambda, y_P) - \widetilde{\psi}_{yyy}(\lambda, y_P)\big]\, d\lambda\, d\mu\, d\nu. \end{aligned} \tag{2.14}$$

Since all the terms on right hand sides of (2.11), (2.12), (2.13) are known therefore $\widetilde{\phi}_{xx}, \widetilde{\phi}_x$ and $\widetilde{\phi}$ are known along PQ and hence we know $\widetilde{\phi}, \nabla\widetilde{\phi}$ and $\Delta\widetilde{\phi}$ at $Q(x, y_P)$.

Now from the point Q we draw the line QR where $R(x, y) \in \bar{\Omega}$ is an arbitrary point. Again, since $(\widetilde{\phi}, \widetilde{\psi})$ satisfies (1.2)(b); therefore

$$\widetilde{\phi}_{xxy} = \frac{1}{2}[\widetilde{\psi}_{xxx} - \widetilde{\psi}_{xyy}], \tag{2.15}$$

which on integration, along QR, gives

$$\widetilde{\phi}_{xx}(x, y) = \widetilde{\phi}_{xx}(x, y_P) + \frac{1}{2}\int_{y_P}^{y} [\widetilde{\psi}_{xxx}(x, \lambda) - \widetilde{\psi}_{xyy}(x, \lambda)] d\lambda, \tag{2.16}$$

$$\widetilde{\phi}_{x}(x, y) = \widetilde{\phi}_{x}(x, y_P) + \frac{1}{2}\int_{y_P}^{y} [\widetilde{\psi}_{xx}(x, \lambda) - \widetilde{\psi}_{yy}(x, \lambda)] d\lambda. \tag{2.17}$$

But the following expression from (1.2)(a)

$$\widetilde{\phi}_{yy} = \widetilde{\phi}_{xx} + 2\widetilde{\psi}_{xy}, \tag{2.18}$$

on integration along QR gives

$$\widetilde{\phi}_{y}(x, y) = \widetilde{\phi}_{y}(x, y_P) + \int_{y_P}^{y} [\widetilde{\phi}_{xx}(x, \lambda) + 2\widetilde{\psi}_{xy}(x, \lambda)]\, d\lambda. \tag{2.19}$$

Using (2.10) and (2.16) the expression (2.19) takes the form

$$\begin{aligned} \widetilde{\phi}_{y}(x, y) &= \phi_y^P + \frac{1}{2}\int_{x_P}^{x} [\widetilde{\psi}_{xx}(\lambda, y_P) - \widetilde{\psi}_{yy}(\lambda, y_P)] d\lambda + (y - y_P)\widetilde{\phi}_{xx}(x, y_P) \\ &\quad + \frac{1}{2}\int_{y_P}^{y}\int_{y_P}^{\mu} [\widetilde{\psi}_{xxx}(x, \lambda) - \widetilde{\psi}_{xyy}(x, \lambda)] d\lambda\, d\mu + 2\int_{y_P}^{y} \widetilde{\psi}_{xy}(x, \lambda) d\lambda. \end{aligned} \tag{2.20}$$

Integrating along QR we obtain from (2.20) as follows.

$$\begin{aligned} \widetilde{\phi}(x, y) &= \widetilde{\phi}(x, y_P) + (y - y_P)\phi_y^P + \frac{1}{2}(y - y_P)^2\widetilde{\phi}_{xx}(x, y_P) \\ &\quad + \frac{1}{2}(y - y_P)\int_{x_P}^{x} [\widetilde{\psi}_{xx}(\lambda, y_P) - \widetilde{\psi}_{yy}(\lambda, y_P)] d\lambda \\ &\quad + \frac{1}{2}\int_{y_P}^{y}\int_{y_P}^{\nu}\int_{y_P}^{\mu} [\widetilde{\psi}_{xxx}(x, \lambda) - \widetilde{\psi}_{xyy}(x, \lambda)] d\lambda\, d\mu d\nu \\ &\quad + 2\int_{y_P}^{y}\int_{y_P}^{\mu} \widetilde{\psi}_{xy}(x, \lambda) d\lambda\, d\mu. \end{aligned} \tag{2.21}$$

Using (2.12) and (2.14) we finally obtain the following expression for $\widetilde{\phi}(x,y)$ at an arbitrary point $(x,y)\in\bar{\Omega}$.

$$\begin{aligned}
&\widetilde{\phi}(x,y)\\
&=\phi^P+(x-x_P)\phi^P_x+(y-y_P)\phi^P_y+\frac{1}{2}[(x-x_P)^2+(y-y_P)^2]\phi^P_{yy}\\
&\quad-(y-y_P)^2\widetilde{\psi}_{xy}(x,y_P)+\frac{1}{2}(y-y_P)\int_{x_P}^{x}[\widetilde{\psi}_{xx}(\lambda,y_P)-\widetilde{\psi}_{yy}(\lambda,y_P)]\,d\lambda\\
&\quad+\frac{1}{4}(y-y_P)^2\int_{x_P}^{x}[\widetilde{\psi}_{xxy}(\lambda,y_P)-\widetilde{\psi}_{yyy}(\lambda,y_P)]d\lambda\\
&\quad-2\int_{x_P}^{x}\int_{x_P}^{\mu}\widetilde{\psi}_{xy}(\lambda,y_P)\,d\lambda\,d\mu+2\int_{y_P}^{y}\int_{y_P}^{\mu}\widetilde{\psi}_{xy}(x,\lambda)d\lambda\,d\mu\\
&\quad+\frac{1}{2}\int_{x_P}^{x}\int_{x_P}^{\nu}\int_{x_P}^{\mu}[\widetilde{\psi}_{xxy}(\lambda,y_P)-\widetilde{\psi}_{yyy}(\lambda,y_P)]d\lambda\ d\mu\ d\nu\\
&\quad+\frac{1}{2}\int_{y_P}^{y}\int_{y_P}^{\nu}\int_{y_P}^{\mu}[\widetilde{\psi}_{xxx}(x,\lambda)-\widetilde{\psi}_{xyy}(x,\lambda)]d\lambda\ d\mu\,d\nu.
\end{aligned}\tag{2.22}$$

Obviously we have obtained an explicit representation for $\widetilde{\phi}$ in terms of the point conditions and $\widetilde{\psi}$, on the assumption that $(\widetilde{\phi},\widetilde{\psi})$ satisfies (1.2) and (2.1)–(2.2). Next we show that $(\widetilde{\phi},\widetilde{\psi})$ actually satisfies the Bitsadze system (1.2) and the conditions (2.2).

From expression (2.22) it is easy to verify that $\widetilde{\phi}(x_P,y_P)=\phi^P$. We use (2.13) in (2.17) to obtain

$$\begin{aligned}
\widetilde{\phi}_x(x,y)&=\phi^P_x+\int_{x_P}^{x}[\phi^P_{yy}+\frac{1}{2}\int_{x_P}^{\mu}[\widetilde{\psi}_{xxy}(\lambda,y_P)-\widetilde{\psi}_{yyy}(\lambda,y_P)]d\lambda]\,d\mu\\
&\quad-2\int_{x_P}^{x}\widetilde{\psi}_{xy}(\lambda,y_P)\,d\lambda+\frac{1}{2}\int_{y_P}^{y}[\widetilde{\psi}_{xx}(x,\lambda)-\widetilde{\psi}_{yy}(x,\lambda)]d\lambda,
\end{aligned}$$

and it can be easily verified that $\widetilde{\phi}_x(x_P,y_P)=\phi^P_x$. Similarly from (2.10) and (2.20) we have

$$\widetilde{\phi}_y(x,y)=\phi^P_y+\frac{1}{2}\int_{x_P}^{x}[\widetilde{\psi}_{xx}(\lambda,y_P)-\widetilde{\psi}_{yy}(\lambda,y_P)]\,d\lambda+\int_{y_P}^{y}[\widetilde{\phi}_{xx}(x,\lambda)+2\widetilde{\psi}_{xy}(x,\lambda)]\,d\lambda,$$

and it follows that $\widetilde{\phi}_y(x_P,y_P)=\phi^P_y$. Again, from (2.12)and (2.16) we obtain

$$\begin{aligned}
\widetilde{\phi}_{xx}(x,y)&=\phi^P_{yy}+\frac{1}{2}\int_{x_P}^{x}[\widetilde{\psi}_{xxy}(\lambda,y_P)-\widetilde{\psi}_{yyy}(\lambda,y_P)]\,d\lambda-2\widetilde{\psi}_{xy}(x,y_P)\\
&\quad+\frac{1}{2}\int_{y_P}^{y}[\widetilde{\psi}_{xxx}(x,\lambda)-\widetilde{\psi}_{xyy}(x,\lambda)]\,d\lambda,
\end{aligned}$$

which at P yields

$$\widetilde{\phi}_{xx}(x_P,y_P)=\phi^P_{yy}-2\widetilde{\psi}_{xy}(x_P,y_P),\tag{2.23}$$

and from (2.7)(a) we obtain $\widetilde{\phi}_{xx}(x_P,y_P)=\phi^P_{xx}$. Also from (2.18) it is obvious that

$$\widetilde{\phi}_{yy}(x_P,y_P)=\widetilde{\phi}_{xx}(x_P,y_P)+2\widetilde{\psi}_{xy}(x_P,y_P),\tag{2.24}$$

and (2.23)–(2.24) yield $\widetilde{\phi}_{yy}(x_P,y_P)=\phi^P_{yy}$.

Now we verify that $\widetilde{\phi}(x,y)$ satisfies (1.2)(a). Using (2.10) in (2.20) and then differentiating with respect to x we obtain

$$\begin{aligned}\widetilde{\phi}_{xy}(x,y) &= \frac{1}{2}[\widetilde{\psi}_{xx}(x,y_P)-\widetilde{\psi}_{yy}(x,y_P)] + \frac{1}{2}(y-y_P)[\widetilde{\psi}_{xxy}(x,y_P)-\widetilde{\psi}_{yyy}(x,y_P)]\\ &\quad - 2(y-y_P)\widetilde{\psi}_{xxy}(x,y_P) + \frac{1}{2}\int_{y_p}^{y}\int_{y_P}^{\mu}[\widetilde{\psi}_{xxxx}(x,\lambda)-\widetilde{\psi}_{xxyy}(x,\lambda)]\,d\lambda\,d\mu\\ &\quad + 2\widetilde{\psi}_{xx}(x,y) - 2\widetilde{\psi}_{xx}(x,y_P),\end{aligned}$$

which, since $\Delta^2\widetilde{\psi}=0$, can be simplified as

$$\begin{aligned}&\widetilde{\phi}_{xy}(x,y)\\ &= -\frac{1}{2}[3\widetilde{\psi}_{xx}(x,y_P)+\widetilde{\psi}_{yy}(x,y_P)] - \frac{1}{2}(y-y_P)[3\widetilde{\psi}_{xxy}(x,y_P)+\widetilde{\psi}_{yyy}(x,y_P)]\\ &\quad - \frac{1}{2}[3\widetilde{\psi}_{xx}(x,y_P)+\widetilde{\psi}_{yy}(x,y)] + \frac{1}{2}[3\widetilde{\psi}_{xx}(x,y_P)+\widetilde{\psi}_{yy}(x,y_P)]\\ &\quad + \frac{1}{2}(y-y_P)[3\widetilde{\psi}_{xxy}(x,y_P)+\widetilde{\psi}_{yyy}(x,y_P)] + 2\widetilde{\psi}_{xx}(x,y),\end{aligned} \tag{2.25}$$

and we obtain

$$\widetilde{\phi}_{xy}(x,y) = \frac{1}{2}[\widetilde{\psi}_{xx}(x,y)-\widetilde{\psi}_{yy}(x,y)]. \tag{2.26}$$

Then, to verify that $\widetilde{\phi}(x,y)$ satisfies (1.2)(b), we use (2.22) to obtain

$$\begin{aligned}&\widetilde{\phi}_{xx}(x,y)-\widetilde{\phi}_{yy}(x,y)\\ &= -(y-y_P)^2\widetilde{\psi}_{xxxy}(x,y_P) + \frac{1}{2}(y-y_P)[\widetilde{\psi}_{xxx}(x,y_P)-\widetilde{\psi}_{xyy}(x,y_P)]\\ &\quad + \frac{1}{4}(y-y_P)^2[\widetilde{\psi}_{xxxy}(x,y_P)-\widetilde{\psi}_{xyyy}(x,y_P)]\\ &\quad + 2\int_{y_P}^{y}\int_{y_P}^{\mu}\widetilde{\psi}_{xxxy}(x,\lambda)d\lambda\,d\mu + \frac{1}{2}\int_{x_P}^{x}[\widetilde{\psi}_{xxy}(\lambda,y_P)-\widetilde{\psi}_{yyy}(\lambda,y_P)]\,d\lambda\\ &\quad + \frac{1}{2}\int_{y_P}^{y}\int_{y_P}^{\nu}\int_{y_P}^{\mu}[\widetilde{\psi}_{xxxxx}(x,\lambda)-\widetilde{\psi}_{xxxyy}(x,\lambda)]\,d\lambda\,d\mu\,d\nu\\ &\quad - \frac{1}{2}\int_{x_P}^{x}[\widetilde{\psi}_{xxy}(\lambda,y_P)-\widetilde{\psi}_{yyy}(\lambda,y_P)]d\lambda - 2\widetilde{\psi}_{xy}(x,y)\\ &\quad - \frac{1}{2}\int_{y_P}^{y}[\widetilde{\psi}_{xxx}(x,\lambda)-\widetilde{\psi}_{xyy}(x,\lambda)]\,d\lambda,\end{aligned}$$

which can further be simplified to obtain

$$\begin{aligned}&\widetilde{\phi}_{xx}(x,y)-\widetilde{\phi}_{yy}(x,y)\\ &= -\frac{1}{4}(y-y_P)^2[3\widetilde{\psi}_{xxxy}(x,y_P)+\widetilde{\psi}_{xyyy}(x,y_P)]\\ &\quad - \frac{1}{2}(y-y_P)[3\widetilde{\psi}_{xxx}(x,y_P)+\widetilde{\psi}_{xyy}(x,y_P)]\\ &\quad - \frac{1}{2}\int_{y_P}^{y}[3\widetilde{\psi}_{xxx}(x,\lambda)+\widetilde{\psi}_{xyy}(x,\lambda)]\,d\lambda + \frac{1}{2}(y-y_P)[3\widetilde{\psi}_{xxx}(x,y_P)+\widetilde{\psi}_{xyy}(x,y_P)]\\ &\quad + \frac{1}{4}(y-y_P)^2[3\widetilde{\psi}_{xxxy}(x,y_P)+\widetilde{\psi}_{xyyy}(x,y_P)]\end{aligned}$$

$$-2\widetilde{\psi}_{xy}(x,y)+\frac{1}{2}\int_{y_P}^{y}[3\widetilde{\psi}_{xxx}(x,\lambda)+\widetilde{\psi}_{xyy}(x,\lambda)]d\lambda,$$

and finally we have

$$\widetilde{\phi}_{xx}(x,y)-\widetilde{\phi}_{yy}(x,y)=-2\widetilde{\psi}_{xy}(x,y),$$

which completes the proof. □

Conclusion. It has been proved by construction that there exists a unique solution $(\widetilde{\phi},\widetilde{\psi})$ in $C^4(\Omega)\times C^4(\Omega)$ to the Stokes-Bitsadze system (1.2) subject to the boundary conditions (2.1) along with additional single point conditions (2.2).

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Muhammad Tahir
Mathematics Department, HITEC University, Taxila Cantonment, Pakistan
E-mail address: mtahir@hitecuni.edu.pk