SIMULTANEOUS AND NON-SIMULTANEOUS BLOW-UP AND
UNIFORM BLOW-UP PROFILES FOR REACTION-DIFFUSION
SYSTEM

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Abstract. This article concerns the blow-up solutions of a reaction-diffusion
system with nonlocal sources, subject to the homogeneous Dirichlet boundary
conditions. The criteria used to identify simultaneous and non-simultaneous
blow-up of solutions by using the parameters $p$ and $q$ in the model are proposed.
Also, the uniform blow-up profiles in the interior domain are established.

1. Introduction and description of results

In this article, we investigate the following reaction-diffusion system with nonlo-
cal sources
\begin{align*}
  u_t &= \Delta u + \|uw\|_p^p, \quad (x, t) \in \Omega \times (0, T), \\
  v_t &= \Delta v + \|uw\|_q^q, \quad (x, t) \in \Omega \times (0, T) \\
  u(x, 0) &= u_0(x), \quad v(x, 0) = v_0(x), \quad x \in \Omega, \quad (1.3) \\
  u(x, t) &= 0, \quad v(x, t) = 0, \quad (x, t) \in \partial \Omega \times (0, T), \quad (1.4)
\end{align*}
where $\Omega = B_R = \{|x| < R\} \subset \mathbb{R}^N$ ($N \geq 1$), $\alpha, \beta \geq 1$, $p, q > 0$, and the continuous
functions $u_0(x), v_0(x)$ are nonnegative, nontrivial, radially symmetric, decreasing
with $|x|$, and vanish on $\partial B_R$, where $\| \cdot \|_q^q = \int_{\Omega} |\cdot|^q \, dx$.

Nonlinear parabolic systems (1.1)-(1.4) can be used to describe some reaction
diffusion phenomena, such as heat propagations in a two-component combustible
mixture [3], chemical reactions [6], interaction of two biological groups without
self-limiting [10], etc., where $u$ and $v$ represent the temperatures of two different
materials during a propagation, the thicknesses of two kinds of chemical reactants,
the densities of two biological groups during a migration, etc. Using the methods
of [7, 12, 4] we know that (1.1)-(1.4) has a local nonnegative classical solution.
Moreover, if $p, q \geq 1$, then the uniqueness holds.

In recent years, many results on blow-up solutions have been obtained for the
nonlinear parabolic system. We will recall several results in the following. As for the
other related works on the global existence and blow-up of solutions of the nonlinear
parabolic system, they can be found in [15, 16, 14] and references therein.

2000 Mathematics Subject Classification. 35B33, 35B40, 35K55, 35K57.
Key words and phrases. Simultaneous and non-simultaneous blow-up;
uniform blow-up profile; reaction-diffusion system; nonlocal sources.
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Li, Huang and Xie in [8] and Deng, Li and Xie in [2] considered the following two systems, respectively,

\[ u_t = \Delta u + \int_{\Omega} u^m(x, t)v^n(x, t) \, dx, \quad v_t = \Delta v + \int_{\Omega} u^p(x, t)v^q(x, t) \, dx, \]

with \( x \in \Omega, \ t > 0; \) and

\[ u_t = \Delta u^m + a\|v\|_p^p, \quad v_t = \Delta v^n + b\|u\|_q^q, \quad (x, t) \in \Omega \times (0, T). \]

The authors showed some results on the global solutions, the blow-up solutions and the blow-up profiles. In 2002, Zheng, Zhao and Chen in [18] studied the problem

\[ u_t = \Delta u + f_1(u, v), \quad v_t = \Delta v + f_2(u, v), \quad (x, t) \in \Omega \times (0, T) \quad (1.5) \]

with homogeneous Dirichlet boundary conditions, where

\[ f_1(u, v) = e^{mu(x, t) + pv(x, t)}, \quad f_2(u, v) = e^{qu(x, t) + v(x, t)}. \]

The simultaneous blow-up rates are obtained for radially symmetric blow-up solutions in the exponent region \( \{ 0 \leq m < q, 0 \leq n < p \}. \)

Later, Zhao and Zheng in [17], Li and Wang in [9] studied the localized problem (1.5) with the more general \( \Omega \subset \mathbb{R}^N \) and

\[ f_1(u, v) = e^{mu(x_0, t) + pv(x_0, t)}, \quad f_2(u, v) = e^{qu(x_0, t) + v(x_0, t)}, \quad x_0 \in \Omega. \]

The critical blow-up exponents were discussed. Uniform blow-up profiles for simultaneous blow-up solutions were proved in the exponent region \( \{ 0 \leq m \leq q, 0 \leq n \leq p \}. \)

Our present work is motivated by the above mentioned papers, the main purpose of this paper is to identify the simultaneous and non-simultaneous blow-up of the solutions and establish the uniform blow-up profiles for the system (1.1)–(1.4).

For convenience, we introduce a pair of parameters \( \sigma \) and \( \theta \), the solution of

\[ \begin{pmatrix} p - 1 \\ q \\ q - 1 \end{pmatrix} \begin{pmatrix} \sigma \\ \theta \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \]

(1.6)

namely,

\[ \sigma = \frac{p - (q - 1)}{p + q - 1}, \quad \theta = \frac{q - (p - 1)}{p + q - 1}. \]

(1.7)

This paper is organized as follows. In the next Section, we investigate the simultaneous and non-simultaneous blow-up of the solutions for the system (1.1)–(1.4), and give the blow-up criteria. In Section 3, we deal with the blow-up rates of the solutions.

2. SIMULTANEOUS AND NON-SIMULTANEOUS BLOW-UP

In this section, we discuss the simultaneous and non-simultaneous blow-up phenomena for the system (1.1)–(1.4), and propose a complete and optimal classification to identify the simultaneous and non-simultaneous blow-up solutions.

For problem (1.1)–(1.4), because of the nonlinear sources, there exist solution \((u, v)\) that blow up in finite time, \( T \), if and only if the exponents \( p, q \) verify any of conditions, \( p > 1, q > 1 \) or \( pq > (q - 1)(p - 1) \). In particular, the component \( u \) (or \( v \)) can blow up for the large initial data if \( p > q - 1 \) (or \( q > p - 1 \)), see [9, 12]. So there may be non-simultaneous blow-up, that is to say that one component blows...
up while the other remains bounded. On the other hand, the simultaneous blow-up means that
\[
\limsup_{t \to T} \|u(\cdot, t)\|_{\infty} = \limsup_{t \to T} \|v(\cdot, t)\|_{\infty} = +\infty.
\]
Assume the initial data \(u_0(x), v_0(x)\) satisfy
\[
\Delta u_0(x) + \|u_0 v_0\|_p^p - \varepsilon \varphi(x) u_0^p(0) v_0^p(0) \geq 0, \quad x \in B_R, \quad (2.1)
\]
\[
\Delta v_0(x) + \|u_0 v_0\|_q^q - \varepsilon \varphi(x) u_0^q(0) v_0^q(0) \geq 0, \quad x \in B_R \quad (2.2)
\]
for some a constant \(\varepsilon \in (0, 1)\), where \(\varphi(x)\) is the first eigenfunction of
\[-\Delta \varphi = \lambda \varphi, \quad x \in B_R; \quad \varphi = 0, \quad x \in \partial B_R, \]
normalized by \(\|\varphi\|_{\infty} = 1, \varphi > 0\) in \(B_R\). In addition, by using the methods in [16], it is easy to check that \(u_t, v_t \geq 0\) for \((x, t) \in B_R \times (0, T)\) by the comparison principle.

Our results about the simultaneous and non-simultaneous blow-up criteria are as follows.

**Theorem 2.1.** If \(p + q > 1\), then there exists initial data such that the non-simultaneous blow-up occurs in \((1.1)-(1.4)\) if and only if \(\sigma < 0\) (or \(\theta < 0\)) (for \(v\) blowing up alone, respectively).

**Theorem 2.2.** If \(p + q > 1\), then any blow-up in \((1.1)-(1.4)\) is non-simultaneous if and only if \(\sigma \geq 0\) with \(\theta < 0\) (for \(u\) blowing up alone), or \(\theta \geq 0\) with \(\sigma < 0\) (for \(v\) blowing up alone).

**Corollary 2.3.** If \(p + q > 1\), then any blow-up in \((1.1)-(1.4)\) is simultaneous if and only if \(\sigma \geq 0\) and \(\theta \geq 0\).

Similar to the study in [8], it is seen that

**Corollary 2.4.** All solutions are global in \((1.1)-(1.4)\) if and only if \(\sigma < 0\) and \(\theta < 0\) (i.e., \(p + q < 1\)).

In summary, the complete and optimal classification for simultaneous and non-simultaneous blow-up solutions of the problem \((1.1)-(1.4)\) can be described by Figure 1.

![Figure 1. Regions of simultaneous and non-simultaneous blow-up](image-url)
sufficient help to the blow-up of \( v \) (with small \( v_0 \)), while \( q < p - 1 \) ensures that \( v \) can provide effective help to the blow-up of \( u \), but \( v \) remains bounded. Before we give the proof of Theorem 2.1, we first introduce the following lemma. Let \( \phi(x, t) \) satisfy
\[
\phi_t = \Delta \phi, \ (x, t) \in B_R \times (0, T); \quad \phi = 0, \ (x, t) \in \partial B_R \times (0, T)
\]
with
\[
\phi(x, 0) = \varphi(x), \quad x \in B_R.
\]

**Lemma 2.5.** Under conditions \([2.1]\) and \([2.2]\), the solution \((u, v)\) of \([1.1]-[1.4]\) satisfies
\[
\begin{align*}
&u_t(x, t) \geq \varepsilon \phi(x, t) u^p(0, t) v^q(0, t), \quad (x, t) \in B_R \times [0, T), \quad \text{(2.3)} \\
v_t(x, t) \geq \varepsilon \phi(x, t) u^p(0, t) v^q(0, t), \quad (x, t) \in B_R \times [0, T). \quad \text{(2.4)}
\end{align*}
\]

**Proof.** Since the proofs of the inequalities \([2.3]\) and \([2.4]\) are similar, we prove only \([2.3]\). Let
\[
J(x, t) = u_t(x, t) - \varepsilon \phi(x, t) u^p(0, t) v^q(0, t).
\]
It is easy to check that for \( \varepsilon \) small enough since \( u_t, v_t \geq 0 \), we obtain
\[
\begin{align*}
J_t - \Delta J &= \left( \| uv \|_p \right)_t - \varepsilon \phi(u^p(0, t) v^q(0, t)) \right)_t \geq 0, \quad (x, t) \in B_R \times (0, T), \\
J(x, t) &= 0, \quad (x, t) \in \partial B_R \times (0, T), \\
J(x, 0) &= \Delta u_0(x) + \| u_0 v_0 \|^p - \varepsilon \varphi(x) u_0^p(0) v_0^q(0) \geq 0, \quad x \in B_R.
\end{align*}
\]
Consequently, \([2.3]\) is true by the comparison principle. \( \square \)

**Proof of Theorem 2.1.** Without loss of generality, we only prove that there exist suitable initial data such that \( u \) blows up while \( v \) remains bounded if and only if \( \theta < 0 \).

Assume \( \theta < 0 \), namely, \( p - 1 > q \) and \( p > 1 \) by Figure 1 and \([1.7]\). From \([2.3]\), we obtain that
\[
u_t(0, t) \geq \varepsilon \phi(0, T) u^p(0, t) v_0^q(0), \quad t \in [0, T). \quad \text{(2.5)}
\]
Integrating the above inequality \([2.5]\) from \( t \) to \( T \), we have the estimate for \( u \) as follows
\[
u(0, t) \leq \left( \varepsilon (p - 1) \phi(0, T) v_0^q(0) \right)^{-1/(p - 1)} (T - t)^{-1/(p - 1)}, \quad t \in [0, T). \quad \text{(2.6)}
\]
At the same time, since the initial data \((u_0, v_0)\) is radially symmetric and non-increasing, therefore the \((u, v)\) is also radial symmetrical and non-increasing; i.e., \( u_r(r, t), v_r(r, t) \leq 0 \) for \( r \in [0, R] \). Thus, \( u(x, t) \) and \( v(x, t) \) always reach their maxima at \( x = 0 \), which means that
\[
\Delta u(0, t) \leq 0, \quad \Delta v(0, t) \leq 0.
\]
Hence, from \([1.1]\) and \([1.2]\), we know that there exist constants \( C_1, C_2 > 0 \) such that
\[
\begin{align*}
u_t(0, t) &\leq \| uv \|_p^p \leq C_1 u^p(0, t) v^q(0, t), \quad t \in [0, T) \\
v_t(0, t) &\leq \| uv \|_q^q \leq C_2 u^p(0, t) v^q(0, t), \quad t \in [0, T). \quad \text{(2.7)}
\end{align*}
\]
Let
\[
\Gamma(x, y, t, s) = \frac{1}{[4\pi(t - s)]^{N/2}} \exp \left\{ -\frac{|x - y|^2}{4(t - s)} \right\}
\]
be the fundamental solution of the heat equation. Suppose that \((\bar{u}_0, \bar{v}_0)\) is a pair of initial data such that the solution of \([1.1]-[1.4]\) blows up. Fix radially symmetrical...
$v_0(\geq \tilde{v}_0)$ in $B_R$ and take constant $M_1 > v_0(x)$. By the proof of [11, Theorem 1.1], we know that if $u_0$ is large with $v_0$ fixed then $T$ becomes small. Therefore, let $u_0(\geq \tilde{u}_0)$ be large such that $T$ becomes small and satisfies

$$M_1 \geq v_0(0) + \frac{p - 1}{p - 1 - q} \left( \varepsilon(p - 1) \phi(0, T)v_0^p(0) \right)^{-\frac{q}{p - 1 - q}} T^{-\frac{p - 1 - q}{p - 1}} \|M_1\|_\beta^q,$$

where $\|M_1\|_\beta^q = (\int_{\Omega_1} M_1^q \, dx)^{1/q}$. Consider the following auxiliary problem

$$\bar{v}_t = \Delta \bar{v} + \left( \varepsilon(p - 1) \phi(0, T)v_0^p(0) \right)^{-\frac{q}{p - 1 - q}} (T - t)^{-\frac{p - 1 - q}{p - 1}} \|M_1\|_\beta^q, \quad (x, t) \in B_R \times (0, T),$$

$$\bar{v}(x, t) = 0, \quad (x, t) \in \partial B_R \times (0, T),$$

$$\bar{v}(x, 0) = v_0(x), \quad x \in B_R.$$

Since $p - 1 > q$, we obtain by Green’s identity that

$$\bar{v} \leq v_0(0) + \frac{p - 1}{p - 1 - q} \left( \varepsilon(p - 1) \phi(0, T)v_0^p(0) \right)^{-\frac{q}{p - 1 - q}} T^{-\frac{p - 1 - q}{p - 1}} \|M_1\|_\beta^q \leq M_1,$$

and hence $\bar{v}$ satisfies

$$\bar{v}_t \geq \Delta \bar{v} + \left( \varepsilon(p - 1) \phi(0, T)v_0^p(0) \right)^{-\frac{q}{p - 1 - q}} (T - t)^{-\frac{p - 1 - q}{p - 1}} \|\bar{v}(x, t)\|_\beta^q.$$

On the other hand, $v$ satisfies

$$v_t \geq \Delta v + \left( \varepsilon(p - 1) \phi(0, T)v_0^p(0) \right)^{-\frac{q}{p - 1 - q}} (T - t)^{-\frac{p - 1 - q}{p - 1}} \|v(x, t)\|_\beta^q.$$

Therefore, by the comparison principle, we conclude $v \leq \bar{v} \leq M_1$.

Now assume that $u$ blows up while $v$ remains bounded. By (2.7) we have

$$u_t(0, t) \leq Cu^p(0, t), \quad \text{for } t \in [0, T).$$

This implies $p > 1$ and the estimate for $u$ that

$$u(0, t) \geq \left( C(p - 1)^{-1/(p-1)}(T - t)^{-1/(p-1)} \right).$$

Therefore, by using (2.4), we have

$$v_t(0, t) \geq \varepsilon \phi(0, T) \left( C(p - 1)^{-\frac{q}{p - 1}} v_0^q(0)(T - t)^{-\frac{p - 1 - q}{p - 1}} \right).$$

By integrating, we obtain that

$$v(0, t) \geq v_0(0) + \varepsilon \phi(0, T) \left( C(p - 1)^{-\frac{q}{p - 1}} v_0^q(0) \right) \int_0^t (T - s)^{-\frac{p - 1 - q}{p - 1}} \, ds. \quad (2.8)$$

The boundedness of $v$ requires $p - 1 > q$ from (2.8), that is $\theta < 0$. Thus, the proof is complete.

**Proof of Theorem 2.2** We only treat the case of $u$ blowing up and $v$ remains bounded.

Assume $\sigma \geq 0$ with $\theta < 0$; that is $p \geq q - 1, q < p - 1$ and $p > 1$ by Figure 1 and (1.7). From (2.3) and (2.7), we have

$$v^{p-q}(0, t)v_t(0, t) \leq \frac{C_2}{\varepsilon \phi(0, T)} u^{p-q}(0, t)u_t(0, t), \quad t \in [0, T). \quad (2.9)$$

By Theorem 2.1 it is impossible for $v$ blowing up alone under $\sigma \geq 0$ with $\theta < 0$. Then we show that $v$ is bounded. In fact, by integrating the inequality (2.9) from 0 to $t$, we have

$$v^{p-q+1}(0, t) \leq C - Cu^{-(p-q-1)}(0, t)$$

for some $C > 0$. Therefore, we can get the boundedness of $v(0, t)$. 

Now, assume that any blow-up must be the case for \( u \) blowing up alone. This requires \( \theta < 0 \) by Theorem 2.1. Again by Theorem 2.1, if in addition \( \sigma < 0 \), there exists the initial data such that \( v \) blows up alone. Therefore, it has to be satisfied that \( \sigma \geq 0 \). Then, the proof is complete. \( \square \)

3. Uniform Blow-up Profiles

In this section, we study the uniform blow-up profiles for system (1.1)–(1.4). At first, the following result of Souplet for a single diffusion equation with nonlocal nonlinear sources \( \text{[13, Theorem 4.1]} \) will play a basic role in our discussion.

**Lemma 3.1.** Let \( u \in C^{2,1}(\bar{\Omega} \times (0, T^*)) \) be a solution of the problem
\[
\begin{align*}
    u_t &= \Delta u + g(t), \quad (x, t) \in \Omega \times (0, T^*), \\
    u(x, t) &= 0, \quad (x, t) \in \partial \Omega \times (0, T^*), \\
    u(x, 0) &= u_0(x), \quad x \in \Omega,
\end{align*}
\]
where \( g(t) \) is nonnegative and may depend on the solution \( u \). Then
\[
\lim_{t \to T^*} \| u(\cdot, t) \|_{\infty} = +\infty
\]
if and only if \( \int_0^t g(s) \, ds = +\infty \). Furthermore, if (3.1) is fulfilled, then
\[
\lim_{t \to T^*} \frac{u(x, t)}{G(t)} = \lim_{t \to T^*} \frac{\| u(\cdot, t) \|_{\infty}}{G(t)} = 1
\]
uniformly on compact subsets of \( \Omega \), where \( G(t) = \int_0^t g(s) \, ds \).

For convenience, we denote
\[
\begin{align*}
    f(t) &= \| uv \|_a^p, \quad g(t) = \| uv \|_b^q, \\
    F(t) &= \int_0^t f(s) \, ds, \quad G(t) = \int_0^t g(s) \, ds.
\end{align*}
\]
According to the Lemma 3.1, we have the following result.

**Lemma 3.2.** Assume \( u, v \in C^{2,1}(\Omega \times [0, T]) \) are the solutions of (1.1)–(1.4). If \( u \) and \( v \) blow up simultaneously in the finite time \( T^* \), then we have
\[
\lim_{t \to T^*} \frac{u(x, t)}{F(t)} = 1, \quad \lim_{t \to T^*} \frac{v(x, t)}{G(t)} = 1
\]
uniformly on compact subsets of \( \Omega \), and
\[
\lim_{t \to T^*} F(t) = \lim_{t \to T^*} G(t) = \infty.
\]

We remark that if we assume that only \( u \) (or \( v \)) blows up in finite time \( T^* \), then the above conclusions about \( u \) (or \( v \)) and \( F \) (or \( G \)) are also valid.

Throughout this section the notation \( f(t) \sim g(t) \) is used to describe such functions \( f(t) \) and \( g(t) \) satisfying \( f(t)/g(t) \to 1 \) as \( t \to T^* \). When \( u \) and \( v \) blow up simultaneously, we have the following results about the uniform blow-up profiles for \( u \) and \( v \).

**Theorem 3.3.** Let \( (u, v) \) be a solution of (1.1)–(1.4) with simultaneous blow-up time \( T^* \). Then the following limits hold uniformly on any compact subset of \( \Omega \):

1. If \( \sigma > 0 \) and \( \theta > 0 \), then
\[
\lim_{t \to T^*} u(x, t)(T^* - t)^{\sigma} = \left( \frac{\| \Omega \|^{p/\alpha}}{\sigma} \left( \frac{\| \Omega \|^{q/\beta} - \frac{p}{q} \alpha \sigma}{\theta} \right)^{\rho/(p+1-\rho)} \right)^{-\sigma}, \quad (3.2)
\]
Hence, from (3.13) and Lemma 3.2, we have

\[ \lim_{t \to T^*} v(x, t)(T^* - t)^\theta = \left( \frac{\Omega |q/\beta - \frac{\theta}{\sigma} q/(q+1-p)}{\theta} \right)^{-\theta}. \]  

(3.3)

(2) If \( \sigma = 0 \), then

\[ \lim_{t \to T^*} u^2(x, t) \ln(T^* - t)^{-1} = \frac{2}{p} |\Omega|^{\frac{q}{\sigma}}. \]  

(3.4)

\[ \lim_{t \to T^*} v^p(x, t) (\ln v(x, t))^{\frac{q}{2}} (T^* - t) = \frac{1}{p} |\Omega|^{-q/\alpha} (2 |\Omega|^{\frac{q}{\sigma}})^{-q/2}. \]  

(3.5)

(3) If \( \theta = 0 \), then we have

\[ \lim_{t \to T^*} u^q(x, t) (\ln u(x, t))^{\frac{q}{2}} (T^* - t) = \frac{1}{q} |\Omega|^{-p/\alpha} (2 |\Omega|^{\frac{q}{\sigma}})^{-p/2}, \]  

(3.6)

\[ \lim_{t \to T^*} v^2(x, t) \ln(T^* - t)^{-1} = \frac{2}{q} |\Omega|^{\frac{q}{\sigma}}. \]  

(3.7)

Proof. From Lemma 3.2, we know that \( u(x, t) \sim F(t) \) and \( v(x, t) \sim G(t) \), then

\[ \lim_{t \to T^*} \frac{u^\alpha(x, t)}{F^\alpha(t)} = \lim_{t \to T^*} \frac{v^\alpha(x, t)}{G^\alpha(t)} = 1, \]

\[ \lim_{t \to T^*} \frac{u^\beta(x, t)}{F^\beta(t)} = \lim_{t \to T^*} \frac{v^\beta(x, t)}{G^\beta(t)} = 1. \]

By the Lebesgue dominated convergence theorem, we find that

\[ F'(t) = f(t) = \|uv\|_\alpha^\gamma \sim |\Omega|^{\frac{p}{\alpha}} F^p(t) G^p(t), \]  

(3.8)

\[ G'(t) = g(t) = \|uv\|_\beta^\gamma \sim |\Omega|^{\frac{q}{\beta}} F^q(t) G^q(t). \]  

(3.9)

Hence,

\[ F^{q-p} dF \sim |\Omega|^{\frac{q}{\beta} - \frac{p}{\alpha}} G^{p-q} dG. \]  

(3.10)

(1) Note that the conditions \( \sigma > 0 \) and \( \theta > 0 \) imply that \( p + 1 > q, q + 1 > p \) since \( p + q > 1 \). Integrating (3.10) from 0 to \( t \), we obtain

\[ F^{q+1-p}(t) \sim |\Omega|^{\frac{q}{\beta} - \frac{p}{\alpha}} \frac{q+1-p}{p+1-q} G^{p+1-q}(t) = |\Omega|^{\frac{q}{\beta} - \frac{p}{\alpha}} G^{p+1-q}(t). \]  

(3.11)

Combining (3.9) and (3.11), we can obtain

\[ G'(t) \sim |\Omega|^{\frac{q}{\beta}} |\Omega|^{\frac{q}{\beta} - \frac{p}{\alpha}} G^{p+1-q}(t)^{\frac{q}{\beta}} G^{\frac{2q}{p+1-q}}(t). \]  

(3.12)

Since

\[ 1 - \frac{2q}{q + 1 - p} = -\frac{p + q - 1}{q + 1 - p} = -\frac{1}{\theta} < 0 \]

and \( \lim_{t \to T^*} G(t) = \infty \), by integrating (3.12), we obtain

\[ G(t) \sim \left( \frac{|\Omega|^{q/\beta}}{\theta} \left(|\Omega|^{\frac{q}{\beta} - \frac{p}{\alpha}} G^{p+1-q}ight)^{\frac{q}{\beta}} \right)^{-\theta} (T^* - t)^{-\theta}. \]  

(3.13)

From (3.13) and Lemma 3.2, we have

\[ \lim_{t \to T^*} v(x, t)(T^* - t)^\theta = \left( \frac{|\Omega|^{q/\beta}}{\theta} \left(|\Omega|^{\frac{q}{\beta} - \frac{p}{\alpha}} G^{p+1-q}ight)^{\frac{q}{\beta}} \right)^{-\theta}, \]

which holds uniformly on the compact subsets of \( \Omega \).
Combining (3.8) and (3.11), and applying the similar proofs of \( F \) and \( u \), we obtain that
\[
\lim_{t \to T^*} u(x, t)(T^* - t)^\sigma = \left( \frac{|\Omega|^{p/\alpha}}{\sigma} \left( \frac{|\Omega|^{\frac{p}{q} - \frac{p}{q}}}{\frac{p}{q}} \right) \right)^{-\sigma}
\]
holds uniformly on the compact subsets of \( \Omega \).

(2) When \( \sigma = 0 \), or \( p + 1 = q \), noticing (3.9) and (3.10), we see that
\[
G'(t) = |\Omega|^{q/\beta} (2|\Omega|^{\frac{p}{q} - \frac{q}{q}})^{q/2} G^q(t) (\ln G(t))^{q/2}.
\]
Note that \( \lim_{t \to T^*} G(t) = \infty \), integrating (3.14) from \( t \) to \( T^* \) asserts
\[
\int_{G(t)}^{\infty} \frac{1}{s^q(\ln s)^{q/2}} ds \sim |\Omega|^{q/\beta} (2|\Omega|^{\frac{p}{q} - \frac{q}{q}})^{q/2} (T^* - t).
\]
Furthermore,
\[
\lim_{t \to T^*} \frac{\int_{G(t)}^{\infty} s^{-q}(\ln s)^{-q/2} ds}{G^{1-q}(\ln G(t))^{-q/2}} = \lim_{G \to \infty} \frac{\int_{G}^{\infty} s^{-q}(\ln s)^{-q/2} ds}{G^{1-q}(\ln G)^{-q/2}} = \frac{1}{q-1} = \frac{1}{p}.
\]
That is to say that
\[
p \int_{G(t)}^{\infty} s^{-q}(\ln s)^{-q/2} ds \sim G^{1-q}(\ln G(t))^{-q/2} = G^{-p}(\ln G(t))^{-q/2} = G^{-p}(\ln G(t))^{-q/2}.
\]
By (3.15) and (3.16), it indicates
\[
G^{-p}(\ln G(t))^{-q/2} \sim p|\Omega|^{q/\beta} (2|\Omega|^{\frac{p}{q} - \frac{q}{q}})^{q/2} (T^* - t).
\]
Since \( \lim_{t \to T^*} v(x, t) = \infty \) uniformly on the compact subset of \( \Omega \) and \( \lim_{t \to T^*} G(t) = \infty \), we may claim that the following equivalent is valid uniformly on the compact subset of \( \Omega \),
\[
v(x, t) \sim G(t) \Rightarrow \ln v(x, t) \sim \ln G(t).
\]
And thus by (3.17), we reach the conclusion
\[
v^{-p}(x, t)(\ln v(x, t))^{-q/2} \sim p|\Omega|^{q/\beta} (2|\Omega|^{\frac{p}{q} - \frac{q}{q}})^{q/2} (T^* - t).
\]
Then uniformly on the compact subsets of \( \Omega \), it yields
\[
\lim_{t \to T^*} v^{-p}(x, t)(\ln v(x, t))^{q/2}(T^* - t) = \frac{1}{p} |\Omega|^{-q/\beta} (2|\Omega|^{\frac{p}{q} - \frac{q}{q}})^{-q/2}.
\]
Since
\[
\ln G(t) \sim \frac{1}{2} |\Omega|^{\frac{p}{q} - \frac{q}{q}} F(t),
\]
it follows from (3.8) and (3.17) that
\[
F'(t)F^{-p}(t) \sim |\Omega|^{p/\alpha} G^{-p}(t) \sim \frac{F^{-q}(t)}{p(T^* - t)} |\Omega|^{\frac{p}{q} - \frac{q}{q}}.
\]
In view of (3.18), we have
\[
1/2 F^2(t) \sim \frac{1}{p} |\Omega|^{\frac{p}{q} - \frac{q}{q}} |\ln(T^* - t)|.
\]
Therefore, by Lemma 3.2 we obtain
\[
\theta^2(x, t) \sim \frac{2}{p} |\Omega|^{\frac{p}{q} - \frac{q}{q}} |\ln(T^* - t)|;
\]
that is to say
\[
\lim_{t \to T^*} u^2(x,t) |\ln(T^* - t)|^{-1} = \frac{2}{p} |\Omega|^\frac{2}{p} - \frac{2}{p}
\]
holds uniformly on the compact subsets of \(\Omega\).

(3) When \(\theta = 0\), the proof is similar to that of the case (2). Then, the proof is completed. \(\square\)

Acknowledgements. This work was supported by the NNSF of China. The authors want to thank the anonymous referees for their helpful suggestions.

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