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REGULARITY OF RANDOM ATTRACTORS FOR STOCHASTIC SEMILINEAR DEGENERATE PARABOLIC EQUATIONS

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Abstract. We consider the stochastic semilinear degenerate parabolic equation

$$du + [-\operatorname{div}(\sigma(x)\nabla u) + f(u) + \lambda u]dt = gdt + \sum_{j=1}^{m} h_j d\omega_j$$

in a bounded domain $\mathcal{O} \subset \mathbb{R}^N$, with the nonlinearity satisfies an arbitrary polynomial growth condition. The random dynamical system generated by the equation is shown to have a random attractor $\{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ in $\mathcal{D}_0^1(\mathcal{O},\sigma) \cap L^p(\mathcal{O})$. The results obtained improve some recent ones for stochastic semilinear degenerate parabolic equations.

1. Introduction

It is known that the asymptotic behavior of random dynamical systems generated by stochastic partial differential equations can be determined by random attractors. The concept of random attractors, which is an extension of the well-known concept of global attractors [12], was introduced in [13, 14] and has been proved useful in the understanding of the dynamics of random dynamical systems. In recent years, many mathematicians paid their attention to the existence of random attractors for stochastic parabolic equations with additive or multiplicative noise, both in bounded domains [6, 11, 18, 19] and in unbounded domains [8, 22, 23]. However, up to the best of our knowledge, little seems to be known for random attractors for degenerate parabolic equations.

In this paper, we consider the stochastic semilinear degenerate parabolic equation

$$\begin{gathered} du + [-\operatorname{div}(\sigma(x)\nabla u) + f(u) + \lambda u]dt = gdt + \sum_{j=1}^{m} h_j d\omega_j, x \in \mathcal{O}, t > 0, \\ u|_{\partial\mathcal{O}} = 0, t > 0, \\ u|_{t=0} = u_0, \end{gathered} \tag{1.1}$$

where $\mathcal{O} \subset \mathbb{R}^N (N \geq 2)$ is a bounded domain with smooth boundary $\partial\mathcal{O}$, $\lambda > 0$, and $\{\omega_j\}_{=1}^m$ are independent two-sided real-valued Wiener processes on a probability space which will be specified later. To study problem (1.1), we assume that

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the diffusion coefficient $\sigma(x)$, the nonlinearity $f(\cdot)$, the external force g, and the functions $\{h_j\}_{j=1}^m$ satisfy the following hypotheses:

- (H1) The function $\sigma : \mathcal{O} \to \mathbb{R}$ is a non-negative measurable function such that $\sigma \in L^1_{\rm loc}(\mathcal{O})$ and for some $\alpha \in (0,2)$, $\liminf_{x\to z} |x-z|^{-\alpha}\sigma(x) > 0$ for every $z \in \overline{\mathcal{O}}$;
- (F1) The function $f \in C^1(\mathbb{R},\mathbb{R})$ satisfies a dissipativeness and growth condition of polynomial type; that is, there is a number $p \geq 2$ such that for all $u \in \mathbb{R}$,

$$f(u)u \geq C_1|u|^p - C_2, \tag{1.2}$$
$$|f(u)| \leq C_3|u|^{p-1} + C_4, \tag{1.3}$$
$$f'(u) \geq -\ell, \tag{1.4}$$

  where C_i, $i = 1,2,3,4$, and ℓ are positive constants;
- (G1) $g \in L^2(\mathcal{O})$;
- (H2) The functions h_j, $j = 1,\dots,m$, belong to $L^{2p-2}(\mathcal{O}) \cap \mathrm{Dom}(A) \cap D^p(A)$, where $Au = -\operatorname{div}(\sigma(x)\nabla u)$, $\mathrm{Dom}(A) = \{u \in \mathcal{D}^1_0(\mathcal{O},\sigma) : Au \in L^2(\mathcal{O})\}$, and $D^p(A) = \{u \in \mathcal{D}^1_0(\mathcal{O},\sigma) : \int_{\mathcal{O}} |Au|^p dx < +\infty\}$.

Here the degeneracy of problem (1.1) is considered in the sense that the measurable, non-negative diffusion coefficient $\sigma(\cdot)$, is allowed to have at most a finite number of (essential) zeroes at some points. For the physical motivation of the assumption $(\mathcal{H}_\alpha)$, we refer the reader to [9, 16, 17].

In the deterministic case, problem (1.1) can be derived as a simple model for neutron diffusion (feedback control of nuclear reactor) (see [15]). In this case u and σ stand for the neutron flux and neutron diffusion respectively. The existence and regularity of global attractors/pullback attractors for problem (1.1) in the deterministic case has been studied extensively in both autonomous case [3, 5, 16, 17] and non-autonomous case [1, 2].

The existence of a random attractor in $L^2(\mathcal{O})$ for the random dynamical system generated by problem (1.1) has been studied recently by Kloeden and Yang in [24]. The aim of this paper is to study the regularity of this random attractor. More precisely, we will prove the existence of random attractors in the spaces $L^p(\mathcal{O})$ and $\mathcal{D}^1_0(\mathcal{O},\sigma)$, and these random attractors of course concide the random attractor obtained in [24] because of the uniqueness of random attractors. To do this, we exploit and develope the asymptotic *a priori* estimate method introduced the first time in [20, 25] for autonomous deterministic equations to the random framework. It is noticed that this method has been developed to study the regularity of the pullback attractor for problem (1.1) in the deterministic case in some recent works [1, 2]. The theory of pullback attractors has shown to be very useful in the understanding of the dynamics of non-autonomous dynamical systems [10].

The paper is organized as follows. In Section 2, for convenience of the reader, we recall some basic results on function spaces and the theory of random dynamical systems. In Section 3, we prove the existence of a random attractor in $L^p(\mathcal{O})$ for the random dynamical system generated by problem (1.1). The existence of a random attractor in $\mathcal{D}^1_0(\mathcal{O},\sigma)$ is proved in the last section. The results obtained improve some recent results for semilinear degenerate stochastic parabolic equations in [24], and as far as we know, the existence of a random attractor in $H^1_0(\mathcal{O})$, which is formally obtained when $\sigma = 1$, is even new for stochastic reaction-diffusion equations.

2. Preliminaries

2.1. **Function spaces and operators.** We recall some basic results on the function spaces which we will use. Let $N \geq 2$, $\alpha \in (0,2)$, and

$$2^*_\alpha = \begin{cases} \frac{4}{\alpha} & \text{if } N = 2 \\ \frac{2N}{N-2+\alpha} \in \Big(2, \frac{2N}{N-2}\Big) & \text{if } N \geq 3. \end{cases}$$

The exponent 2^*_α has the role of the critical exponent in the Sobolev embedding below.

The natural energy space for problem (1.1) involves the space $\mathcal{D}^1_0(\mathcal{O},\sigma)$ defined as the completion of $C^\infty_0(\mathcal{O})$ with respect to the norm

$$\|u\|_{\mathcal{D}^1_0(\mathcal{O},\sigma)} := \Big(\int_{\mathcal{O}} \sigma(x)|\nabla u|^2 dx\Big)^{1/2}.$$

The space $\mathcal{D}^1_0(\mathcal{O},\sigma)$ is a Hilbert space with respect to the scalar product

$$((u,v)) := \int_{\mathcal{O}} \sigma(x)\nabla u \nabla v dx.$$

The following lemma comes from [9, Proposition 3.2].

Lemma 2.1. *Assume that $\mathcal{O}$ is a bounded domain in $\mathbb{R}^N$, $N \geq 2$, and σ satisfies $(\mathcal{H}_\alpha)$. Then the following embeddings hold:*
 (i) *$\mathcal{D}^1_0(\mathcal{O},\sigma) \hookrightarrow L^{2^*_\alpha}(\mathcal{O})$ continuously;*
 (ii) *$\mathcal{D}^1_0(\mathcal{O},\sigma) \hookrightarrow L^p(\mathcal{O})$ compactly if $p \in [1, 2^*_\alpha)$.*

Under condition $(\mathcal{H}_\alpha)$, it is well-known [1, 4] that $Au = -\operatorname{div}(\sigma(x)\nabla u)$ with the domain

$$\operatorname{Dom}(A) = \{u \in \mathcal{D}^1_0(\mathcal{O},\sigma) : Au \in L^2(\Omega)\}$$

is a positive self-adjoint linear operator with an inverse compact. Thus, there exists a complete orthonormal system of eigenvectors (e_j, λ_j) such that

$$\begin{gathered}(e_j, e_k) = \delta_{jk} \quad \text{and} \quad -\operatorname{div}(\sigma(x)\nabla e_j) = \lambda_j e_j, \quad j,k = 1,2,3,\dots, \\ 0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots, \quad \lambda_j \to +\infty \text{ as } j \to +\infty.\end{gathered}$$

Noting that

$$\lambda_1 = \inf\Big\{\frac{\|u\|^2_{\mathcal{D}^1_0(\mathcal{O},\sigma)}}{\|u\|^2_{L^2(\mathcal{O})}} : u \in \mathcal{D}^1_0(\mathcal{O},\sigma), u \neq 0\Big\},$$

we have

$$\|u\|^2_{\mathcal{D}^1_0(\mathcal{O},\sigma)} \geq \lambda_1 \|u\|^2_{L^2(\mathcal{O})}, \quad \text{for all } u \in \mathcal{D}^1_0(\mathcal{O},\sigma).$$

We also define $D^p(A) = \{u \in \mathcal{D}^1_0(\mathcal{O},\sigma) : Au \in L^p(\mathcal{O})\}$.

2.2. **Random dynamical systems.** Here, we recall some basic concepts on the theory of random attractors for random dynamical systems (RDS for short); for more details, we refer the reader to [6, 13].

Let $(X, \|\cdot\|_X)$ be a separable Banach space with Borel σ-algebra $\mathcal{B}(X)$, and let $(\Omega, \mathcal{F}, P)$ be a probability space.

Definition 2.2. $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$ is called a metric dynamical system if $\theta : \mathbb{R}\times\Omega \to \Omega$ is $(\mathcal{B}(\mathbb{R}) \times \mathcal{F}, \mathcal{F})$-measurable, θ_0 is the identity on Ω, $\theta_{s+t} = \theta_t\theta_s$ for all $s,t \in \mathbb{R}$, and $\theta_t(P) = P$ for all $t \in \mathbb{R}$.

Definition 2.3. A function $\phi : \mathbb{R}^+ \times \Omega \times X \to X$ is called a random dynamical system on a metric dynamical system $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$ if for P-a.e. $\omega \in \Omega$,

(i) $\phi(0, \omega, \cdot)$ is the identity of X;
(ii) $\phi(t + s, \omega, x) = \phi(t, \theta_t\omega, \phi(s, \omega, x))$ for all $t, s \in \mathbb{R}^+, x \in X$.

Moreover, ϕ is said to be continuous if $\phi(t, \omega, \cdot) : X \to X$ is continuous for all $t \in \mathbb{R}^+$ and for P-a.e. $\omega \in \Omega$.

We need the following definition about tempered random set.

Definition 2.4. A random bounded set $\{B(\omega)\}_{\omega\in\Omega}$ of X is called tempered with respect to $(\theta_t)_{t\in\mathbb{R}}$ if for P-a.e. $\omega \in \Omega$,

$$\lim_{t\to\infty} e^{-\beta t} d(B(\theta_{-t}\omega)) = 0 \quad \text{for all } \beta > 0,$$

where $d(B) = \sup_{x\in B} \|x\|_X$.

Hereafter, we assume that ϕ is a random dynamical system on $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$ and denote by $\mathcal{D}$ a collection of tempered random subsets of X.

Definition 2.5. A random set $\{K(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ is said to be a random absorbing set for ϕ in $\mathcal{D}$ if for every $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ and P-a.e. $\omega \in \Omega$, there exists $t_B(\omega) > 0$ such that

$$\phi(t, \theta_{-t}\omega, B(\theta_{-t}\omega)) \subset K(\omega) \quad \text{for all } t \geq t_B(\omega).$$

Definition 2.6. A random dynamical system ϕ is called $\mathcal{D}$-pullback asymptotically compact in X if for P-a.e. $\omega \in \Omega$, $\{\phi(t_n, \theta_{-t_n}\omega, x_n)\}_{n\geq 1}$ has a convergent subsequence in X for any $t_n \to \infty$, and $x_n \in B(\theta_{-t_n}\omega)$ with $B \in \mathcal{D}$.

Definition 2.7. A random set $\{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ of X is called a $\mathcal{D}$-random attractor for ϕ if the following conditions are satisfied, for P-a.e. $\omega \in \Omega$,

(i) $\mathcal{A}(\omega)$ is compact, and the map $\omega \mapsto d(x, \mathcal{A}(\omega))$ is measurable for every $x \in X$;
(ii) $\{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ is invariant; that is,
$$\phi(t, \omega, \mathcal{A}(\omega)) = \mathcal{A}(\theta_t\omega) \quad \text{for all } t \geq 0;$$
(iii) $\{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ attracts every set in $\mathcal{D}$; i.e., for every $\{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$,
$$\lim_{t\to\infty} \operatorname{dist}_X(\phi(t, \theta_{-t}\omega, B(\theta_{-t}\omega)), \mathcal{A}(\omega)) = 0,$$
where dist_X is the Hausdorff semi-distance of X,
$$\operatorname{dist}_X(A, B) = \sup_{x\in A} \inf_{y\in B} \|x - y\|_X \quad \text{where } A, B \subset X.$$

The following result was proved in [7, 13].

Theorem 2.8 ([7, 13])**.** *Assume that ϕ is a continuous RDS which has a random absorbing set $\{K(\omega)\}_{\omega\in\Omega}$. If ϕ is pullback asymptotically compact, then it possesses a random attractor $\{\mathcal{A}(\omega)\}_{\omega\in\Omega}$, where*

$$\mathcal{A}(\omega) = \cap_{\tau\geq 0}\overline{\cup_{t\geq\tau}\phi\left(t, \theta_{-t}\omega, K(\theta_{-t}\omega)\right)}.$$

As we know, the continuity of the RDS corresponding to (1.1) in $L^p(\mathcal{O})$ and in $\mathcal{D}_0^1(\mathcal{O}, \sigma)$ is not known, thus, we cannot apply Theorem 2.8 to prove the existence of random attractors in these spaces. Fortunately, in [18], the authors have proved that the existence of random attractors can be obtained under weaker assumptions on the continuity of the RDS, more precisely, we only need the RDS to be *quasi-continuous.*

Definition 2.9 ([18]). A RDS ϕ is called to be *quasi-continuous* if for P-a.e. $\omega \in \Omega$, $\phi(t_n, \omega, x_n) \rightharpoonup \phi(t, \omega, x)$ whenever $\{(t_n, x_n)\}$ is a sequence in $\mathbb{R}^+ \times X$ such that $\{\phi(t_n, \omega, x_n)\}$ is bounded and $(t_n, x_n) \to (t, x)$ as $n \to \infty$.

The following lemma gives us a criteria to check the quasi-continuity of a RDS.

Lemma 2.10 ([18]). *Let X, Y be two Banach spaces with the dual spaces X^*, Y^*, respectively, and assume that*

(i) *the embedding $i : X \to Y$ is densely continuous;*
(ii) *the adjoint operator $i^* : Y^* \to X^*$ is dense; i.e., $i^*(Y^*)$ is dense in X^*.*

If ϕ is continuous in Y, then ϕ is quasi-continuous in X.

In this article, we will use the following result on the existence of random attractors for quasi-continuous dynamical systems.

Theorem 2.11 ([18]). *Let ϕ be a quasi-continuous RDS which has a random absorbing set $\{K(\omega)\}_{\omega\in\Omega}$ in X. Assume also that ϕ is pullback asymptotically compact in X. Then, ϕ has a unique random attractor $\{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ in X. Moreover, we have*

$$\mathcal{A}(\omega) = \cap_{\tau\geq 0}\overline{\cup_{t\geq\tau}\phi\left(t, \theta_{-t}\omega, K(\theta_{-t}\omega)\right)}^{\mathbf{weak}}.$$

In what follows, for brevity, we will denote by $|\cdot|_p$ and $\|\cdot\|$ the norms in $L^p(\mathcal{O})$ and $\mathcal{D}_0^1(\mathcal{O}, \sigma)$ respectively. The inner product in $L^2(\mathcal{O})$ will be written as $(\cdot,\cdot)$. The letter C stands for an arbitrary constant which can be different from line to line or even in the same line, $\mathcal{D}$ and $\mathcal{D}_p$ denote the collection of all tempered random subsets of $L^2(\mathcal{O})$ and $L^p(\mathcal{O})$ respectively

3. Existence of a random attractor in $L^p(\mathcal{O})$

We consider the canonical probability space $(\Omega, \mathcal{F}, P)$, where

$$\Omega = \{\omega = (\omega_1, \omega_2, \dots, \omega_m) \in C(\mathbb{R}; \mathbb{R}^m) : \omega(0) = 0\},$$

and $\mathcal{F}$ is the Borel σ-algebra induced by the compact open topology of Ω, while P is the corresponding Wiener measure on $(\Omega, \mathcal{F})$. Then, we identify ω with

$$W(t) = (\omega_1(t), \omega_2(t), \dots, \omega_m(t)) = \omega(t) \quad \text{for } t \in \mathbb{R}.$$

We define the time shift by $\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t), t \in \mathbb{R}$. Then, $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$ is a metric dynamical system.

We now want to establish a random dynamical system corresponding to (1.1). For this purpose, we need to convert the stochastic equation with an additive noise into a deterministic equation with random parameters.

Given $j = 1, \dots, m$, consider the stochastic stationary solution of the one-dimensional Ornstein-Uhlenbeck equation

$$dz_j + \lambda z_j dt = d\omega_j(t). \tag{3.1}$$

One may check that a solution to (3.1) is given by

$$z_j(t) = z_j(\theta_t\omega_j) = -\lambda \int_{-\infty}^{0} e^{\lambda\tau}(\theta_t\omega_j)(\tau)d\tau, \quad t \in \mathbb{R}.$$

From Definition 2.4, the random variable $|z_j(\omega_j)|$ is tempered and $z_j(\theta_t\omega_j)$ is P-a.e. continuous. Therefore, it follows from [6, Proposition 4.3.3] that there exists a tempered function $r(\omega) > 0$ such that

$$\sum_{j=1}^{m} \left(|z_j(\omega_j)|^2 + |z_j(\omega_j)|^p + |z_j(\omega_j)|^{2p-2}\right) \le r(\omega), \tag{3.2}$$

where $r(\omega)$ satisfies, for P-a.e. $\omega \in \Omega$,

$$r(\theta_t\omega) \le e^{\frac{\lambda}{2}|t|} r(\omega), \quad t \in \mathbb{R}. \tag{3.3}$$

Combining (3.2) and (3.3), it implies that

$$\sum_{j=1}^{m} \left(|z_j(\theta_t\omega_j)|^2 + |z_j(\theta_t\omega_j)|^p + |z_j(\theta_t\omega_j)|^{2p-2}\right) \le e^{\frac{\lambda}{2}|t|} r(\omega), \quad t \in \mathbb{R}.$$

Putting $z(\theta_t\omega) = \sum_{j=1}^{m} h_j z_j(\theta_t\omega_j)$, by (3.1) we have

$$dz + \lambda z dt = \sum_{j=1}^{m} h_j d\omega_j.$$

Since $h_j \in L^{2p-2}(\mathcal{O}) \cap \operatorname{Dom}(A) \cap D^p(A)$, we have

$$\begin{aligned} p(\theta_t\omega) &= \|z(\theta_t\omega)\|^2 + |z(\theta_t\omega)|_p^p + |z(\theta_t\omega)|_{2p-2}^{2p-2} + |Az(\theta_t\omega)|_2^2 + |Az(\theta_t\omega)|_p^p \\ &\le Ce^{\frac{\lambda}{2}|t|} r(\omega). \end{aligned} \tag{3.4}$$

To show that the problem (1.1) generates a random dynamical system, we let $v(t) = u(t) - z(\theta_t\omega)$ where u is a solution of (1.1). Then v satisfies

$$v_t + Av + f(v + z(\theta_t\omega)) + \lambda v = g - Az(\theta_t\omega), \tag{3.5}$$

where $Au = -\operatorname{div}(\sigma(x)\nabla u)$. By the Galerkin method, one can show that if f satisfies (1.2)-(1.3), then for P-a.e. $\omega \in \Omega$ and for all $v_0 \in L^2(\mathcal{O})$, (3.5) has a unique solution $v(\cdot, \omega, v_0) \in C([0,T]; L^2(\mathcal{O})) \cap L^2(0,T; \mathcal{D}_0^1(\mathcal{O},\sigma))$ with $v(0,\omega,v_0) = v_0$ for every $T > 0$. Let $u(t,\omega,u_0) = v(t,\omega,u_0 - z(\omega)) + z(\theta_t\omega)$, then u is the solution of (1.1). We now define a mapping $\phi : \mathbb{R}^+ \times \Omega \times L^2(\mathcal{O}) \to L^2(\mathcal{O})$ by

$$\phi(t,\omega,u_0) = u(t,\omega,u_0) = v(t,\omega,u_0 - z(\omega)) + z(\theta_t\omega).$$

By Definition 2.3, ϕ is a random dynamical system associated to problem (1.1). The following result was proved in [24].

Lemma 3.1. [24] *Under assumptions* (H1), (F1), (G1), (H2), *the RDS ϕ corresponding to* (1.1) *is continuous in $L^2(\mathcal{O})$. Moreover, ϕ possesses a random absorbing set in $\mathcal{D}_0^1(\mathcal{O},\sigma)$, that is, for any $\{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$, there exists $T_1 > 0$ such that, for P-a.e. $\omega \in \Omega$,*

$$\|\phi(t,\theta_{-t}\omega,u_0(\theta_{-t}\omega))\|^2 \le C\left(1 + r(\omega)\right),$$

for all $t \ge T_1$ and $u_0(\theta_{-t}\omega) \in B(\theta_{-t}\omega)$.

Since $\mathcal{D}_0^1(\mathcal{O},\sigma) \hookrightarrow L^2(\mathcal{O})$ compactly, we see that the RDS ϕ corresponding to problem (1.1) possesses a random attractor in $L^2(\mathcal{O})$. To prove the existence of a random attractor in $L^p(\mathcal{O})$, we will use the following results.

Lemma 3.2. [23] *Let ϕ be a continuous random dynamical system (RDS) on $L^2(\mathcal{O})$ and an RDS on $L^p(\mathcal{O})$, where $2 \le p \le \infty$. Assume that ϕ has a $\mathcal{D}$-random attractor. Then ϕ has a $\mathcal{D}_p$-random attractor if and only if the following conditions hold:*

(i) *ϕ has a $\mathcal{D}_p$-random absorbing set $\{K_0(\omega)\}_{\omega\in\Omega}$;*
(ii) *for any $\epsilon > 0$ and every $\{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$, there exist positive constants $M = M(\epsilon, B, \omega)$ and $T = T(\epsilon, B, \omega)$ such that, for all $t \geq T$,*

$$\sup_{u_0(\omega)\in B(\omega)} \int_{\mathcal{O}(|\Psi(t)u_0(\theta_{-t}\omega)|\geq M)} |\Psi(t)u_0(\theta_{-t}\omega)|^p dx \leq \frac{\epsilon^p}{2^{p+2}},$$

where $\Psi(t) = \phi(t, \theta_{-t}\omega)$ and

$$\mathcal{O}(|\Psi(t)u_0(\theta_{-t}\omega)| \geq M) = \{x \in \mathcal{O} : |\Psi(t)u_0(\theta_{-t}\omega)(x)| \geq M\}.$$

Moreover, the $\mathcal{D}$-random attractor and the $\mathcal{D}_p$-random attractor are identical in the set inclusion-relation sense.

Lemma 3.3 ([24])**.** *Let assumptions* (H1), (F1), (G1), (H2) *hold, and let $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ and $u_0(\omega) \in B(\omega)$. Then for P-a.e. $\omega \in \Omega$, there exists $T = T(B, \omega) > 0$ such that for all $t \geq T$,*

$$\int_t^{t+1} |u(s, \theta_{-t-1}\omega, u_0(\theta_{-t-1}\omega))|_p^p ds \leq c(1 + r(\omega)).$$

We now show that ϕ processes a $\mathcal{D}_p$-random absorbing set $\{K_0(\omega)\}_{\omega\in\Omega}$, which belong to $\mathcal{D}_p$ and absorbs every random set of $\mathcal{D}$ in the topology of $L^p(\mathcal{O})$.

Lemma 3.4. *Assume that* (H1), (F1), (G1), (H2) *hold. Let $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ and $u_0(\omega) \in B(\omega)$. Then for P-a.e. $\omega \in \Omega$, for all $t \geq T$,*

$$\int_t^{t+1} |v(s, \theta_{-t-1}\omega, u_0(\theta_{-t-1}\omega) - z(\theta_{-t-1}\omega))|_p^p ds \leq c(1 + r(\omega)),$$

where c is a positive constant and $r(\omega)$ is a tempered random function in (3.2)*.*

Proof. Note that

$$v(s, \theta_{-t-1}\omega, u_0(\theta_{-t-1}\omega) - z(\theta_{-t-1}, \omega)) = u(s, \theta_{-t-1}\omega, u_0(\theta_{-t-1}\omega)) - z(\theta_{s-t-1}\omega).$$

Then by Lemma 3.3 and (3.2)-(3.3), we have, with $z(\theta_t\omega) = \sum_{j=1}^m h_j z_j(\theta_t\omega_j)$ and $h_j \in L^{2p-2}(\mathcal{O}) \cap \mathrm{Dom}(A) \cap D^p(A)$,

$$\begin{aligned}
&\int_t^{t+1} |v(s, \theta_{-t-1}\omega, u_0(\theta_{-t-1}\omega) - z(\theta_{-t-1}\omega))|_p^p ds \\
&= \int_t^{t+1} |u(s, \theta_{-t-1}\omega, u_0(\theta_{-t-1}\omega)) - z(\theta_{s-t-1}\omega)|_p^p ds \\
&\leq 2^{p-1}\Big(\int_t^{t+1} |u(s, \theta_{-t-1}\omega, u_0(\theta_{-t-1}\omega))|_p^p ds + \int_t^{t+1} |z(\theta_{s-t-1}\omega)|_p^p ds \Big) \\
&\leq 2^{p-1}\Big(c(1 + r(\omega)) + c\int_{-1}^0 |z(\theta_s\omega)|_p^p ds \Big) \\
&\leq 2^{p-1}\Big(c(1 + r(\omega)) + c\int_{-1}^0 \sum_{j=1}^m |z_j(\theta_s\omega_j)|^p ds \Big) \\
&\leq 2^{p-1}\Big(c(1 + r(\omega)) + c\int_{-1}^0 r(\theta_s\omega) ds \Big) \\
&\leq 2^{p-1}\Big(c(1 + r(\omega)) + cr(\omega)\int_{-1}^0 e^{-\frac{\lambda}{2}s} ds \Big)
\end{aligned}$$

$$\le c(1 + r(\omega))$$

for all $t \ge T(B,\omega)$, where $T(B,\omega) > 0$ is in Lemma 3.3. $\square$

Lemma 3.5. *Assume that* (H1), (F1), (G1), (H2) *hold. Let $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ and $u_0(\omega) \in B(\omega)$. Then for P-a.e $\omega \in \Omega$, there exists $T = T(B,\omega) > 0$ such that, for all $t \ge T$,*

$$|u(t, \theta_{-t}\omega, u_0(\theta_{-t}\omega))|_p^p \le c(1 + r(\omega)).$$

In particular, for $\omega \in \Omega, K_0(\omega) = \{u \in L^p(\mathcal{O}) : |u|_p^p \le c(1 + r(\omega))\}$ is a $\mathcal{D}_p$-random absorbing set in $\mathcal{D}_p$ for ϕ.

Proof. Multiplying (3.5) with $|v|^{p-2}v$ and then integrating over $\mathcal{O}$, we have

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}|v|_p^p + \lambda|v|_p^p + \int_{\mathcal{O}} \sigma(x)|\nabla v|^2|v|^{p-2}dx + \int_{\mathcal{O}} f(v(t) + z(\theta_t\omega))|v|^{p-2}vdx \\
&= \int_{\mathcal{O}} (g(x) - Az(\theta_t\omega))|v|^{p-2}vdx.
\end{aligned} \tag{3.6}$$

To estimate the nonlinearity, we have

$$\begin{aligned}
f(v + z(\theta_t\omega))v &= f(u)u - f(u)z(\theta_t\omega) \\
&\ge C_1|u|^p - C_2 - (C_3|u|^{p-1} + C_4)z(\theta_t\omega).
\end{aligned}$$

Using Young's inequality, we obtain

$$\begin{gathered}
C_3|u|^{p-1}z(\theta_t\omega) \le \frac{1}{2}C_1|u|^p + C|z(\theta_t\omega)|^p \\
C_4 z(\theta_t\omega) \le \frac{1}{2}C_4^2 + \frac{1}{2}|z(\theta_t\omega)|^2.
\end{gathered}$$

Hence,

$$f(v + z(\theta_t\omega))v \ge \frac{1}{2}C_1|u|^p - C(|z(\theta_t\omega)|^p + |z(\theta_t\omega)|^2) - C.$$

By Hölder's inequality, $|u|^p \ge 2^{1-p}|v|^p - |z(\theta_t\omega)|^p$, then it implies that

$$f(v + z(\theta_t\omega))v \ge \frac{C_1}{2^p}|v|^p - C(|z(\theta_t\omega)|^p + |z(\theta_t\omega)|^2) - C|v|^{p-2}, \tag{3.7}$$

from which it follows by Young's inequality that

$$\begin{aligned}
&f(v + z(\theta_t\omega))|v|^{p-2}v \\
&\ge \frac{C_1}{2^p}|v|^{2p-2} - C|z(\theta_t\omega)|^p|v|^{p-2} - C|z(\theta_t\omega)|^2v^{p-2} - C|v|^{p-2} \\
&\ge \frac{C_1}{2^p}|v|^{2p-2} - \frac{C_1}{2^{p+1}}|v|^{2p-2} - C|z(\theta_t\omega)|^{2p-2} - \frac{\lambda(p-1)}{2p}|v|^p - C|z(\theta_t\omega)|^p \\
&\quad - \frac{\lambda(p-1)}{2p}|v|^p - C \\
&\ge \frac{C_1}{2^{p+1}}|v|^{2p-2} - \frac{\lambda(p-1)}{p}|v|^p - C(|z(\theta_t\omega)|^{2p-2} + |z(\theta_t\omega)|^p) - C.
\end{aligned} \tag{3.8}$$

So, we finally obtain the estimate of the nonlinearity as follows

$$\begin{aligned}
&\int_{\mathcal{O}} f(v + z(\theta_t\omega))|v|^{p-2}vdx \\
&\ge \frac{C_1}{2^{p+1}}|v|_{2p-2}^{2p-2} - \frac{\lambda(p-1)}{p}|v|_p^p - C(|z(\theta_t\omega)|_{2p-2}^{2p-2} + |z(\theta_t\omega)|_p^p) - C|\mathcal{O}|.
\end{aligned} \tag{3.9}$$

On the other hand, the term on the right-hand side of (3.6) is bounded by

$$|g|_2.|v|_{2p-2}^{p-1} + |Az(\theta_t\omega)|_2.|v|_{2p-2}^{p-1} \le \frac{C_1}{2^{p+2}}|v|_{2p-2}^{2p-2} + c|Az(\theta_t\omega)|_2^2 + c|g|_2^2. \tag{3.10}$$

Then it follows from (3.6) and (3.9)-(3.10) that

$$\frac{d}{dt}|v|_p^p + \lambda|v|_p^p + c|v|_{2p-2}^{2p-2} \le c_1(|z(\theta_t\omega)|_{2p-2}^{2p-2} + |z(\theta_t\omega)|_2^2 + |Az(\theta_t\omega)|_2^2) + c_0. \tag{3.11}$$

From (3.11) we have

$$\frac{d}{dt}|v|_p^p \le p(\theta_t\omega) + c_0. \tag{3.12}$$

We let $T(B,\omega)$ be the same as in Lemma 3.4 and $t \ge T(B,\omega)$. Integrating (3.12) from s to $t+1$, where $s \in (t, t+1)$, we obtain

$$|v(t+1,\omega,v_o(\omega))|_p^p \le \int_t^{t+1} p(\theta_\tau\omega)d\tau + |v(s,\omega,v_0(\omega))|_p^p + c_0. \tag{3.13}$$

By replacing ω by $\theta_{-t-1}\omega$ and then integrating from t to $t+1$ in (3.13), it yields that

$$\begin{aligned} &|v(t+1,\theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega))|_p^p \\ &\le \int_t^{t+1} p(\theta_{\tau-t-1}\omega)d\tau + \int_t^{t+1} |v(s,\theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega))|_p^p ds + c_0. \end{aligned} \tag{3.14}$$

By employing Lemma 3.4 and together with (3.4), it follows from (3.14) that

$$\begin{aligned} |v(t+1,\theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega))|_p^p &\le \int_{-1}^0 p(\theta_\tau\omega)d\tau + c(1+r(\omega)) \\ &\le c_3 r(\omega)\int_{-1}^0 e^{-\frac{1}{2}\lambda\tau}d\tau + c(1+r(\omega)) \\ &\le c(1+r(\omega)). \end{aligned}$$

Therefore, there exists $T_1(B,\omega) > 0$ such that, for all $t \ge T_1(B,\omega)$,

$$|v(t,\theta_{-t}\omega, u_0(\theta_{-t}\omega))|_p^p \le c(1+r(\omega)),$$

from which and (3.2), it follows that for all $t \ge T_1(B,\omega)$,

$$\begin{aligned} |u(t,\theta_{-t}\omega, u_0(\theta_{-t}\omega))|_p^p &= |v(t,\theta_{-t}\omega, v_0(\theta_{-t}\omega) + z(\omega))|_p^p \\ &\le 2^{p-1}(|v(t,\theta_{-t-1}\omega, v_0(\theta_{-t}\omega))|_p^p + |z(\omega)|_p^p) \\ &\le c2^{p-1}(1+r(\omega)) + 2^{p-1}|z(\omega)|_p^p \\ &\le c(1+r(\omega)). \end{aligned} \tag{3.15}$$

Given $\omega \in \Omega$, denote

$$K_0(\omega) = \{u \in L^p(\mathcal{O}) : |u|_p^p \le c(1+r(\omega))\}.$$

Then $\{K_0(\omega)\}_{\omega\in\Omega} \in \mathcal{D}_p$. Moreover, (3.15) indicates that $\{K_0(\omega)\}_{\omega\in\Omega}$ is a $\mathcal{D}_p$-random absorbing set in $\mathcal{D}_p$ for ϕ, which completes the proof. □

Lemma 3.6. *Assume that* (H1), (F1), (G1), (H2) *hold. Let* $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ *and* $u_0(\omega) \in B(\omega)$. *Then for* P*-a.e.* $\omega \in \Omega$, *there exists* $T = T(B,\omega) > 0$ *such that, for all* $t \ge T$ *and* $s \in [t, t+1]$,

$$|v(s,\theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega))|_2^2 \le c(1+r(\omega)),$$

where $v_0(\omega) = u_0(\omega) - z(\omega)$.

Proof. Using a similar argument as given in [24, Lemma 6.1], we obtain

$$|v(s,\omega,v_o(\omega))|_2^2 \le e^{-\lambda s}|v_0(\omega)|_2^2 + \int_0^s e^{\lambda(\tau-s)}p(\theta_\tau\omega)d\tau + \frac{c}{\lambda}. \tag{3.16}$$

We choose $s\in[t,t+1]$. By replacing ω by $\theta_{-t-1}\omega$ in (3.16), we obtain, with (3.3)

$$\begin{aligned}
&|v(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))|_2^2\\
&\le e^{-\lambda s}|v_0(\theta_{-t-1}\omega)|_2^2 + \int_0^s e^{\lambda(\tau-s)}p(\theta_{\tau-t-1}\omega)d\tau + \frac{c}{\lambda}\\
&\le e^{\lambda}e^{-\lambda(t+1)}|v_0(\theta_{-t-1}\omega)|_2^2 + \int_0^{t+1} e^{\lambda(\tau-t)}p(\theta_{\tau-t-1}\omega)d\tau + \frac{c}{\lambda}\\
&\le e^{\lambda}e^{-\lambda(t+1)}|v_0(\theta_{-t-1}\omega)|_2^2 + \int_{-t-1}^0 e^{\lambda(\tau+1)}p(\theta_\tau\omega)d\tau + \frac{c}{\lambda}\\
&\le e^{\lambda}\Big(e^{-\lambda(t+1)}|v_0(\theta_{-t-1}\omega)|_2^2 + c\int_{-t-1}^0 e^{\frac{\lambda}{2}\tau}r(\omega)d\tau\Big) + \frac{c}{\lambda}\\
&\le e^{\lambda}\Big(e^{-\lambda(t+1)}|v_0(\theta_{-t-1}\omega)|_2^2 + \frac{2c}{\lambda}r(\omega)\Big) + \frac{c}{\lambda}\\
&\le 2e^{\lambda}e^{-\lambda(t+1)}\left(|u_0(\theta_{-t-1}\omega)|_2^2 + |z(\theta_{-t-1}\omega)|_2^2\right) + \frac{2ce^{\lambda}}{\lambda}r(\omega) + \frac{c}{\lambda}.
\end{aligned} \tag{3.17}$$

Note that $\{B(\omega)\}_{\omega\in\Omega}\in\mathcal{D}$ and $|z(\omega)|_2^2$ is also tempered. Then for $u_0(\theta_{-t-1}\omega)\in B(\theta_{-t-1}\omega)$, there exists $T=T(B,\omega)$ such that, for all $t\ge T$,

$$2e^{\lambda}e^{-\lambda(t+1)}\left(|u_0(\theta_{-t-1}\omega)|_2^2 + |z(\theta_{-t-1}\omega)|_2^2\right) \le c(1+r(\omega)). \tag{3.18}$$

Hence, it follows from (3.17) and (3.18) that for all $t\ge T$ and $s\in[t,t+1]$,

$$|v(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))|_2^2 \le c(1+r(\omega)),$$

which completes the proof. □

Lemma 3.7. *Assume that* (H1), (F1), (G1), (H2) *hold. Let* $B=\{B(\omega)\}_{\omega\in\Omega}\in\mathcal{D}$ *and* $u_0(\omega)\in B(\omega)$. *Then for every* $\epsilon>0$ *and* P*-a.e.* $\omega\in\Omega$, *there exist* $T=T(B,\omega)>0$ *and* $M=M(\epsilon,B,\omega)$ *such that for all* $t\ge T$ *and* $s\in[t,t+1]$,

$$m(\mathcal{O}|v(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))|\ge M) < \epsilon,$$

where $v_0(\omega)=u_0(\omega)-z(\omega)$ *and* $m(e)$ *is the Lebesgue measure of* $e\subset\mathbb{R}^N$.

Proof. By Lemma 3.6, there exists a random variable $M_0=M_0(\omega)$ such that, for every $B=\{B(\omega)\}_{\omega\in\Omega}\in\mathcal{D}$, we can find a constant $T=T(B,\omega)$ such that for all $t\ge T$ and $s\in[t,t+1]$,

$$|v(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega)|_2^2 \le M_0$$

with $v_0(\omega)=u_0(\omega)-z(\omega)$ and $u_0\in B(\omega)$. On the other hand, for any fixed $M>0$,

$$\begin{aligned}
&|v(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))|_2^2\\
&= \int_{\mathcal{O}} |v(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}))|^2dx\\
&\ge \int_{\mathcal{O}(|v(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))|\ge M)} |v(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}))|^2dx\\
&\ge M^2 m(\mathcal{O}(|v(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))|\ge M)).
\end{aligned} \tag{3.19}$$

Then for any $\epsilon > 0$, by (3.19), we obtain that $m(\mathcal{O}|v(s, \theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega))| \geq M) < \epsilon$ provided that we choose $M > \big(\frac{M_0}{\epsilon}\big)^{1/2}$. ☐

By a technique similar to that in [23, Lemma 4.6], we can prove the following lemma.

Lemma 3.8. *Assume that* (H1), (F1), (G1), (H2) *hold. Let* $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$, *then for every* $\epsilon > 0$ *and* P*-a.e.* $\omega \in \Omega$, *there exists* $T = T(\epsilon, B, \omega) > 0, M_1 = M_1(\epsilon, B, \omega)$ *and* $M_2 = M_2(\epsilon, B, \omega)$ *such that for all* $t \geq T$,

$$\int_{\mathcal{O}(|u(t,\theta_{-t-1}\omega,u_0(\theta_{-t-1}\omega))|\geq M_1)} |u(t, \theta_{-t-1}\omega, u_0(\theta_{-t-1}\omega))|^2 dx \leq \epsilon,$$

$$\int_{\mathcal{O}(|v(t,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))|\geq M_2)} |v(t, \theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega))|^2 dx \leq \epsilon, \tag{3.20}$$

where $v_0(\omega) = u_0(\omega) - z(\omega)$.

Lemma 3.9. *Assume that* (H1), (F1), (G1), (H2) *hold. Let* $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$, *then for every* $\epsilon > 0$ *and* P*-a.e* $\omega \in \Omega$, *there exist* $T = T(\epsilon, B, \omega) > 0, M = M(\epsilon, B, \omega)$ *such that for all* $t \geq T$,

$$\int_{\mathcal{O}(|u(t,\theta_{-t}\omega,u_0(\theta_{-t}\omega))|\geq M)} |u(t, \theta_{-t}\omega, u_0(\theta_{-t}\omega))|^p dx \leq \epsilon. \tag{3.21}$$

Proof. For any fixed $\epsilon > 0$, there exists $\delta > 0$ such that for any $e \subset \mathcal{O}$ with $m(e) \leq \delta$, we have

$$\int_e |g|^2 dx < \epsilon. \tag{3.22}$$

In particular, by our assumptions $h_j \in L^{2p-2}(\mathcal{O}) \cap \mathrm{Dom}(A) \cap D^p(A)$ for $j = 1, 2, \dots, m$, there exists $\delta_2 = \delta_2(\epsilon) > 0$ such that, for any $e \subset \mathbb{R}^N$ with $m(e) \leq \delta_2$,

$$\int_e (|h_j(x)|^{2p-2} + |h_j(x)|^p + |h_j(x)|^2 + |Ah_j(x)|^2) dx < \frac{\epsilon}{r(\omega)}. \tag{3.23}$$

On the other hand, from Lemma 3.7, we know that for every $u_0(\omega) \in B(\omega)$, there exists $T_1 \geq T$ and M_3 such that for all $t \geq T_1$ and $s \in [t, t+1]$,

$$m(\mathcal{O}(|v(s, \theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega))| \geq M_3)) \leq \min\{\epsilon, \delta_1, \delta_2\}. \tag{3.24}$$

Then inequalities (3.22)-(3.23) hold for $e = \mathcal{O}(|v(s, \theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega))| \geq M_3)$. Let now $M = \max(M_1, M_2, M_3), F = |z(\omega)|_\infty$, and $t \geq T_1$. By a similar computation as in [23, Lemma 4.6], we can show that F is finite for P-a.e. $\omega \in \Omega$. Multiplying (3.5) with $(v - M)_+$ and then integrating over $\mathcal{O}$, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|(v - M)_+|_2^2 + \lambda \int_{\mathcal{O}} v(v - M)_+ dx + \int_{\mathcal{O}} \sigma(x)|\nabla(v - M)_+|^2 dx \\ &+ \int_{\mathcal{O}} f(v + z(\theta_t\omega))(v - M)_+ dx \\ &= \int_{\mathcal{O}} (g - Az(\theta_t\omega))(v - M)_+ dx, \end{aligned} \tag{3.25}$$

where

$$(v - M)_+ = \begin{cases} v - M & \text{if } v \geq M, \\ 0 & \text{if } v \leq M. \end{cases}$$

We now estimate all terms of (3.25). First, we have

$$\int_{\mathcal{O}} \sigma(x)|\nabla(v-M)_+|^2 dx \geq 0, \tag{3.26}$$

$$\lambda \int_{\mathcal{O}} v(v-M)_+ dx \geq \lambda |(v-M)_+|_2^2. \tag{3.27}$$

From (3.7), we find that

$$\begin{aligned}
&f(v+z(\theta_t\omega))(v-M)\\
&= f(v-M+z(\theta_t\omega)+M)(v-M)\\
&\geq \frac{C_1}{2^p}|v-M|^p - C(|z(\theta_t\omega)+M|^p + |z(\theta_t\omega)+M|^2) - C\\
&\geq \frac{C_1}{2^p}|v-M|^p - C(|z(\theta_t\omega)|^p + |z(\theta_t\omega)|^2) - C,
\end{aligned} \tag{3.28}$$

which gives

$$\begin{aligned}
&\int_{\mathcal{O}(v\geq M)} f(v+z(\theta_t\omega))(v-M)dx\\
&\geq c_1 \int_{\mathcal{O}(v\geq M)} |v-M|^p dx - C\int_{\mathcal{O}(v\geq M)} (|z(\theta_t\omega)|^p + |z(\theta_t\omega)|^2)dx\\
&\quad - Cm(\mathcal{O}(v\geq M)),
\end{aligned} \tag{3.29}$$

where $c_1 = \frac{C_1}{2^p}$. By Young's inequality, we have

$$\int_{\mathcal{O}(v\geq M)} (g - Az(\theta_t\omega))(v-M)_+ dx \leq \lambda|(v-M)_+|_2^2 + c\int_{\mathcal{O}(v\geq M)} (|g|^2 + |Az(\theta_t\omega)|^2)dx. \tag{3.30}$$

Then it follows from (3.25)-(3.30) that

$$\begin{aligned}
&\frac{d}{dt}|(v-M)_+|_2^2 + 2c_1\int_{\mathcal{O}(v\geq M)} |v-M|^p dx\\
&\leq 2C\int_{\mathcal{O}(v\geq M)} (|z(\theta_t\omega)|^p + |z(\theta_t\omega)|^2 + |Az(\theta_t\omega)|^2)dx\\
&\quad + \int_{\mathcal{O}(v\geq M)} 2cg^2 dx + Cm(\mathcal{O}(v\geq M)).
\end{aligned} \tag{3.31}$$

Replacing t by τ and then integrating (3.31) for τ from t to $t+1$, it yields

$$\begin{aligned}
&\int_t^{t+1}\int_{\mathcal{O}(v(\tau,\omega,v_0(\omega))\geq M)} |v(\tau,\omega,v_0(\omega)-M)|^p dx d\tau\\
&\leq c_1\int_t^{t+1}\int_{\mathcal{O}(v(\tau,\omega,v_0(\omega))\geq M)} (|z(\theta_\tau\omega)|^p + |z(\theta_\tau\omega)|^2 + |Az(\theta_\tau\omega)|^2)dx d\tau\\
&\quad + c_2\int_t^{t+1}\int_{\mathcal{O}(v(\tau,\omega,v_0(\omega))\geq M)} cg^2 dx d\tau + c_3\int_t^{t+1} m(\mathcal{O}(v\geq M))d\tau\\
&\quad + |(v(t,\omega,v_0(\omega)) - M)_+|_2^2.
\end{aligned} \tag{3.32}$$

Let $D_1(\tau)=\mathcal{O}(v(\tau,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\geq M)$. Replacing ω by $\theta_{-t-1}\omega$ in (3.32), we see that

$$\begin{aligned}
&\int_t^{t+1}\int_{D_1(\tau)}|v(\tau,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega)-M)|^p dxd\tau\\
&\leq c_1\int_t^{t+1}\int_{D_1(\tau)}(|z(\theta_{\tau-t-1}\omega)|^p+|z(\theta_{\tau-t-1}\omega)|^2+|Az(\theta_{\tau-t-1}\omega)|^2)dxd\tau\\
&\quad+c_2\int_t^{t+1}\int_{D_1(\tau)}cg^2dxd\tau+c_3\int_t^{t+1}m(\mathcal{O}(v\geq M))d\tau\\
&\quad+|(v(t,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))-M)_+|_2^2.
\end{aligned}\tag{3.33}$$

By (3.20), together with (3.22) and (3.24), we have

$$c_2\int_t^{t+1}\int_{D_1(\tau)}cg^2dxd\tau+|(v(t,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))-M)_+|_2^2\leq c\epsilon,\tag{3.34}$$

where c is a generic positive constant independent of ϵ. By (3.23) and using Hölder's inequality repeatedly, we have the following bound for the first term on the right - hand side of (3.33),

$$\begin{aligned}
&c_1\int_t^{t+1}\int_{D_1(\tau)}(|z(\theta_{\tau-t-1}\omega)|^p+|z(\theta_{\tau-t-1}\omega)|^2+|Az(\theta_{\tau-t-1}\omega)|^2)dxd\tau\\
&\leq c_1m^{p-2}\int_t^{t+1}\int_{D_1(\tau)}\Big(\sum_{j=1}^m|h_j|^p\sum_{j=1}^m|z_j(\theta_{\tau-t-1}\omega_j)|^p\\
&\quad+\sum_{j=1}^m|h_j|^2\sum_{j=1}^m|z_j(\theta_{\tau-t-1}\omega_j)|^2+\sum_{j=1}^m|Ah_j|^2\sum_{j=1}^m|z_j(\theta_{\tau-t-1}\omega_j)|^2\Big)dxd\tau\\
&\leq\frac{3cm^{p-2}\epsilon}{r(\omega)}\int_t^{t+1}\Big(\sum_{j=1}^m|z_j(\theta_{-\tau-t-1}\omega_j)|^p+\sum_{j=1}^m|z_j(\theta_{-\tau-t-1}\omega_j)|^2\Big)d\tau\\
&\leq\frac{3cm^{p-1}\epsilon}{r(\omega)}\int_t^{t+1}p(\theta_{\tau-t-1}\omega)d\tau\\
&\leq\frac{3cm^{p-1}\epsilon}{r(\omega)}\int_{-1}^0p(\theta_\tau\omega)d\tau\\
&\leq\frac{3cm^{p-1}\epsilon}{r(\omega)}\int_{-1}^0r(\omega)e^{-\frac12\lambda\tau}d\tau\leq c\epsilon.
\end{aligned}\tag{3.35}$$

Then by (3.24), it follows from (3.33)-(3.35) that for all $t\geq T_1$,

$$\int_t^{t+1}\int_{D_1(\tau)}|v(\tau,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega)-M)|^p dxd\tau\leq c\epsilon.\tag{3.36}$$

We then take the inner product of (3.5) with $(v-M)_+^{p-1}$ to find that

$$\begin{aligned}
&\frac1p\frac{d}{dt}|(v-M)_+|_p^p+\lambda\int_{\mathcal{O}}v(v-M)_+^{p-1}dx\\
&+(p-1)\int_{\mathcal{O}}\sigma(x)|\nabla(v-M)_+|^2|(v-M)_+|^{p-2}dx\\
&+\int_{\mathcal{O}}f(v+z(\theta_t\omega))(v-M)_+^{p-1}dx
\end{aligned}$$

$$= \int_{\mathcal{O}} (g - Az(\theta_t\omega))(v-M)_+^{p-1} dx.$$

If $v \geq M$, then by (3.8), we have

$$\begin{aligned}
&f(v + z(\theta_t\omega))(v-M)^{p-1} \\
&= f(v - M + z(\theta_t\omega) + M)(v-M)^{p-1} \\
&\geq \frac{C_1}{2^{p+1}}|v-M|^{2p-2} - \lambda|v-M|^p - C(|z(\theta_t\omega)+M|^{2p-2} + |z(\theta_t\omega)+M|^p) - C \\
&\geq \frac{C_1}{2^{p+1}}|v-M|^{2p-2} - \lambda|v-M|^p - C(|z(\theta_t\omega)|^{2p-2} + |z(\theta_t\omega)|^p) - C,
\end{aligned} \tag{3.37}$$

from which we have the following bounds for the nonlinearity

$$\begin{aligned}
&\int_{\mathcal{O}} f(v + z(\theta_t\omega))(v-M)_+^{p-1} dx \\
&\geq \frac{C_1}{2^{p+1}}|(v-M)_+|_{2p-2}^{2p-2} - \lambda|(v-M)_+|_p^p \\
&\quad - C\int_{\mathcal{O}} (|z(\theta_t\omega)|^{2p-2} + |z(\theta_t\omega)|^p) dx - cm(\mathcal{O}(v \geq M)).
\end{aligned} \tag{3.38}$$

On the other hand, we have

$$\lambda \int_{\mathcal{O}} v(v-M)_+^{p-1} dx \geq \lambda|(v-M)_+|_p^p, \tag{3.39}$$

$$\int_{\mathcal{O}} \sigma(x)|\nabla(v-M)_+|^2|(v-M)_+|^{p-2} dx \geq 0. \tag{3.40}$$

By Young's inequality, we deduce that

$$\int_{\mathcal{O}} (g - Az(\theta_t\omega))(v-M)_+^{p-1} dx \leq \frac{C_1}{2^{p+1}}|(v-M)_+|_{2p-2}^{2p-2} + c\int_{\mathcal{O}(v\geq M)} |g|^2 + |Az(\theta_t\omega)|^2 dx. \tag{3.41}$$

Thus from (3.37) - (3.41), it follows that

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}|(v-M)_+|_p^p &\leq c_1 \int_{\mathcal{O}(v\geq M)} (|z(\theta_t\omega)|^{2p-2} + |z(\theta_t\omega)|^p + |Az(\theta_t\omega)|^2) dx \\
&\quad + c_2 \int_{\mathcal{O}(v\geq M)} g^2 dx + c_3 m(\mathcal{O}(v \geq M)).
\end{aligned} \tag{3.42}$$

Replacing t by τ and then integrating (3.42) for τ from s to $t+1$ with $s \in [t, t+1]$, we obtain that

$$\begin{aligned}
&|v(t+1, \omega, v_0(\omega)) - M)|_p^p \\
&\leq c_1 \int_t^{t+1} \int_{\mathcal{O}(v\geq M)} (|z(\theta_\tau\omega)|^{2p-2} + |z(\theta_\tau\omega)|^p + |Az(\theta_\tau\omega)|^2) dx d\tau \\
&\quad + c_2 \int_t^{t+1} \int_{\mathcal{O}(v\geq M)} g^2 dx d\tau + c_3 \int_t^{t+1} m(\mathcal{O}(v \geq M)) d\tau \\
&\quad + |(v(s, \omega, v_0(\omega)) - M)_+|_p^p.
\end{aligned} \tag{3.43}$$

We first replace ω by $\theta_{-t-1}\omega$, then integrate (3.43) for s in the interval $[t, t+1]$ to find that

$$
\begin{aligned}
&|(v(t+1,\theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega)) - M)_+|_p^p \\
&\le c_1 \int_t^{t+1} \int_{D_1(\tau)} (|z(\theta_{\tau-t-1}\omega)|^{2p-2} + |z(\theta_{\tau-t-1}\omega)|^p + |Az(\theta_{\tau-t-1}\omega)|^2) dx d\tau \\
&\quad + c_2 \int_t^{t+1} \int_{D_1(\tau)} g^2 dx d\tau + c_3 \int_t^{t+1} m(D_1(\tau)) d\tau \\
&\quad + \int_t^{t+1} |(v(s,\theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega)) - M)_+|_p^p ds,
\end{aligned} \tag{3.44}
$$

where $D_1(\tau) = \mathcal{O}(v(\tau, \theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega)) \ge M$. Then it follows from (3.22), (3.24) and (3.36) that

$$
\begin{aligned}
&c_2 \int_t^{t+1} \int_{\mathcal{D}_1(\tau)} g^2 dx d\tau + c_3 \int_t^{t+1} m(D_1(\tau)) d\tau \\
&+ \int_t^{t+1} |(v(s,\theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega)) - M)_+|_p^p ds \le c\epsilon,
\end{aligned} \tag{3.45}
$$

and by similar argument as (3.35), we have

$$
c_1 \int_t^{t+1} \int_{\mathcal{D}_1(\tau)} (|z(\theta_{\tau-t-1}\omega)|^{2p-2} + |z(\theta_{\tau-t-1}\omega)|^p + |Az(\theta_{\tau-t-1}\omega)|^2) dx d\tau \le c\epsilon. \tag{3.46}
$$

Hence, from (3.44) - (3.46) we obtain that for all $t \ge T_1$

$$
|(v(t+1,\theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega)) - M)_+|_p^p \le c\epsilon, \tag{3.47}
$$

and then we deduce that for all $t \ge T_1 + 1$,

$$
\int_{D_2(t)} |v(t,\theta_{-t}\omega, v_0(\theta_{-t}\omega))|^p dx \le c\epsilon, \tag{3.48}
$$

where $D_2(t) = \mathcal{O}(v(t,\theta_{-t}\omega, v_0(\theta_{-t}\omega)) \ge 2M)$. Note that $u(t,\theta_{-t}, u_0(\theta_{-t}\omega)) = v(t,\theta_{-t}\omega, v_0(\theta_{-t}\omega)) + z(\omega)$. Then we see that

$$
\mathcal{O}(|u(t,\theta_{-t}, u_0(\theta_{-t}\omega))| \ge 2M + F) \subset \mathcal{O}(|v(t,\theta_{-t}\omega, v_0(\theta_{-t}\omega))| \ge 2M) = D_2(t),
$$

where $F = |z(\omega)|_\infty$. This, together with (3.24) and (3.48), gives that for all $t \ge T_1 + 1$

$$
\begin{aligned}
&\int_{\mathcal{O}(u(t,\theta_{-t}\omega, u_0(\theta_{-t}\omega)) \ge 2M+F)} |u(t,\theta_{-t}\omega, u_0(\theta_{-t}\omega))|^p dx \\
&\le 2^{p-1} \Big(\int_{D_2(t)} |v(t,\theta_{-t}\omega, v_0(\theta_{-t}\omega))|^p dx + \int_{D_2(t)} |z(\omega)|^p dx \Big) \\
&\le 2^{p-1}(c\epsilon + F^p m(D_2(t))) \le c\epsilon.
\end{aligned} \tag{3.49}
$$

Repeating the same arguments above, just taking $(v+M)_-$ and $|(v+M)_-|^{p-2}(v+M)_-$ instead of $(v-M)_+$ and $(v-M)_+^{p-1}$, respectively, where $(v+M)_-$ is the negative part of $v+M$, we can deduce that

$$
\int_{\mathcal{O}(|u(t,\theta_{-t}\omega, u_0(\theta_{-t}\omega))| \le -2M-F)} |u(t,\theta_{-t}\omega, u_0(\theta_{-t}\omega))|^p dx \le c\epsilon. \tag{3.50}
$$

Then the result (3.21) follows from (3.49) and (3.50). $\square$

Theorem 3.10. *Assume that* (H1), (F1), (G1), (H2) *hold. Then the RDS ϕ generated by* (3.5) *has a unique $\mathcal{D}_p$-random attractor $\{\mathcal{A}_p(\omega)\}_{\omega\in\Omega}$ which is a compact and invariant tempered random subset of $L^p(\mathcal{O})$ attracting every tempered random subset of $L^2(\mathcal{O})$. Furthermore, $\mathcal{A}_p(\omega)=\mathcal{A}(\omega)$, where $\{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ is the random attractor in $L^2(\mathcal{O})$.*

4. Existence of a random attractor in $\mathcal{D}_0^1(\mathcal{O},\sigma)$

We denote by

$$B^*(\omega)=\{u\in L^p(\mathcal{O})\cap\mathcal{D}_0^1(\mathcal{O},\sigma):|u|_p^p+\|u\|^2\le c(1+r(\omega))\} \tag{4.1}$$

for $\omega\in\Omega$. By Lemma 3.1 and Lemma 3.5 we see that $\{B^*(\omega)\}_{\omega\in\Omega}$ is a random absorbing set for ϕ in $L^p(\mathcal{O})\cap\mathcal{D}_0^1(\mathcal{O},\sigma)$. In the next lemma, we show that we can take initial data in $\{B^*(\omega)\}_{\omega\in\Omega}$ to obtain the pullback asymptotic compactness of ϕ.

Lemma 4.1. *Assume that $\{B^*(\omega)\}_{\omega\in\Omega}$ is a random absorbing in $L^p(\mathcal{O})\cap\mathcal{D}_0^1(\mathcal{O},\sigma)$ for the RDS ϕ. Then ϕ is pullback asymptotically compact if for P-a.e. $\omega\in\Omega$, $\{\phi(t_n,\theta_{-t_n}\omega,x_n)\}$ whenever $t_n\to+\infty$ and $x_n\in B^*(\theta_{-t_n}\omega)$.*

Proof. Take an arbitrary random set $\{B(\omega)\}_{\omega\in\Omega}\in\mathcal{D}$, a sequence $t_n\to+\infty$ and $y_n\in B(\theta_{-t_n}\omega)$. We have to prove that $\{\phi(t_n,\theta_{-t_n}\omega,y_n)\}$ is precompact.

Since $\{B^*(\omega)\}$ is a random absorbing for ϕ, then there exists $T>0$ such that, for all $\omega\in\Omega$,

$$\phi(t,\theta_{-t}\omega,B(\theta_{-t}\omega))\subset B^*(\omega)\quad\text{for all } t\ge T. \tag{4.2}$$

Because $t_n\to+\infty$, we can choose $n_1\ge1$ such that $t_{n_1}-1\ge T$. Applying (4.2) for $t=t_{n_1}-1$ and $\omega=\theta_{-1}\omega$, we find that

$$x_1:=\phi(t_{n_1}-1,\theta_{-t_{n_1}}\omega,y_{n_1})\in\phi(t_{n_1}-1,\theta_{-t_{n_1}}\omega,B(\theta_{-t_{n_1}}\omega))\subset B^*(\theta_{-1}\omega). \tag{4.3}$$

Similarly, we can choose a subsequence $\{n_k\}$ of $\{n\}$ such that $n_1<n_2<\dots<n_k\to+\infty$ such that

$$x_k:=\phi(t_{n_k}-k,\theta_{-t_{n_k}}\omega,y_{n_k})\in B^*(\theta_{-k}\omega). \tag{4.4}$$

Hence, by the assumption we conclude that

$$\text{the sequence } \{\phi(k,\theta_{-k}\omega,x_k)\} \text{ is precompact.} \tag{4.5}$$

On the other hand, by (4.4)

$$\begin{aligned}\phi(k,\theta_{-k}\omega,x_k)&=\phi(k,\theta_{-k}\omega,\phi(t_{n_k}-k,\theta_{-t_{n_k}}\omega,y_{n_k}))\\&=\phi(t_{n_k},\theta_{-t_{n_k}}\omega,y_{n_k}),\quad\forall k\ge1.\end{aligned} \tag{4.6}$$

Combining (4.5), (4.6) we obtain that the sequence $\{\phi(t_{n_k},\theta_{-t_{n_k}}\omega,y_{n_k})\}$ is precompact, thus $\{\phi(t_n,\theta_{t_n}\omega,y_n)\}$ is precompact. This completes the proof. $\square$

Lemma 4.2. *There exists $T>0$ such that, for P-a.e. $\omega\in\Omega$*

$$\int_0^t e^{-\lambda(t-s)}|v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))|_{2p-2}^{2p-2}ds\le C(1+r(\omega)), \tag{4.7}$$

for all $t\ge T$ and all $u_0(\theta_{-t}\omega)\in B^*(\theta_{-t}\omega)$.

Proof. We recall here inequality (3.11),

$$\frac{d}{dt}|v|_p^p + \lambda|v|_p^p + c|v|_{2p-2}^{2p-2} \le c_1(|z(\theta_t\omega)|_{2p-2}^{2p-2} + |z(\theta_t\omega)|_2^2 + |Az(\theta_t\omega)|_2^2) + c_0. \tag{4.8}$$

Multiplying (4.8) by $e^{\lambda t}$ and integrating over $(0,t)$, we have

$$\begin{aligned}
&|v(t,\omega,v_0(\omega))|_p^p + c\int_0^t e^{-\lambda(t-s)}|v(s,\omega,v_0(\omega))|_{2p-2}^{2p-2}ds \\
&\le e^{-\lambda t}|v_0(\omega)|_p^p + c_1\int_0^t e^{-\lambda(t-s)}(|z(\theta_s\omega)|_{2p-2}^{2p-2} + |z(\theta_s\omega)|_2^2 + |Az(\theta_s\omega)|_2^2)ds \\
&\quad + c_0\int_0^t e^{-\lambda(t-s)}ds.
\end{aligned} \tag{4.9}$$

We replace ω by $\theta_{-t}\omega$ in (4.9) to obtain

$$\begin{aligned}
&c\int_0^t e^{-\lambda(t-s)}|v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))|_{2p-2}^{2p-2}ds \\
&\le e^{-\lambda t}|v_0(\theta_{-t}\omega)|_p^p + c_1\int_{-t}^0 e^{\lambda s}(|z(\theta_s\omega)|_{2p-2}^{2p-2} + |z(\theta_s\omega)|_2^2 + |Az(\theta_s\omega)|_2^2)ds + \frac{c_0}{\lambda} \\
&\le e^{-\lambda t}|u_0(\theta_{-t}\omega) - z(\theta_{-t}\omega)|_p^p + c_1\int_{-t}^0 p(\theta_s\omega)ds + \frac{c_0}{\lambda} \\
&\le 2^p(e^{-\lambda t}|u_0(\theta_{-t}\omega)|_p^p + e^{-\lambda t}|z(\theta_{-t}\omega)|_p^p) + \frac{2c_1 r(\omega)}{\lambda} + \frac{c_0}{\lambda}.
\end{aligned} \tag{4.10}$$

Since $u_0(\theta_{-t}\omega) \in B^*(\theta_{-t}\omega)$ and $|z(\omega)|$ is tempered, we have

$$\lim_{t\to+\infty} e^{-\lambda t}|u_0(\theta_{-t}\omega)|_p^p = \lim_{t\to+\infty} e^{-\lambda t}|z(\theta_{-t}\omega)|_p^p = 0.$$

Hence, from (4.10), we can choose T large enough such that

$$\int_0^t e^{-\lambda(t-s)}|v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))|_{2p-2}^{2p-2}ds \le c(1+r(\omega)), \forall t \ge T.$$

□

Lemma 4.3. *For all $t \ge T$ and all $u_0(\theta_{-t}\omega) \in B(\theta_{-t}\omega)$, we have*

$$\int_0^t e^{-\lambda(t-s)}\int_{\mathcal{O}} |f(u(s,\theta_{-t}\omega,u_0(\theta_{-t}\omega)))|^2\,dx\,ds \le C(1+r(\omega)). \tag{4.11}$$

Proof. By condition (1.3) of f, we find that

$$\begin{aligned}
&\int_{\mathcal{O}} |f(u(s,\theta_{-t}\omega,u_0(\theta_{-t}\omega)))|^2dx \\
&\le C_3^2\int_{\mathcal{O}} |u(s,\theta_{-t}\omega,u_0(\theta_{-t}\omega))|^{2p-2}dx + C_4^2|\Omega| \\
&\le C_3^2 2^{2p-2}\int_{\mathcal{O}} \left(|v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))|^{2p-2} + |z(\theta_{s-t}\omega)|^{2p-2}\right)dx + C_4^2|\Omega| \\
&\le C\left(|v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))|_{2p-2}^{2p-2} + |z(\theta_{s-t}\omega)|_{2p-2}^{2p-2} + 1\right) \\
&\le C\left(|v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))|_{2p-2}^{2p-2} + p(\theta_{s-t}\omega) + 1\right).
\end{aligned} \tag{4.12}$$

Thus,

$$
\begin{aligned}
&\int_0^t e^{-\lambda(t-s)}\int_{\mathcal{O}}|f(u(s,\theta_{-t}\omega,u_0(\theta_{-t}\omega)))|^2\,dx\,ds\\
&\le C\int_0^t e^{-\lambda(t-s)}\Big(|v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))|_{2p-2}^{2p-2}+p(\theta_{s-t}\omega)+1\Big)\,ds\\
&\le C\int_0^t e^{-\lambda(t-s)}|v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))|_{2p-2}^{2p-2}ds+C\int_{-t}^0 e^{\lambda\tau}p(\theta_\tau\omega)d\tau+C \qquad (4.13)\\
&\le C\int_0^t e^{-\lambda(t-s)}|v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))|_{2p-2}^{2p-2}ds+C\int_{-t}^0 e^{\lambda\tau}e^{-\frac{\lambda}{2}\tau}r(\omega)d\tau+C\\
&\le C\int_0^t e^{-\lambda(t-s)}|v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))|_{2p-2}^{2p-2}ds+C(1+r(\omega)).
\end{aligned}
$$

By (4.7) and (4.13), we obtain (4.11). □

Lemma 4.4. *Let $\tau\in\mathbb{R}$. If a function $h:\mathbb{R}\to\mathbb{R}^+$ satisfies that*

$$\sup_{t\ge\tau}\int_\tau^t e^{-\mu(t-s)}h(s)ds<+\infty,\quad \text{for some } \mu>0,$$

then we have

$$\lim_{\gamma\to\infty}\sup_{t\ge\tau}\int_\tau^t e^{-\gamma(t-s)}h(s)ds=0.$$

Proof. The idea of the proof follows from [21]. First, we prove that, for any $\epsilon>0$, there exists $\eta>0$ such that

$$\sup_{r\ge\tau}\int_r^{r+\eta}e^{-\mu(t-s)}h(s)ds<\epsilon.$$

Indeed, if not, there exist ϵ_0 and $r_n\ge\tau,\eta_n>0$ and $\eta_n\to0^+$, as $n\to\infty$, such that

$$\int_{r_n}^{r_n+\eta_n}e^{-\mu(t-s)}h(s)ds\ge\epsilon_0\quad\text{for all } n\ge1.$$

If $\{r_n\}_{n\ge1}$ is bounded, there exists a convergent subsequence $\{r_{n_k}\}$ of $\{r_n\}$ and $r'\in\mathbb{R}$ such that $\lim_{k\to\infty}r_{n_k}=r'$. We have

$$\epsilon_0\le\lim_{k\to\infty}\int_{r_{n_k}}^{r_{n_k}+\eta_{n_k}}e^{-\mu(t-s)}h(s)ds=\int_{r'}^{r'}e^{-\mu(t-s)}h(s)ds=0,$$

this contradicts to $\epsilon_0>0$.

If $r_n\to+\infty$, then we obtain

$$\epsilon_0\le\int_{r_n}^{r_n+\eta_n}e^{-\mu(t-s)}h(s)ds\le\int_{r_n}^{+\infty}e^{-\mu(t-s)}h(s)ds\to0,\ \text{as } n\to+\infty,$$

we also have a contradiction.

Next, by the above result, for given $\epsilon>0$ we can get $\eta>0$ such that

$$\sup_{r\ge\tau}\int_r^{r+\eta}e^{-\mu(t-s)}h(s)ds\le\epsilon.$$

We choose $k \in \mathbb{N}$ such that $t - k\eta \geq \tau \geq t - (k+1)\eta$ to have

$$\begin{aligned}
&\int_\tau^t e^{-\gamma(t-s)} h(s)ds \\
&= \int_{t-\eta}^t e^{-(\gamma-\mu)(t-s)} e^{-\mu(t-s)} h(s)ds + \int_{t-2\eta}^{t-\eta} e^{-(\gamma-\mu)(t-s)} e^{-\mu(t-s)} h(s)ds \\
&\quad + \dots + \int_\tau^{t-k\eta} e^{-(\gamma-\mu)(t-s)} e^{-\mu(t-s)} h(s)ds \\
&\leq \int_{t-\eta}^t e^{-\mu(t-s)} h(s)ds + e^{-(\gamma-\mu)\eta} \int_{t-2\eta}^{t-\eta} e^{-\mu(t-s)} h(s)ds \\
&\quad + \dots + e^{-k(\gamma-\mu)\eta} \int_\tau^{t-k\eta} e^{-\mu(t-s)} h(s)ds \\
&\leq \epsilon \Big(1 + e^{-(\gamma-\mu)\eta} + e^{-2(\gamma-\mu)\eta} + \dots + e^{-k(\gamma-\mu)\eta}\Big) \\
&\leq \frac{\epsilon}{1 - e^{-(\gamma-\mu)\eta}} \to \epsilon
\end{aligned}$$

as $\gamma \to +\infty$, uniformly in t and in τ. This completes the proof. $\square$

The following lemma is the key to prove the pullback asymptotic compactness of the random dynamical system.

Lemma 4.5. *For any $\eta > 0$, there exist $t_0 > 0$ and $m \in \mathbb{N}^*$ such that*

$$\|(Id_{\mathcal{D}_0^1(\mathcal{O},\sigma)} - P_m)v(t, \theta_{-t}\omega, v_0(\theta_{-t}\omega))\|^2 \leq \eta, \quad \forall t \geq t_0, \forall u_0(\theta_{-t}\omega) \in B^*(\theta_{-t}\omega), \tag{4.14}$$

where P_m is a canonical projector from $\mathcal{D}_0^1(\mathcal{O},\sigma)$ onto an m-dimensional subspace.

Proof. We denote by $H_m = \operatorname{span}\{e_1, e_2, \dots, e_m\}$, where $\{e_j\}_{j\geq1}$ are eigenvalues of the operator $A = -\operatorname{div}(\sigma(x)\nabla)$ with Dirichlet boundary condition. For any v solution to (3.5), we write $v = P_m v + (Id - P_m)v = v_1 + v_2$. Multiplying (3.5) by Av_2 then integrating over $\mathcal{O}$, we find that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|v_2\|^2 + |Av_2|_2^2 + \int_\mathcal{O} f(v + z(\theta_t\omega))(Av_2)dx + \lambda\|v_2\|^2 \\
&= \int_\mathcal{O} (g - Az(\theta_t\omega))(Av_2)dx.
\end{aligned} \tag{4.15}$$

Using the Cauchy inequality, we have

$$\int_\mathcal{O} (g - Az(\theta_t\omega))(Av_2)dx \leq 2(|g|_2^2 + |Az(\theta_t\omega)|_2^2) + \frac{1}{4}|Av_2|_2^2, \tag{4.16}$$

and

$$\int_\mathcal{O} f(v + z(\theta_t\omega))(Av_2)dx \leq \int_\mathcal{O} |f(v + z(\theta_t\omega))|^2 dx + \frac{1}{4}|Av_2|_2^2. \tag{4.17}$$

Combining (4.15)-(4.17) and noting that $|Av_2|_2^2 \geq \lambda_{m+1}\|v_2\|^2$, we obtain

$$\frac{d}{dt}\|v_2\|^2 + \lambda_{m+1}\|v_2\|^2 \leq C\Big(1 + |Az(\theta_t\omega)|_2^2 + \int_\mathcal{O} |f(v + z(\theta_t\omega))|^2 dx\Big). \tag{4.18}$$

By Gronwall's inequality,

$$\|v_2(t, \omega, v_0(\omega))\|^2$$

$$\le e^{-\lambda_{m+1}t}\|v_0(\omega)\|^2 + C\int_0^t e^{-\lambda_{m+1}(t-s)}\Big(1+|Az(\theta_s\omega)|_2^2 + \int_{\mathcal{O}}|f(u)|^2dx\Big)ds.$$

Replacing ω by $\theta_{-t}\omega$ leads to

$$\begin{aligned}
&\|v_2(t,\theta_{-t}\omega, v_0(\theta_{-t}\omega))\|^2 \\
&\le e^{-\lambda_{m+1}t}\|v_0(\theta_{-t}\omega)\|^2 + C\int_0^t e^{-\lambda_{m+1}(t-s)}\Big(1+|Az(\theta_{s-t}\omega)|_2^2 \\
&\quad + \int_{\mathcal{O}}|f(u(s,\theta_{-t}\omega,u_0(\theta_{-t}\omega)))|^2dx\Big)ds.
\end{aligned} \tag{4.19}$$

We need to estimate all terms on the right hand side of (4.19). First,

$$e^{-\lambda_{m+1}t}\|v_0(\theta_{-t}\omega)\|^2 \le 2e^{-\lambda_{m+1}t}(\|u_0(\theta_{-t}\omega)\|^2 + \|z(\theta_{-t}\omega)\|^2) \to 0, \tag{4.20}$$

as $t, m \to \infty$ since $u_0(\theta_{-t}\omega) \in B^*(\theta_{-t}\omega)$ and $\|z(\omega)\|^2$ is tempered. Second,

$$\int_0^t e^{-\lambda_{m+1}(t-s)}ds = \frac{1}{\lambda_{m+1}}(1-e^{-\lambda_{m+1}t}) \to 0 \quad \text{as } m \to \infty. \tag{4.21}$$

Third,

$$\begin{aligned}
\int_0^t e^{-\lambda_{m+1}(t-s)}|Az(\theta_{s-t}\omega)|_2^2ds &\le \int_{-t}^0 e^{\lambda_{m+1}\tau}p(\theta_\tau\omega)d\tau \\
&\le \int_{-t}^0 e^{\lambda_{m+1}\tau}e^{\frac{-\lambda}{2}\tau}r(\omega)d\tau \\
&\le \frac{r(\omega)}{\lambda_{m+1}-\frac{\lambda}{2}}(1-e^{-(\lambda_{m+1}-\frac{\lambda}{2})t}) \to 0,
\end{aligned} \tag{4.22}$$

as $m \to \infty$. Finally, due to Lemmas 4.3 and 4.4, we have

$$\lim_{m\to\infty}\int_0^t e^{-\lambda_{m+1}(t-s)}\int_{\mathcal{O}}|f(u(s,\theta_{-t}\omega,u_0(\theta_{-t}\omega)))|^2\,dx\,ds = 0. \tag{4.23}$$

Applying (4.20)-(4.23) to (4.19), we obtain (4.14). □

Theorem 4.6. *Suppose that assumptions* (H1), (F1), (G1), (H2) *hold. Then the random dynamical system generated by* (1.1) *possesses a compact random attractor* $\mathcal{A} = \{A(\omega)\}_{\omega\in\Omega}$ *in* $\mathcal{D}_0^1(\mathcal{O},\sigma)$.

Proof. By Lemma 3.1, ϕ is quasi-continuous in $\mathcal{D}_0^1(\mathcal{O},\sigma)$ and has a random absorbing set in $\mathcal{D}_0^1(\mathcal{O},\sigma)$. Due to Theorem 2.11, we remain to prove the pullback asymptotic compactness of ϕ in $\mathcal{D}_0^1(\mathcal{O},\sigma)$. Using Lemma 4.1, we have to show that $\{\phi(t_n,\theta_{-t_n}\omega,u_0(\theta_{-t_n}\omega))\}$ is precompact in $\mathcal{D}_0^1(\mathcal{O},\sigma)$ for any $t_n \to +\infty$ and $u_0(\theta_{-t_n}\omega) \in B^*(\theta_{-t_n}\omega)$. For any given $\epsilon > 0$, since $t_n \to +\infty$, we can apply Lemma 4.5 to find that there exist $N_1 > 0$ and $m \in \mathbb{N}$ such that

$$\|(\mathrm{Id}_{\mathcal{D}_0^1(\mathcal{O},\sigma)} - P_m)\phi(t_k,\theta_{-t_k}\omega,u_0(\theta_{-t_k}\omega))\| \le \epsilon, \quad \forall k \ge N_1. \tag{4.24}$$

From Lemma 3.1, since $t_n \to +\infty$ and $u_0(\theta_{-t_n}\omega) \in B^*(\theta_{-t_n}\omega)$, we conclude that $\{\phi(t_n,\theta_{-t_n}\omega,u_0(\theta_{-t_n}\omega))\}$ is bounded in $\mathcal{D}_0^1(\mathcal{O},\sigma)$.

Thus, $\{P_m\phi(t_n,\theta_{-t_n}\omega,u_0(\theta_{-t_n}\omega))\}$ is bounded in $P_m(\mathcal{D}_0^1(\mathcal{O},\sigma))$. Because the set $P_m(\mathcal{D}_0^1(\mathcal{O},\sigma))$ is a finite dimensional subspace of $\mathcal{D}_0^1(\mathcal{O},\sigma)$, we can assume that $\{P_m\phi(t_n,\theta_{-t_n}\omega,u_0(\theta_{-t_n}\omega))\}$ is a Cauchy sequence. Thus, there exists $N_2 > 0$ satisfying

$$\|P_m\phi(t_k,\theta_{-t_k}\omega,u_0(\theta_{-t_k}\omega)) - P_m\phi(t_l,\theta_{-t_l}\omega,u_0(\theta_{-t_l}\omega))\| \le \epsilon, \tag{4.25}$$

for all $k, l \geq N_2$. Now, we set $N = \max\{N_1, N_2\}$. Hence, from (4.24) and (4.25), we find that, for all $k, l \geq N$,

$$\begin{aligned} &\|\phi(t_k, \theta_{-t_k}\omega, u_0(\theta_{-t_k}\omega)) - \phi(t_l, \theta_{-t_l}\omega, u_0(\theta_{-t_l}\omega))\| \\ &\leq \|P_m\phi(t_k, \theta_{-t_k}\omega, u_0(\theta_{-t_k}\omega)) - P_m\phi(t_l, \theta_{-t_l}\omega, u_0(\theta_{-t_l}\omega))\| \\ &\quad + \|(Id_{\mathcal{D}_0^1(\mathcal{O},\sigma)} - P_m)\phi(t_k, \theta_{-t_k}\omega, u_0(\theta_{-t_k}\omega))\| \\ &\quad + \|(Id_{\mathcal{D}_0^1(\mathcal{O},\sigma)} - P_m)\phi(t_l, \theta_{-t_l}\omega, u_0(\theta_{-t_l}\omega))\| \leq 3\epsilon. \end{aligned} \tag{4.26}$$

This show that $\{\phi(t_n, \theta_{-t_n}\omega, u_0(\theta_{-t_n}\omega))\}$ is precompact in $\mathcal{D}_0^1(\mathcal{O}, \sigma)$, and thus it completes the proof. $\square$

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