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EXISTENCE OF POSITIVE SOLUTIONS FOR NONLINEAR FRACTIONAL SYSTEMS IN BOUNDED DOMAINS

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ABSTRACT. We prove the existence of positive continuous solutions to the nonlinear fractional system

$$(-\Delta|_D)^{\alpha/2}u + \lambda g(.,v) = 0,$$
$$(-\Delta|_D)^{\alpha/2}v + \mu f(.,u) = 0,$$

in a bounded $C^{1,1}$-domain D in $\mathbb{R}^n$ $(n \geq 3)$, subject to Dirichlet conditions, where $0 < \alpha \leq 2$, λ and μ are nonnegative parameters. The functions f and g are nonnegative continuous monotone with respect to the second variable and satisfying certain hypotheses related to the Kato class.

1. Introduction and statement of main results

Let $\chi = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, P^x)$ be a Brownian motion in $\mathbb{R}^n$, $n \geq 3$ and $\pi = (\Omega, \mathcal{G}, T_t)$ be an $\frac{\alpha}{2}$-stable process subordinator starting at zero, where $0 < \alpha \leq 2$ and such that χ and π are independent. Let D be a bounded $C^{1,1}$-domain in $\mathbb{R}^n$ and Z_α^D be the subordinate killed Brownian motion process. This process is obtained by killing χ at τ_D, the first exit time of χ from D giving the process χ^D and then subordinating this killed Brownian motion using the $\alpha/2$-stable subordinator T_t. For more description of the process Z_α^D we refer to [7, 9, 14, 15]. Note that the infinitesimal generator of the process Z_α^D is the fractional power$(-\Delta|_D)^{\alpha/2}$ of the negative Dirichlet Laplacian in D, which is a prototype of non-local operator and a very useful object in analysis and partial differential equations, see, for instance [13, 16].

In this article, we will deal with the existence of positive continuous solutions for the nonlinear fractional system

$$\begin{gathered}
(-\Delta|_D)^{\alpha/2}u + \lambda g(.,v) = 0 \quad \text{in } D, \text{ in the sense of distributions} \\
(-\Delta|_D)^{\alpha/2}v + \mu f(.,u) = 0 \quad \text{in } D, \text{ in the sense of distributions} \\
\lim_{x\to z\in\partial D} \frac{u(x)}{M_\alpha^D 1(x)} = \varphi(z), \quad \lim_{x\to z\in\partial D} \frac{v(x)}{M_\alpha^D 1(x)} = \psi(z),
\end{gathered} \tag{1.1}$$

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where λ, μ are nonnegative parameters, φ, ψ are positive continuous functions on ∂D and $M_\alpha^D 1$ is the nonnegative harmonic function with respect to Z_α^D given by the formula (see [7, Theorem 3.1],

$$M_\alpha^D 1(x) = \frac{1-\frac{\alpha}{2}}{\Gamma(\frac{\alpha}{2})}\int_0^\infty t^{-2+\frac{\alpha}{2}}(1-P_t^D 1(x))dt, \tag{1.2}$$

where $(P_t^D)_{t>0}$ is the semi-group corresponding to the killed Brownian motion χ^D.

Note that from [15, remark 3.3], there exists a constant $C>0$ such that

$$\frac{1}{C}\big(\delta(x)\big)^{\alpha-2} \le M_\alpha^D 1(x) \le C\big(\delta(x)\big)^{\alpha-2}, \quad \text{for all } x\in D, \tag{1.3}$$

where $\delta(x)$ denotes the Euclidian distance from x to the boundary of D.

In the classical case (i.e. $\alpha = 2$), there exist a lot of work related to the existence and nonexistence of solutions for the problem (1.1); see for example, the papers of Cirstea and Radulescu [3], Ghanmi et al [6], Ghergu and Radulescu [8], Lair and Wood [10, 11] and references therein. Most of the studies of these papers turn about the existence or the nonexistence of positive radial ones. In [11], the authors studied the system (1.1) with $\alpha = 2$, in the case $\mu f(.,u) = pu^s$, $\lambda g(.,v) = qv^r$, $s>0$, $r>0$ and p,q are nonnegative continuous and not necessarily radial. They showed that entire positive bounded solutions exist if p and q satisfy the following condition

$$p(x)+q(x) \le C|x|^{-(2+\gamma)}$$

for some positive constant γ and $|x|$ large.

Throughout this article, we denote by G_α^D the Green function of Z_α^D. We recall the following interesting sharp estimates on G_α^D due to [14]. Namely, there exists a positive constant $C>0$ such that for all x,y in D, we have

$$\frac{1}{C}H(x,y) \le G_\alpha^D(x,y) \le CH(x,y), \tag{1.4}$$

where

$$H(x,y) = \frac{1}{|x-y|^{n-\alpha}}\min\Big(1, \frac{\delta(x)\delta(y)}{|x-y|^2}\Big).$$

We also denote by $M_\alpha^D\varphi$ the unique positive continuous solution of

$$\begin{gathered}(-\Delta|_D)^{\alpha/2}u = 0 \quad \text{in } D, \text{ in the sense of distributions}\\ \lim_{x\to z\in\partial D}\frac{u(x)}{M_\alpha^D 1(x)} = \varphi(z),\end{gathered} \tag{1.5}$$

which is given (see [7]) by

$$M_\alpha^D\varphi(x) = \frac{1}{\Gamma(\alpha/2)}E^x(\varphi(X_{\tau_D})\tau_D^{\frac{\alpha}{2}-1}). \tag{1.6}$$

We aim at giving two existence results for (1.1) as f and g are nondecreasing or nonincreasing with respect to the second variable. More precisely, to state our first existence result, we assume that $f,g : D\times[0,\infty)\to[0,\infty)$ are Borel measurable functions satisfying

(H1) The functions f and g are continuous and nondecreasing with respect to the second variable.

(H2) The functions
$$\widetilde{p}(y) := \frac{1}{M_\alpha^D\psi(y)} f(y, M_\alpha^D\varphi(y)) \quad \text{and} \quad \widetilde{q}(y) := \frac{1}{M_\alpha^D\varphi(y)} g(y, M_\alpha^D\psi(y))$$
belong to the Kato class $K_\alpha(D)$, defined below.

Definition 1.1 ([5]). A Borel measurable function q in D belongs to the Kato class $K_\alpha(D)$ if
$$\lim_{r\to 0}\Big(\sup_{x\in D}\int_{(|x-y|\le r(\cap D}\frac{\delta(y)}{\delta(x)}G_\alpha^D(x,y)|q(y)|dy\Big) = 0.$$

This class is quite rich, it contains for example any function belonging to $L^s(D)$, with $s > n/\alpha$ (see Example 2.1 below). On the other hand, it has been shown in [5], that
$$x \to \big(\delta(x)\big)^{-\gamma} \in K_\alpha(D), \quad \text{for } \gamma < \alpha. \tag{1.7}$$
For more examples of functions belonging to $K_\alpha(D)$, we refer to [5]. Note that for the classical case (i.e. $\alpha = 2$), the class $K_2(D)$ was introduced and studied in [12].

Our first existence result is the following.

Theorem 1.2. *Assume that* (H1), (H2) *are satisfied. Then there exist two constants $\lambda_0 > 0$ and $\mu_0 > 0$ such that for each $\lambda \in [0, \lambda_0)$ and each $\mu \in [0, \mu_0)$, problem* (1.1) *has a positive continuous solution such that*
$$\begin{gathered}(1-\frac{\lambda}{\lambda_0})M_\alpha^D\varphi \le u \le M_\alpha^D\varphi \quad \text{in } D,\\ (1-\frac{\mu}{\mu_0})M_\alpha^D\psi \le v \le M_\alpha^D\psi \quad \text{in } D.\end{gathered}$$
In particular $\lim_{x\to z\in\partial D} u(x) = \infty$ *and* $\lim_{x\to z\in\partial D} v(x) = \infty$.

We note that in [6], the authors studied a problem similar to (1.1) for the case $\alpha = 2$. They have obtained positive continuous bounded solution (u, v). Here, we are interesting in the fractional setting.

As second existence result, we aim at proving the existence of blow-up positive continuous solutions for the system
$$\begin{gathered}(-\Delta|_D)^{\alpha/2}u + p(x)g(v) = 0 \quad \text{in } D, \text{ in the sense of distributions}\\ (-\Delta|_D)^{\alpha/2}v + q(x)f(u) = 0 \quad \text{in } D, \text{ in the sense of distributions}\\ \lim_{x\to z\in\partial D}\frac{u(x)}{M_\alpha^D 1(x)} = \varphi(z), \quad \lim_{x\to z\in\partial D}\frac{v(x)}{M_\alpha^D 1(x)} = \psi(z),\end{gathered} \tag{1.8}$$
where φ, ψ are positive continuous functions on ∂D and p, q are nonnegative Borel measurable functions in D. To this end, we fix ϕ a positive continuous functions on ∂D, we put $h_0 = M_\alpha^D\phi$ and we assume the following:

(H3) The functions $f, g : (0,\infty) \to [0,\infty)$ are continuous and nonincreasing.

(H4) The functions $p_0 := p\frac{f(h_0)}{h_0}$ and $q_0 := q\frac{g(h_0)}{h_0}$ belongs to the class $K_\alpha(D)$.

As a typical example of nonlinearity f and p satisfying (H3)-(H4), we have $f(t) = t^{-\nu}$, for $\nu > 0$, and p a nonnegative Borel measurable function such that
$$p(x) \le \frac{C}{\big(\delta(x)\big)^r}, \quad \text{for all } x \in D,$$
for some $C > 0$ and $r + (1+\nu)(\alpha - 2) < \alpha$.

Indeed, since there exists a constant $c > 0$, such that for all $x \in D$, $h_0(x) \geq c\big(\delta(x)\big)^{\alpha-2}$, we deduce by (1.7), that the function $p_0 := p\frac{f(h_0)}{h_0} \in K_\alpha(D)$. Using the Schauder's fixed point theorem, we prove the following result.

Theorem 1.3. *Under the assumptions* (H3), (H4)*, there exists a constant $c > 1$ such that if $\varphi \geq c\phi$ and $\psi \geq c\phi$ on ∂D, then problem* (1.8) *has a positive continuous solution (u, v) satisfying for each $x \in D$,*

$$\begin{gathered} h_0 \leq u \leq M_\alpha^D \varphi \quad \text{in } D, \\ h_0 \leq v \leq M_\alpha^D \psi \quad \text{in } D. \end{gathered}$$

In particular $\lim_{x\to z\in\partial D} u(x) = \infty$ *and* $\lim_{x\to z\in\partial D} v(x) = \infty$.

This result extends the one of Athreya [1], who considered the problem

$$\begin{gathered} \Delta u = g(u), \quad \text{in } \Omega \\ u = \varphi \quad \text{on } \partial\Omega, \end{gathered} \tag{1.9}$$

where Ω is a simply connected bounded C^2-domain and $g(u) \leq \max(1, u^{-\alpha})$, for $0 < \alpha < 1$. Then he proved that there exists a constant $c > 1$ such that if $\varphi \geq c\widetilde{h_0}$ on $\partial\Omega$, where $\widetilde{h_0}$ is a fixed positive harmonic function in Ω, problem $(*)$ has a positive continuous solution u such that $u \geq \widetilde{h_0}$.

The content of this article is organized as follows. In Section 2, we collect some properties of functions belonging to the Kato class $K_\alpha(D)$, which are useful to establish our results. Our main results are proved in Section 3.

As usual, let $B^+(D)$ be the set of nonnegative Borel measurable functions in D. We denote by $C_0(D)$ the set of continuous functions in $\overline{D}$ vanishing continuously on ∂D. Note that $C_0(D)$ is a Banach space with respect to the uniform norm $\|u\|_\infty = \sup_{x\in D}|u(x)|$. The letter C will denote a generic positive constant which may vary from line to line. When two positive functions ρ and θ are defined on a set S, we write $\rho \approx \theta$ if the two sided inequality $\frac{1}{C}\theta \leq \rho \leq C\theta$ holds on S. For $\rho \in B^+(D)$, we define the potential kernel G_α^D of Z_α^D by

$$G_\alpha^D \rho(x) := \int_D G_\alpha^D(x, y)\rho(y)dy, \quad \text{for } x \in D$$

and we denote by

$$a_\alpha(\rho) := \sup_{x,y\in D} \int_D \frac{G_\alpha^D(x, z)G_\alpha^D(z, y)}{G_\alpha^D(x, y)}\rho(y)dy. \tag{1.10}$$

2. The Kato class $K_\alpha(D)$

Example 2.1. For $s > \frac{n}{\alpha}$, we have $L^s(D) \subset K_\alpha(D)$. Indeed, let $0 < r < 1$ and $q \in L^s(D)$ with $s > \frac{n}{\alpha}$. Using (1.4), there exists a constant $C > 0$, such that for each $x, y \in D$

$$\frac{\delta(y)}{\delta(x)} G_\alpha^D(x, y) \leq C\frac{1}{|x - y|^{n-\alpha}}. \tag{2.1}$$

This fact and the Hölder inequality imply that

$$\int_{B(x,r)\cap D} \Big(\frac{\delta(y)}{\delta(x)}\Big) G_\alpha^D(x, y)|q(y)|dy$$

$$
\begin{aligned}
&\le C\int_{B(x,r)\cap D}\frac{|q(y)|}{|x-y|^{n-\alpha}}dy\\
&\le C\Big(\int_D |q(y)|^s dy\Big)^{1/s}\Big(\int_{B(x,r)}|x-y|^{(\alpha-n)\frac{s}{s-1}}dy\Big)^{\frac{s-1}{s}}\\
&\le C\Big(\int_0^r t^{(\alpha-n)\frac{s}{s-1}+n-1}dt\Big)^{\frac{s-1}{s}}\to 0,
\end{aligned}
$$

as $r\to 0$, since $(\alpha-n)\frac{s}{s-1}+n-1>-1$ when $s>\frac{n}{\alpha}$.

Proposition 2.2 ([5]). *Let q be a function in $K_\alpha(D)$, then we have*

(i) $a_\alpha(q)<\infty$.

(ii) *Let h be a positive excessive function on D with respect to Z_α^D. Then we have*

$$\int_D G_\alpha^D(x,y)h(y)|q(y)|dy\le a_\alpha(q)h(x). \tag{2.2}$$

Furthermore, for each $x_0\in\overline{D}$, we have

$$\lim_{r\to 0}\Big(\sup_{x\in D}\frac{1}{h(x)}\int_{B(x_0,r)\cap D}G_\alpha^D(x,y)h(y)|q(y)|dy\Big)=0. \tag{2.3}$$

(iii) *The function $x\to\big(\delta(x)\big)^{\alpha-1}q(x)$ is in $L^1(D)$.*

Lemma 2.3. *Let q be a nonnegative function in $K_\alpha(D)$, then the family of functions*

$$\Lambda_q=\Big\{\frac{1}{M_\alpha^D\varphi(x)}\int_D G_\alpha^D(x,y)M_\alpha^D\varphi(y)\rho(y)dy,\ |\rho|\le q\Big\}$$

is uniformly bounded and equicontinuous in $\overline{D}$. Consequently Λ_q is relatively compact in $C_0(D)$.

Proof. Taking $h\equiv M_\alpha^D\varphi$ in (2.2), we deduce that for ρ such that $|\rho|\le q$ and $x\in D$, we have

$$\Big|\int_D\frac{G_\alpha^D(x,y)}{M_\alpha^D\varphi(x)}M_\alpha^D\varphi(y)\rho(y)dy\Big|\le\int_D\frac{G_\alpha^D(x,y)}{M_\alpha^D\varphi(x)}M_\alpha^D\varphi(y)q(y)dy\le a_\alpha(q)<\infty. \tag{2.4}$$

So the family Λ_q is uniformly bounded.

Next we aim at proving that the family Λ_q is equicontinuous in $\overline{D}$. Let $x_0\in\overline{D}$ and $\varepsilon>0$. By (2.3), there exists $r>0$ such that

$$\sup_{z\in D}\frac{1}{M_\alpha^D\varphi(z)}\int_{B(x_0,2r)\cap D}G_\alpha^D(z,y)M_\alpha^D\varphi(y)q(y)dy\le\frac{\varepsilon}{2}.$$

If $x_0\in D$ and $x,x'\in B(x_0,r)\cap D$, then for ρ such that $|\rho|\le q$, we have

$$
\begin{aligned}
&\Big|\int_D\frac{G_\alpha^D(x,y)}{M_\alpha^D\varphi(x)}M_\alpha^D\varphi(y)\rho(y)dy-\int_D\frac{G_\alpha^D(x',y)}{M_\alpha^D\varphi(x')}M_\alpha^D\varphi(y)\rho(y)dy\Big|\\
&\le\int_D\Big|\frac{G_\alpha^D(x,y)}{M_\alpha^D\varphi(x)}-\frac{G_\alpha^D(x',y)}{M_\alpha^D\varphi(x')}\Big|M_\alpha^D\varphi(y)q(y)dy\\
&\le 2\sup_{z\in D}\int_{B(x_0,2r)\cap D}\frac{1}{M_\alpha^D\varphi(z)}G_\alpha^D(z,y)M_\alpha^D\varphi(y)q(y)dy\\
&\quad+\int_{(|x_0-y|\ge 2r)\cap D}\Big|\frac{G_\alpha^D(x,y)}{M_\alpha^D\varphi(x)}-\frac{G_\alpha^D(x',y)}{M_\alpha^D\varphi(x')}\Big|M_\alpha^D\varphi(y)q(y)dy
\end{aligned}
$$

$$\le \varepsilon + \int_{(|x_0-y|\ge 2r)\cap D} \Big|\frac{G_\alpha^D(x,y)}{M_\alpha^D\varphi(x)} - \frac{G_\alpha^D(x',y)}{M_\alpha^D\varphi(x')}\Big| M_\alpha^D\varphi(y)q(y)dy.$$

On the other hand, for every $y \in B^c(x_0,2r)\cap D$ and $x,x' \in B(x_0,r)\cap D$, by using (1.4) and the fact that $M_\alpha^D\varphi(z) \approx (\delta(z))^{\alpha-2}$, we have

$$\begin{aligned}
&\Big|\frac{1}{M_\alpha^D\varphi(x)}G_\alpha^D(x,y) - \frac{1}{M_\alpha^D\varphi(x')}G_\alpha^D(x',y)\Big| M_\alpha^D\varphi(y)\\
&\le \frac{M_\alpha^D\varphi(y)}{M_\alpha^D\varphi(x)}G_\alpha^D(x,y) + \frac{M_\alpha^D\varphi(y)}{M_\alpha^D\varphi(x')}G_\alpha^D(x',y)\\
&\le C\Big[\frac{\big(\delta(x)\big)^{3-\alpha}\big(\delta(y)\big)^{\alpha-1}}{|x-y|^{n+2-\alpha}} + \frac{\big(\delta(x')\big)^{3-\alpha}\big(\delta(y)\big)^{\alpha-1}}{|x'-y|^{n+2-\alpha}}\Big]\\
&\le C\Big[\frac{1}{|x-y|^{n+2-\alpha}} + \frac{1}{|x'-y|^{n+2-\alpha}}\Big](\delta(y))^{\alpha-1}\\
&\le C\big(\delta(y)\big)^{\alpha-1}.
\end{aligned}$$

Now since $x \mapsto \frac{1}{M_\alpha^D\varphi(x)}G_\alpha^D(x,y)$ is continuous outside the diagonal and $q \in K_\alpha(D)$, we deduce by the dominated convergence theorem and Proposition 2.2 (iii), that

$$\int_{(|x_0-y|\ge 2r)\cap D} \Big|\frac{G_\alpha^D(x,y)}{M_\alpha^D\varphi(x)} - \frac{G_\alpha^D(x',y)}{M_\alpha^D\varphi(x')}\Big| M_\alpha^D\varphi(y)q(y)dy \to 0 \quad \text{as } |x-x'| \to 0.$$

If $x_0 \in \partial D$ and $x \in B(x_0,r)\cap D$, then

$$\Big|\int_D \frac{G_\alpha^D(x,y)}{M_\alpha^D\varphi(x)} M_\alpha^D\varphi(y)\rho(y)dy\Big| \le \frac{\varepsilon}{2} + \int_{(|x_0-y|\ge 2r)\cap D} \frac{G_\alpha^D(x,y)}{M_\alpha^D\varphi(x)} M_\alpha^D\varphi(y)q(y)dy.$$

Now, since $\frac{G_\alpha^D(x,y)}{M_\alpha^D\varphi(x)} \to 0$ as $|x-x_0| \to 0$, for $|x_0-y| \ge 2r$, then by same argument as above, we obtain

$$\int_{(|x_0-y|\ge 2r)\cap D} \frac{G_\alpha^D(x,y)}{M_\alpha^D\varphi(x)} M_\alpha^D\varphi(y)q(y)dy \to 0 \quad \text{as } |x-x_0| \to 0.$$

So the family Λ_q is equicontinuous in $\overline{D}$. Therefore by Ascoli's theorem, the family Λ_q becomes relatively compact in $C_0(D)$. □

3. Proofs of Theorems 1.2 and 1.3

Proof of Theorem 1.2. Put

$$\lambda_0 := \inf_{x\in D} \frac{M_\alpha^D\varphi(x)}{G_\alpha^D(g(.,M_\alpha^D\psi))(x)}, \quad \mu_0 := \inf_{x\in D} \frac{M_\alpha^D\psi(x)}{G_\alpha^D(f(.,M_\alpha^D\varphi))(x)}.$$

Using (H2) and (2.2) we deduce that $\lambda_0 > 0$ and $\mu_0 > 0$.

Let $\lambda \in [0,\lambda_0)$ and $\mu \in [0,\mu_0)$. Then for each $x \in D$, we have

$$\lambda_0 G_\alpha^D(g(.,M_\alpha^D\psi))(x) \le M_\alpha^D\varphi(x)$$
$$\mu_0 G_\alpha^D(f(.,M_\alpha^D\varphi))(x) \le M_\alpha^D\psi(x).$$

So we define the sequences $(u_k)_{k\ge 0}$ and $(v_k)_{k\ge 0}$ by

$$v_0 = 1,$$
$$u_k(x) = 1 - \frac{\lambda}{M_\alpha^D\varphi(x)} \int_D G_\alpha^D(x,y) g\big(y, v_k(y) M_\alpha^D\psi(y)\big)dy,$$

$$v_{k+1}(x) = 1 - \frac{\mu}{M_\alpha^D \psi(x)} \int_D G_\alpha^D(x,y) f\big(y, u_k(y) M_\alpha^D \varphi(y)\big) dy.$$

By induction, we can see that

$$\begin{gathered} 0 < (1 - \frac{\lambda}{\lambda_0}) \leq u_k \leq 1, \\ 0 < (1 - \frac{\mu}{\mu_0}) \leq v_{k+1} \leq 1. \end{gathered}$$

Next, we prove that the sequence $(u_k)_{k\geq 0}$ is nondecreasing and the sequence $(v_k)_{k\geq 0}$ is nonincreasing. Indeed, we have

$$v_1 - v_0 = -\frac{\mu}{M_\alpha^D \psi} G_\alpha^D(f(., u_0 M_\alpha^D \varphi)) \leq 0$$

and therefore by (H1), we obtain that

$$u_1 - u_0 = \frac{\lambda}{M_\alpha^D \varphi} G_\alpha^D[g(., v_0 M_\alpha^D \psi) - g(., v_1 M_\alpha^D \psi)] \geq 0.$$

By induction, we assume that $u_k \leq u_{k+1}$ and $v_{k+1} \leq v_k$. Then we have

$$v_{k+2} - v_{k+1} = \frac{\mu}{M_\alpha^D \psi} G_\alpha^D[f(., u_k M_\alpha^D \varphi) - f(., u_{k+1} M_\alpha^D \varphi)] \leq 0$$

and

$$u_{k+2} - u_{k+1} = \frac{\lambda}{M_\alpha^D \varphi} G_\alpha^D[g(., v_{k+1} M_\alpha^D \psi) - g(., v_{k+2} M_\alpha^D \psi)] \geq 0.$$

Therefore, the sequences $(u_k)_{k\geq 0}$ and $(v_k)_{k\geq 0}$ converge respectively to two functions $\widetilde{u}$ and $\widetilde{v}$ satisfying

$$\begin{gathered} 0 < (1 - \frac{\lambda}{\lambda_0}) \leq \widetilde{u} \leq 1, \\ 0 < (1 - \frac{\mu}{\mu_0}) \leq \widetilde{v} \leq 1. \end{gathered} \tag{3.1}$$

On the other hand, using (H1), Proposition 2.2 and the dominate convergence theorem, we deduce that

$$\begin{gathered} \widetilde{u}(x) = 1 - \frac{\lambda}{M_\alpha^D \varphi(x)} \int_D G_\alpha^D(x,y) g(y, \widetilde{v}(y) M_\alpha^D \psi(y)) dy, \\ \widetilde{v}(x) = 1 - \frac{\mu}{M_\alpha^D \psi(x)} \int_D G_\alpha^D(x,y) f(y, \widetilde{u}(y) M_\alpha^D \varphi(y)) dy. \end{gathered}$$

Now by using (H1), (H2) and similar arguments as in the proof of Lemma 2.3, we deduce that $\widetilde{u}$ and $\widetilde{v}$ belongs to $C(\overline{D})$.

Put $u = \widetilde{u} M_\alpha^D \varphi$ and $v = \widetilde{v} M_\alpha^D \psi$. Then u and v are continuous in D and satisfy

$$\begin{gathered} u(x) = M_\alpha^D \varphi(x) - \lambda \int_D G_\alpha^D(x,y) g(y, v(y)) dy \\ v(x) = M_\alpha^D \psi(x) - \mu \int_D G_\alpha^D(x,y) f\big(y, u(y)\big) dy. \end{gathered} \tag{3.2}$$

In addition, since for each $x \in D$, $f\big(y, u(y)\big) \leq C\big(\delta(y)\big)^{\alpha-2} \widetilde{p}(y)$ and $g\big(y, u(y)\big) \leq C\big(\delta(y)\big)^{\alpha-2} \widetilde{q}(y)$, we deduce by Proposition 2.2 *(iii)* that the map $y \to f\big(y, u(y)\big) \in L^1_{\rm loc}(D)$ and $y \to g\big(y, u(y)\big) \in L^1_{\rm loc}(D)$. On the other hand, by (3.2), we have that $G_\alpha^D f(., u) \in L^1_{\rm loc}(D)$ and $G_\alpha^D g(., v) \in L^1_{\rm loc}(D)$. Hence, applying $(-\Delta|_D)^{\alpha/2}$ on both sides of (3.2), we conclude by [9, p. 230] that (u, v) is the required solution. □

Example 3.1. Let $\nu \geq 0$, $\sigma \geq 0$, $r+(1-\sigma)(\alpha-2)<\alpha$ and $\beta+(1-\nu)(\alpha-2)<\alpha$. Let p and q be two positive Borel measurable functions such that

$$p(x) \leq C\big(\delta(x)\big)^{-r}, \quad q(x) \leq C\big(\delta(x)\big)^{-\beta} \quad \text{for all } x \in D.$$

Let φ and ψ be positive continuous functions on ∂D. Therefore by Theorem 1.2, there exist two constants $\lambda_0>0$ and $\mu_0>0$ such that for each $\lambda \in [0,\lambda_0)$ and each $\mu \in [0,\mu_0)$, the problem

$$\begin{gathered}
(-\Delta|_D)^{\alpha/2}u+\lambda p(x)v^{\sigma}=0 \quad \text{in } D, \text{ in the sense of distributions} \\
(-\Delta|_D)^{\alpha/2}v+\mu q(x)u^{\nu}=0 \quad \text{in } D, \text{ in the sense of distributions} \\
\lim_{x\to z\in\partial D}\frac{u(x)}{M_\alpha^D 1(x)}=\varphi(z), \quad \lim_{x\to z\in\partial D}\frac{v(x)}{M_\alpha^D 1(x)}=\psi(z),
\end{gathered}$$

has a positive continuous solution (u,v) such that

$$\begin{gathered}
(1-\frac{\lambda}{\lambda_0})M_\alpha^D\varphi \leq u \leq M_\alpha^D\varphi \quad \text{in } D, \\
(1-\frac{\mu}{\mu_0})M_\alpha^D\psi \leq v \leq M_\alpha^D\psi \quad \text{in } D.
\end{gathered}$$

In particular, $\lim_{x\to z\in\partial D}u(x)=\infty$ and $\lim_{x\to z\in\partial D}v(x)=\infty$.

Proof of Theorem 1.3. Let $c:=1+a_\alpha(p_0)+a_\alpha(q_0)$, where $a_\alpha(p_0)$ and $a_\alpha(q_0)$ are the constant defined by the formula (1.10). We recall that from (H4) and Proposition 2.2 (i), we have $a_\alpha(p_0)<\infty$ and $a_\alpha(q_0)<\infty$. Let φ,ψ be positive continuous functions on ∂D such that $\varphi \geq c\phi$ and $\psi \geq c\phi$ on ∂D. It follows from the integral representation of $M_\alpha^D\varphi(x)$ and $M_\alpha^D\psi(x)$ (see [5, p. 265]), that for each $x \in D$ we have

$$M_\alpha^D\varphi(x) \geq ch_0(x) \quad \text{and} \quad M_\alpha^D\psi(x) \geq ch_0(x). \tag{3.3}$$

Let Λ be the nonempty closed convex set given by

$$\Lambda=\big\{\omega \in C(\overline{D}): \frac{h_0}{M_\alpha^D\varphi} \leq \omega \leq 1\big\}.$$

We define the operator T on Λ by

$$T(\omega)=1-\frac{1}{M_\alpha^D\varphi}G_\alpha^D(pf\left[M_\alpha^D\psi-G_\alpha^D(qg(\omega M_\alpha^D\varphi))\right]). \tag{3.4}$$

We will prove that T has a fixed point. Since for $\omega \in \Lambda$, we have $\omega \geq \frac{h_0}{M_\alpha^D\varphi}$, then we deduce from hypotheses (H3), (H4) and (2.2) that

$$G_\alpha^D(qg(\omega M_\alpha^D\varphi)) \leq G_\alpha^D(qg(h_0))=G_\alpha^D(q_0h_0) \leq a_\alpha(q_0)h_0. \tag{3.5}$$

So by using (3.3) and (3.5), we obtain

$$\begin{aligned}
M_\alpha^D\psi-G_\alpha^D(qg(\omega M_\alpha^D\varphi)) &\geq M_\alpha^D\psi-a_\alpha(q_0)h_0 \\
&\geq ch_0-a_\alpha(q_0)h_0 \\
&=(1+a_\alpha(p_0))h_0 \\
&\geq h_0>0.
\end{aligned}$$

Hence, by using again (H3), (H4) and (2.2), we deduce that

$$G_\alpha^D(pf\left[M_\alpha^D\psi-G_\alpha^D(qg(\omega M_\alpha^D\varphi))\right]) \leq G_\alpha^D(pf(h_0))=G_\alpha^D(p_0h_0) \leq a_\alpha(p_0)h_0. \tag{3.6}$$

Using the fact that $M_\alpha^D\varphi \approx h_0$ and Lemma 2.3, we deduce that the family of functions

$$\Big\{\frac{1}{M_\alpha^D\varphi}G_\alpha^D\big(pf\left[M_\alpha^D\psi - G_\alpha^D(qg(\omega M_\alpha^D\varphi))\right]\big) : \omega\in\Lambda\Big\}$$

is relatively compact in $C_0(D)$. Therefore, the set $T\ \Lambda$ is relatively compact in $C(\overline{D})$.

Next, we shall prove that T maps Λ into it self.

Since $M_\alpha^D\psi - G_\alpha^D(qg(\omega M_\alpha^D\varphi)) \geq h_0 > 0$, we have for all $\omega\in\Lambda$, $T\omega \leq 1$. Moreover, form (3.6), we obtain $T\omega \geq 1 - \frac{a_\alpha(p_0)h_0}{M_\alpha^D\varphi} \geq \frac{h_0}{M_\alpha^D\varphi}$, which proves that $T(\Lambda)\subset\Lambda$.

Now, we shall prove the continuity of the operator T in Λ in the supremum norm. Let $(\omega_k)_{k\in\mathbb{N}}$ be a sequence in Λ which converges uniformly to a function ω in Λ. Then, for each $x\in D$, we have

$$\begin{aligned}|T\omega_k(x) - T\omega(x)| \leq \frac{1}{M_\alpha^D\varphi(x)}G_\alpha^D\Big[p\Big|&f(M_\alpha^D\psi - G_\alpha^D(qg(\omega_k M_\alpha^D\varphi)))\\ &- f(M_\alpha^D\psi - G_\alpha^D(qg(\omega M_\alpha^D\varphi)))\Big|\Big](x).\end{aligned}$$

On the other hand, by similar arguments as above, we have

$$\begin{aligned}&p\Big|f(M_\alpha^D\psi - G_\alpha^D(qg(\omega_k M_\alpha^D\varphi))) - f(M_\alpha^D\psi - G_\alpha^D(qg(\omega M_\alpha^D\varphi)))\Big|\\ &\leq p\Big[f(M_\alpha^D\psi - G_\alpha^D(qg(\omega_k M_\alpha^D\varphi))) + f(M_\alpha^D\psi - G_\alpha^D(qg(\omega M_\alpha^D\varphi)))\Big]\\ &\leq 2p_0h_0.\end{aligned}$$

By the fact that $M_\alpha^D\varphi \approx h_0$, (2.2) and the dominated convergence theorem, We conclude that for all $x\in D$,

$$T\omega_k(x)\to T\omega(x) \quad \text{as } k\to+\infty.$$

Consequently, as $T(\Lambda)$ is relatively compact in $C(\overline{D})$, we deduce that the pointwise convergence implies the uniform convergence, namely,

$$\|T\omega_k - T\omega\|_\infty \to 0 \quad \text{as } k\to+\infty.$$

Therefore, T is a continuous mapping from Λ into itself. So, since $T(\Lambda)$ is relatively compact in $C(\overline{D})$, it follows that T is compact mapping on Λ.

Finally, the Schauder fixed-point theorem implies the existence of a function $\omega\in\Lambda$ such that $\omega = T\omega$. Put

$$u(x) = \omega(x)M_\alpha^D\varphi(x) \quad \text{and} \quad \upsilon(x) = M_\alpha^D\psi(x) - G_\alpha^D(qg(u))(x), \quad \text{for } x\in D.$$

Then (u,υ) satisfies

$$\begin{aligned}u(x) &= M_\alpha^D\varphi(x) - G_\alpha^D(pf(\upsilon))(x),\\ \upsilon(x) &= M_\alpha^D\psi(x) - G_\alpha^D(qg(u))(x).\end{aligned}$$

Finally, we verify that (u,υ) is the required solution. □

Example 3.2. Let $\nu>0$, $\sigma>0$, $r+(1+\nu)(\alpha-2)<\alpha$ and $\beta+(1+\sigma)(\alpha-2)<\alpha$. Let p and q be two nonnegative Borel measurable functions such that

$$p(x)\leq C\big(\delta(x)\big)^{-r}, \quad q(x)\leq C\big(\delta(x)\big)^{-\beta} \quad \text{for all } x\in D.$$

Let φ, ψ and ϕ be positive continuous functions on ∂D. Then there exists a constant $c > 1$ such that if $\varphi \geq c\phi$ and $\psi \geq c\phi$ on ∂D, then the problem

$$
\begin{gathered}
(-\Delta|_D)^{\alpha/2} u + p(x) v^{-\sigma} = 0 \quad \text{in } D, \text{ in the sense of distributions} \\
(-\Delta|_D)^{\alpha/2} v + q(x) u^{-\nu} = 0 \quad \text{in } D, \text{ in the sense of distributions} \\
\lim_{x \to z \in \partial D} \frac{u(x)}{M_\alpha^D 1(x)} = \varphi(z), \quad \lim_{x \to z \in \partial D} \frac{v(x)}{M_\alpha^D 1(x)} = \psi(z),
\end{gathered}
$$

has a positive continuous solution (u, v) satisfying that for each $x \in D$,

$$
\begin{gathered}
M_\alpha^D \phi \leq u \leq M_\alpha^D \varphi \quad \text{in } D, \\
M_\alpha^D \phi \leq v \leq M_\alpha^D \psi \quad \text{in } D.
\end{gathered}
$$

In particular $u(x) \approx \big(\delta(x)\big)^{\alpha-2} \approx v(x)$ in D.

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