Electronic Journal of Differential Equations, Vol. 2012 (2012), No. 216, pp. 1–14.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

NONTRIVIAL SOLUTIONS FOR NONLINEAR PROBLEMS WITH ONE SIDED RESONANCE

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Abstract. We find nontrivial smooth solutions for nonlinear elliptic Dirichlet problems driven by the p-Laplacian ($1 < p < \infty$), when one sided resonance occurs at the principal spectral interval.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ ($N \geq 1$) be a bounded domain with a C^2-boundary $\partial\Omega$. We consider the nonlinear Dirichlet problem

$$\begin{gathered} -\Delta_p u(z) = f(z, u(z)) \quad \text{a.e. in } \Omega, \\ u|_{\partial\Omega} = 0. \end{gathered} \tag{1.1}$$

Here Δ_p denotes the p-Laplacian differential operator defined by

$$\Delta_p u(z) = \operatorname{div}(\|Du(z)\|^{p-2} Du(z)), \quad \text{where } 1 < p < \infty.$$

The aim of this article is to derive nontrivial smooth solutions for (1.1), when one sided resonance occurs. Namely, asymptotically as $|x| \to \infty$, the quotient $\frac{f(z,x)}{|x|^{p-2}x}$ lies in the principal spectral interval $[\lambda_1, \lambda_2)$ and possibly interacts λ_1. Here λ_1, λ_2 are the first and the second eigenvalue respectively of the negative p-Laplacian with Dirichlet boundary conditions, denoted henceforth by $-\Delta_p^D$.

Starting with the celebrated paper of Landesman-Lazer [11], many authors have proved existence results for resonant elliptic boundary-value problems (see, e.g. [3, 5, 12, 19, 20, 21, 22] and the references therein). These works have established the existence of one solution or one nontrivial solution or multiple solutions of (1.1), under Landesman-Lazer (LL)-type conditions on the nonlinearity. For the use of the minimax method or the degree theory, one can refer for example to [3], [12], [19]. Another method used to deal with the resonance problem is the well-known Morse theory (see, e.g. [5, 12, 22]). Leray Schauder degree theory and saddle point theorem are also used to deal with the resonance problem when the nonlinearity is unbounded (see, e.g. [20, 21]).

2000 *Mathematics Subject Classification.* 35J80, 35J85, 58E05.

Key words and phrases. One sided resonance; principal spectral interval; Cerami-condition; critical groups; Morse theory.

Submitted March 21, 2012. Published November 29, 2012.

In the present work we do not use LL-type conditions and our hypotheses are in principle easier to verify. Our approach combines variational methods based on the critical point theory, together with techniques from Morse theory.

2. Mathematical Background

Let X be a Banach space and X^* its topological dual. By $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair (X, X^*). By " $\xrightarrow{w}$ " and " $\to$ " we denote the weak and strong convergence respectively, on X.

We say that a map $A : X \to X^*$ is of type $(S)_+$, if for each sequence $\{x_n\}_{n\geq 1} \subseteq X$ such that

$$x_n \xrightarrow{w} x \text{ in } X \text{ and } \limsup_{n\to\infty} \langle A(x_n), x_n - x \rangle \leq 0,$$

one has $x_n \to x$ in X.

Let $\Omega \subseteq \mathbb{R}^N$ $(N \geq 1)$ be a bounded domain with a C^2-boundary $\partial\Omega$. In the analysis of problem (1.1), we will use the Sobolev space $W_0^{1,p}(\Omega)$ $(1 < p < \infty)$ which is the closure with respect to the Sobolev norm of the linear space

$$C_0^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) : u|_{\partial\Omega} = 0\}.$$

Let $A : W_0^{1,p}(\Omega) \to (W_0^{1,p}(\Omega))^*$ be the operator, defined by

$$\langle A(x), y \rangle = \int_\Omega \|Dx(z)\|^{p-2} (Dx(z), Dy(z))_{\mathbb{R}^N}\, dz, \quad \text{for all } x, y \in W_0^{1,p}(\Omega).$$

Then A is of type $(S)_+$. (Here $(\cdot, \cdot)_{\mathbb{R}^N}$ denotes the usual inner product in $\mathbb{R}^N$ and Dx is the gradient of x).

Next, let us recall a few basic definitions and facts from critical point theory and from Morse theory.

Let $\varphi \in C^1(X)$. We say that φ satisfies the *Palais-Smale condition*, if every sequence $\{x_n\}_{n\geq 1} \subseteq X$ such that

$$\sup_n |\varphi(x_n)| < \infty \quad \text{and} \quad \|\varphi'(x_n)\|_* \to 0, \quad \text{as } n \to \infty,$$

has a strongly convergent subsequence.

A similar compactness condition which is weaker than PS-condition is the *Cerami condition*. Namely, we say that φ satisfies the *Cerami condition*, if every sequence $\{x_n\}_{n\geq 1} \subseteq X$ such that

$$\sup_n |\varphi(x_n)| < \infty \quad \text{and} \quad (1 + \|x_n\|)\|\varphi'(x_n)\|_* \to 0, \quad \text{as } n \to \infty,$$

admits a strongly convergent subsequence.

For each $c \in \mathbb{R}$, we introduce the sets

$$\varphi^c = \{x \in X : \varphi(x) \leq c\} \quad \text{(the sublevel set of } \varphi \text{ at } c)$$
$$K_\varphi = \{x \in X : \varphi'(x) = 0\} \quad \text{(the critical set of } \varphi).$$

Let (Y_1, Y_2) be a topological pair with $Y_1 \subseteq Y_2 \subseteq X$. For every integer $k \geq 0$, by $H_k(Y_2, Y_1)$ we denote the k^{th}-relative singular homology group of (Y_1, Y_2) with integer coefficients. Special case: $H_k(X, \emptyset) = \delta_{k,0}\mathbb{Z}, k \geq 0$.

If $x_0 \in X$ is an isolated critical point of φ with $\varphi(x_0) = c$, then the critical groups of φ at x_0 are defined by

$$C_k(\varphi, x_0) = H_k\big(\varphi^c \cap U, \varphi^c \cap U \setminus \{x_0\}\big) \quad \text{for all } k \geq 0,$$

where U is a neighborhood of x_0 such that $K_\varphi \cap \varphi^c \cap U = \{x_0\}$ (see [5, 16]). The excision property of singular homology implies that the above definition is independent of the particular neighborhood U we use.

Now, suppose that $\varphi \in C^1(X)$ satisfies the Palais-Smale or the Cerami-condition and $\inf \varphi(K_\varphi) > -\infty$. Let $c < \inf \varphi(K_\varphi)$. The critical groups of φ at infinity are defined by

$$C_k(\varphi, \infty) = H_k(X, \varphi^c) \quad \text{for all } k \geq 0$$

(see [4]).

The second deformation theorem (see, e.g. [7]) implies that this definition is independent of the particular choice of the level $c < \inf \varphi(K_\varphi)$.

If $C_k(\varphi, \infty) \neq 0$, for some $k \geq 0$, then there exists a critical point $x \in X$ of φ, such that $C_k(\varphi, x) \neq 0$.

Finally, let us recall some basic facts about the spectrum of the negative Dirichlet p-Laplacian with weight m, denoted by $(-\Delta_p^D, m)$. So, let

$$L^\infty(\Omega)_+ = \{m \in L^\infty(\Omega) : m(z) \geq 0 \text{ a.e. in } \Omega\},$$

let $m \in L^\infty(\Omega)_+ \setminus \{0\}$ and consider the weighted nonlinear eigenvalue problem

$$\begin{gathered} -\Delta_p u(z) = \widehat{\lambda} m(z) |u(z)|^{p-2} u(z), \quad \text{a.e. in } \Omega, \\ u|_{\partial\Omega} = 0, \quad \widehat{\lambda} \in \mathbb{R}. \end{gathered} \tag{2.1}$$

By an eigenvalue of $(-\Delta_p^D, m)$ we mean a real number $\widehat{\lambda}$, such that (2.1) has a nontrivial solution u. Nonlinear regularity theory (see e.g. [7, pp. 737-738]) implies that $u \in C_0^1(\overline{\Omega})$. The least $\widehat{\lambda} \in \mathbb{R}$ for which (2.1) has a nontrivial solution is the first eigenvalue of $(-\Delta_p^D, m)$ and it is denoted by $\widehat{\lambda}_1(m)$. We recall some basic properties of $\widehat{\lambda}_1(m)$:

- $\widehat{\lambda}_1(m) > 0$.
- $\widehat{\lambda}_1(m)$ is isolated (i.e., there exists $\varepsilon > 0$ such that $(\widehat{\lambda}_1(m), \widehat{\lambda}_1(m) + \varepsilon)$ contains no eigenvalues).
- $\widehat{\lambda}_1(m)$ is simple (i.e., the corresponding eigenspace is one-dimensional).
- $\widehat{\lambda}_1(m)$ is characterized by the Rayleigh quotient:

$$\widehat{\lambda}_1(m) = \inf \big\{ \frac{\|Du\|_p^p}{\int_\Omega m|u|^p \, dz} : u \in W_0^{1,p}(\Omega), u \not\equiv 0 \big\}.$$

The above is attained on the one dimensional eigenspace of $\widehat{\lambda}_1(m)$. Let $\widehat{u}_1$ be a normalized eigenfunction of $\widehat{\lambda}_1(m)$, i.e.,

$$\int_\Omega m|\widehat{u}_1|^p dz = 1.$$

We already know that $\widehat{u}_1 \in C_0^1(\overline{\Omega})$ and from the Rayleigh quotient, it is clear that $\widehat{u}_1$ does not change sign, so we may assume that $\widehat{u}_1(z) \geq 0$, for all $z \in \overline{\Omega}$. Using the nonlinear maximum principle of Vázquez [23], we obtain that $\widehat{u}_1(z) > 0$, for all $z \in \overline{\Omega}$. It turns out that for each $\widehat{\lambda}_1(m)-$ eigenfunction u we have that $u(z) \neq 0$, for all $z \in \overline{\Omega}$. For more details we refer for example to [1, 7, 14, 15].

Since $-\Delta_p^D$ is $(p-1)$-homogeneous operator, the Ljusternik-Schnirelmann theory implies that we have a whole strictly increasing sequence $\{\widehat{\lambda}_k(m)\}_{k\geq 1}$ of eigenvalues such that

$$\widehat{\lambda}_k(m) \to +\infty, \quad \text{as } k \to +\infty$$

(see [6]). These eigenvalues are called the "LS-eigenvalues" of $(-\Delta_p^D, m)$.

We know that $\widehat{\lambda}_2(m)$ is the second eigenvalue of $(-\Delta_p^D, m)$; i.e., $\widehat{\lambda}_2(m) > \widehat{\lambda}_1(m)$ and there are no eigenvalues between $\widehat{\lambda}_1(m)$ and $\widehat{\lambda}_2(m)$.

Viewed as functions of the weight $m \in L^\infty(\Omega)_+ \setminus \{0\}$, the eigenvalues $\widehat{\lambda}_1(m)$ and $\widehat{\lambda}_2(m)$ are continuous functions and exhibit certain monotonicity properties, namely:

- If $m(z) \le \widetilde{m}(z)$, a.e. on Ω, with strict inequality on a set of positive measure, then $\widehat{\lambda}_1(\widetilde{m}) < \widehat{\lambda}_1(m)$.
- If $m(z) < \widetilde{m}(z)$, a.e. on Ω, then $\widehat{\lambda}_2(\widetilde{m}) < \widehat{\lambda}_2(m)$ (see [2]).

Special cases: If $m \equiv 1$, then we write $\widehat{\lambda}_k(m) = \lambda_k, k \ge 1$ and λ_k is the k-th eigenvalue of the negative Dirichlet p-Laplacian $-\Delta_p^D$.

If $m \equiv \lambda_k$ for some $k \ge 1$, then clearly $\widehat{\lambda}_k(\lambda_k) = 1$.

3. Main result

In this section we establish the existence of at least one nontrivial smooth solution of the problem (1.1), when one-sided resonance occurs at the principal spectral interval $[\lambda_1, \lambda_2)$ of $-\Delta_p^D$.

The hypotheses on the reaction $f(z,x)$ are:

(H) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z,0) = 0$ for a.a. $z \in \Omega$,

(i)
$$|f(z,x)| \le \alpha(z) + c_1|x|^{p-1}$$
for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, with $\alpha \in L^\infty(\Omega)_+, c_1 > 0$.

(ii)
$$\lambda_1 \le \liminf_{|x|\to\infty} \frac{f(z,x)}{|x|^{p-2}x} \le \limsup_{|x|\to\infty} \frac{f(z,x)}{|x|^{p-2}x} < \lambda_2, \quad \text{uniformly for a.a.} z \in \Omega.$$

(iii) If $F(z,x) = \int_0^x f(z,s)ds$, then
$$\lim_{|x|\to\infty} [f(z,x)x - pF(z,x)] = +\infty, \quad \text{uniformly for a.a.} z \in \Omega.$$

(iv) There exist $\tau, \sigma \in (1,p)$, $\delta_0 > 0$, $c_2 > 0$ such that for almost all $z \in \Omega$ and for all $|x| \le \delta_0$, we have
$$F(z,x) \ge c_2|x|^\tau, \quad \sigma F(z,x) \ge f(z,x)x.$$

Note that Hypothesis H(ii) implies that we have one-sided resonance at the principal spectral interval $[\lambda_1, \lambda_2)$ of $-\Delta_p^D$. On the other hand, hypothesis H(iii) enables us to avoid conditions of Landesman-Lazer type which are usually imposed on the nonlinearity when one deals with problems at resonance.

Remark 3.1. Each weak solution $u \in W_0^{1,p}(\Omega)$ of problem (1.1) is smooth; i.e., $u \in C_0^1(\overline{\Omega})$. This follows from the nonlinear regularity theory (see [10], [13]) and from the fact that the function α in hypothesis H(i) lies in $L^\infty(\Omega)_+$.

Example 3.2. The following function satisfies H(i)-(iv) (for the sake of simplicity, we drop the z-dependence):
$$f(x) = \begin{cases} \lambda_1|x|^{p-2}x - |x|^{\tau-2}x, & \text{if } |x| > 1 \\ \lambda_1|x|^{\tau-2}x - |x|^{p-2}x, & \text{if } |x| \le 1 \end{cases}$$

with $1 < \tau < p < \infty$. Indeed, H(i) is easily checked whereas $\lim_{|x|\to\infty} \frac{f(x)}{|x|^{p-2}x} = \lambda_1$ and hence H(ii) holds. Moreover, for $|x| > 1$ and for some $c_3 > 0$ we have

$$xf(x) - pF(x) = (\frac{p}{\tau} - 1)|x|^\tau - c_3 \to +\infty, \quad \text{as } |x| \to \infty$$

and thus, H(iii) also holds. Finally, to obtain H(iv) choose

$$\sigma \in (\tau, p), \quad c_2 \in (0, \frac{\lambda_1}{\tau}) \quad \text{and} \quad \delta_0 \in (0,1) \quad \text{with} \quad \delta_0^{p-\tau} < p(\frac{\lambda_1}{\tau} - c_2).$$

Then for $|x| \le \delta_0$ we have

$$\begin{gathered}
\sigma F(x) - xf(x) = \lambda_1(\frac{\sigma}{\tau} - 1)|x|^\tau + (1 - \frac{\sigma}{p})|x|^p \ge 0, \\
F(x) = \frac{\lambda_1}{\tau}|x|^\tau - \frac{|x|^p}{p} = |x|^\tau(\frac{\lambda_1}{\tau} - \frac{|x|^{p-\tau}}{p}) \ge c_2|x|^\tau.
\end{gathered}$$

In [20], $f(x)$ is unbounded for $x < 0$ and bounded for $x \ge 0$. For the function f defined above we have that $f(+\infty) = +\infty$.

Now we set $g(x) = f(x) - \lambda_1|x|^{p-2}x, x \in \mathbb{R}$. Under the classic versions of the LL - conditions, the limits $g(\pm\infty)$ are real numbers (see for example [3], [11]). Unlike these works, the above defined function g satisfies $g(\pm\infty) = \mp\infty$.

Moreover, *generalized* LL - conditions are used in [12, 19, 21, 22] in the semilinear case ($p = 2$). In all these works, the function g satisfies the following condition:

> For each sequence $\{w_n\} \subseteq W_0^{1,2}(\Omega)$ with
>
> $$\|w_n\| \to \infty, \quad \frac{\|P_1 w_n\|}{\|w_n\|} \to 1,$$
>
> we have that
>
> $$\limsup_n \int_\Omega g(w_n(z))\frac{P_1 w_n(z)}{\|P_1 w_n\|}dz > 0,$$
>
> where P_1 is the projection operator from $W_0^{1,2}(\Omega)$ onto the principal eigenspace of $-\Delta^D$.

In our example this condition fails. To see this, let $\widehat{u}_1$ be the normalized positive smooth principal eigenfunction of $-\Delta_p^D$ and set $w_n = n\widehat{u}_1$, $n \ge 1$. Clearly, $\|w_n\| \to \infty$ and $\|P_1 w_n\|/\|w_n\| = 1$, for all $n \ge 1$ (note that for $p \ne 2$, the projection P_1 is still well defined). On the other hand, for all $n > 1/\min_{\overline{\Omega}} \widehat{u}_1$ we have

$$\int_\Omega g(w_n(z))\frac{P_1 w_n(z)}{\|P_1 w_n\|}dz = -n^{\tau-1}\int_\Omega \widehat{u}_1(z)^\tau dz \to -\infty, \quad \text{as } n \to \infty.$$

We introduce the energy functional

$$\varphi(u) = \frac{1}{p}\|Du\|_p^p - \int_\Omega F(z, u(z))dz, \quad u \in W_0^{1,p}(\Omega).$$

Under hypothesis H(i), $\varphi \in C^1(W_0^{1,p}(\Omega))$ and each weak solution of the problem (1.1) is a critical point of φ.

Since $f(z,0) = 0$, a.e. in Ω, the origin 0 is trivially a critical point of φ. We search for nontrivial critical points of φ. For this purpose, we are going to compute the critical groups

$$C_k(\varphi, \infty), \quad C_k(\varphi, 0), \quad k \ge 0.$$

First, we compute the critical groups of φ at infinity. In this direction, we prove an auxiliary result which slightly extends [18, Lemma 2.4] (the latter is formulated in Hilbert spaces).

Proposition 3.3. *Let X be a Banach space and $(t,u) \to h_t(u)$ be a homotopy which belongs to $C^1([0,1]\times X)$ and it is bounded. Suppose that*

(i) *there exists $R>0$ s.t. for all $t\in[0,1]$,*
$$K_{h_t} \subseteq \overline{B}_R = \{x \in X : \|x\| \le R\}$$
(ii) *the maps $u \to \partial_t h_t(u)$ and $u \to h_t'(u)$ are both locally Lipschitz*
(iii) *h_0 and h_1 both satisfy the C-condition*
(iv) *there exist $\beta \in \mathbb{R}$ and $\delta > 0$ s.t.*
$$h_t(u) \le \beta \Rightarrow (1+\|u\|)\|h_t'(u)\|_* \ge \delta \quad \text{for all } t \in [0,1].$$

Then $C_k(h_0,\infty) = C_k(h_1,\infty)$, for all $k \ge 0$.

Proof. By the hypothesis $h \in C^1([0,1]\times X)$, we know that it admits a pseudogradient vector field $\widehat{v} = (v_0, v) : [0,1] \times (X \setminus \overline{B}_R) \to [0,1]\times X$. Moreover, taking into account the construction of the pseudogradient vector field, we know that $v_0 = \partial_t h_t$. Also, by definition $(t,u) \to v_t(u)$ is locally Lipschitz and in fact for every $t \in [0,1]$, $v_t(\cdot)$ is a pseudogradient vector field for the functional $h_t(\cdot)$. So, for every $(t,u) \in [0,1]\times (X\setminus \overline{B}_R)$ we have
$$\langle h_t'(u), v_t(u)\rangle \ge \|h_t'(u)\|_*^2. \tag{3.1}$$
The map
$$X \setminus \overline{B}_R \ni u \to -\frac{|\partial_t h_t(u)|}{\|h_t'(u)\|_*^2} v_t(u) = w_t(u) \in X$$
is well defined and locally Lipschitz. Since by hypothesis $(t,u)\to h_t(u)$ is bounded, we can find $\eta \le \beta$ s.t.
$$\eta < \inf[h_t(u) : t\in[0,1], \|u\| \le R].$$
We choose $\eta \le \beta$ s.t. $h_0^\eta \ne \emptyset$ or $h_1^\eta \ne \emptyset$, (if no such η can be found, then $C_k(h_0,\infty) = C_k(h_1,\infty) = H_k(X,\emptyset) = \delta_{k,0}\mathbb{Z}$ for all $k\ge 0$ and so we are done). To fix things, we assume that $h_0^\eta \ne \emptyset$ and choose $y \in h_0^\eta$. We consider the following Cauchy problem
$$\frac{d\xi}{dt} = w_t(\xi) \quad t\in[0,1], \quad \xi(0) = y. \tag{3.2}$$
Since w_t is locally Lipschitz, this Cauchy problem admits a unique local flow (see [7, p. 618]). We have
$$\begin{aligned} \frac{d}{dt}h_t(\xi) &= \langle h_t'(\xi), \frac{d\xi}{dt}\rangle + \partial_t h_t(\xi) \\ &= \langle h_t'(\xi), w_t(\xi)\rangle + \partial_t h_t(\xi) \quad \text{(see (3.2))} \\ &\le -|\partial_t h_t(\xi)| + \partial_t h_t(\xi) \le 0 \end{aligned}$$
(see (3.1)). This implies that the mapping $t \mapsto h_t(\xi(t,y))$ is non-increasing. Therefore,
$$\begin{gathered} h_t(\xi(t,y)) \le h_0(\xi(0,y)) = h_0(y) \le \eta \le \beta, \\ \Rightarrow (1+\|\xi(t,y)\|)\|h_t'(\xi(t,y))\|_* \ge \delta \end{gathered}$$
(by hypothesis); therefore, $h_t'(\xi(t,y)) \ne 0$.

This shows that the flow $\xi(\cdot, y)$ is global on $[0,1]$. Then $\xi(1,y)$ is a homeomorphism between h_0^η and a subset of h_1^η. Reversing the time ($t \to 1-t$), we show that h_1^η is a homeomorphism to a subset of h_0^η. Therefore h_0^η and h_1^η are homotopy equivalent and so

$$\begin{gathered} H_k(X, h_0^\eta) = H_k(X, h_1^\eta) \quad \text{for all } k \geq 0, \\ \Rightarrow C_k(h_0, \infty) = C_k(h_1, \infty) \quad \text{for all } k \geq 0. \end{gathered}$$

□

To proceed, let $\widehat{u}_1$ be a λ_1-eigenfunction of $-\Delta_p^D$ with $\|\widehat{u}_1\|_p = 1$. Consider the set

$$V = \{u \in W_0^{1,p}(\Omega) : \int_\Omega \widehat{u}_1^{p-1} u dz = 0\}.$$

Then V is a closed linear subspace of $W_0^{1,p}(\Omega)$ and we have

$$W_0^{1,p}(\Omega) = \mathbb{R}\widehat{u}_1 \oplus V.$$

We introduce the quantity

$$\lambda_V = \inf\big\{\frac{\|Du\|_p^p}{\|u\|_p^p} : u \in V, u \neq 0\big\}.$$

We know that $\lambda_1 < \lambda_V \leq \lambda_2$ (see [8, Lemma 3.3]).

Let $\mu \in (\lambda_1, \lambda_V)$ and consider the C^1-functional $\psi : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\psi(u) = \frac{1}{p}\|Du\|_p^p - \frac{\mu}{p}\|u\|_p^p \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Using standard arguments we may show that ψ has the following properties:

- 0 is the unique critical point of ψ.
- ψ satisfies the Palais-Smale condition.
- $\psi_{|\mathbb{R}\widehat{u}_1}$ is anticoercive, $\psi_{|V}$ is coercive.

The last two properties yield

$$C_1(\psi, \infty) \neq 0 \tag{3.3}$$

(see [4, Proposition 3.8]).

We intend to prove the following statement.

Proposition 3.4. *Under hypotheses* H(i), (ii), (iii), *we have*

$$C_k(\varphi, \infty) \simeq C_k(\psi, \infty), \quad k \geq 0.$$

For the proof of Proposition 3.4 we shall need the following result.

Proposition 3.5. *Assume that hypotheses* H(i), (ii), (iii) *hold. We consider the homotopy $h : [0,1] \times W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by*

$$h(t,u) = (1-t)\varphi(u) + t\psi(u) \quad \textit{for all } (t,u) \in [0,1] \times W_0^{1,p}(\Omega).$$

Let $\{u_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$, $\{t_n\}_{n\geq 1} \subseteq [0,1]$ be sequences such that

$$t_n \to t, \quad (1 + \|u_n\|)\|h_u'(t_n, u_n)\|_* \to 0, \quad \|u_n\| \to +\infty.$$

Then by passing to subsequences, we obtain

$$t_n \to 0, \quad |u_n(z)| \to +\infty, \ \textit{a.e. in } \Omega, \quad h(t_n, u_n) \to +\infty.$$

Proof. By the convergence

$$(1+\|u_n\|)\|h'_u(t_n,u_n)\|_* \to 0$$

we have

$$\big|\langle A(u_n),h\rangle-(1-t_n)\int_\Omega f(z,u_n)hdz-t_n\mu\int_\Omega |u_n|^{p-2}u_nhdz\big| \le \frac{\varepsilon_n\|h\|}{1+\|u_n\|} \tag{3.4}$$

for all $h\in W_0^{1,p}(\Omega)$, with $\varepsilon_n\to 0^+$.

We set $y_n=\frac{u_n}{\|u_n\|}$, $n\ge 1$. Then $\|y_n\|=1$ for all $n\ge 1$ and so we may assume that

$$y_n \xrightarrow{w} y \text{ in } W_0^{1,p}(\Omega), \quad y_n\to y \text{ in } L^p(\Omega), \quad y_n(z)\to y(z), \text{ a.e. in } \Omega. \tag{3.5}$$

Dividing both sides of (3.4) by $\|u_n\|^{p-1}$ we have

$$\begin{aligned} &\big|\langle A(y_n),h\rangle-(1-t_n)\int_\Omega \frac{f(z,u_n)}{\|u_n\|^{p-1}}hdz-t_n\mu\int_\Omega |y_n|^{p-2}y_nhdz\big| \\ &\le \frac{\varepsilon_n\|h\|}{(1+\|u_n\|)\|u_n\|^{p-1}}, \quad \text{for all } n\ge 1. \end{aligned} \tag{3.6}$$

Hypothesis H(i) implies that the sequence

$$\Big\{\frac{f(\cdot,u_n(\cdot))}{\|u_n\|^{p-1}}\Big\}_{n\ge 1}\subseteq L^{p'}(\Omega), \quad 1/p+1/p'=1,$$

is bounded. Thus, we may assume that it is weakly convergent in $L^{p'}(\Omega)$. Using hypothesis H(iii) and reasoning as in [17, Proposition 5], we may find $\xi\in L^\infty(\Omega)_+$ such that

$$\frac{f(\cdot,u_n(\cdot))}{\|u_n\|^{p-1}} \xrightarrow{w} \xi|y|^{p-2}y \text{ in } L^{p'}(\Omega) \quad \text{and} \quad \lambda_1\le\xi(z)<\lambda_2 \text{ a.e. in } \Omega. \tag{3.7}$$

In (3.6) we choose $h=y_n-y\in W_0^{1,p}(\Omega)$, pass to the limit as $n\to\infty$ and use (3.5). Then

$$\lim_{n\to\infty}\langle A(y_n),y_n-y\rangle=0,$$

which implies $y_n\to y$ in $W_0^{1,p}(\Omega)$ (since A is of type $(S)_+$). Then

$$\|y\|=1. \tag{3.8}$$

So, if in (3.6) we pass to the limit as $n\to\infty$ and use (3.7) and (3.8), then

$$\langle A(y),h\rangle=(1-t)\int_\Omega \xi|y|^{p-2}yhdz+t\mu\int_\Omega |y|^{p-2}yhdz \quad \text{for all } h\in W_0^{1,p}(\Omega),$$

which implies

$$A(y)=\xi_t|y|^{p-2}y \quad \text{with } \xi_t=(1-t)\xi+t\mu;$$

therefore,

$$-\Delta_p y(z)=\xi_t(z)|y(z)|^{p-2}y(z) \text{ a.e. in } \Omega, \quad u=0 \text{ on } \partial\Omega. \tag{3.9}$$

Note that $\lambda_1\le\xi_t(z)<\lambda_2$ a.e. in Ω (recall that $t\in[0,1],\lambda_1<\mu<\lambda_2$). If $\xi_t\not\equiv\lambda_1$, then the monotonicity properties of the weighted eigenvalues (see Section 2) yield

$$\widehat{\lambda}_1(\xi_t)<\widehat{\lambda}_1(\lambda_1)=1, \quad \widehat{\lambda}_2(\xi_t)>\widehat{\lambda}_2(\lambda_2)=1;$$

therefore, $y\equiv 0$ (see (3.9)) which contradicts (3.8).

Thus, $\xi_t \equiv \lambda_1$, so $t = 0$ and $\xi \equiv \lambda_1$. It follows from (3.9) that y is a λ_1-eigenfunction and hence, $y(z) \neq 0$, a.e. in Ω. Consequently,

$$|u_n(z)| = \|u_n\||y_n(z)| \to +\infty, \quad \text{a.e. in } \Omega. \tag{3.10}$$

It remains to show that

$$h(t_n, u_n) \to +\infty.$$

Indeed, the convergence

$$(1 + \|u_n\|)\|h'_u(t_n, u_n)\|_* \to 0$$

implies that

$$\langle h'_u(t_n, u_n), u_n\rangle \to 0.$$

Moreover, (3.10) combined with hypothesis H(iii) and also with Fatou's lemma gives

$$\int_\Omega [u_n(z)f(z, u_n(z)) - pF(z, u_n(z))]dz \to +\infty.$$

Now the conclusion follows from the formula

$$ph(t_n, u_n) = \langle h'_u(t_n, u_n), u_n\rangle + (1 - t_n)\int_\Omega [u_n(z)f(z, u_n(z)) - pF(z, u_n(z))]dz,$$

$n \geq 1$. □

Corollary 3.6. *Under hypotheses* H(i), (ii), (iii)*, the energy functional* φ *satisfies the Cerami condition.*

Proof. Suppose that $\{u_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ satisfies

$$\sup_n |\varphi(u_n)| < \infty, \quad (1 + \|u_n\|)\|\varphi'(u_n)\|_* \to 0.$$

We claim that $\{u_n\}_{n\geq 1}$ is bounded in $W_0^{1,p}(\Omega)$. Indeed, if this is not the case, then by passing to subsequences we may assume that

$$\|u_n\| \to +\infty.$$

Now we observe that $\varphi(u) = h(0, u)$, for all $u \in W_0^{1,p}(\Omega)$. Applying Proposition 3.5 and by passing to subsequences we deduce that $\varphi(u_n) = h(0, u_n) \to +\infty$ (false). This proves our claim, i.e., $\{u_n\}_{n\geq 1}$ is bounded in $W_0^{1,p}(\Omega)$.

Therefore, we may assume that

$$u_n \xrightarrow{w} u \text{ in } W_0^{1,p}(\Omega) \quad \text{and} \quad u_n \to u \text{ in } L^p(\Omega). \tag{3.11}$$

Then (3.11) in conjunction with hypothesis H(i) and also with the convergence $\|\varphi'(u_n)\|_* \to 0$ yields

$$\int_\Omega f(\cdot, u(\cdot))(u_n - u)dz \to 0, \quad \langle \varphi'(u_n), u_n - u\rangle \to 0.$$

But

$$\langle A(u_n), u_n - u\rangle - \int_\Omega f(\cdot, u(\cdot))(u_n - u)dz = \langle \varphi'(u_n), u_n - u\rangle, \quad n \geq 1,$$

so,

$$\lim_{n\to\infty} \langle A(u_n), u_n - u\rangle = 0, \Rightarrow u_n \to u \quad \text{in } W_0^{1,p}(\Omega)$$

(since A is of type $(S)_+$). □

Proof of Proposition 3.4. We consider the homotopy $h : [0,1] \times W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$h(t,u) = (1-t)\varphi(u) + t\psi(u) \quad \text{for all } (t,u) \in [0,1] \times W_0^{1,p}(\Omega).$$

Clearly, $h(0,\cdot) = \varphi, h(1,\cdot) = \psi$. By Proposition 3.3, it suffices to show that there exist $\beta \in \mathbb{R}, \delta > 0$, such that for all $t \in [0,1]$, $u \in W_0^{1,p}(\Omega)$,

$$h(t,u) \leq \beta \Rightarrow (1+\|u\|)\|h'_u(t,u)\|_* > \delta.$$

Suppose that this is not the case. Then we may find

$$\{t_n\}_{n\geq 1} \subseteq [0,1], \quad \{u_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega),$$

such that

$$t_n \to t \in [0,1], \quad (1+\|u_n\|)\|h'_u(t_n,u_n)\|_* \to 0, \quad h(t_n,u_n) \to -\infty.$$

Now Proposition 3.5 guarantees that $\{u_n\}_{n\geq 1}$ is bounded so, we may assume that (3.11) holds. Applying (3.4) for $h = u_n - u$ and passing to the limit as $n \to +\infty$, we obtain

$$\lim_{n\to\infty} \langle A(u_n), u_n - u\rangle = 0$$

which implies $u_n \to u$ in $W_0^{1,p}(\Omega)$ (since A is of type $(S)_+$). Therefore, $h(t_n,u_n) \to h(t,u)$, which is a contradiction. $\square$

Next, we compute the critical groups of φ at zero. Without loss of generality we may assume that 0 is an isolated critical point of φ (otherwise we can produce a whole sequence of distinct critical points of φ, so we are done). We start with two lemmas.

Lemma 3.7. *Let $g \in C^1([0,1])$ such that either $g(1) < 0$ or $g(1) = 0, g'(1) > 0$. If $g(\widehat{t}) > 0$, for some $\widehat{t} \in (0,1)$, then there exists $\widehat{t_2} \in (\widehat{t},1)$, such that*

$$g(\widehat{t_2}) = 0, \quad g'(\widehat{t_2}) \leq 0.$$

Proof. We claim that

$$g(\widehat{t_1}) = 0, \quad \text{for some } \widehat{t_1} \in (\widehat{t},1).$$

Indeed, this is clear from Bolzano's theorem, in the case $g(1) < 0$.

Suppose now that $g(1) = 0, g'(1) > 0$. By continuity of g', we may find $\theta \in (0,1)$, such that

$$0 < \widehat{t} < \theta < 1, \quad g' > 0 \quad \text{on } [\theta,1].$$

Since $g(1) = 0$, we obtain that $g < 0$ on $[\theta,1)$ and the claim follows again from Bolzano's theorem.

To proceed, we set

$$\widehat{t_2} = \min\{t \in [\widehat{t},1] : g(t) = 0\}.$$

Then

$$\widehat{t} < \widehat{t_2} \leq \widehat{t_1}, \quad g(\widehat{t_2}) = 0, \quad g(t) \neq 0, \quad \text{for all } t \in [\widehat{t},\widehat{t_2}).$$

But since $g(\widehat{t}) > 0$, the continuity of g gives $g(t) > 0$ for all $t \in [\widehat{t},\widehat{t_2})$. Then

$$g'(\widehat{t_2}) = \lim_{t\to\widehat{t_2}^-} \frac{g(t)}{t-\widehat{t_2}} \leq 0,$$

which completes the proof. $\square$

Lemma 3.8. *Let X be a Banach space and $\varphi \in C^1(X), \rho > 0$ such that*

$$\langle \varphi'(u), u \rangle > 0, \quad \textit{for all } u \in \overline{B}_\rho \setminus \{0\} \textit{ with } \varphi(u) = 0.$$

Then

(i) *for each $u \in \varphi^0 \cap \overline{B}_\rho$, we have $[0, u] \subseteq \varphi^0$, where*

$$[0, u] = \{tu : t \in [0, 1]\}.$$

(ii) *the set $\varphi^0 \cap \overline{B}_\rho$ is contractible.*

(Here $\overline{B}_\rho$ is the closed ball centered at the origin with radius ρ and φ^0 is the sublevel set of φ at 0.)

Proof. (i) Suppose on the contrary that

$$\varphi(\widehat{t}u) > 0, \quad \text{for some } u \in (\varphi^0 \cap \overline{B}_\rho) \setminus \{0\},\ \widehat{t} \in (0, 1).$$

Define $g(t) = \varphi(tu)$, $t \in [0, 1]$. Then $g(\widehat{t}) > 0$.
If $\varphi(u) < 0$, then $g(1) < 0$.
If $\varphi(u) = 0$, then $g(1) = 0$ and

$$g'(1) = \langle \varphi'(u), u \rangle > 0.$$

Hence, g satisfies the hypotheses of Lemma 3.7, so we may find $\widehat{t}_2 \in (\widehat{t}, 1)$, such that

$$g(\widehat{t}_2) = 0, \qquad g'(\widehat{t}_2) \le 0.$$

But then

$$0 < \langle \varphi'(\widehat{t}_2 u, \widehat{t}_2 u \rangle = \widehat{t}_2 g'(\widehat{t}_2) \le 0,$$

which is a contradiction.
(ii) Define the homotopy $h : [0, 1] \times (\varphi^0 \cap \overline{B}_\rho) \to \varphi^0 \cap \overline{B}_\rho$ by

$$h(t, u) = (1 - t)u.$$

Due to (i), h is well defined whereas it is clearly continuous. Since $h(1, u) = 0$ for all $u \in \varphi^0 \cap \overline{B}_\rho$, we derive that the set $\varphi^0 \cap \overline{B}_\rho$ is contractible in itself. □

Proposition 3.9. *Under hypotheses* H(i), H(iv)*, we have*

$$C_k(\varphi, 0) = 0, \quad \textit{for all } k \ge 0.$$

Proof. From hypothesis H(iv), we can find $c_3, c_4 > 0$ such that

$$F(z, x) \ge c_3 |x|^\tau - c_4 |x|^r \quad \text{for all } z \in \Omega, \text{ all } x \in \mathbb{R}, \tag{3.12}$$

with $p < r < p^*$ (p^* denotes the critical Sobolev exponent).
Claim 1: There exists $\rho \in (0, 1)$ small such that

$$\langle \varphi'(u), u \rangle > 0, \quad \text{for all } u \in \overline{B}_\rho \setminus \{0\} \text{ with } \varphi(u) = 0.$$

To see this, choose $u \in W_0^{1,p}(\Omega) \setminus \{0\}$ such that $\varphi(u) = 0$. Then

$$\begin{aligned}
\langle \varphi'(u), u\rangle &= \|Du\|_p^p - \int_\Omega f(z,u)u dz \\
&= (1 - \frac{\sigma}{p})\|Du\|_p^p + \int_\Omega (\sigma F(z,u) - f(z,u)u)dz \quad (\text{since } \varphi(u) = 0) \\
&= (1 - \frac{\sigma}{p})\|Du\|_p^p + \int_{\{|u|\le\delta_0\}} (\sigma F(z,u) - f(z,u)u)dz \\
&\quad + \int_{\{|u|>\delta_0\}} (\sigma F(z,u) - f(z,u)u)dz.
\end{aligned} \tag{3.13}$$

By hypothesis H(iv), we have

$$\int_{\{|u|\le\delta_0\}} (\sigma F(z,u) - f(z,u)u)dz \ge 0. \tag{3.14}$$

Moreover, hypothesis H(i) implies

$$\int_{\{|u|>\delta_0\}} (\sigma F(z,u) - f(z,u)u)dz \ge -c_5\|u\|_r^r \tag{3.15}$$

for some $c_5 > 0$ and with $p < r < p^*$.

Returning to (3.13) and use (3.14), (3.15) with the embedding $W_0^{1,p}(\Omega) \subseteq L^r(\Omega)$, to obtain

$$\langle \varphi'(u), u\rangle \ge (1 - \frac{\sigma}{p})\|Du\|_p^p - c_6\|Du\|_p^r \quad \text{for some } c_6 > 0.$$

Now Claim 1 follows easily from the last inequality, because of the fact that $\sigma < p < r$.

Taking into account Claim 1 in conjunction with Lemma 3.8(ii) we deduce that

$$\varphi^0 \cap \overline{B}_\rho \quad \text{is contractible.}$$

Claim 2: For each $u \in W_0^{1,p}(\Omega) \setminus \{0\}$, there exists $t^* = t^*(u) \in (0,1)$ small such that

$$\varphi(tu) < 0 \quad \text{for all } t \in (0, t^*).$$

Indeed, for $t > 0$ and $u \in W_0^{1,p}(\Omega)$, we have

$$\begin{aligned}
\varphi(tu) &= \frac{t^p}{p}\|Du\|_p^p - \int_\Omega F(z,tu)dz \\
&\le \frac{t^p}{p}\|Du\|_p^p - c_3 t^\tau \|u\|_\tau^\tau + c_4 t^r \|u\|_r^r \quad (\text{see } (3.12)).
\end{aligned}$$

Then Claim 2 follows from the fact that $\tau < p < r$.

Claim 3: Let $\rho > 0$ be as postulated in Claim 1. Then for each $u \in \overline{B}_\rho$ with $\varphi(u) > 0$, there exists a unique $t(u) \in (0,1)$ such that

$$\varphi(t(u)u) = 0.$$

To prove this, let $u \in \overline{B}_\rho$ be fixed with $\varphi(u) > 0$. Then Claim 2 combined with Bolzano's theorem yield

$$\varphi(t(u)u) = 0, \quad \text{for some } t(u) \in (0,1).$$

We need to show that this $t(u) \in (0,1)$ is unique. We argue by contradiction. So, suppose we can find

$$0 < t_1(u) < t_2(u) < 1 \quad \text{such that } \varphi(t_1(u)u) = \varphi(t_2(u)u) = 0.$$

Then we have $\varphi(tt_2(u)u) \le 0$ for all $t \in [0,1]$ (see Claim 1 and Lemma 3.8(i)). Hence $\frac{t_1(u)}{t_2(u)} \in (0,1)$ is a maximizer of the function $t \to \varphi(tt_2(u)u)$, $t \in [0,1]$. Therefore

$$\frac{d}{dt}\varphi(tt_1(u)u)|_{t=1} = \frac{t_1(u)}{t_2(u)}\frac{d}{dt}\varphi(tt_2(u)u)\big|_{t=\frac{t_1(u)}{t_2(u)}} = 0.$$

But

$$\frac{d}{dt}\varphi(tt_1(u)u)|_{t=1} = \langle \varphi'(t_1(u)u), t_1(u)u\rangle > 0,$$

by Claim 1. Thus, we arrived at a contradiction and the proof of Claim 3 is complete.

Summarizing the above arguments we obtain the following:

- For each $u \in \overline{B}_\rho$ with $\varphi(u) \le 0$, we have that $\varphi \le 0$ on $[0,u]$. Moreover, the set $\varphi^0 \cap \overline{B}_\rho$ is contractible.
- For each $u \in \overline{B}_\rho \setminus \{0\}$ with $\varphi(u) > 0$, there exists a unique $t(u) \in (0,1)$ such that

$$\varphi(t(u)u) = 0, \quad \varphi < 0 \text{ on } (0, t(u)u), \quad \varphi > 0 \text{ on } (t(u)u, u].$$

To proceed, let $q : \overline{B}_\rho \setminus \{0\} \to (0,1]$ be defined by

$$q(u) = \begin{cases} 1 & \text{if } u \in \overline{B}_\rho \setminus \{0\}, \varphi(u) \le 0 \\ t(u) & \text{if } u \in \overline{B}_\rho \setminus \{0\}, \varphi(u) > 0. \end{cases}$$

According to the previous discussion, q is well-defined and the implicit function theorem implies that q is continuous.

Let $Q : \overline{B}_\rho \setminus \{0\} \to (\varphi^0 \cap \overline{B}_\rho) \setminus \{0\}$ be defined by

$$Q(u) = q(u)u.$$

Clearly, Q is continuous and $Q|_{(\varphi^0\cap\overline{B}_\rho)\setminus\{0\}} = \mathrm{id}|_{(\varphi^0\cap\overline{B}_\rho)\setminus\{0\}}$. It follows that $(\varphi^0 \cap \overline{B}_\rho) \setminus \{0\}$ is a retract of $\overline{B}_\rho \setminus \{0\}$. Since $W_0^{1,p}(\Omega)$ is infinite dimensional, the set $\overline{B}_\rho \setminus \{0\}$ is contractible in itself, hence so is the set $(\varphi^0 \cap \overline{B}_\rho) \setminus \{0\}$. Finally, since both $\varphi^0 \cap \overline{B}_\rho$ and $(\varphi^0 \cap \overline{B}_\rho) \setminus \{0\}$ are contractible, we conclude that

$$C_k(\varphi, 0) = H_k(\varphi^0 \cap \overline{B}_\rho, (\varphi^0 \cap \overline{B}_\rho) \setminus \{0\}) = 0 \quad \text{for all } k \ge 0$$

(see Granas -Dugundji [9, p. 389]). □

Now we are ready to state and prove our existence result.

Theorem 3.10. *Under hypotheses* H(i)-(iv), *problem* (1.1) *has at least one nontrivial smooth solution.*

Proof. By Proposition 3.4 we obtain that

$$C_1(\varphi, \infty) \simeq C_1(\psi, \infty) \ne 0$$

(see (3.3)), which implies that

$$C_1(\varphi, u) \ne 0, \quad \text{for some } u \in K_\varphi.$$

Clearly, u is a smooth weak solution to the problem (see remark 3.1). On the other hand, Proposition 3.9 says that $C_1(\varphi, 0) = 0$. Hence, $u \not\equiv 0$. □

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