

OPTIMAL CONTROL APPLIED TO NATIVE-INVASIVE SPECIES COMPETITION VIA A PDE MODEL

WANDI DING, VOLODYMYR HRYNKIV, XIAOYU MU

ABSTRACT. We consider an optimal control problem of a system of parabolic partial differential equations modelling the competition between an invasive and a native species. The motivating example is cottonwood-salt cedar competition, where the effect of disturbance in the system (such as flooding) is taken to be a control variable. Flooding being detrimental at low and high levels, and advantageous at medium levels led us to consider the quadratic growth function of the control. The objective is to maximize the native species and minimize the invasive species while minimizing the cost of implementing the control. An existence result for an optimal control is given. Numerical examples are presented to illustrate the results.

1. INTRODUCTION

The problem of biological invasion as well as the interaction between the invasive and native species has drawn a great deal of attention from both biologists and mathematicians. The invasion of natural ecosystems by exotic species is an important part of global environmental change and poses a major threat to biodiversity [19]. In this context, understanding the underlying dynamics between all elements of the ecosystem plays an important role in eradication and control of the invasive species.

In this paper we consider an optimal control problem for a system of parabolic partial differential equations (PDEs) modelling the competition between an invasive and a native species. As a motivating example, we take cottonwood - salt cedar competition, where the effect of disturbance in the system (such as flooding) is taken to be a control.

Very few mathematical models have been applied to the cottonwood - salt cedar competition. We would like to point out the ordinary differential equations (ODEs) model for the cottonwood - salt cedar competition in [9] and the use of the optimal control approach there to find the optimal flooding pattern. A new control analysis result was developed there for the existence of an optimal control in the case of quadratic functions of the control appearing in both ODEs, under a restriction

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on the coefficients of the quadratics. A more detailed biological background on cottonwood - salt cedar competition and its management can be found in [9].

To understand the impact of habitat heterogeneity and other spatial features on the management of invasive-native species, we consider a model which includes explicit representation of space. The competition model as well as an optimal control problem considered in this paper are extensions of those in [9] in that respect that now they include space variable hence leading to a system of PDEs. PDEs are one of the standard modelling frameworks to account for spatial effects in ecological systems [4]. Optimal control of PDEs involves deriving an optimal spatial and temporal strategy, and the corresponding techniques are just beginning to be applied to natural resource problems with appropriate numerical algorithms for specific scenarios [3, 5, 8, 18, 11]. The theoretical foundation of optimal control of models with underlying dynamics given by PDEs was developed by Lions [17] whereas to derive optimal control solutions for problems in which the state variable dynamics is governed by ODEs one needs to use Pontryagin's Maximum Principle [23]. Even though there is no full generalization of Pontryagin's Maximum Principle to partial differential equations, the control theory and corresponding analysis for many special cases of PDEs have been developed [1, 2, 7, 10, 12, 13, 14, 16].

In this paper we consider a model with diffusive and convective terms since many plants rely on animals/insects/humans and wind as their dispersal agents. A more detailed justification as well as description of models that account for different dispersal mechanisms for plants can be found in [15, 20, 21, 22, 25] and references therein.

The paper is organized as follows: in section 2, we introduce our system of partial differential equations and set up the optimal control problem. In section 3 we prove the existence and uniqueness of solution to the state system. In section 4, we present the existence of an optimal control in the case of quadratic functions of the control appearing in both PDEs. In sections 5 and 6 we establish the adjoint system and characterize the optimal control. Section 7 deals with the uniqueness of the optimal control. Finally, in section 8 we give some numerical examples to illustrate the results.

2. THE MODEL

The state variables $N_1(x, t)$ and $N_2(x, t)$ represent the population densities (expressed as seedlings per unit of area) of a native and an invasive species, respectively. Define $Q = \Omega \times (0, T)$, where the domain $\Omega \in \mathbb{R}^n$ is bounded and smooth and T denotes time. Let the PDE operators $\mathbf{L}_k, k = 1, 2$ be

$$\mathbf{L}_k N_k = \sum_{i,j=1}^n \left(d_{ij}^k(x, t)(N_k)_{x_i} \right)_{x_j} - \sum_{i=1}^n r_i^k(x, t)(N_k)_{x_i}; \quad k = 1, 2. \quad (2.1)$$

The general model for the population dynamics is given by

$$\begin{aligned} (N_1)_t &= \mathbf{L}_1 N_1 + \left(\theta_1(t, u(x, t)) - a_{11} N_1 \right) N_1 - a_{12} N_1 N_2, \\ (N_2)_t &= \mathbf{L}_2 N_2 + \left(\theta_2(t, u(x, t)) - a_{22} N_2 \right) N_2 - a_{21} N_1 N_2, \quad \text{in } Q. \end{aligned} \quad (2.2)$$

The diffusion coefficients are $d_{ij}^k(x, t)$, $i, j = 1, \dots, n$, $k = 1, 2$, the convective coefficients are $r_i^k(x, t)$, $i = 1, \dots, n$, $k = 1, 2$, and the interaction coefficients are $a_{ij} \geq 0$, $i, j = 1, 2$, indicating how species j affects species i .

The quadratic functions

$$\theta_i(t, u(x, t)) = \left(a_i u^2(x, t) + b_i u(x, t) \right) \chi_\Gamma + c_i, \quad i = 1, 2, \quad (2.3)$$

are the intrinsic growth rates, with constants a_i, b_i, c_i , ($i = 1, 2$) and the effect of disturbance in the system (such as flooding) is modelled by a control variable $u(x, t)$. Here χ_Γ is the characteristic function of the set Γ , which is given by

$$\Gamma = \cup_{i=1}^T [\sigma_i, \tau_i],$$

with the intervals $[\sigma_i, \tau_i]$ being in the i -th year. Thus, on the set $(0, T) \setminus \Gamma$, the control does not appear in the state system, which amounts to saying that the control does not apply during those time intervals. With our motivating example in mind, this means that only flooding during the spring thaw when cottonwoods release their seeds is allowed (see [9]). With flooding being detrimental at low and high levels and being advantageous at medium levels, we consider quadratic growth functions of the control.

The control set is

$$U = \{u \in L^\infty(\Omega \times \Gamma) : 0 \leq u(x, t) \leq M\}, \quad (2.4)$$

where the constant $M > 0$.

The growth rate parameters a_i, b_i , and c_i , are chosen to fit a specific scenario. In the cottonwood - salt cedar situation, the parameters are chosen so that with little or no flooding the salt cedar population $N_2(x, t)$ has a higher growth rate than the cottonwood population $N_1(x, t)$ (see [9]).

The initial and boundary conditions are:

$$\begin{aligned} N_1(x, 0) &= N_{10}(x), & N_2(x, 0) &= N_{20}(x), & x &\in \Omega, \\ N_1(x, t) &= N_2(x, t) = 0, & (x, t) &\in \partial\Omega \times (0, T), \end{aligned} \quad (2.5)$$

where the initial populations $N_{10}(x), N_{20}(x)$ are given.

The goal is to maximize

$$J(u) = \int_{\Omega} \left(AN_1(x, T) - BN_2(x, T) \right) dx - \iint_{\Omega \times \Gamma} \left(B_1 u(x, t) + B_2 u^2(x, t) \right) dx dt, \quad (2.6)$$

where the constants $A, B \geq 0$, the cost coefficients are $B_1, B_2 \geq 0$. This means we want to maximize the native population $N_1(x, t)$ and minimize the invasive population $N_2(x, t)$ at the final time T , while minimizing the cost of applying the control. The costs can include the actual financial impact to carry out the management plan but also may consider the negative impact on the environment or economic development.

3. EXISTENCE OF THE STATE SYSTEM

The following assumptions are used this article:

- (A1) Ω is a smooth bounded domain in \mathbb{R}^n ,
- (A2) $N_{10}(x), N_{20}(x) \in L^\infty(\Omega)$ and $N_{10}(x), N_{20}(x) \geq 0$ on Ω ,
- (A3) $r_i^k \in C^1(\bar{Q})$ and $d_{ij}^k \in L^\infty(Q)$, $d_{ij}^k = d_{ji}^k$, for $i, j = 1, \dots, n$; $k = 1, 2$,
- (A4) $\sum_{i,j=1}^n d_{ij}^k(x, t) \xi_i \xi_j \geq \mu |\xi|^2$, for $k = 1, 2$, $\mu > 0$, for all $(x, t) \in Q$, $\xi \in \mathbb{R}^n$.
- (A5) The underlying state space for system (2.2) is $V := L^2(0, T; H_0^1(\Omega))$.

We define the following bilinear form for $v, \psi \in V$ and a.e. $0 \leq t \leq T$,

$$B^k[v, \psi; t] = \int_{\Omega} \sum_{i,j=1}^n d_{ij}^k(x, t) v_{x_i} \psi_{x_j} - \sum_{i=1}^n (r_i^k(x, t) \psi)_{x_i} v \, dx, \quad k = 1, 2. \quad (3.1)$$

Definition 3.1. A pair of functions $u, v \in L^2(0, T; H_0^1(\Omega))$ is a weak solution of system (2.2) and (2.5) provided that:

- (1) $u_t, v_t \in L^2(0, T; H^{-1}(\Omega))$,
- (2)

$$\begin{aligned} \int_0^T \{ \langle (N_1)_t, \phi_1 \rangle + B^1[N_1, \phi_1; t] \} &= \int_Q \left((\theta_1(t, u) - a_{11}N_1)N_1 - a_{12}N_1N_2 \right) \phi_1 \\ \int_0^T \{ \langle (N_2)_t, \phi_2 \rangle + B^2[N_2, \phi_2; t] \} &= \int_Q \left((\theta_2(t, u) - a_{22}N_2)N_2 - a_{21}N_1N_2 \right) \phi_2 \end{aligned} \quad (3.2)$$

for all $\phi_1, \phi_2 \in V$, and

- (3) $N_1(x, 0) = N_{10}(x)$, $N_2(x, 0) = N_{20}(x)$.

Here $\langle \cdot, \cdot \rangle$ denotes the duality between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$. Also, note that since $N_1, N_2 \in C([0, T]; L^2(\Omega))$ it follows that condition (3) makes sense [6].

Theorem 3.2. *Given $u \in U$, there exists a unique $(N_1, N_2) \in (V \times V) \cap L^\infty(Q)$ which is the solution of the system (2.2) and (2.5).*

Proof. To obtain the existence of the solution to the system (2.2)-(2.5), we construct two sequences by means of iteration. To obtain L^∞ bounds, we need supersolutions and subsolutions for the N_1, N_2 iterates. Let \bar{N}_1, \bar{N}_2 be the solutions of the problem

$$\begin{aligned} (\bar{N}_1)_t &= \mathbf{L}_1(\bar{N}_1) + \theta_1(t, u(x, t))\bar{N}_1, \quad \text{in } Q, \\ (\bar{N}_2)_t &= \mathbf{L}_2(\bar{N}_2) + \theta_2(t, u(x, t))\bar{N}_2, \\ \bar{N}_1(x, 0) &= N_{10}(x), \quad \bar{N}_2(x, 0) = N_{20}(x), \quad x \in \Omega, \\ \bar{N}_1(x, t) &= \bar{N}_2(x, t) = 0, \quad (x, t) \in \partial\Omega \times (0, T). \end{aligned} \quad (3.3)$$

By standard maximum principle arguments [6], \bar{N}_1, \bar{N}_2 are bounded in $L^\infty(Q)$. Define $N_1^0 = \bar{N}_1$, and N_2^0 to be the solution of

$$(N_2^0)_t = \mathbf{L}_2(N_2^0) + (\theta_2(t, u(x, t)) - a_{22}\bar{N}_2)N_2^0 - a_{21}N_1^0N_2^0 \quad \text{in } Q, \quad (3.4)$$

with $N_2^0(x, 0) = N_{20}(x)$, $x \in \Omega$, and $N_2^0(x, t) = 0$, $(x, t) \in \partial\Omega \times (0, T)$. By the weak maximum principle $0 \leq N_2^0(x, t) \leq \bar{N}_2(x, t)$ on Ω . For $k = 1, 2, 3, \dots$, we define N_1^k, N_2^k to be the solutions of the following problems, respectively:

$$\begin{aligned} (N_1^k)_t - \mathbf{L}_1N_1^k + RN_1^k &= (\theta_1(t, u) - a_{11}N_1^{k-1})N_1^{k-1} - a_{12}N_1^{k-1}N_2^{k-1} + RN_1^{k-1}, \\ (N_2^k)_t - \mathbf{L}_2N_2^k + RN_2^k &= (\theta_2(t, u) - a_{22}N_2^{k-1})N_2^{k-1} - a_{21}N_1^{k-1}N_2^{k-1} + RN_2^{k-1}, \end{aligned} \quad (3.5)$$

with $N_1^k(x, 0) = N_{10}(x)$, $N_2^k(x, 0) = N_{20}(x)$, for $x \in \Omega$, and $N_1^k(x, t) = N_2^k(x, t) = 0$, on $\partial\Omega \times (0, T)$, and the constant R is chosen to satisfy $R \geq 2a_{11}\bar{N}_1 + a_{12}\bar{N}_2 - \theta_1$ and $R \geq 2a_{22}\bar{N}_2 + a_{21}\bar{N}_1 - \theta_2$. Let $k = 1$ in equation (3.5), subtracting equation (3.5) from (3.3), we have

$$(N_1^0 - N_1^1)_t - \mathbf{L}_1(N_1^0 - N_1^1) + R(N_1^0 - N_1^1) = a_{11}N_1^0N_1^0 + a_{12}N_1^0N_2^0 \geq 0.$$

By comparison results, we have $N_1^0 \geq N_1^1$. Similarly, let $k = 1$ in (3.5), subtracting equation (3.4) and (3.5), we obtain

$$(N_2^0 - N_2^1)_t - \mathbf{L}_2(N_2^0 - N_2^1) + R(N_2^0 - N_2^1) = -a_{22}(\bar{N}_2 - N_2^0)N_2^0 \leq 0.$$

Since $\bar{N}_2 \geq N_2^0$, then $N_2^0 \leq N_2^1$. Let $k = 1$ and $k = 2$ in (3.5). Then we have

$$\begin{aligned} & (N_1^1 - N_1^2)_t - \mathbf{L}_1(N_1^1 - N_1^2) + R(N_1^1 - N_1^2) \\ &= \theta_1(t, u(x, t))(N_1^0 - N_1^1) - a_{11}(N_1^0 N_1^0 - N_1^1 N_1^1) - a_{12}(N_1^0 N_2^0 - N_1^1 N_2^1) \\ &\quad + R(N_1^0 - N_1^1) \\ &\geq \theta_1(t, u(x, t))(N_1^0 - N_1^1) - a_{11}(N_1^0 - N_1^1)(N_1^0 + N_1^1) - a_{12}N_2^1(N_1^0 - N_1^1) \\ &\quad + R(N_1^0 - N_1^1) \\ &\geq (N_1^0 - N_1^1) \left(\theta_1(t, u(x, t)) - a_{11}(N_1^0 + N_1^1) - a_{12}N_2^1 + R \right) \geq 0, \end{aligned}$$

where we used $R > 2a_{11}\bar{N}_1 + a_{12}\bar{N}_2 - \theta_1$, $N_1^1 \leq N_1^0$ and $N_2^1 \geq N_2^0$. By comparison results, $N_1^1 \geq N_1^2$. Similarly, we obtain $N_2^1 \leq N_2^2$.

By an induction argument coupled with comparison results, we can prove that there exist $N_1^k, N_2^k, \tilde{N}_1, \tilde{N}_2$, such that the following monotone pointwise convergence hold $N_1^k \searrow \tilde{N}_1, N_2^k \nearrow \tilde{N}_2$ in Q , and $0 \leq N_1^k \leq \tilde{N}_1, 0 \leq N_2^k \leq \tilde{N}_2$ for $k = 1, 2, \dots$

From the boundedness of N_1^k and N_2^k in V and (3.5) (see similar arguments in [13]), we obtain that $\|(N_1)_t^k\|, \|(N_2)_t^k\|$ are bounded in $L^2(0, T; H^{-1}(\Omega))$. Hence, using weak compactness, we have

$$(N_1^k)_t \rightharpoonup (\tilde{N}_1)_t, \quad (N_2^k)_t \rightharpoonup (\tilde{N}_2)_t.$$

Since $L^2(0, T; H_0^1(\Omega)) \subset\subset L^2(Q)$ [26], we have $N_1^k \rightarrow \tilde{N}_1, N_2^k \rightarrow \tilde{N}_2$ strongly in $L^2(Q)$.

Now we outline the proof of uniqueness. Let $(\tilde{N}_1, \tilde{N}_2)$ be another solution to the state system (2.2) with the initial and boundary conditions $\tilde{N}_1(x, 0) = N_{10}(x), \tilde{N}_2(x, 0) = N_{20}(x), x \in \Omega, \tilde{N}_1(x, t) = \tilde{N}_2(x, t) = 0, (x, t) \in \partial\Omega \times (0, T)$.

Let $\tilde{N}_1 = e^{\lambda t} \tilde{w}_1, \tilde{N}_2 = e^{\lambda t} \tilde{w}_2, N_1 = e^{\lambda t} w_1$, and $N_2 = e^{\lambda t} w_2$, where $\lambda > 0$ is to be chosen later. After making this change of variables in the corresponding equations, we obtain

$$\begin{aligned} & \int_0^T \langle (w_1)_t, \phi_1 \rangle dt + \int_Q \lambda w_1 \phi_1 dx dt \\ &+ \int_Q \sum_{i,j=1}^n d_{ij}^1(x, t) (w_1)_{x_i} (\phi_1)_{x_j} - \int_Q \sum_{i=1}^n (r_i^1(x, t) \phi_1)_{x_i} w_1 \\ &= \int_Q \left((\theta_1(t, u) - e^{\lambda t} a_{11} w_1) w_1 - e^{\lambda t} a_{12} w_1 w_2 \right) \phi_1, \end{aligned}$$

where $\phi_1, \phi_2 \in V$. Similar equations can be written for w_2, \tilde{w}_1 , and \tilde{w}_2 . Define $Q_1 := \Omega \times (0, T_1)$ for $0 < T_1 \leq T$. Subtracting the equation for \tilde{w}_i from the equation for w_i , using $w_i - \tilde{w}_i$ as test functions, using uniform ellipticity, regularity of the coefficients, and Cauchy inequality, after standard manipulations we will obtain

$$(\lambda - c_1 e^{\lambda T_1} - c_2) \int_{Q_1} \{ (w_1 - \tilde{w}_1)^2 + (w_2 - \tilde{w}_2)^2 \} dx dt \leq 0,$$

where c_i depends only on the coefficients and L^∞ bounds of w_i, \tilde{w}_i . Choosing λ large enough and T_1 small, we can obtain the uniqueness on $\Omega \times (0, T_1)$. By stacking time intervals, we can obtain the uniqueness on $\Omega \times (0, T)$. \square

4. EXISTENCE OF AN OPTIMAL CONTROL

To prove the existence of an optimal control we borrow the technique used in [9] and adopt it for our case. Notice here we also include the space variable so we have different function spaces to consider. First, we rewrite the state system (2.2) as follows

$$(N_1)_t = \theta_1(t, u(x, t))N_1 + g_1, \quad (N_2)_t = \theta_2(t, u(x, t))N_2 + g_2, \quad (4.1)$$

where

$$g_1 = \mathbf{L}_1 N_1 - (a_{11} N_1^2 + a_{12} N_1 N_2), \quad g_2 = \mathbf{L}_2 N_2 - (a_{22} N_2^2 + a_{21} N_1 N_2).$$

Note that (4.1) is understood in the weak sense (3.2).

Denote

$$N = \begin{bmatrix} N_1(x, t) \\ N_2(x, t) \end{bmatrix}, \quad f(t, N, u) = \begin{bmatrix} \theta_1(t, u(x, t))N_1 \\ \theta_2(t, u(x, t))N_2 \end{bmatrix}, \quad g(t, N) = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}.$$

Observe that for given $t \in [0, T]$, we have $f : [0, T] \times (H_0^1(\Omega))^2 \times U \rightarrow (H_0^1(\Omega))^2$, [16, (P3) on p.103].

Next we will rewrite the objective functional (2.6). Let $f^0 : [0, T] \times U \rightarrow \mathbb{R}$ be defined as follows $f^0(t, u) := \int_{\Omega} (B_1 u + B_2 u^2) \chi_{\Gamma} dx$. Also, define

$$g^0(N(T)) := \int_{\Omega} [AN_1(x, T) - BN_2(x, T)] dx.$$

Therefore, we obtain $J(u) = g^0(N(T)) - \int_0^T f^0(t, u) dt$. We rewrite the state system as follows:

$$\frac{\partial N}{\partial t} = f(t, N, u) + g(t, N), \quad (4.2)$$

which is to be understood in the weak sense (3.2).

Theorem 4.1. *Suppose there exists $\kappa \geq 0$ such that*

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \kappa \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \quad B_1 \kappa - B_2 \leq 0. \quad (4.3)$$

Then there exists an optimal control u in U with the corresponding states N_1, N_2 that maximizes the objective functional (2.6).

Proof. For given $(t, N) \in [0, T] \times (H_0^1(\Omega))^2$, define the set

$$\mathcal{E}(t, N) := \left\{ (z^0, z) \in \mathbb{R} \times (H_0^1(\Omega))^2 : z^0(t) \geq f^0(t, u), \right. \\ \left. z(t) = f(t, N, u) \text{ for some } 0 \leq u(x, t) \leq M \right\}.$$

We break the proof into two steps: Step 1. If, for a.e. $t \in [0, T]$ and each $N \in (H_0^1(\Omega))^2$, the set \mathcal{E} is convex and closed, then there exists an optimal control. Step 2. The set \mathcal{E} is convex and closed.

Proof of Step 1. Let $\{u^n\}$ be a maximizing sequence and N_1^n, N_2^n be the states corresponding to u^n . From the *a priori* estimates, we have

$$N_1^n \rightarrow \hat{N}_1 \quad \text{and} \quad N_2^n \rightarrow \hat{N}_2 \quad \text{in } L^2(Q). \quad (4.4)$$

Then, it follows that for a.e. $t \in [0, T]$

$$(z_n^0(t), z_n(t)) \equiv \left(f^0(t, N^n(t), u^n(t)), f(t, N^n(t), u^n(t)) \right) \in \mathcal{E}(t, N^n(t)).$$

Since $N^n(x, t)$ is bounded in $(L^2(0, T; H_0^1(\Omega)))^2$, it follows that $\{z_n(\cdot)\}$ converges weakly to some \hat{z} in $(L^2(0, T; H_0^1(\Omega)))^2$. Then by Mazur's theorem we can find a convex combination $\alpha_{ij} \geq 0$, and $\sum_{i \geq j} \alpha_{ij} = 1$, such that $\psi_j(\cdot) \equiv \sum_{i \geq j} \alpha_{ij} z_i(\cdot) \rightarrow \hat{z}$ in $L^2(0, T; H_0^1(\Omega))$. This implies that $\psi_j(t) \rightarrow \hat{z}(t)$ in $H_0^1(\Omega)$ a.e. $t \in [0, T]$. If we set $\psi_j^0(\cdot) \equiv \sum_{i \geq j} \alpha_{ij} z_i^0(\cdot)$, then we will get $\hat{z}^0(t) \equiv \liminf_{j \rightarrow \infty} \psi_j^0(t) \geq 0$ a.e. $t \in [0, T]$ [16, (P3), p.103], $f^0(t, u) \geq 0$. Then, since the set $\mathcal{E}(t, N)$ is closed and convex (i.e., has Cesari property), it will follow that $(\hat{z}^0(t), \hat{z}(t)) \in \mathcal{E}(t, N)$ a.e. $t \in [0, T]$. After passing to the limit in the state system, we obtain

$$\begin{aligned} \frac{\partial \hat{N}}{\partial t} &= \hat{z}(t) + g(t, \hat{N}), \\ \sup_{u \in U} J(u(\cdot)) &= \lim_{n \rightarrow \infty} J(u^n) \leq g^0(\hat{N}(T)) - \int_0^T \hat{z}^0(t) dt. \end{aligned}$$

Since $(\hat{z}^0(t), \hat{z}(t)) \in \mathcal{E}(t, \hat{N})$ a.e. $t \in [0, T]$, it follows that there exists $\hat{u}(\cdot)$ such that $\hat{z}^0(t) \geq f^0(t, \hat{u}(t))$ and $\hat{z}(t) = f(t, \hat{N}(t), \hat{u}(t))$ a.e. $t \in [0, T]$. This implies that $\hat{N}(\cdot)$ is the state that corresponds to the control $\hat{u}(\cdot)$, and

$$\begin{aligned} \sup_{u \in U} J(u(\cdot)) &\leq g^0(\hat{N}(T)) - \int_0^T \hat{z}^0(t) dt \\ &\leq g^0(\hat{N}(T)) - \int_0^T f^0(t, \hat{u}(t)) dt = J(\hat{u}(\cdot)). \end{aligned}$$

Hence, $\hat{u}(\cdot)$ is an optimal control. This completes the proof of Step 1.

Proof of Step 2. Since the map $u \mapsto (f^0(t, u), f(t, N, u))$ is continuous, it follows that the set $\mathcal{E}(t, N)$ is closed. To obtain the convexity, observe that condition (4.3) gives

$$f(t, N, u) = (\kappa + u)\chi_\Gamma(t) \begin{bmatrix} b_1 N_1 \\ b_2 N_2 \end{bmatrix} + \begin{bmatrix} c_1 N_1 \\ c_2 N_2 \end{bmatrix}. \tag{4.5}$$

When $t \in [0, T] \setminus \Gamma$, then the set $\mathcal{E}(t, N) = [0, \infty) \times \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$ is trivially convex.

Therefore, we consider $t \in \Gamma$. Denote $\varphi(u) := \kappa u^2 + u$. As $\kappa \geq 0$, it follows that $\varphi'(u) = 2\kappa + 1 \geq 0$, for $u \in [0, 1]$. This implies that $\varphi(\cdot)$ is convex, monotone increasing, and

$$\varphi([0, 1]) = [\varphi(0), \varphi(1)] = [0, \kappa + 1], \tag{4.6}$$

which is convex. Hence, $f(t, N, [0, 1])$ is convex. For arbitrary $(z^0, z), (\zeta^0, \zeta) \in \mathcal{E}(t, N)$ and $\lambda \in (0, 1)$, we have some $u, v \in [0, 1]$ such that

$$z = f(t, N, u), \quad z^0 \geq f^0(t, u), \quad \zeta = f(t, N, v), \quad \zeta^0 \geq f^0(t, v).$$

The convexity of $\varphi(\cdot)$ together with (4.6) imply that there exists $w \in [0, 1]$ such that

$$\varphi(\lambda u + (1 - \lambda)v) \leq \lambda \varphi(u) + (1 - \lambda)\varphi(v) = \varphi(w) \tag{4.7}$$

which leads to

$$\begin{aligned} \lambda z + (1 - \lambda)\zeta &\equiv \lambda f(t, N, u) + (1 - \lambda)f(t, N, v) \\ &= [\lambda\varphi(u) + (1 - \lambda)\varphi(v)] \begin{bmatrix} b_1 N_1 \\ b_2 N_2 \end{bmatrix} \\ &= \varphi(w) \begin{bmatrix} b_1 N_1 \\ b_2 N_2 \end{bmatrix} = f(t, N, w). \end{aligned} \quad (4.8)$$

By the monotonicity of $u \mapsto \varphi(u)$ and the first inequality in (4.7), we obtain

$$\lambda u + (1 - \lambda)v \leq w. \quad (4.9)$$

Moreover, the equality in (4.7) implies that

$$\kappa[\lambda u^2 + (1 - \lambda)v^2 - w^2] + \lambda u + (1 - \lambda)v - w = 0. \quad (4.10)$$

Hence, when $\kappa > 0$, taking into account (4.3), (4.9), and (4.10), we get

$$\begin{aligned} &\lambda z^0 + (1 - \lambda)\zeta^0 - f^0(t, w) \\ &\geq \lambda f^0(t, u) + (1 - \lambda)f^0(t, v) - f^0(t, w) \\ &= B_2[\lambda u^2 + (1 - \lambda)v^2 - w^2] + B_1[\lambda u + (1 - \lambda)v - w] \\ &= \left(B_1 - \frac{B_2}{\kappa}\right)[\lambda u + (1 - \lambda)v - w] \geq 0. \end{aligned}$$

In the case when $\kappa = 0$, (4.10) becomes $w = \lambda u + (1 - \lambda)v$, which leads to (due to convexity of the map $u \mapsto u^2$)

$$\begin{aligned} \lambda z^0 + (1 - \lambda)\zeta^0 - f^0(t, w) &\geq \lambda f^0(t, u) + (1 - \lambda)f^0(t, v) - f^0(t, w) \\ &= B_2\{\lambda u^2 + (1 - \lambda)v^2 - [\lambda u + (1 - \lambda)v]^2\} \geq 0. \end{aligned}$$

Hence, it follows that

$$\lambda(z^0, z) + (1 - \lambda)(\zeta^0, \zeta) \in \mathcal{E}(t, N),$$

which proves the convexity of the set $\mathcal{E}(t, N)$. \square

We remark that the constraints on the parameters obtained in (4.3) are identical to those in [9].

5. SENSITIVITY

To characterize the optimal control, we need to differentiate the objective functional with respect to the control u . Since $N_i = N_i(u)$, $i = 1, 2$ is involved in $J(u)$, we first must prove appropriate differentiability of the mapping $u \rightarrow N_i(u)$, $i = 1, 2$ whose derivative is called the *sensitivity*.

Theorem 5.1. *The mapping $u \mapsto N(u) = (N_1(u), N_2(u))$ is differentiable in the following sense:*

$$\begin{aligned} \frac{N_1(u + \varepsilon h) - N_1(u)}{\varepsilon} &\xrightarrow{w} \Psi_1 \text{ in } L^2(0, T; H_0^1(\Omega)), \\ \frac{N_2(u + \varepsilon h) - N_2(u)}{\varepsilon} &\xrightarrow{w} \Psi_2 \text{ in } L^2(0, T; H_0^1(\Omega)), \end{aligned} \quad (5.1)$$

as $\varepsilon \rightarrow 0$ for any $u \in U$ and $h \in L^\infty(Q)$ such that $u + \varepsilon h \in U$ for small ε . Moreover, the sensitivities, $\Psi_1 \in L^2(0, T; H_0^1(\Omega))$ and $\Psi_2 \in L^2(0, T; H_0^1(\Omega))$ satisfy

$$\begin{aligned} (\Psi_1)_t &= \mathbf{L}_1 \Psi_1 + ((a_1 u^2 + b_1 u)\chi_\Gamma + c_1)\Psi_1 \\ &\quad + (2a_1 u + b_1)hN_1 - 2a_{11}N_1\Psi_1 - a_{12}(N_1\Psi_2 + N_2\Psi_1), \\ (\Psi_2)_t &= \mathbf{L}_2 \Psi_2 + ((a_2 u^2 + b_2 u)\chi_\Gamma + c_2)\Psi_2 \\ &\quad + (2a_2 u + b_2)hN_2 - 2a_{22}N_2\Psi_2 - a_{21}(N_1\Psi_2 + N_2\Psi_1). \end{aligned} \quad (5.2)$$

$$\Psi_1(x, 0) = \Psi_2(x, 0) = 0, \quad x \in \Omega,$$

$$\Psi_1(x, t) = \Psi_2(x, t) = 0, \quad (x, t) \in \partial\Omega \times (0, T).$$

Proof. Define $N_1^\varepsilon = N_1(u + \varepsilon h)$, $N_2^\varepsilon = N_2(u + \varepsilon h)$. Let $N_1^\varepsilon = e^{\lambda t}W_1^\varepsilon$, $N_2^\varepsilon = e^{\lambda t}W_2^\varepsilon$, $N_1 = e^{\lambda t}W_1$, and $N_2 = e^{\lambda t}W_2$, where $\lambda > 0$ is to be chosen later. After making this change of variables in (2.2), we obtain

$$\begin{aligned} &\int_0^T \langle (W_1)_t, \phi_1 \rangle dt + \int_Q \lambda W_1 \phi_1 \\ &+ \int_Q \sum_{i,j=1}^n d_{ij}^1(x, t)(W_1)_{x_i}(\phi_1)_{x_j} - \int_Q \sum_{i=1}^n (r_i^1(x, t)\phi_1)_{x_i} W_1 \\ &= \int_Q \left((\theta_1(t, u) - e^{\lambda t}a_{11}W_1)W_1 - e^{\lambda t}a_{12}W_1W_2 \right) \phi_1, \end{aligned}$$

where $\phi_1 \in V$.

Similar equations can be written for W_2 , W_1^ε and W_2^ε . We derive our estimate on $Q_1 := \Omega \times (0, T_1)$. Then subtracting the equation for W_1 from the equation for W_1^ε and dividing by ε , and taking $\phi_1 = \frac{W_1^\varepsilon - W_1}{\varepsilon}$, using uniform ellipticity, regularity of the coefficients, and Cauchy inequality where necessary, we obtain the estimate

$$\begin{aligned} &\frac{1}{2} \int_{\Omega \times \{t=T_1\}} \left| \frac{W_1^\varepsilon - W_1}{\varepsilon} \right|^2 dx + \lambda \int_{Q_1} \left| \frac{W_1^\varepsilon - W_1}{\varepsilon} \right|^2 + \frac{\mu}{2} \int_{Q_1} \left| \nabla \left(\frac{W_1^\varepsilon - W_1}{\varepsilon} \right) \right|^2 \\ &\leq C_1 \int_{Q_1} \left| \frac{W_1^\varepsilon - W_1}{\varepsilon} \right|^2 dx dt + C_2 e^{\lambda T_1} \int_{Q_1} \left| \frac{W_1^\varepsilon - W_1}{\varepsilon} \right|^2 |W_1 + W_1^\varepsilon| \\ &\quad + C_3 \int_{Q_1} |W_1^\varepsilon| \left| \frac{W_1^\varepsilon - W_1}{\varepsilon} \right| dx dt + C_4 e^{\lambda T_1} \int_{Q_1} |W_2| \left| \frac{W_1^\varepsilon - W_1}{\varepsilon} \right|^2 dx dt \\ &\quad + C_5 e^{\lambda T_1} \int_{Q_1} |W_1^\varepsilon| \left| \frac{W_2^\varepsilon - W_2}{\varepsilon} \right| \left| \frac{W_1^\varepsilon - W_1}{\varepsilon} \right| dx dt \\ &\leq C_6 e^{\lambda T_1} \int_{Q_1} \left| \frac{W_1^\varepsilon - W_1}{\varepsilon} \right|^2 dx dt + C_7 \int_{Q_1} |h|^2 dx dt. \end{aligned}$$

A similar estimate can be derived for the quotient $\frac{W_2^\varepsilon - W_2}{\varepsilon}$. Combining the estimates for both quotients gives

$$\begin{aligned} &\frac{1}{2} \int_{\Omega \times \{t=T_1\}} \left| \frac{W_1^\varepsilon - W_1}{\varepsilon} \right|^2 + \left| \frac{W_2^\varepsilon - W_2}{\varepsilon} \right|^2 dx \\ &+ (\lambda - C_9 e^{\lambda T_1} - C_8) \int_{Q_1} \left| \frac{W_1^\varepsilon - W_1}{\varepsilon} \right|^2 + \left| \frac{W_2^\varepsilon - W_2}{\varepsilon} \right|^2 dx dt \\ &+ \frac{\mu}{2} \int_{Q_1} \left| \nabla \left(\frac{W_1^\varepsilon - W_1}{\varepsilon} \right) \right|^2 + \frac{\mu}{2} \int_{Q_1} \left| \nabla \left(\frac{W_2^\varepsilon - W_2}{\varepsilon} \right) \right|^2 \end{aligned}$$

$$\leq C_{10} \int_{Q_1} |h|^2 dx dt,$$

where C_i only depends on the coefficients and L^∞ bounds of W_i, W_i^ε . Choosing λ large enough and T_1 small, we get the estimate

$$\left\| \frac{W_i^\varepsilon - W_i}{\varepsilon} \right\|_{L^2(0, T_1; H_0^1(\Omega))} \leq C, \quad i = 1, 2,$$

where $C > 0$ denotes a generic constant. By stacking time intervals, we can obtain the estimate on $\Omega \times (0, T)$. This will give us the desired estimate for $\frac{N_i^\varepsilon - N_i}{\varepsilon}$. Hence, we conclude that there exist $\Psi_i \in V, i = 1, 2$, such that on a subsequence $\frac{N_i^\varepsilon - N_i}{\varepsilon} \rightharpoonup \Psi_i$ in $L^2(0, T; H_0^1(\Omega))$. Using the state system (2.2) and the above estimate it can be shown that

$$\left\| \left(\frac{N_i^\varepsilon - N_i}{\varepsilon} \right)_t \right\|_{L^2(0, T; H^{-1}(\Omega))} \leq C,$$

which implies that $\left(\frac{N_i^\varepsilon - N_i}{\varepsilon} \right)_t \xrightarrow{w} (\Psi_i)_t$ in $L^2(0, T; H^{-1}(\Omega))$. Using the compactness result by Simon [26], we have $\frac{N_i^\varepsilon - N_i}{\varepsilon} \rightarrow \Psi_i$ strongly in $L^2(Q)$. Now, passing to the limit in the weak form of the PDEs satisfied by the quotients $\frac{N_i^\varepsilon - N_i}{\varepsilon}, i = 1, 2$, we obtain (5.2). \square

To rewrite the sensitivity system in matrix form, let

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}, \quad (5.3)$$

where $M_{11} = -((a_1 u^2 + b_1 u)\chi_\Gamma + c_1) + 2a_{11}N_1 + a_{12}N_2$, $M_{12} = a_{12}N_1$, $M_{21} = a_{21}N_2$, and $M_{22} = -((a_2 u^2 + b_2 u)\chi_\Gamma + c_2) + 2a_{22}N_2 + a_{21}N_1$. Then

$$\mathbf{L} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \begin{pmatrix} (\Psi_1)_t - \mathbf{L}_1 \Psi_1 \\ (\Psi_2)_t - \mathbf{L}_2 \Psi_2 \end{pmatrix} + M \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}. \quad (5.4)$$

6. THE ADJOINT SYSTEM

Now we are ready to characterize the optimal control, by deriving the optimality system through differentiating $J(u)$ with respect to u at an optimal control. Define the adjoint system as

$$\mathbf{L}^* \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} A \\ -B \end{pmatrix}, \quad (6.1)$$

where

$$\mathbf{L}^* \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} -p_t - \sum_{i,j=1}^n \left(d_{ij}^1(x, t) p_{x_i} \right)_{x_j} - \sum_{i=1}^n r_i^1(x, t) p_{x_i} \\ -q_t - \sum_{i,j=1}^n \left(d_{ij}^2(x, t) q_{x_i} \right)_{x_j} - \sum_{i=1}^n r_i^2(x, t) q_{x_i} \end{pmatrix} + M^T \begin{pmatrix} p \\ q \end{pmatrix}, \quad (6.2)$$

where M^T is the transpose of the matrix M . The terminal conditions are

$$p(x, T) = A, \quad q(x, T) = -B, \quad x \in \Omega. \quad (6.3)$$

So given an optimal control u^* and the corresponding states N_1^* and N_2^* , the adjoint variables p and q satisfy

$$\begin{aligned} \mathbf{L}_1^* p &= \left((a_1 (u^*)^2 + b_1 u^*) \chi_\Gamma + c_1 - 2a_{11}N_1^* - a_{12}N_2^* \right) p - a_{21}N_2^* q, \quad \text{in } \Omega, \\ p &= 0, \quad \text{on } \partial\Omega \times (0, T), \quad p(x, T) = A, \quad \text{for } x \in \Omega, \end{aligned}$$

$$\begin{aligned} \mathbf{L}_2^* q &= \left((a_2(u^*)^2 + b_2 u^*) \chi_\Gamma + c_2 - 2a_{22} N_2^* - a_{21} N_1^* \right) q - a_{12} N_1^* p, \quad \text{in } \Omega, \\ q &= 0, \quad \text{on } \partial\Omega \times (0, T), \quad q(x, T) = -B, \quad \text{for } x \in \Omega. \end{aligned}$$

Theorem 6.1. *If there exists $\kappa \geq 0$, such that $B_1 \kappa - B_2 < 0$, $a_i = \kappa b_i$, $i = 1, 2$, then given an optimal control u^* and the corresponding states N_1^* and N_2^* , there exist solutions p and q to the adjoint system. Moreover, let $S = \{(x, t) : B_2 - a_1 N_1^* p - a_2 N_2^* q = 0\}$ and $m(S)$ be a Lebesgue measure of S , then this optimal control u^* is characterized by the following:*

- (1) if $m(S) > 0$, then $u^* = M$ on S ;
- (2) if $m(S) = 0$, then for $(x, t) \notin S$,

$$u^* = \min \left\{ M, \max \left\{ 0, \frac{b_1 N_1^* p + b_2 N_2^* q - B_1}{2B_2 - 2a_1 N_1^* p - 2a_2 N_2^* q} \right\} \right\}, \quad (6.4)$$

and it holds on $\Omega \times \Gamma$ a.e.

Proof. Since the adjoint system is linear, by the standard theory of parabolic equations there exists a weak solution (p, q) satisfying the adjoint system. Let $N_i^\varepsilon = N_i(u^* + \varepsilon h)$, $i = 1, 2$, with $u^* + \varepsilon h \in U$ and $h \in L^\infty(Q)$. Then

$$\begin{aligned} 0 &\geq \lim_{\varepsilon \rightarrow 0^+} \frac{J(u^* + \varepsilon h) - J(u^*)}{\varepsilon} \\ &= \lim_{\varepsilon \rightarrow 0^+} \int_\Omega \left(A \frac{N_1^\varepsilon - N_1}{\varepsilon}(x, T) - B \frac{N_2^\varepsilon - N_2}{\varepsilon}(x, T) \right) - \iint_{\Omega \times \Gamma} (B_1 h + 2B_2 u^* h) \\ &= \int_\Omega \left(A \Psi_1(x, T) - B \Psi_2(x, T) \right) dx - \iint_{\Omega \times \Gamma} (B_1 + 2B_2 u^*) h dx dt \\ &= \int_{\Omega \times (0, T)} (p, q) \mathbf{L} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} dx - \iint_{\Omega \times \Gamma} (B_1 + 2B_2 u^*) h dx dt \\ &= \iint_{\Omega \times \Gamma} (p, q) \begin{pmatrix} (2a_1 u^* + b_1) h N_1^* \\ (2a_2 u^* + b_2) h N_2^* \end{pmatrix} dx dt - \iint_{\Omega \times \Gamma} (B_1 + 2B_2 u^*) h dx dt \\ &= \iint_{\Omega \times \Gamma} \left((2a_1 u^* + b_1) N_1^* p + (2a_2 u^* + b_2) N_2^* q - B_1 - 2B_2 u^* \right) h dx dt \\ &= \iint_{\Omega \times \Gamma} \left(u^* (2a_1 N_1^* p + 2a_2 N_2^* q - 2B_2) + b_1 N_1^* p + b_2 N_2^* q - B_1 \right) h dx dt. \end{aligned} \quad (6.5)$$

Observe that we used (6.1) so that

$$\iint_{\Omega \times (0, T)} (\Psi_1, \Psi_2) \mathbf{L}^* \begin{pmatrix} p \\ q \end{pmatrix} = \iint_{\Omega \times (0, T)} (p, q) \mathbf{L} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} \quad (6.6)$$

in weak sense. Also we used the sensitivity system (5.4).

Define $S = \{(x, t) : B_2 - a_1 N_1^* p - a_2 N_2^* q = 0\}$. Let $m(S) > 0$ and h have support on S . Recalling that $a_i = \kappa b_i$, $i = 1, 2$, it follows from (6.5) that

$$\begin{aligned} 0 &\geq \iint_S \kappa (b_1 N_1^* p + b_2 N_2^* q - B_1) h dx dt \\ &= \iint_S (\kappa b_1 N_1^* p + \kappa b_2 N_2^* q - \kappa B_1) h dx dt \\ &= \iint_S (a_1 N_1^* p + a_2 N_2^* q - \kappa B_1) h dx dt = \iint_S (B_2 - \kappa B_1) h dx dt, \end{aligned}$$

if $\kappa B_1 < B_2$, then $h \leq 0$, which implies $u^* = M$ on S .

Otherwise, if $m(S) = 0$, then for $(x, t) \notin S$, on the set $0 < u < M$, we choose variation h with support on this set and h to be of any sign, which gives $(2a_1u^* + b_1)N_1^*p + (2a_2u^* + b_2)N_2^*q - B_1 - 2B_2u^* = 0$. On the set where $u = 0$, we choose $h \geq 0$, which implies $(2a_1u^* + b_1)N_1^*p + (2a_2u^* + b_2)N_2^*q - B_1 - 2B_2u^* \leq 0$. Similarly when $u = M$, we choose $h \leq 0$ which implies $(2a_1u^* + b_1)N_1^*p + (2a_2u^* + b_2)N_2^*q - B_1 - 2B_2u^* \geq 0$. This can be written in the compact form as

$$u^* = \min \left\{ M, \max \left\{ 0, \frac{b_1 N_1^* p + b_2 N_2^* q - B_1}{2B_2 - 2a_1 N_1^* p - 2a_2 N_2^* q} \right\} \right\}. \quad (6.7)$$

□

We remark that characterization of the optimal control (6.4) is identical to the one in [9] except that now it includes space.

7. UNIQUENESS OF THE OPTIMAL CONTROL

The state equations (2.2)-(2.5), the adjoint equations (6.1)-(6.3), and the optimal control characterization (6.4) together are called the optimality system (OS). The weak solutions of the optimality system exist by Theorems 3.2 and 5.1, at the same time the existence of an optimal control u and corresponding states N_1, N_2 implies the existence of p and q .

Now we will prove the uniqueness of the optimal control.

Theorem 7.1. *For T sufficiently small, there is a unique optimal control.*

Proof. We show uniqueness by showing strict convexity of the map

$$u \in U \rightarrow J(u).$$

This convexity follows from showing for all $u, v \in U$, and $0 < \varepsilon < 1$,

$$g''(\varepsilon) < 0,$$

where $g(\varepsilon) = J(\varepsilon u + (1 - \varepsilon)v) = J(v + \varepsilon(u - v))$. To calculate

$$g'(\varepsilon) = \lim_{\delta \rightarrow 0} \frac{J(v + (\varepsilon + \delta)(u - v)) - J(v + \varepsilon(u - v))}{\delta}, \quad (7.1)$$

denote

$$\tilde{N}_i^\varepsilon = \tilde{N}_i(v + \varepsilon(u - v)), \quad \tilde{N}_i^{\varepsilon+\delta} = \tilde{N}_i(v + (\varepsilon + \delta)(u - v)), \quad i = 1, 2. \quad (7.2)$$

Using the same argument like in Theorem 5.1, we have

$$\frac{\tilde{N}_1^{\varepsilon+\delta} - \tilde{N}_1^\varepsilon}{\delta} \underset{w}{\rightharpoonup} \tilde{\Psi}_1, \quad \frac{\tilde{N}_2^{\varepsilon+\delta} - \tilde{N}_2^\varepsilon}{\delta} \underset{w}{\rightharpoonup} \tilde{\Psi}_2 \quad \text{in } L^2(0, T; H_0^1(\Omega)) \quad (7.3)$$

as $\delta \rightarrow 0$, and $\tilde{\Psi}_1$ satisfies

$$\begin{aligned} (\tilde{\Psi}_1^\varepsilon)_t &= \mathbf{L}_1 \tilde{\Psi}_1^\varepsilon + \left[\left(a_1(v + \varepsilon(u - v))^2 + b_1(v + \varepsilon(u - v)) \right) \chi_\Gamma + c_1 \right] \tilde{\Psi}_1^\varepsilon \\ &\quad + \left[2a_1(v + \varepsilon(u - v)) \chi_\Gamma + b_1 \right] (u - v) \tilde{N}_1^\varepsilon - 2a_{11} \tilde{N}_1^\varepsilon \tilde{\Psi}_1^\varepsilon \\ &\quad - a_{12} (\tilde{N}_1^\varepsilon \tilde{\Psi}_2^\varepsilon + \tilde{N}_2^\varepsilon \tilde{\Psi}_1^\varepsilon), \\ \tilde{\Psi}_1^\varepsilon(x, 0) &= \tilde{\Psi}_2^\varepsilon(x, 0) = 0, \quad x \in \Omega, \\ \tilde{\Psi}_1^\varepsilon(x, t) &= \tilde{\Psi}_2^\varepsilon(x, t) = 0, \quad (x, t) \in \partial\Omega \times (0, T). \end{aligned}$$

A similar equation can be written for $\tilde{\Psi}_2$. Estimating like in Theorem 5.1, we obtain for $0 < s \leq T$,

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} (\tilde{\Psi}_1^\varepsilon(x, s))^2 + (\tilde{\Psi}_2^\varepsilon(x, s))^2 dx + B[\tilde{\Psi}_1^\varepsilon, \tilde{\Psi}_1^\varepsilon; t] + B[\tilde{\Psi}_2^\varepsilon, \tilde{\Psi}_2^\varepsilon; t] \\ & \leq C_1 \int_0^s \int_{\Omega} ((\tilde{\Psi}_1^\varepsilon)^2 + (\tilde{\Psi}_2^\varepsilon)^2) dx dt + C_2 \int_0^T (u - v)^2 dt, \end{aligned}$$

where C_1 only depends on the coefficients and L^∞ bounds for \tilde{N}_i^ε , $i = 1, 2$. Using Gronwall's inequality, we have

$$\sup_{0 \leq s \leq T} \int_{\Omega} (\tilde{\Psi}_1^\varepsilon(x, s))^2 + (\tilde{\Psi}_2^\varepsilon(x, s))^2 dx \leq C_3 \int_0^T (u - v)^2 dt.$$

To calculate $g''(\varepsilon)$, we need a second derivative of N_i with respect to the control. Similar *a priori* estimates imply that

$$\frac{\tilde{\Psi}_1^{\varepsilon+\eta} - \tilde{\Psi}_1^\varepsilon}{\eta} \xrightarrow{w} \tilde{\sigma}_1^\varepsilon, \quad \frac{\tilde{\Psi}_2^{\varepsilon+\eta} - \tilde{\Psi}_2^\varepsilon}{\eta} \xrightarrow{w} \tilde{\sigma}_2^\varepsilon \quad \text{in } L^2(0, T; H_0^1(\Omega)), \quad (7.4)$$

as $\eta \rightarrow 0$, and $\tilde{\sigma}_1^\varepsilon$ satisfies

$$\begin{aligned} (\tilde{\sigma}_1^\varepsilon)_t &= \mathbf{L}_1 \tilde{\sigma}_1^\varepsilon + \left[(a_1(v + \varepsilon(u - v))^2 + b_1(v + \varepsilon(u - v))) \chi_\Gamma + c_1 \right] \tilde{\sigma}_1^\varepsilon \\ &+ 2 \left[2a_1(v + \varepsilon(u - v)) + b_1 \right] (u - v) \tilde{\Psi}_1^\varepsilon + 2a_1(u - v)^2 \tilde{N}_1^\varepsilon \\ &- 2a_{11}((\tilde{\Psi}_1^\varepsilon)^2 + \tilde{N}_1^\varepsilon) - a_{12}(2\tilde{\Psi}_1^\varepsilon \tilde{\Psi}_2^\varepsilon + \tilde{N}_1^\varepsilon \tilde{\sigma}_2^\varepsilon + \tilde{N}_2^\varepsilon \tilde{\sigma}_1^\varepsilon), \\ \tilde{\sigma}_1^\varepsilon(x, 0) &= \tilde{\sigma}_2^\varepsilon(x, 0) = 0, \quad x \in \Omega, \\ \tilde{\sigma}_1^\varepsilon(x, t) &= \tilde{\sigma}_2^\varepsilon(x, t) = 0, \quad (x, t) \in \partial\Omega \times (0, T). \end{aligned}$$

A similar equation can be written for $\tilde{\sigma}_2^\varepsilon$.

Estimating and using Gronwall's inequality, after standard manipulations we obtain

$$\sup_{0 \leq s \leq T} \int_{\Omega} (\tilde{\sigma}_1^\varepsilon(x, s))^2 + (\tilde{\sigma}_2^\varepsilon(x, s))^2 dx \leq C_6 \left(\int_0^T (u - v)^2 dt \right)^2. \quad (7.5)$$

Continuing from (7.1), we are ready to calculate derivatives of g . We can show that

$$\begin{aligned} g'(\varepsilon) &= \int_{\Omega} A \tilde{\Psi}_1^\varepsilon(x, T) - B \tilde{\Psi}_2^\varepsilon(x, T) dx \\ &- \iint_{\Omega \times \Gamma} B_1(u - v) + 2B_2(v + \varepsilon(u - v))(u - v) dx dt. \end{aligned}$$

Similarly, for the second derivative, we have

$$g''(\varepsilon) = \lim_{\eta \rightarrow 0} \frac{g'(\varepsilon + \eta) - g'(\varepsilon)}{\eta} \leq \int_{\Omega} (2B_2T - C_7)(u - v)^2 dx,$$

which gives $g''(\varepsilon) < 0$ for T sufficiently small. □

8. NUMERICAL RESULTS

In this section we will solve the optimality system (8.1)–(8.5) numerically by the following iteration method. 1. Initialization: choose initial guesses for N_{10} , N_{20} and u_0 . 2. Discretization: use the finite difference method to discretize state and adjoint equations to nonlinear algebraic system. 3. Iteration: u_n is known, (a) solve discretized (8.1) for states N_1, N_2 ;

- (b) solve discretized (8.3) for adjoints p, q ;
 (c) update the control by entering new state and adjoint values into the characterization of the optimal control (8.5).
 4. Repeat step 3 if a stopping criterion is not satisfied.

Recall that according to our model we use the control on the time interval $\Gamma = \cup_{i=1}^T [\sigma_i, \tau_i]$, with T denoting the number of years of controls and intervals $[\sigma_i, \tau_i]$ being in the i -th year.

The spatial domain $\Omega = [0, L]$ is an interval in \mathbb{R} with $L > 0$. Thus, the original nonlinear system of parabolic PDE becomes:

$$\begin{aligned} (N_1)_t &= d^1(N_1)_{xx} - r^1(N_1)_x + \left(\theta_1(t, u(x, t)) - a_{11}N_1\right)N_1 - a_{12}N_1N_2, \\ (N_2)_t &= d^2(N_2)_{xx} - r^2(N_2)_x + \left(\theta_2(t, u(x, t)) - a_{22}N_2\right)N_2 - a_{21}N_1N_2, \end{aligned} \quad (8.1)$$

where

$$\theta_i(t, u(x, t)) = \left(a_i u(x, t)^2 + b_i u(x, t)\right)\chi_\Gamma + c_i, \quad i = 1, 2.$$

Here d^i, r^i, a_{ij} , $i, j = 1, 2$ are constants. Flooding is allowed during spring thaw when cottonwood seeds are dispersed from trees. The initial and boundary conditions are:

$$\begin{aligned} N_1(x, 0) &= N_{10}(x), \quad N_2(x, 0) = N_{20}(x), \quad x \in [0, L], \\ N_1(0, t) &= N_2(0, t) = 0, \quad N_1(L, t) = N_2(L, t) = 0, \quad t \in (0, T). \end{aligned} \quad (8.2)$$

The adjoint equations are

$$\begin{aligned} -p_t &= d^1 p_{xx} + r^1 p_x + \left(\theta_1(t, u(x, t)) - 2a_{11}N_1 - a_{12}N_2\right)p - a_{21}N_2q, \\ -q_t &= d^2 q_{xx} + r^2 q_x + \left(\theta_2(t, u(x, t)) - 2a_{22}N_2 - a_{21}N_1\right)q - a_{12}N_1p, \end{aligned} \quad (8.3)$$

with the terminal and boundary conditions

$$\begin{aligned} p(x, T) &= A, \quad q(x, T) = -B, \quad x \in [0, L], \\ p(0, t) &= q(0, t) = 0, \quad p(L, t) = q(L, t) = 0, \quad t \in (0, T). \end{aligned} \quad (8.4)$$

The characterization for the optimal control is

$$u^* = \min \left\{ M, \max \left\{ 0, \frac{b_1 N_1^* p + b_2 N_2^* q - B_1}{2B_2 - 2a_{11}N_1^* p - 2a_{22}N_2^* q} \right\} \right\}. \quad (8.5)$$

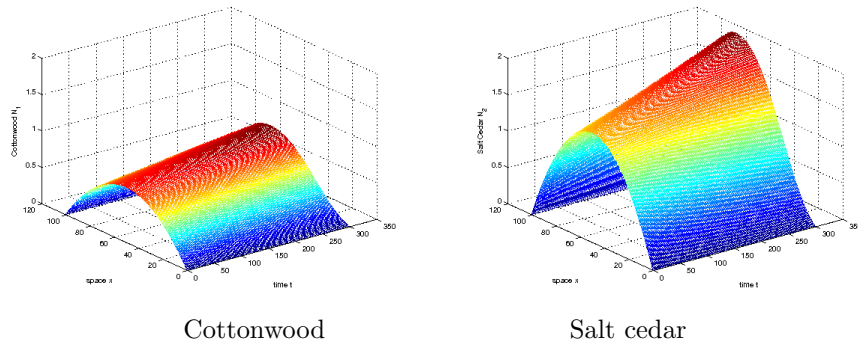
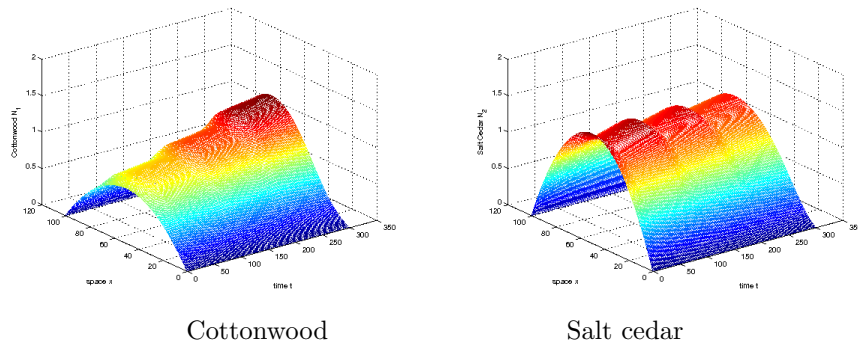
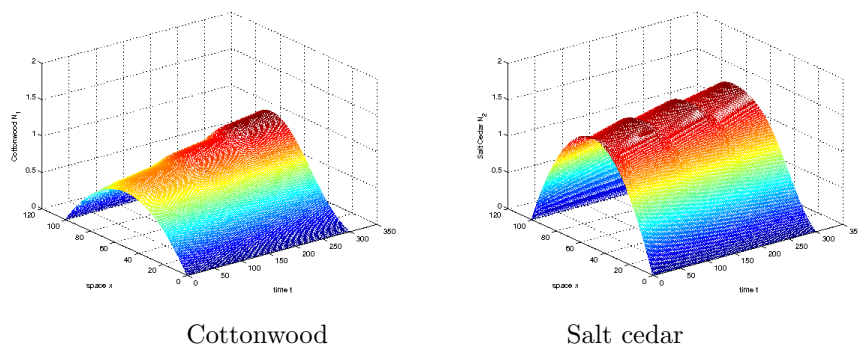
Note that in our numerical simulations, we always have $m(S) = 0$, where S is defined in Theorem 6.1. We will solve equations (8.1)–(8.5).

In the cottonwood - salt cedar scenario, the quadratic growth the functions $\theta_i(t, u(x, t))$, $i = 1, 2$, represent reasonable qualitative behavior of the newly developed seedling communities. To model this interaction, we choose $\theta_1 < \theta_2$ when $u = 0$ and

$$\theta_1 = 0.1u^2 + u + 0.2, \quad \theta_2 = -0.1u^2 - u + 0.4,$$

where $\kappa = 0.1$ which satisfies the restriction in (4.3).

Since the salt cedar currently takes over the cottonwood, our initial conditions are $N_1(x, 0) = x(1 - x)$, $N_2(x, 0) = 3x(1 - x)$. The upper bound for the control is $M = 1$, the length of the one dimensional interval is $L = 1$. We want to see the effect of the growth functions θ_i , so we choose the diffusion and convection coefficients $d^k = 1$, $r^k = 1$, $k = 1, 2$. The interaction coefficients a_{ij} represent how species j affects species i and we choose $a_{11} = 0.005$, $a_{12} = 0.001$, $a_{21} = 0.12$,

FIGURE 1. Cottonwood and salt cedar without control, $L = 1$, $T = 3$ FIGURE 2. Cottonwood and salt cedar with $B_2 = 5$, $B_1 = 1$ FIGURE 3. Cottonwood and salt cedar with $B_2 = 10$, $B_1 = 1$

and $a_{22} = 0.02$, since Sher et al. [24] found that cottonwood density affects the population of both species (in a negative way) more than the salt cedar density.

Without any flooding (control), cottonwood density increases slowly and the salt cedar density increases rapidly. See Figure 1.

Consider the case when $A = 10, B = 1$. First we pick $B_1 = 1, B_2 = 5$. To correspond to the spring thaw, flooding is only allowed during the first 1/4 of the year in order to illustrate the idea for a period of 3 years. We can see the cottonwood density increases while the salt cedar decreases and the time when this happens corresponds to the flooding time every year. See Figure 2.

Note that we need to choose parameters B_1, B_2 carefully, because of the restriction in (4.3). We want to see how the parameter B_2 , the quadratic cost coefficient in $J(u)$, affects the optimal flooding strategy. We fix $B_1 = 1$, as we increase the values of B_2 , flooding levels are decreasing, since it is more expensive to apply the flooding. See Figures 2 and 3. We can see in Figure 3 cottonwood density is still increasing and salt cedar is decreasing but not as much as in Figure 2 when it is much cheaper to apply the control. Figure 4 gives the corresponding optimal flooding strategies for $B_1 = 1$ and $B_2 = 5, 10$, we can examine that as B_2 increases, optimal control strategy is decreasing.

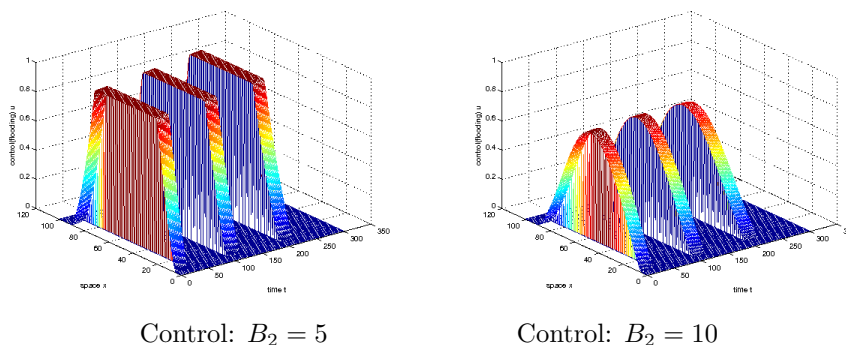


FIGURE 4. Control - flood: $B_2 = 5, 10$, fixed $B_1 = 1$

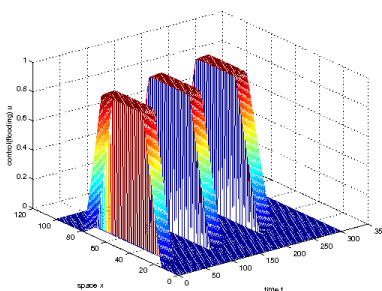


FIGURE 5. Control - flood: $B_2 = 5, B_1 = 5$

We also want to study the effect of the parameter B_1 - the linear cost coefficient in $J(u)$. We fix $B_2 = 5$ and compare $B_1 = 1$ with $B_1 = 5$ to observe as B_1 increases, the flooding is decreasing. See Figures 4(a) and 5 for the comparison of the optimal flooding.

In summary, the control (flooding) is focused near the middle of the region due to the Dirichlet boundary conditions we imposed on our problem. When increasing the cost coefficients B_1 or B_2 , the flooding level is decreasing. Compared with Figure 1 without any control, Figures 2 and 3 gave good illustrations that flooding can indeed control the invasive species N_2 and help the growth of the native species N_1 .

Conclusions. This project shows that the optimal control theory can be an appropriate tool for designing the intervention strategy of the invasive - native species interaction. We proved existence of the optimal control when the control is quadratic in the growth function in the PDE system under certain conditions on the coefficients. We gave numerical examples for different parameter values that can help natural resource managers to apply the most appropriate and cost-effective control methods to the invasive - native species scenario.

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WANDI DING

DEPARTMENT OF MATHEMATICAL SCIENCES AND COMPUTATIONAL SCIENCE PROGRAM, MIDDLE TENNESSEE STATE UNIVERSITY, MURFREESBORO, TN 37132, USA

E-mail address: wandi.ding@mtsu.edu

VOLODYMYR HRYNKIV

DEPARTMENT OF COMPUTER AND MATHEMATICAL SCIENCES, UNIVERSITY OF HOUSTON - DOWNTOWN, HOUSTON, TX 77002, USA

E-mail address: HrynkivV@uhd.edu

XIAOYU MU

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TENNESSEE, KNOXVILLE, TN 37996-1320, USA

E-mail address: xiaoyumoon@gmail.com