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EXISTENCE OF SOLUTIONS TO NONLOCAL ELLIPTIC EQUATIONS WITH DISCONTINUOUS TERMS

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ABSTRACT. In this article, we study the existence of nonnegative solutions for the elliptic partial differential equation

$$-[M(\|u\|_{1,p}^p)]^{1,p}\Delta_p u = f(x,u) \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega,$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $f : \overline{\Omega}\times\mathbb{R}^+ \to \mathbb{R}$ is a discontinuous nonlinear function.

1. Introduction

This article concerns the existence of solution to the elliptic problem

$$\begin{gathered} -[M(\|u\|_{1,p}^p)]^{1,p}\Delta_p u = f(x,u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $f : \overline{\Omega} \times \mathbb{R}^+ \to \mathbb{R}$ is a discontinuous function, $M : \mathbb{R}^+ \to \mathbb{R}$, $\mathbb{R}^+ = [0,\infty)$, Δ_p is the p-Laplacian

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u), \quad p > 1,$$

and $\|\cdot\|_{1,p}$ is the usual norm

$$\|u\|_{1,p}^p = \int_\Omega |\nabla u|^p$$

in the Sobolev space $W_0^{1,p}(\Omega)$.

The interest of the mathematicians on the so called nonlocal problems like (1.1) (nonlocal because of the presence of the term $M(\|u\|_{1,p}^p)$, has increased because they represent a variety of relevant physical and engineering situations and requires a nontrivial apparatus to solve them.

More precisely, we study the existence of nonnegative nontrivial solutions of the problem

$$\begin{gathered} -[M(\|u\|_{1,p}^p)]^{p-1}\Delta_p u = \lambda H(u-a)u^q + h(x)u^s \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.2}$$

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where $M : \mathbb{R}^+ \to \mathbb{R}$ is a continuous function, $1 < q+1 < p < s+1 < p^* = \frac{pN}{N-P}$, $a > 0$ and $\lambda > 0$ are real parameters, $h : \Omega \to (0, \infty)$ is a positive measurable function, $h \in L^\infty(\Omega)$ and H is the Heaviside function

$$H(t) = \begin{cases} 0 & \text{if } t \le 0, \\ 1 & \text{if } t > 0. \end{cases}$$

We assume the following conditions:

(H1) There exist $m_1, t_1 > 0$ such that $M(t) \ge m_1$ if $0 \le t \le t_1$;
(H2) There exist $m_2, t_2 > 0$ such that $0 < M(t) \le m_2$ if $t \ge t_2$;
(H3) $\lim_{t\to\infty} [M(t^p)]^{p-1} t^{(p-1)-q} = +\infty$;
(H4) M is non-increasing and $M(t) > 0$ for all $t > 0$.

Problems involving discontinuous nonlinearity appears in several physical situations. Among these, we may cite electrical phenomena, plasma physics, free boundary value problems, etc. The reader may consult Ambrosetti-Calahorrano-Dobarro [5], Ambrosetti-Turner [6], Arcoya-Calahorrano [7], Arcoya-Diaz-Tello [8], Badialle [9], [10], and the references therein. Some physical problems are related to discontinuous surface

$$\Gamma_a(u) = \{x \in \Omega; u(x) = a\}$$

which causes difficulties in analyzing this kind of problems.

When $M \equiv 1$, (1.2) becomes a local problem, and has been widely studied. In particular, Alves-Bertone [1] and Alves-Bertone-Goncalves [2] use variants of the Mountain Pass Theorem (for locally Lipschitz functionals), the Ekeland Variational Principle, and the Subdifferential Calculus. On the other hand, after the work by Alves, Correa and Matofu [3] several papers appeared dealing with nonlocal problems with variational techniques; see for example [13, 14, 15, 16, 17, 20, 21, 24, 25].

This article maybe the the first study of a nonlocal problem with variational techniques for a non-differentiable functional. We consider the non-differentiable functional

$$I_{\lambda,a}(u) = \frac{1}{p}\widehat{M}(\|u\|_{1,p}^p) - \lambda\psi(u) - \frac{1}{s+1}\int_\Omega h(x)(u^+)^{s+1},$$

defined on $W_0^{1,p}(\Omega)$, where

$$\widehat{M}(t) = \int_0^t [M(s)]^{p-1} ds, \quad \psi(u) = \int_\Omega F(u),$$
$$F(u) = \int_0^u f(t)dt, \quad f(t) = H(t-a)(t^+)^q, \quad t^+ = \max\{0, t\}.$$

By a solution to (1.2) we mean a function $u \in W_0^{1,p}(\Omega) \cap W^{p,\frac{q+1}{q}}(\Omega)$ satisfying

$$-[M(\|u\|_{1,p}^p)]^{p-1}\Delta_p u(x) - h(x)u(x)^s \in \lambda[\underline{f}(u(x)), \overline{f}(u(x))] \quad \text{a.e. in } \Omega, \tag{1.3}$$

where $f(t) = H(t-a)(t^+)^q$ is nondecreasing,

$$\overline{f}(t) = \lim_{\delta\to 0^+} f(t+\delta), \quad \underline{f}(t) = \lim_{\delta\to 0^+} f(t-\delta).$$

Let us consider the level set

$$\Gamma_a(u) = \{x \in \Omega; u(x) = a\}.$$

Note that if $|\Gamma_a(u)| = 0$, then u satisfies

$$-\,[M(\|u\|_{1,p}^p)]^{p-1}\Delta_p u(x) = \lambda H(u(x)-a)u(x)^q + h(x)u(x)^s \quad \text{a.e. in } \Omega. \tag{1.4}$$

Clearly, a solution in the sense of (1.3) is also a solution in the sense of (1.4). The main result of this work is as follows.

Theorem 1.1. *Suppose that M satisfies assumptions* (H1)–(H4) *and $h \in L^\infty(\Omega)$. Then there are $\lambda^* > 0$ and $a^* > 0$ such that for $\lambda \in (0, \lambda^*)$ and $a \in (0, a^*)$, Problem* (1.2) *possesses at least two nontrivial and nonnegative solutions u_1 and u_2 satisfying*

- (i) $|\Gamma_a(u_i)| = 0$, $i = 1, 2$;
- (ii) $I_{\lambda,a}(u_2) < 0 < I_{\lambda,a}(u_1)$;
- (iii) $|\{x \in \Omega; u_i(x) > a\}| > 0$, $i = 1, 2$.

2. Abstract Framework

In this section we establish some basic results on critical point theory for locally Lipschitz functionals, developed by Chang [11] based on Convex Analysis and on the Subdifferential Calculus by Clarke [12].

Definition 2.1. Let X be a Banach space. We say that the functional $I : X \to \mathbb{R}$ is locally Lipschitz ($I \in \mathrm{Lip}_{\mathrm{loc}}(X, \mathbb{R})$) if, given $u \in X$, there exist a neighborhood $V \equiv V_u \subset X, u \in V$ and a constant $k \equiv k_V > 0$ such that

$$|I(v_2) - I(v_1)| \le k\|v_2 - v_1\|, \quad v_1, v_2 \in V.$$

Definition 2.2. The directional derivative of the locally Lipschitz functional $I : X \to \mathbb{R}$ at $u \in X$ in the direction $v \in X$ is defined by

$$I^0(u; v) = \limsup_{h\to 0, \lambda\to 0^+} \frac{I(u + h + \lambda v) - I(u + h)}{\lambda}.$$

We may prove that $I^0(u; v)$ is subadictive and positively homogeneous; that is,

$$I^0(u; v_1 + v_2) \le I^0(u; v_1) + I^0(u; v_2)$$

and

$$I^0(u; \lambda v) = \lambda I^0(u; v)$$

for all $u, v_1, v_2 \in X$ and $\lambda > 0$.

Using these properties, it follows that

$$|I^0(u, v_1) - I^0(u, v_2)| \le K|v_1 - v_2|, \quad K \equiv K_u > 0.$$

Consequently, $I^0(u, \cdot)$ is continuous and, because it is also convex, we may consider its subdifferential at $z \in X$ which is given by

$$\partial I^0(u; z) = \{\mu \in X^*; I^0(u; v) \ge I^0(u; z) + \langle \mu, v - z\rangle, v \in X\},$$

where X^* is the topological dual of X and $\langle \cdot, \cdot\rangle$ is the duality pairing between X^* and X.

Definition 2.3. The generalized gradient of $I \in \mathrm{Lip}_{\mathrm{loc}}(X, \mathbb{R})$ at $u \in X$ is defined as being the set

$$\partial I(u) = \{\mu \in X^*; I^0(u; v) \ge \langle \mu, v\rangle, \text{ for all } v \in X\}.$$

Since $I^0(u;0) = 0$, it follows that $\partial I(u) = \partial I^0(u;0)$. Furthermore, for all $v \in X$, we have
$$I^0(u;v) = \max\{\langle \mu, v\rangle; v \in \partial I(u)\}.$$
An important property of the generalized gradient is as follows: if $u \in X$, then $\partial I(u)$ is a convex, nonempty and weak* − compact. In particular, there is $\omega \in \partial I(u)$ such that
$$m(u) = \min\{\|\omega\|_*; \omega \in \partial I(u)\}.$$
The reader may find more properties on this subject in [12] and [18]. We note that $\partial I(u) = \{I'(u)\}$ when $I \in C^1(X, \mathbb{R})$.

Definition 2.4. A sequence $(u_n) \subset X$ is a Palais-Smale sequence at the level c ($(PS)_c$), if
$$I(u_n) \to c, \quad m(u_n) \to 0.$$

Definition 2.5. We say that the functional $I \in \mathrm{Lip}_{\mathrm{loc}}(X, \mathbb{R})$ satisfies the Palais-Smale condition at the level c, if any $(PS)_c$ sequence possesses a strongly convergent subsequence.

The proof of our main result rests heavily on the following version of the Mountain Pass Theorem for $\mathrm{Lip}_{\mathrm{loc}}$ functionals whose proof may be found in Chang [11]. Its proof uses an appropriate version of the Deformation Lemma whose proof is found in [18].

We say that $u_0 \in X$ is a critical point of I if $0 \in \partial I(u_0)$. Clearly, every local minimum (maximum) point is a critical point.

Theorem 2.6. Let $I \in \mathrm{Lip}_{\mathrm{loc}}(X, \mathbb{R})$ be a functional such that $I(0) = 0$ and suppose that:

(i) There are constants $\eta > 0$ and $\rho > 0$ such that $I(u) > \eta$, for $\|u\| = \rho, u \in X$;
(ii) There is $e \in X$, with $\|e\| > \rho$, such that $I(e) < 0$.

If, in addition, I satisfies the Palais-Smale condition at the level
$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)),$$
where
$$\Gamma = \{\gamma \in C([0,1], X), \gamma(0) = 0 \text{ and } \gamma(1) = e\},$$
then $c > 0$ is a critical value of I.

3. Preliminary Results

In this section we establish some results for proving the main result of this article.

Lemma 3.1. *There is $\lambda^* > 0$ such that the functional $I_{\lambda,a}$ satisfies the geometric conditions (i) and (ii) of Mountain Pass Theorem 2.6, for all $a > 0$*

Proof. (i) Let us consider $u \in W_0^{1,p}(\Omega)$ such that $0 < \|u\|_{1,p}^p = r < t_1$. Then, by (H1),
$$I_{\lambda,a}(u) \geq \frac{m_1^{p-1}}{p}\|u\|_{1,p}^p - \lambda \int_\Omega \int_0^u H(t-a)(t^+)^q \,dt\,dx - \frac{1}{s+1}\int_\Omega h(x)(u^+)^{s+1}$$
Noticing that $H \leq 1$ and $u^+ \leq |u|$, we obtain
$$I_{\lambda,a}(u) \geq \frac{m_1^{p-1}}{p}\|u\|_{1,p}^p - \frac{\lambda}{q+1}|u|_{q+1}^{q+1} - \frac{1}{s+1}\int_\Omega h(x)(u^+)^{s+1}.$$

From the Sobolev immersions and from the fact that $h \in L^\infty(\Omega)$, we obtain

$$I_{\lambda,a}(u) \geq \frac{m_1^{p-1}}{p} r^p - \frac{C_1 \lambda}{q+1} r^{q+1} - C_2 r^{s+1}.$$

Choosing $r > 0$ sufficiently small, there exists

$$\lambda^* = \frac{m_1^{p-1}(q+1)}{4pC_1 r^{(q+1)-p}}$$

such that $I_{\lambda,a}(u) \geq \eta > 0$ for $\|u\|_{1,p} = r$, for all $\lambda \in (0, \lambda^*)$, and for some $\eta > 0$.

(ii) Let φ be a function in $C_0^\infty(\Omega), \varphi > 0$ in Ω. Hence, for $t > 0$ with $\|t\varphi\|_{1,p}^p > t_2$, it follows from (H2) and recalling that $H \geq 0$, we have

$$I_{\lambda,a}(t\varphi) \leq \frac{1}{p} m_2^{p-1} t^p \|\varphi\|_{1,p}^p - \frac{1}{s+1} \int_\Omega h(x)(u^+)^{s+1} + \widetilde{C}$$

and because $1 < q+1 < p < s+1$, we have $I_{\lambda,a}(t\varphi) \to -\infty$ as $t \to +\infty$. This completes the proof. □

Remark 3.2. Using Lemma 3.1 we may infer, from the Mountain Pass Theorem for $\mathrm{Lip}_{\mathrm{loc}}$ functionals, the existence of a sequence $(u_n) \subset W_0^{1,p}(\Omega)$ such that $I_{\lambda,a}(u_n) \to c$ and $m(u_n) \to 0$.

The proof of the following lemma can be found in [11].

Lemma 3.3. *If $u \in W_0^{1,p}(\Omega)$ and $\omega \in \partial\psi(u)$, then*

$$\omega(x) \in [\underline{f}(u(x)), \overline{f}(u(x))] \quad \text{a.e. in } \Omega.$$

In what follows, for the functional $I_{\lambda,a}$, we will use notation

$$I_{\lambda,a}(u) = \phi(u) - \lambda\psi(u) - J(u),$$

where

$$\phi(u) = \frac{1}{p}\widehat{M}(\|u\|_{1,p}^p), \quad \psi(u) = \int_\Omega F(u), \quad J(u) = \frac{1}{s+1}\int_\Omega h(x)(u^+)^{s+1}.$$

Lemma 3.4. *The functional $I_{\lambda,a}$ satisfies the Palais-Smale condition.*

Proof. Let $(u_n) \subset W_0^{1,p}(\Omega)$ be a sequence satisfying $I_{\lambda,a}(u_n) \to c$ and $m(u_n) \to 0$.

For the rest of this article, consider $(\omega_n) \subset (W_0^{1,p}(\Omega))^*$ be such that $m(u_n) = \|\omega_n\|_*$ and

$$\omega_n = \phi'(u_n) - \lambda\rho_n - J'(u_n) \tag{3.1}$$

with $(\rho_n) \subset \partial\psi(u_n)$.

Claim 3.5. *The sequence $(u_n) \subset W_0^{1,p}(\Omega)$ is bounded.*

Indeed, from (3.1),

$$\langle \omega_n + \lambda\rho_n, u_n \rangle = [M(\|u_n\|_{1,p}^p)]^{p-1}\|u_n\|_{1,p}^p - \int_\Omega h(x)(u_n^+)^s u_n$$

and so

$$\begin{aligned} &I_{\lambda,a}(u_n) - \frac{1}{s+1}\langle \omega_n + \lambda\rho_n, u_n \rangle \\ &= \frac{1}{p}\widehat{M}(\|u_n\|_{1,p}^p) - \lambda\int_\Omega F(u_n) - \frac{1}{s+1}\int_\Omega h(x)(u_n^+)^{s+1} \end{aligned}$$

$$- \frac{1}{s+1}[M(\|u_n\|_{1,p}^p)]^{p-1}\|u_n\|_{1,p}^p + \frac{1}{s+1}\int_\Omega h(x)(u_n^+)^s u_n.$$

Writing $u_n = u_n^+ - u_n^-$, we obtain

$$\frac{1}{s+1}\int_\Omega h(x)(u_n^+)^s u_n = \frac{1}{s+1}\int_\Omega h(x)(u_n^+)^{s+1},$$

which implies

$$\begin{aligned} &I_{\lambda,a}(u_n) - \frac{1}{s+1}\langle \omega_n + \lambda\rho_n, u_n\rangle \\ &= \frac{1}{p}\widehat{M}(\|u_n\|_{1,p}^p) - \lambda\int_\Omega F(u_n) - \frac{1}{s+1}\int_\Omega h(x)(u_n^+)^{s+1} \\ &\quad - \frac{1}{s+1}[M(\|u_n\|_{1,p}^p)]^{p-1}\|u_n\|_{1,p}^p + \frac{1}{s+1}\int_\Omega h(x)(u_n^+)^{s+1}, \end{aligned}$$

from which it follows that

$$\begin{aligned} &I_{\lambda,a}(u_n) - \frac{1}{s+1}\langle \omega_n + \lambda\rho_n, u_n\rangle \\ &= \frac{1}{p}\widehat{M}(\|u_n\|_{1,p}^p) - \frac{1}{s+1}[M(\|u_n\|_{1,p}^p)]^{p-1}\|u_n\|_{1,p}^p - \lambda\int_\Omega F(u_n). \end{aligned} \tag{3.2}$$

Since (u_n) is a $(PS)_c$ sequence, there exists a constant $C_2 > 0$ such that $|I_{\lambda,a}(u_n)| \le C_2$ for all $n \in \mathbb{N}$. Because

$$\frac{1}{s+1}[-\langle \omega_n, u_n\rangle] \le \frac{1}{s+1}|\langle \omega_n, u_n\rangle| \le \frac{1}{s+1}\|\omega_n\|_*\|u_n\|_{1,p} \le C_3\|u_n\|_{1,p},$$

by (3.2), we obtain

$$\begin{aligned} I_{\lambda,a}(u_n) - \frac{1}{s+1}\langle \omega_n + \lambda\rho_n, u_n\rangle &\le C_2 + \frac{1}{s+1}|\langle \omega_n, u_n\rangle| + \frac{1}{s+1}|\langle \lambda\rho_n, u_n\rangle| \\ &\le C_2 + C_3\|u_n\|_{1,p} + \frac{\lambda}{s+1}|\langle \rho_n, u_n\rangle|. \end{aligned}$$

Since $(\rho_n) \subset \partial\psi(u_n)$, $\langle \rho_n, v\rangle \le \psi^0(u_n, v)$ for all $v \in W_0^{1,p}(\Omega)$.

Using arguments found in [11] and [12], we can show that

$$\begin{aligned} \langle \rho_n, u_n\rangle &\le \psi^0(u_n; u_n) \\ &\le \int_{\{u_n<0\}} \underline{f}(u_n)u_n + \int_{\{u_n>0\}} \overline{f}(u_n)u_n \\ &\le \int_{\{u_n>0\}} |u_n|^{q+1} \le \int_\Omega |u_n|^{q+1} \\ &\le C_4\|u_n\|_{1,p}^{q+1}. \end{aligned}$$

and

$$I_{\lambda,a}(u_n) - \frac{1}{s+1} l\angle \omega_n + \lambda\rho_n, u_n\rangle \le C_2 + C_3\|u_n\|_{1,p} + C_5\|u_n\|_{1,p}^{q+1}. \tag{3.3}$$

From inequalities (3.2)-(3.3), it follows that

$$\begin{aligned} &\frac{1}{p}\widehat{M}(\|u_n\|_{1,p}^p) - \frac{1}{s+1}[M(\|u_n\|_{1,p}^p)]^{p-1}\|u_n\|_{1,p}^p \\ &\le C_2 + C_3\|u_n\|_{1,p} + C_5\|u_n\|_{1,p}^{q+1} + \lambda\int_\Omega\int_0^{u_n}(t^+)^q\,dt\,dx. \end{aligned}$$

Using the Sobolev immersions,

$$\begin{aligned}&\frac{1}{p}\int_0^{\|u_n\|_{1,p}^p}[M(s)]^{p-1}ds-\frac{1}{s+1}[M(\|u_n\|_{1,p}^p)]^{p-1}\|u_n\|_{1,p}^p\\&\le C_2+C_3\|u_n\|_{1,p}+C_7\|u_n\|_{1,p}^{q+1}.\end{aligned}\tag{3.4}$$

From the continuity of M^{p-1} on $[0,\|u_n\|_{1,p}^p]$ and in view of Mean Value Theorem for integrals, there exists ξ_n, $0<\xi_n<\|u_n\|_{1,p}^p$, such that

$$\frac{1}{p}\int_0^{\|u_n\|_{1,p}^p}[M(s)]^{p-1}ds=[M(\xi_n)]^{p-1}\|u_n\|_{1,p}^p.$$

Since M is a nonincreasing function, we obtain

$$[M(\xi_n)]^{p-1}\ge[M(\|u_n\|_{1,p}^p)]^{p-1}\tag{3.5}$$

and

$$\begin{aligned}&\frac{1}{p}[M(\|u_n\|_{1,p}^p)]^{p-1}\|u_n\|_{1,p}^p-\frac{1}{s+1}[M(\|u_n\|_{1,p}^p)]^{p-1}\|u_n\|_{1,p}^p\\&\le C_2+C_3\|u_n\|_{1,p}+C_7\|u_n\|_{1,p}^{q+1},\end{aligned}$$

from which

$$\big(\frac{1}{p}-\frac{1}{s+1}\big)[M(\|u_n\|_{1,p}^p)]^{p-1}\|u_n\|_{1,p}^{(p-1)-q}\le\frac{C_2}{\|u_n\|_{1,p}^{q+1}}+\frac{C_3}{\|u_n\|_{1,p}^{q}}+C_7.$$

From (H3), we conclude that (u_n) is bounded. Showing the claim.

Since $\{u_n\}$ is $(P.S)$ sequence, using standard arguments, we can assume, without loss of generality, that $u_n\ge 0$ for all $x\in\Omega$.

As (u_n) is bounded and using the reflexivity of $W_0^{1,p}(\Omega)$ there are $u_1\in W_0^{1,p}(\Omega)$ and $\vartheta\in\mathbb{R}$ such that, up to a subsequence,

$$\|u_n\|_{1,p}^p\to\vartheta^p,\quad u_n\rightharpoonup u_1\quad\text{in } W_0^{1,p}(\Omega).$$

Consequently, $u_1\ge 0$.

Let us now show that $u_n\to u_1$ in $W_0^{1,p}(\Omega)$. From the continuity of M and $\|u_n\|_{1,p}^p\to\vartheta^p$, we obtain $M(\|u_n\|_{1,p}^p)\to M(\vartheta^p)$ and because $M(\vartheta^p)>0$, there is $K>0$ such that

$$M(\|u_n\|_{1,p}^p)\ge K>0\quad\text{for } n \text{ large enough}.$$

Using the well known Simon inequality (see [23]), we obtain

$$\begin{aligned}&K^{p-1}C_p\int_\Omega|\nabla u_n-\nabla u_1|^p\\&\le[M(\|u_n\|^p)]^{p-1}\int_\Omega\langle|\nabla u_n|^{p-2}\nabla u_n-|\nabla u_1|^{p-2}\nabla u_1,\nabla u_n-\nabla u_1\rangle\end{aligned}$$

and so

$$\begin{aligned}&K^{p-1}C_p\int_\Omega|\nabla u_n-\nabla u_1|^p\\&\le[M(\|u_n\|_{1,p}^p)]^{p-1}\|u_n\|_{1,p}^p-[M(\|u_n\|_{1,p}^p)]^{p-1}\int_\Omega|\nabla u_n|^{p-2}\nabla u_n\nabla u_1\\&\quad-[M(\|u_n\|_{1,p}^p)]^{p-1}\int_\Omega|\nabla u_1|^{p-2}\nabla u_1\nabla u_n+[M(\|u_n\|_{1,p}^p)]^{p-1}\|u_1\|_{1,p}^p.\end{aligned}$$

Noticing that

$$[M(\|u_n\|_{1,p}^p)]^{p-1}\|u_1\|_{1,p}^p - [M(\|u_n\|_{1,p}^p)]^{p-1}\int_\Omega |\nabla u_1|^{p-2}\nabla u_1 \nabla u_n = o_n(1),$$

we obtain

$$\begin{aligned}&K^{p-1}C_p\int_\Omega |\nabla u_n - \nabla u_1|^p\\ &\leq [M(\|u_n\|_{1,p}^p)]^{p-1}\|u_n\|_{1,p}^p - [M(\|u_n\|_{1,p}^p)]^{p-1}\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla u_1 + o_n(1).\end{aligned}$$

We point out that

$$\begin{gathered}\int_\Omega h(x)u_n^{s+1} \to \int_\Omega h(x)u_1^{s+1}\\ \int_\Omega h(x)u_1^{s}u_1 \to \int_\Omega h(x)u_1^{s+1}\end{gathered}$$

$$\begin{aligned}|\langle\rho_n, u_1\rangle - \langle\rho_n, u_n\rangle| &= |\langle\rho_n, u_n\rangle - \langle\rho_n, u_1\rangle|\\ &= |\langle\rho_n, u_n - u_1\rangle|\\ &\leq \|\rho_n\|_*|u_n - u_1|_p \to 0,\end{aligned}$$

From (3.1) and boundedness of $\{u_n\}$, it follows that (ρ_n) is bounded in $(W_0^{1,p}(\Omega))^*$, and since $u_n \to u_1$ in $L^\alpha(\Omega)$, $1 \leq \alpha < p^*$ we obtain

$$\lambda(\langle\rho_n, u_1\rangle - \langle\rho_n, u_n\rangle) = o_n(1).$$

We may write

$$\begin{aligned}&K^{p-1}C_p\int_\Omega |\nabla u_n - \nabla u_1|^p\\ &\leq [M(\|u_n\|_{1,p}^p)]^{p-1}\|u_n\|_{1,p}^p - [M(\|u_n\|_{1,p}^p)]^{p-1}\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla u_1\\ &\quad - \int_\Omega h(x)u_n^{s+1} + \int_\Omega h(x)u_n^s u_1 - \lambda\langle\rho_n, u_n\rangle + \lambda\langle\rho_n, u_1\rangle + o_n(1)\\ &= \langle\omega_n, u_n\rangle - \rangle\omega_n, u_1\rangle = o_n(1).\end{aligned}$$

Hence, $u_n \to u_1$ in $W_0^{1,p}(\Omega)$ which shows that $I_{\lambda,a}$ satisfies the $(PS)_c$ condition. □

4. Proof of the Theorem 1.1

Part I: Multiplicity of solutions. Using Lemmas 3.1 and 3.4, from the Mountain Pass Theorem for $\mathrm{Lip}_{\mathrm{loc}}$ functionals, it follows that u_1 is a critical point of $I_{\lambda,a}$ at the level c; i.e.,

$$I_{\lambda,a}(u_1) = c > 0 \tag{4.1}$$

which implies that $u_1 \not\equiv 0$.

Since $\{u_n\}$ is (P.S) sequence, there are $\{\omega_n\} \subset \partial I_{\lambda,a}(u_n)$ and $\{\rho_n\} \subset \partial\Psi(u_n)$ verifying $\|\omega_n\|_* \to 0$ and

$$\langle\omega_n, \phi\rangle = [M(\|u_n\|_{1,p}^p)]^{p-1}\int_\Omega |\nabla u_n|^{p-2}\nabla u_n\nabla\varphi - \int_\Omega h(x)u_n^s - \lambda\int_\Omega \rho_n\varphi \tag{4.2}$$

where, by Lemma 3.3,

$$\rho_n \in [\underline{f}(u_n(x)), \overline{f}(u_n(x))] \quad \text{a.e in } \Omega. \tag{4.3}$$

The boundedness of $\{u_n\}$ combined with (4.3) implies in particular that $\{\rho_n\}$ is bounded in $L^{\frac{q+1}{q}}(\Omega)$. Thus, there is $\rho_0 \in L^{\frac{q+1}{q}}(\Omega)$ such that, up to a subsequence $\rho_n \rightharpoonup \rho_0$ in $L^{\frac{q+1}{q}}(\Omega)$, or equivalently

$$\int_\Omega \rho_n \varphi \to \int_\Omega \rho_0 \varphi, \quad \forall \varphi \in L^{q+1}(\Omega) \tag{4.4}$$

By [4, Lemma 3.3], $\rho_0(x) \in [\underline{f}(u_1(x)), \overline{f}(u_1(x))]$ in Ω.

Letting $n \to +\infty$ in (4.2), and using (4.4), we obtain the identity

$$[M(\|u_1\|_{1,p}^p)]^{p-1} \int_\Omega |\nabla u_1|^{p-2} \nabla u_1 \nabla \varphi - \int_\Omega h(x) u_1^s \varphi = \lambda \int_\Omega \rho_0 \varphi.$$

Showing that u_1 is a weak solution of the problem

$$\begin{gathered} -[M(\|u_1\|_{1,p}^p)]^{p-1} \Delta_p u_1 - h u_1^s = \lambda \rho_0 \quad \text{in } \Omega \\ u_1 > 0 \quad \text{in } \Omega. \end{gathered}$$

By elliptic regularity, once $\rho_0 \in L^{\frac{q+1}{q}}(\Omega)$, it follows that $u_1 \in W^{p,\frac{q+1}{q}}(\Omega)$ and

$$-[M(\|u_1\|_{1,p}^p)]^{p-1} \Delta_p u_1(x) - h(x) u_1^s(x) = \lambda \rho_0(x) \quad \text{a.e. in } \Omega$$

which implies

$$-[M(\|u_1\|_{1,p}^p)]^{p-1} \Delta_p u_1(x) - h(x) u_1^s(x) \in \lambda [\underline{f}(u_1(x)), \overline{f}(u_1(x))] \quad \text{a.e. in } \Omega.$$

This shows that u_1 is a solution of (1.2).

Proof of (i). Let us show that $|\Gamma_a(u_1)| = 0$, where $\Gamma_a(u_1) = \{x \in \Omega; u_1(x) = a\}$. Let us suppose, by contradiction, that $|\Gamma_a(u_1)| > 0$. From the Morrey-Stampacchia Theorem [22], $-\Delta_p u_1(x) = 0$ a.e. in $\Gamma_a(u_1)$, and so

$$-[M(\|u_1\|_{1,p}^p)]^{p-1} \Delta_p u_1(x) = 0 \quad \text{a.e. in } \Gamma_a(u_1). \tag{4.5}$$

Since u_1 is a critical point, it follows that

$$-[M(\|u_1\|_{1,p}^p)]^{p-1} \Delta_p u_1(x) - h(x) u_1^s(x) \in \lambda [\underline{f}(u_1(x)), \overline{f}(u_1(x))] \quad \text{a.e. in } \Omega.$$

From (4.5), we obtain

$$-h(x) u_1^s(x) \in \lambda [\underline{f}(u_1(x)), \overline{f}(u_1(x))] \quad \text{a.e. in } \Omega.$$

As $0 \le H(u_1 - a)(u_1^+)^q \le (u_1^+)^q$, it follows from the definition of $\underline{f}(u_1(x)), \overline{f}(u_1(x))$ and from the fact that $u_1 \ge 0$, that

$$0 \le \underline{f}(u_1(x)) \le \overline{f}(u_1(x)) \le (u_1)^q.$$

Thus, $-h(x) u_1^s(x) \in [0, \lambda a^q]$ which is impossible. Hence $|\Gamma_a(u_1)| = 0$.

Second Solution (Ekeland Variational Principle). By Lemma 3.1, we obtain $I_{\lambda,a}(u) \ge \eta$ for $0 < \|u\|_{1,p} = \rho$ and so $I_{\lambda,a}(u)$ is bounded from below on $\overline{B}_r$ and so there is $\inf_{\overline{B}_r} I_{\lambda,a}(u)$.

Claim 4.1. *There is $a^* > 0$ such that for $a \in (0, a^*)$ we have $\inf_{B_r(0)} I_{\lambda,a}(u) < 0$.*

Indeed, let us define the auxiliary function

$$\varphi_\tau(x) = \begin{cases} \tau a & |x| \le \frac{1}{2}, \\ 2\tau a(1 - |x|) & \frac{1}{2} \le |x| \le 1, \\ 0 & |x| \ge 1. \end{cases}$$

where $\tau > (q+2)^{1/(q+1)}$. We point out that $\varphi_\tau \in W_0^{1,p}(\Omega)$ and

$$\|\varphi_\tau\|_{1,p}^p = \int_\Omega |\nabla\varphi_\tau|^p = \int_{\{|x|\le 1\}} |\nabla\varphi_\tau|^p = (2\tau a)^p \int_{\{|x|\le 1\}} |\nabla|x|)|^p.$$

Using the change of variables $x = \omega_\tau$, with $\omega \in S^{N-1}$ implies $dx = \tau^{N-1} ds(\omega) d\tau$. Then we obtain $|x| = \tau$. Hence, $\frac{\partial\tau}{\partial x_i} = \frac{x_i}{\tau}$ which implies $|\nabla r| = 1$. In this way,

$$\|\varphi_\tau\|_{1,p}^p \le (2\tau a)^p \alpha_N,$$

where α_N is the volume of the unit ball.

If $a < \frac{\sqrt[p]{r}}{2\tau\sqrt[p]{\alpha_N}} =: a_1$, where r is given by the geometry of the Mountain Pass Theorem, $\varphi_\tau \in B_r$.

On the other hand,

$$\begin{aligned}\int_\Omega \int_0^{\varphi_r} H(t-a)(t^+)^q\,dt\,dx &\ge \int_{\{|x|\le 1/2\}} \int_0^{\varphi_r} H(t-a)(t^+)^q\,dt\,dx \\ &= \frac{a^{q+1}(\tau^{q+1}-1)}{q+1} \int_{\{|x|\le 1/2\}} dx;\end{aligned}$$

that is,

$$\int_\Omega \int_0^{\varphi} H(t-a)(t^+)^q\,dt\,dx \ge \frac{a^{q+1}(\tau^{q+1}-1)}{q+1} C_1$$

and so

$$I_{\lambda,a}(\varphi_\tau) = \frac{1}{p}\widehat{M}(\|\varphi_\tau\|_{1,p}^p) - \lambda \int_\Omega \int_0^{\varphi_\tau} H(t-a)(t^+)^q\,dt\,dx - \frac{t^{s+1}}{s+1}\int_\Omega h(x)(\varphi_\tau^+)^{q+1}.$$

As $\frac{t^{s+1}}{s+1}\int_\Omega h(x)(\varphi_\tau^+)^{q+1} \ge 0$, it follows that

$$I_{\lambda,a}(\varphi_\tau) \ge \frac{1}{p}\widehat{M}(\|\varphi_\tau\|_{1,p}^p) - \frac{\lambda C_1 a^{q+1}(\tau^{q+1}-1)}{q+1}.$$

Because M is a continuous function on $[0, \|\varphi_\tau\|_{1,p}^p]$,

$$\begin{aligned}I_{\lambda,a}(\varphi_\tau) &\le \frac{C_2}{p}\|\varphi_\tau\|_{1,p}^p - \frac{\lambda C_1 a^{q+1}(\tau^{q+1}-1)}{q+1} \\ &\le \frac{C_2}{p} 2^p a^p \tau^p \alpha_N - \frac{\lambda C_1 a^{q+1}(\tau^{q+1}-1)}{q+1} \\ &= a^{q+1}\Big(\frac{C_2 2^p a^{p-(q+1)}\tau^p \alpha_N}{p} - \frac{\lambda C_1(\tau^{q+1}-1)}{q+1}\Big).\end{aligned}$$

We now point out that

$$\frac{C_2 2^p a^{p-(q+1)}\tau^p\alpha_N}{p} - \frac{\lambda C_1(\tau^{q+1}-1)}{q+1} \le 0 \Leftrightarrow a^{p-(q+1)} \le \frac{\lambda C_1(\tau^{q+1}-1)p}{C_2 2^p (q+1)\alpha_N \tau^p}.$$

Setting

$$a_2 = \frac{1}{2}\frac{\lambda C_1(\tau^{q+1}-1)p}{C_2 2^p (q+1)\alpha_N\tau^p}$$

and taking $a^* = a^*(\lambda) = \min\{a_1, a_2\}$ it follows that $\inf_{B_r} I_{\lambda,a} < 0$ for $a \in (0, a^*)$, which proves the claim.

By the Ekeland Variational Principle, there exists $u_\epsilon \in B_r$ such that

$$I_{\lambda,a}(u_\epsilon) < \inf_{B_r} I_{\lambda,a} + \epsilon, \tag{4.6}$$

$$I_{\lambda,a}(u_\epsilon) < I_{\lambda,a}(u) + \epsilon\|u-u_\epsilon\|_{1,p}, \text{ for all } u \in W_0^{1,p}(\Omega), \text{ with } u \neq u_\epsilon. \tag{4.7}$$

Let us choose $\epsilon > 0$ in such a way that

$$0 < \epsilon < \inf_{\partial B_r} I_{\lambda,a} - \inf_{B_r} I_{\lambda,a}$$

and so $u_\epsilon \in B_r$.

Let $\gamma > 0$ be small enough and $v \in W_0^{1,p}(\Omega)$ with $\|v\|_{1,p} < 1$ so that $u_\gamma = u_\epsilon + \gamma v \in B_r$. From (4.7) we have

$$I_{\lambda,a}(u_\epsilon) < I_{\lambda,a}(u_\epsilon + \gamma v) + \epsilon\gamma\|v\|_{1,p}$$

which implies

$$I_{\lambda,a}(u_\epsilon + \gamma v) - I_{\lambda,a}(u_\epsilon) + \epsilon\gamma\|v\|_{1,p} \geq 0.$$

Consequently,

$$-\epsilon\|v\|_{1,p} \leq \frac{I_{\lambda,a}(u_\epsilon + \gamma v) - I_{\lambda,a}(u_\epsilon)}{\gamma}$$

and so

$$-\epsilon\|v\|_{1,p} \leq \limsup_{\gamma\to 0} \frac{I_{\lambda,a}(u_\epsilon + \gamma v) - I_{\lambda,a}(u_\epsilon)}{\gamma} \leq I^0_{\lambda,a}(u_\epsilon; v).$$

Now, since

$$I^0_{\lambda,a}(u;v) = \max_{\mu\in\partial I_{\lambda,a}(u_\epsilon)} \langle \mu, v\rangle, \quad \text{for all } u, v \in W_0^{1,p}(\Omega),$$

it follows that

$$-\epsilon\|v\|_{1,p} \leq I^0_{\lambda,a}(u_\epsilon; v) = \max_{\omega\in\partial I_{\lambda,a}(u_\epsilon)} \langle \omega, v\rangle.$$

Interchanging v and $-v$ we obtain

$$-\epsilon\|v\|_{1,p} \leq \max_{\omega\in\partial I_{\lambda,a}(u_\epsilon)} \langle \omega, -v\rangle = -\min_{\omega\in\partial I_{\lambda,a}(u_\epsilon)} \langle \omega, v\rangle.$$

Therefore,

$$\min_{\omega\in\partial I_{\lambda,a}(u_\epsilon)} \langle \omega, v\rangle \leq \epsilon\|v\|_{1,p}, \quad \text{for all } v \in W_0^{1,p}(\Omega),$$

concluding that

$$\sup_{\|v\|_{1,p}<1} \min_{\omega\in\partial I_{\lambda,a}(u_\epsilon)} \langle \omega, v\rangle \leq \epsilon.$$

By Fan's Min-max theorem, we obtain

$$\min_{\omega\in\partial I_{\lambda,a}(u_\epsilon)} \sup_{\|v\|_{1,p}<1} \langle \omega, v\rangle \leq \epsilon.$$

Which along with (4.6) yields the existence of $u_n \in B_r$ such that

$$I_{\lambda,a}(u_n) \to \widetilde{c}, \quad m(u_n) = \min_{\omega\in\partial I_{\lambda,a}(u_n)} \|\omega\|_* \to 0;$$

that is, (u_n) is a Palais-Smale sequence at the level $\widetilde{c}$.

By lemma 3.4, there exists $u_2 \in W_0^{1,p}(\Omega)$, where, passing to a subsequence if necessary, we obtain

$$u_n \to u_2 \quad \text{in } W_0^{1,p}(\Omega), \tag{4.8}$$

$$I_{\lambda,a}(u_2) = \widetilde{c} = \inf_{B_r(0)} I_{\lambda,a} < 0. \tag{4.9}$$

Thus, u_2 is a local minimum point and consequently is a critical point of $I_{\lambda,a}$. Hence, following the same arguments made before, we have that u_2 is also solution of the problem (1.2) and using the same arguments used in (i) with u_1 solution, we obtain also u_2 satisfy (i).

Proof of (ii): By (4.1) and (4.9), it follows that $I_{\lambda,a}(u_2) < 0 < I_{\lambda,a}(u_1)$.

Proof of (iii): We will show now that $|\{x \in \Omega; u_i(x) > a\}| > 0$, $i = 1, 2$. We begin with the solution u_1 obtained via the Mountain Pass Theorem. Suppose, by contradiction, that $u_1(x) \le a$ a.e, in Ω. So

$$\lambda \int_\Omega \int_0^{u_1} H(t-a)(t^+)^q = 0.$$

By the above equality and since u_1 is critical point of $I_{\lambda,a}$, we obtain

$$[M(\|u_1\|_{1,p}^p)]^{p-1}\|u_1\|_{1,p}^p = \int_\Omega h u_1^{s+1}.$$

Note that

$$\int_\Omega h u_1^{s+1} \le |h|_\infty \int_\Omega u_1^p u_1^{s+1-p}$$

and thus

$$[M(\|u_1\|_{1,p}^p)]^{p-1}\|u_1\|_{1,p}^p \le |h|_\infty \int_\Omega u_1^p u_1^{s+1-p} \le C|h|_\infty a^{s+1-p}\|u_1\|_{1,p}^p$$

which implies

$$[M(\|u_1\|_{1,p}^p)]^{p-1} \le C|h|_\infty a^{s+1-p}.$$

Note that, there exists $C > 0$ such that $\|u_1\| \ge C$. Hence, using (H4); that is, $M(t) > 0$ for all $t \ge 0$, there exists $\widetilde{C} > 0$ such that

$$0 < \widetilde{C} \le [M(\|u_1\|_{1,p}^p)]^{p-1} \le C|h|_\infty a^{s+1-p},$$

for all $a > 0$, which is impossible.

We now consider the solution u_2 obtained via Ekeland Variational Principle. Suppose, by contradiction, that $u_2(x) \le a$ a.e. in Ω. Thus

$$\lambda \int_\Omega \int_0^{u_2} H(t-a)(t^+)q = 0.$$

By the above equality and since u_2 is a critical point of $I_{\lambda,a}$, we obtain

$$[M(\|u_2\|_{1,p}^p)]^{p-1}\|u_2\|_{1,p}^p = \int_\Omega h u_2^{s+1}.$$

We will consider two cases:

Case (1): If $0 < \|u_2\|_{1,p}^p \le t_1$, from (H_1) we have

$$M(\|u_2\|_{1,p}^p) \ge m_1 > 0. \tag{4.10}$$

Hence,

$$[M(\|u_2\|_{1,p}^p)]^{p-1}\|u_2\|_{1,p}^p \le |h|_\infty \int_\Omega u_2^p u_2^{s+1-p} \le C|h|_\infty a^{s+1-p}\|u_2\|_{1,p}^p;$$

that is,

$$[M(\|u_2\|_{1,p}^p)]^{p-1} \le C|h|_\infty a^{s+1-p}.$$

So $[M(\|u_2\|_{1,p}^p)] \to 0$ as $a \to 0$ which cannot happen from (4.10).

Case (2): If $\|u_2\|_{1,p}^p \ge t_1$, then $\widehat{M}(\|u_2\|_{1,p}^p) \ge \widehat{M}(t_1) > 0$, because $\widehat{M}$ is increasing. Moreover,

$$0 \le [M(\|u_2\|_{1,p}^p)]^{p-1}\|u_2\|_{1,p}^p = \int_\Omega h u_2^{s+1} \le |h|_\infty a^{s+1}.$$

Thus $[M(\|u_2\|_{1,p}^p)]^{p-1}\|u_2\|_{1,p}^p \to 0$ as $a \to 0$. Since

$$I_{\lambda,a}(u_2) = \widehat{M}(\|u_2\|_{1,p}^p) - \frac{1}{s+1}\int_\Omega h u_2^{s+1} \geq \widehat{M}(t_1) - [M(\|u_2\|_{1,p}^p)]^{p-1}\|u_2\|_{1,p}^p,$$

we obtain

$$0 < \widehat{M}(t_1) \leq I_{\lambda,a}(u_2) + [M(\|u_2\|_{1,p}^p)]^{p-1}\|u_2\|_{1,p}^p.$$

Because $I_{\lambda,a}(u_2) < 0$, we obtain

$$0 < \widehat{M}(t_1) \leq [M(\|u_2\|_{1,p}^p)]^{p-1}\|u_2\|_{1,p}^p.$$

Hence, as $a \to 0$, we have $\widehat{M}(t_1) = 0$ for $t_1 > 0$, which is an absurd. With this, we conclude the proof of the theorem.

References

[1] C .O. Alves, A. M. Bertone; *A discontinuous problem involving the p-Laplacian*, Electron. J. Differential Equations, vol 2003, no. 42 (2003), 1-10.

[2] C. O. Alves, A .M. Bertone, J. V. Gonçalves; *A variational approach to discontinuous problem with critical Sobolev exponents*, J. Math. Anal. Appl., 265 (2002), 103-127.

[3] C. O. Alves, F. J. S .A. Correa, T. F. Ma; *Positive solutions for a quasilinear elliptic equation of Kirchhoff type*, Comput. Math. Appl., 49 (2005)85-93.

[4] C. O. Alves, R. G. Nascimento; *Existence and concentration of solutions for a class of elliptic problem with discontinuous nonlinearity in* $\mathbb{R}^N$, Math. Scandinavica, to appear.

[5] A. Ambrosetti, M. Calahorrano, F. Dobarro; *Global branching for discontinuous problems*, Comment. Math. Univ. Carolin., 31 (1990), 213-222.

[6] A. Ambrosetti, R. Turner; *Some discontinuous variational problems*, Differential Integral Equations, 1, N. 3 (1998), 341-349.

[7] D. Arcoya, M. Calahorrano; *Some discontinuous variational problems with a quasilinear operator*, J. Math. Anal. Appl., 187(1994), 1059-1072.

[8] D. Arcoya, J. I. Diaz, L. Tello; *S-Shaped bifurcation branch in a quasilinear multivalued model arising in climatology*, J. Differential Equations, 150(1998), 215-225.

[9] M. Badiale; *Critical exponent and discontinuous nonlinearities*, Differential Integral Equations, 6(1993), 1173-1185.

[10] M. Badiale; *Some remarks on elliptic problems with discontinuous nonlinearities*, Rend. Sem. Mat. Univ. Politec. Torino, 51(1993)331-342.

[11] K. C. Chang; *Variational methods for nondifferentiable functionals and their applications to partial differential equations*, J. Math. Anal. Appl., 80 (1981), 102-129.

[12] F. H. Clarke; *Optimization and nonsmooth analysis*, John Wiley & Sons, N.Y. (1983).

[13] F. J. S .A. Correa, D. B. Menezes; *Positive solutions for a class of nonlocal elliptic problems*, 66 (2006), 195-206.

[14] F. J. S. A. Correa, G. M. Figueiredo; *On an elliptic equation of p-Kirchhoff type via variational methods* Bull. Austral. Math. Soc.,72(2) (2006)263-277.

[15] F.J.S.A. Correa & G. M. Figueiredo; *On the existence of positive solution for an elliptic equation of Kirchhoff type via Moser iteration method,* Bound. Value Probl., pages Art. ID 79679, 10 (2006).

[16] F. J. S. A. Correa, G. M. Figueiredo; *On a p-Kirchhoff equation via Krasnoselskii's genus*, Appl. Math. Lett., 22(6) (2009)819-822. Bull. Austral. Math. Soc.,72(2) (2006)263-277.

[17] F. J. S .A. Correa, R. G. Nascimento; *On the existence of solutions of a nonlocal elliptic equation with a p-Kirchhoff-type term*, Int. J. Math. Math. Sci., pages Art. ID 364085, 25 (2008).

[18] M .R. Grossinho, S. A. Tersian; *An introduction to minimax theorems and their applications to differential equations*, Kluwer Academic Press, 2001.

[19] G. Kirchhoff; *Mechanik*, Teubner, Leipzig, 1883.

[20] T. F. Ma; *Remarks on an elliptic equation of Kirchhoff type*, Nonlinear Anal., Vol.63 (2005)1967-1977.

[21] A. Mao, Z. Zhang; *Sign-changing and multiple solutions of Kirchhoff type problems without the P.S. condition*, Nonlinear Anal., Vol.70 (2009), 1275-1287.

[22] C. B. Morrey; *Multiple integrals in calculus of variations*, Springer-Verlag, Berlin (1966).
[23] I. Peral; *Multiplicity of solutions for the p-Laplacian*, Notes of the Second International School in Functional Abalysis and Applications to Differential Equations, ICTP-Trieste, 1997
[24] K. Perera, Zhang; *Nontrivial solutions of Kirchhoff-type problems via the Yang index*, J. Differential Equations, 221, N.1 (2006), 246-255.
[25] B. Ricceri; *On a elliptic Kirchhoff-type problem depending on two parameters*, J. Glob. Opt., 46 (2010)543-549.

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