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NON-TRIVIAL SOLUTIONS FOR TWO-POINT BOUNDARY-VALUE PROBLEMS OF FOURTH-ORDER STURM-LIOUVILLE TYPE EQUATIONS

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Abstract. Using critical point theory due to Bonanno [3], we prove the existence of at least one non-trivial solution for a class of two-point boundary-value problems for fourth-order Sturm-Liouville type equations.

1. Introduction

In this note, we prove the existence of at least one non-trivial solution for the two-point boundary-value problem of fourth-order Sturm-Liouville type:

$$\begin{gathered}(p_i(x)u_i''(x))'' - (q_i(x)u_i'(x))' + r_i(x)u_i(x) = \lambda F_{u_i}(x, u_1, \dots, u_n) \quad x \in (0,1), \\ u_i(0) = u_i(1) = u_i''(0) = u_i''(1) = 0 \end{gathered} \tag{1.1}$$

for $1 \le i \le n$, where $n \ge 1$ is an integer, p_i, q_i, $r_i \in L^\infty([0,1])$ with $p_i^- := \operatorname{ess\,inf}_{x\in[0,1]} p_i(x) > 0$ for $1 \le i \le n$, λ is a positive parameter, $F : [0,1] \times \mathbb{R}^n \to \mathbb{R}$ is a function such that $F(., t_1, \dots, t_n)$ is measurable in $[0,1]$ for all $(t_1, \dots, t_n) \in \mathbb{R}^n$, $F(x, ., \dots, .)$ is C^1 in $\mathbb{R}^n$ for every $x \in [0,1]$ and for every $\varrho > 0$,

$$\sup_{|(t_1,\dots,t_n)| \le \varrho} \sum_{i=1}^{n} |F_{t_i}(x, t_1, \dots, t_n)| \in L^1([0,1]),$$

and F_{u_i} denotes the partial derivative of F with respect to u_i for $1 \le i \le n$.

Due to importance of fourth-order two-point boundary-value problems in describing a large class of elastic deflection, many authors have studied the existence and multiplicity of solutions for such a problem; we refer the reader to [1, 2, 4, 5, 6, 7, 11, 14, 17] and references therein.

In [4], the authors, employing a three critical point theorem due to Bonanno and Marano [8, Theorem 2.6], determined an exact open interval of the parameter λ for which system (1) in the case $n = 1$, admits at least three distinct weak solutions.

The aim of this article is to prove the existence of at least one non-trivial weak solution for (1) for appropriate values of the parameter λ belonging to a precise real interval, which extend the results in [7]. Our motivation comes from the recent

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paper [5]. For basic notation and definitions, and also for a thorough account on the subject, we refer the reader to [9, 12, 13].

2. Preliminaries and basic notation

First we recall for the reader's convenience [16, Theorem 2.5] as given in [3, Theorem 5.1] (see also [3, Proposition 2.1] for related results) which is our main tool to transfer the question of existence of at least one weak solution of (1) to the existence of a critical point of the Euler functional:

For a given non-empty set X, and two functionals $\Phi, \Psi : X \to \mathbb{R}$, we define the following two functions:

$$\beta(r_1, r_2) = \inf_{v\in\Phi^{-1}(]r_1,r_2[)} \frac{\sup_{u\in\Phi^{-1}(]r_1,r_2[)} \Psi(u) - \Psi(v)}{r_2 - \Phi(v)},$$

$$\rho(r_1, r_2) = \sup_{v\in\Phi^{-1}(]r_1,r_2[)} \frac{\Psi(v) - \sup_{u\in\Phi^{-1}(]-\infty,r_1[)} \Psi(u)}{\Phi(v) - r_1}$$

for all $r_1, r_2 \in \mathbb{R}$, $r_1 < r_2$.

Theorem 2.1 ([3, Theorem 5.1])**.** *Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous, coercive and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on X^* and $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Put $I_\lambda = \Phi - \lambda\Psi$ and assume that there are $r_1, r_2 \in \mathbb{R}$, $r_1 < r_2$, such that*

$$\beta(r_1, r_2) < \rho(r_1, r_2).$$

Then, for each $\lambda \in]\frac{1}{\rho(r_1,r_2)}, \frac{1}{\beta(r_1,r_2)}[$ there is $u_{0,\lambda} \in \Phi^{-1}(]r_1, r_2[)$ such that $I_\lambda(u_{0,\lambda}) \le I_\lambda(u)$ $\forall u \in \Phi^{-1}(]r_1, r_2[)$ and $I'_\lambda(u_{0,\lambda}) = 0$.

Let us introduce notation that will be used later. Assume that

$$\min\Big\{\frac{q_i^-}{\pi^2}, \frac{r_i^-}{\pi^4}, \frac{q_i^-}{\pi^2} + \frac{r_i^-}{\pi^4}\Big\} > -p_i^-, \tag{2.1}$$

where

$$p_i^- := \operatorname{ess\,inf}_{x\in[0,1]} p_i(x) > 0, \quad q_i^- := \operatorname{ess\,inf}_{x\in[0,1]} q_i(x), \quad r_i^- := \operatorname{ess\,inf}_{x\in[0,1]} r_i(x),$$

for $1 \le i \le n$. Moreover, set

$$\sigma_i := \min\Big\{\frac{q_i^-}{\pi^2}, \frac{r_i^-}{\pi^4}, \frac{q_i^-}{\pi^2} + \frac{r_i^-}{\pi^4}, 0\Big\}, \quad \delta_i := \sqrt{p_i^- + \sigma_i},$$

for $1 \le i \le n$. Let $Y := H^2([0,1]) \cap H_0^1([0,1])$ be the Sobolev space endowed with the usual norm. We recall the following Poincaré type inequalities (see, for instance, [15, Lemma 2.3]):

$$\|u_i'\|^2_{L^2([0,1])} \le \frac{1}{\pi^2}\|u_i''\|^2_{L^2([0,1])}, \tag{2.2}$$

$$\|u_i\|^2_{L^2([0,1])} \le \frac{1}{\pi^4}\|u_i''\|^2_{L^2([0,1])} \tag{2.3}$$

for all $u_i \in Y$ for $1 \le i \le n$. Therefore, taking into account (2.1)-(2.3), the norm

$$\|u_i\| = \Big(\int_0^1 (p_i(x)|u_i''(x)|^2 + q_i(x)|u_i'(x)|^2 + r_i(x)|u_i(x)|^2)dx\Big)^{1/2}$$

for $1 \le i \le n$ is equivalent to the usual norm and, in particular, one has

$$\|u_i''\|_{L^2([0,1])} \le \frac{1}{\delta_i}\|u_i\| \tag{2.4}$$

for $1 \le i \le n$. We need the following proposition in the proof of Theorem 3.1.

Proposition 2.2 ([4, Proposition 2.1]). *Let $u_i \in Y$ for $1 \le i \le n$. Then*

$$\|u_i\|_\infty \le \frac{1}{2\pi\delta_i}\|u_i\|$$

for $1 \le i \le n$.

Put

$$k_i := \Big(\|p_i\|_\infty + \frac{1}{\pi^2}\|q_i\|_\infty + \frac{1}{\pi^4}\|r_i\|_\infty\Big)^{1/2}$$

for $1 \le i \le n$. It is easy to see that $k_i > 0$ and $\delta_i < k_i$ for $1 \le i \le n$. Set $\underline{\delta} := \min\{\delta_i;\ 1 \le i \le n\}$ and $\overline{k} := \max\{k_i;\ 1 \le i \le n\}$. Here and in the sequel, $X := Y \times \dots \times Y$.

We say that $u = (u_1, \dots, u_n)$ is a weak solution to the (1) if $u = (u_1, \dots, u_n) \in X$ and

$$\begin{aligned}&\sum_{i=1}^n \int_0^1 (p_i(x)u_i''(x)v_i''(x) + q_i(x)u_i'(x)v_i'(x) + r_i(x)u_i(x)v_i(x))\,dx\\ &- \lambda \sum_{i=1}^n \int_0^1 F_{u_i}(x, u_1, \dots, u_n)v_i(x)dx = 0\end{aligned}$$

for every $v = (v_1, \dots, v_n) \in X$. For $\gamma > 0$ we denote the set

$$K(\gamma) = \big\{(t_1, \dots, t_n) \in \mathbb{R}^n :\ \sum_{i=1}^n |t_i| \le \gamma\big\}. \tag{2.5}$$

3. Results

For a given non-negative constant ν and a positive constant τ with $2(\frac{\underline{\delta}\pi\nu}{n})^2 \neq \frac{4096}{54}n(\overline{k}\tau)^2$, put

$$a_\tau(\nu) := \frac{\int_0^1 \sup_{(t_1,\dots,t_n)\in K(\nu)} F(x, t_1, \dots, t_n)dx - \int_{\frac{3}{8}}^{\frac{5}{8}} F(x, \tau, \dots, \tau)dx}{2(\frac{\underline{\delta}\pi\nu}{n})^2 - \frac{4096}{54}n(\overline{k}\tau)^2}$$

where $K(\nu) = \big\{(t_1, \dots, t_n) \in \mathbb{R}^n : \sum_{i=1}^n |t_i| \le \nu\big\}$ (see (2.5)).

We formulate our main result as follows:

Theorem 3.1. *Assume that there exist a non-negative constant ν_1 and two positive constants ν_2 and τ with $\pi\nu_1 < n\sqrt{\frac{4096}{108}n}\tau$ and $n\sqrt{\frac{4096}{108}n}\overline{k}\tau < \underline{\delta}\pi\nu_2$ such that*

- (A1) *$F(x, t_1, \dots, t_n) \ge 0$ for each $(x, t_1, \dots, t_n) \in ([0, 3/8] \cup [5/8, 1]) \times [0, \tau]^n$;*
- (A2) *$a_\tau(\nu_2) < a_\tau(\nu_1)$.*

Then, for each $\lambda \in]\frac{1}{a_\tau(\nu_1)},\ \frac{1}{a_\tau(\nu_2)}[$, system (1) admits at least one non-trivial weak solution $u_0 = (u_{01}, \dots, u_{0n}) \in X$ such that

$$4(\frac{\underline{\delta}\pi\nu_1}{n})^2 < \sum_{i=1}^n \|u_{0i}\|^2 < 4(\frac{\underline{\delta}\pi\nu_2}{n})^2.$$

Proof. To apply Theorem 2.1 to our problem, arguing as in [7, 10], we introduce the functionals $\Phi, \Psi : X \to \mathbb{R}$ for each $u = (u_1, \dots, u_n) \in X$, as follows
$$\Phi(u) = \sum_{i=1}^{n} \frac{\|u_i\|^2}{2},$$
$$\Psi(u) = \int_0^1 F(x, u_1(x), \dots, u_n(x))dx.$$
It is well known that Φ and Ψ are well defined and continuously differentiable functionals whose derivatives at the point $u = (u_1, \dots, u_n) \in X$ are the functionals $\Phi'(u), \Psi'(u) \in X^*$, given by
$$\Phi'(u)(v) = \sum_{i=1}^{n} \int_0^1 \left(p_i(x)u_i''(x)v_i''(x) + q_i(x)u_i'(x)v_i'(x) + r_i(x)u_i(x)v_i(x)\right) dx,$$
$$\Psi'(u)(v) = \int_0^1 \sum_{i=1}^{n} F_{u_i}(x, u_1(x), \dots, u_n(x))v_i(x)dx$$
for every $v = (v_1, \dots, v_n) \in X$, respectively. Moreover, Φ is sequentially weakly lower semicontinuous, Φ' admits a continuous inverse on X^* as well as Ψ is sequentially weakly upper semicontinuous. Furthermore, $\Psi' : X \to X^*$ is a compact operator. Indeed, it is enough to show that Ψ' is strongly continuous on X. For this, for fixed $(u_1, \dots, u_n) \in X$ let $(u_{1m}, \dots, u_{nm}) \to (u_1, \dots, u_n)$ weakly in X as $m \to +\infty$, then we have $(u_{1m}, \dots, u_{nm})$ converges uniformly to $(u_1, \dots, u_n)$ on $[0, 1]$ as $m \to +\infty$(see [18]). Since $F(x, ., \dots, .)$ is C^1 in $\mathbb{R}^n$ for every $x \in [0, 1]$, the derivatives of F are continuous in $\mathbb{R}^n$ for every $x \in [0, 1]$, so for $1 \le i \le n$, $F_{u_i}(x, u_{1m}, \dots, u_{nm}) \to F_{u_i}(x, u_1, \dots, u_n)$ strongly as $m \to +\infty$ which follows $\Psi'(u_{1m}, \dots, u_{nm}) \to \Psi'(u_1, \dots, u_n)$ strongly as $m \to +\infty$. Thus we proved that Ψ' is strongly continuous on X, which implies that Ψ' is a compact operator by Proposition 26.2 of [18]. Set $w(x) = (w_1(x), \dots, w_n(x))$ such that for $1 \le i \le n$,
$$w_i(x) = \begin{cases} -\frac{64\tau}{9}(x^2 - \frac{3}{4}x) & x \in [0, \frac{3}{8}[, \\ \tau & x \in [\frac{3}{8}, \frac{5}{8}], \\ -\frac{64\tau}{9}(x^2 - \frac{5}{4}x + \frac{1}{4}) & x \in]\frac{5}{8}, 1], \end{cases}$$
$r_1 = 2(\frac{\delta\pi\nu_1}{n})^2$ and $r_2 = 2(\frac{\delta\pi\nu_2}{n})^2$. It is easy to verify that $w = (w_1, \dots, w_n) \in X$, and in particular, one has
$$\frac{4096}{27}\delta_i^2\tau^2 \le \|w_i\|^2 \le \frac{4096}{27}k_i^2\tau^2$$
for $1 \le i \le n$. So, from the definition of Φ, we have
$$\frac{4096}{54}n(\underline{\delta}\tau)^2 \le \sum_{i=1}^{n} \frac{4096}{27}\delta_i^2\tau^2 \le \Phi(w) \le \sum_{i=1}^{n} \frac{4096}{27}k_i^2\tau^2 \le \frac{4096}{54}n(\overline{k}\tau)^2.$$
From the conditions $\pi\nu_1 < n\sqrt{\frac{4096}{108}n}\tau$ and $n\sqrt{\frac{4096}{108}n}\overline{k}\tau < \underline{\delta}\pi\nu_2$, we obtain
$$r_1 < \Phi(w) < r_2.$$
Moreover, from Proposition 2.2 one has
$$\sup_{x \in [0,1]} \sum_{i=1}^{n} |u_i(x)|^2 \le \frac{1}{(2\pi\underline{\delta})^2} \sum_{i=1}^{n} \|u_i\|^2$$

for each $u = (u_1, \dots, u_n) \in X$, so from the definition of Φ, we observe that

$$\begin{aligned}\Phi^{-1}(]-\infty, r_2[) &= \{(u_1, \dots, u_n) \in X; \Phi(u_1, \dots, u_n) < r_2\} \\ &= \Big\{(u_1, \dots, u_n) \in X; \sum_{i=1}^n \|u_i\|^2 < 2r_2\Big\} \\ &\subseteq \Big\{(u_1, \dots, u_n) \in X; \sum_{i=1}^n |u_i(x)|^2 < \frac{\nu_2^2}{n^2} \ \text{ for all } x \in [0,1]\Big\} \\ &\subseteq \Big\{(u_1, \dots, u_n) \in X; \sum_{i=1}^n |u_i(x)| < \nu_2 \ \text{ for all } x \in [0,1]\Big\},\end{aligned}$$

from which it follows

$$\begin{aligned}\sup_{(u_1,\dots,u_n)\in\Phi^{-1}(]-\infty,r_2[)} \Psi(u) &= \sup_{(u_1,\dots,u_n)\in\Phi^{-1}(]-\infty,r_2[)} \int_0^1 F(x, u_1(x), \dots, u_n(x))dx \\ &\le \int_0^1 \sup_{(t_1,\dots,t_n)\in K(\nu_2)} F(x, t_1, \dots, t_n)dx.\end{aligned}$$

Since for $1 \le i \le n$, $0 \le w_i(x) \le \tau$ for each $x \in [0,1]$, the condition (A1) ensures that

$$\int_0^{\frac{3}{8}} F(x, w_1(x), \dots, w_n(x))dx + \int_{\frac{5}{8}}^1 F(x, w_1(x), \dots, w_n(x))dx \ge 0.$$

So, one has

$$\begin{aligned}\beta(r_1, r_2) &\le \frac{\sup_{u\in\Phi^{-1}(]-\infty,r_2[)} \Psi(u) - \Psi(w)}{r_2 - \Phi(w)} \\ &\le \frac{\int_0^1 \sup_{(t_1,\dots,t_n)\in K(\nu_2)} F(x, t_1, \dots, t_n)dx - \Psi(w)}{r_2 - \Phi(w)} \\ &\le a_\tau(\nu_2).\end{aligned}$$

On the other hand, by similar reasoning as before, one has

$$\begin{aligned}\rho(r_1, r_2) &\ge \frac{\Psi(w) - \sup_{u\in\Phi^{-1}(]-\infty,r_1[)} \Psi(u)}{\Phi(w) - r_1} \\ &\ge \frac{\Psi(w) - \int_0^1 \sup_{(t_1,\dots,t_n)\in K(\nu_1)} F(x, t_1, \dots, t_n)dx}{\Phi(w) - r_1} \\ &\ge a_\tau(\nu_1).\end{aligned}$$

Hence, from Assumption (A2), one has $\beta(r_1, r_2) < \rho(r_1, r_2)$. Therefore, from Theorem 2.1, taking into account that the weak solutions of the system (1) are exactly the solutions of the equation $\Phi'(u) - \lambda\Psi'(u) = 0$, we have the conclusion. □

Now we point out the following consequence of Theorem 3.1.

Theorem 3.2. *Assume that Assumption (A1) in Theorem 3.1 holds. Suppose that there exist two positive constants ν and τ with $n\sqrt{\frac{4096}{108} n\overline{k}}\tau < \underline{\delta}\pi\nu$ such that*

(A3)

$$\frac{\int_0^1 \sup_{(t_1,\dots,t_n)\in K(\nu)} F(x,t_1,\dots,t_n)dx}{\nu^2} < \frac{108}{4096n^3}(\frac{\underline{\delta}\pi}{\overline{k}})^2 \frac{\int_{\frac{3}{8}}^{\frac{5}{8}} F(x,\tau,\dots,\tau)dx}{\tau^2};$$

(A4) $F(x,0,\dots,0) = 0$ *for every* $x \in [0,1]$

Then, for each

$$\lambda \in \Big] \frac{\frac{4096}{54} n(\overline{k}\tau)^2}{\int_{\frac{3}{8}}^{\frac{5}{8}} F(x,\tau,\dots,\tau)dx}, \frac{2(\frac{\underline{\delta}\pi\nu}{n})^2}{\int_0^1 \sup_{(t_1,\dots,t_n)\in K(\nu)} F(x,t_1,\dots,t_n)dx} \Big[,$$

system (1) *admits at least one non-trivial weak solution* $u_0 = (u_{01},\dots,u_{0n}) \in X$ *such that* $\sum_{i=1}^n \|u_i\|_\infty < \nu$.

Proof. The conclusion follows from Theorem 3.1, by taking $\nu_1 = 0$ and $\nu_2 = \nu$. Indeed, owing to our assumptions, one has

$$\begin{aligned} a_\tau(\nu_2) &< \frac{\Big(1 - \frac{\frac{4096n^3}{108}(\frac{\tau\overline{k}}{\underline{\delta}\pi})^2}{\nu^2}\Big) \int_0^1 \sup_{(t_1,\dots,t_n)\in K(\nu)} F(x,t_1,\dots,t_n)dx}{2(\frac{\underline{\delta}\pi\nu}{n})^2 - \frac{4096}{54} n(\overline{k}\tau)^2} \\ &= \frac{\Big(1 - \frac{\frac{4096}{54} n(\tau\overline{k})^2}{2(\frac{\underline{\delta}\pi\nu}{n})^2}\Big) \int_0^1 \sup_{(t_1,\dots,t_n)\in K(\nu)} F(x,t_1,\dots,t_n)dx}{2(\frac{\underline{\delta}\pi\nu}{n})^2 - \frac{4096}{54} n(\overline{k}\tau)^2} \\ &= \frac{\int_0^1 \sup_{(t_1,\dots,t_n)\in K(\nu)} F(x,t_1,\dots,t_n)dx}{2(\frac{\underline{\delta}\pi\nu}{n})^2}. \end{aligned}$$

On the other hand, taking Assumption (A4) into account, one has

$$\frac{\int_{\frac{3}{8}}^{\frac{5}{8}} F(x,\tau,\dots,\tau)dx}{\frac{4096}{54} n(\overline{k}\tau)^2} = a_\tau(\nu_1).$$

Moreover, since

$$\sup_{x\in[0,1]} \sum_{i=1}^n |u_i(x)|^2 \le \frac{1}{(2\pi\underline{\delta})^2} \sum_{i=1}^n \|u_i\|^2$$

for each $u = (u_1,\dots,u_n) \in X$, an easy computation ensures that $\sum_{i=1}^n \|u_i\|_\infty < \nu$ whenever $\Phi(u) < r_2$. Now, owing to Assumption (A3), it is sufficient to invoke Theorem 3.1 for concluding the proof. □

Now, we point out a simple version of Theorem 3.1 in the case $n = 1$. Let $p_1 = p$, $q_1 = q$, $r_1 = r$, $\delta_1 = \delta$ and $k_1 = k$. Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be an L^2-Carathéodory function. Let F be the function defined by $F(x,t) = \int_0^t f(x,s)ds$ for each $(x,t) \in [0,1] \times \mathbb{R}$. For a given non-negative constant ν and a positive constant τ with $2(\delta\pi\nu)^2 \neq \frac{4096}{54}(k\tau)^2$, put

$$b_\tau(\nu) := \frac{\int_0^1 \sup_{|t|\le\nu} F(x,t)dx - \int_{\frac{3}{8}}^{\frac{5}{8}} F(x,\tau)dx}{2(\delta\pi\nu)^2 - \frac{4096}{54}(k\tau)^2}.$$

Then, we have the following result.

Theorem 3.3. *Assume that there exist a non-negative constant* ν_1 *and two positive constants* ν_2 *and* τ *with* $\pi\nu_1 < \sqrt{\frac{4096}{108}}\tau$ *and* $\sqrt{\frac{4096}{108}}k\tau < \delta\pi\nu_2$ *such that*

(B1) $F(x,t) \geq 0$ *for each* $(x,t) \in ([0,3/8] \cup [5/8,1]) \times [0,\tau]$;;
(B2) $b_\tau(\nu_2) < b_\tau(\nu_1)$.

Then, for each $\lambda \in]\frac{1}{b_\tau(\nu_1)},\ \frac{1}{b_\tau(\nu_2)}[$, *the problem*

$$\begin{gathered} (p(x)u''(x))'' - (q(x)u'(x))' + r(x)u(x) = \lambda f(x,u) \quad x \in (0,1), \\ u(0) = u(1) = u''(0) = u''(1) = 0 \end{gathered} \tag{3.1}$$

admits at least one non-trivial weak solution $u_0 \in Y$ *such that* $2\delta\pi\nu_1 < \|u_0\| < 2\delta\pi\nu_2$.

We remark that if $p(x) = 1$, $q(x) = -A$ and $r(x) = B$ for every $x \in [0,1]$, then Theorem 3.3 gives [7, Theorem 3.1].

The following result gives the existence of at least one non-trivial weak solution in Y to the problem (3.1) in the autonomous case. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Put $F(t) = \int_0^t f(\xi)d\xi$ for all $t \in \mathbb{R}$. We have the following result as a direct consequence of Theorem 3.3.

Theorem 3.4. *Assume that there exist a non-negative constant* ν_1 *and two positive constants* ν_2 *and* τ *with* $\pi\nu_1 < \sqrt{\frac{4096}{108}}\tau$ *and* $\sqrt{\frac{4096}{108}}k\tau < \delta\pi\nu_2$ *such that*

(C1) $f(t) \geq 0$ *for each* $t \in [-\nu_2,\ \max\{\nu_2,\tau\}]$;
(C2) $\frac{F(\nu_2)-\frac{1}{4}F(\tau)}{2(\delta\pi\nu_2)^2-\frac{4096}{54}(k\tau)^2} < \frac{F(\nu_1)-\frac{1}{4}F(\tau)}{2(\delta\pi\nu_1)^2-\frac{4096}{54}(k\tau)^2}$.

Then, for each $\lambda \in]\frac{2(\delta\pi\nu_1)^2-\frac{4096}{54}(k\tau)^2}{F(\nu_1)-\frac{1}{4}F(\tau)},\ \frac{2(\delta\pi\nu_2)^2-\frac{4096}{54}(k\tau)^2}{F(\nu_2)-\frac{1}{4}F(\tau)}[$, *the problem*

$$\begin{gathered} (p(x)u''(x))'' - (q(x)u'(x))' + r(x)u(x) = \lambda f(u) \quad x \in (0,1), \\ u(0) = u(1) = u''(0) = u''(1) = 0 \end{gathered} \tag{3.2}$$

admits at least one non-trivial weak solution $u_0 \in Y$ *such that* $\nu_1 < \frac{\|u_0\|}{2\delta\pi} < \nu_2$.

Proof. Since $\delta < k$, from the conditions $\pi\nu_1 < \sqrt{\frac{4096}{108}}\tau$ and $\sqrt{\frac{4096}{108}}k\tau < \delta\pi\nu_2$, we obtain $2(\delta\pi\nu_1)^2 < \frac{4096}{54}(\delta\tau)^2$ and $\frac{4096}{54}(k\tau)^2 < 2(\delta\pi\nu_2)^2$, and so $2(\delta\pi\nu_1)^2 < 2(\delta\pi\nu_2)^2$, and so $\nu_1 < \nu_2$. Therefore, Assumptions (C1) means $f(t) \geq 0$ for each $t \in [-\nu_1,\ \nu_1]$ and $f(t) \geq 0$ for each $t \in [-\nu_2, \nu_2]$, which follow $\max_t \in [-\nu_1,\ \nu_1]F(t) = F(\nu_1)$ and $\max_t \in [-\nu_2,\ \nu_2]F(t) = F(\nu_2)$. So, from Assumptions (C1) and (C2) we arrive at assumptions (B1) and (B2), respectively. Hence, we achieve the stated assertion by applying Theorem 3.3 observing that the problem (1) reduces to the problem (3.2). $\square$

As an example, we point out the following special case of our main result.

Theorem 3.5. *Let* $g : \mathbb{R} \to \mathbb{R}$ *be a nonnegative continuous function such that* $\lim_{t\to 0^+} \frac{g(t)}{t} = +\infty$. *Then, for each* $\lambda \in]0, 2(\delta\pi)^2 \sup_{\nu>0} \frac{\nu^2}{\int_0^\nu g(\xi)d\xi}[$, *the problem*

$$\begin{gathered} (p(x)u''(x))'' - (q(x)u'(x))' + r(x)u(x) = \lambda g(u) \quad x \in (0,1), \\ u(0) = u(1) = u''(0) = u''(1) = 0 \end{gathered}$$

admits at least one non-trivial weak solution in Y.

Proof. For fixed λ as in the conclusion, there exists positive constant ν such that

$$\lambda < 2(\delta\pi)^2 \frac{\nu^2}{\int_0^\nu g(\xi)d\xi}.$$

Moreover, $\lim_{t\to 0^+}\frac{g(t)}{t}=+\infty$ implies $\lim_{t\to 0^+}\frac{\int_0^t g(\xi)d\xi}{t^2}=+\infty$. Therefore, a positive constant τ satisfying $\sqrt{\frac{4096}{108}}k\tau<\delta\pi\nu$ can be chosen such that

$$\frac{1}{\lambda}\Big(\frac{4\times 4096}{54}k^2\Big)<\frac{\int_0^\tau g(\xi)d\xi}{\tau^2}.$$

Hence, arguing as in the proof of Theorem 3.2, the conclusion follows from Theorem 3.4 with $\nu_1=0$, $\nu_2=\nu$ and $f(t)=g(t)$ for every $t\in\mathbb{R}$. □

Remark 3.6. For fixed ρ put $\lambda_\rho:=2(\delta\pi)^2\sup_{\nu\in]0,\rho[}\frac{\nu^2}{\int_0^\nu g(\xi)d\xi}$. The result of Theorem 3.5 for every $\lambda\in]0,\lambda_\rho[$ holds with $|u_0(x)|<\rho$ for all $x\in[0,1]$ where u_0 is the ensured non-trivial weak solution in Y (see [7, Remark 4.3]).

We close this article by presenting the following examples to illustrate our results.

example 3.7. Consider the problem

$$\begin{gathered}(3e^xu'')''-((x^2-\pi^2)u')'+(x^2-\pi^4)u=\lambda(1+e^{-u^+}(u^+)^2(3-u^+))\quad x\in(0,1),\\ u(0)=u(1)=0,\quad u''(0)=u''(1)=0\end{gathered}\tag{3.3}$$

where $u^+=\max\{u,0\}$. Let

$$g(t)=1+e^{-t^+}(t^+)^2(3-t^+)$$

for all $t\in\mathbb{R}$ where $t^+=\max\{t,0\}$. It is clear that $\lim_{t\to 0^+}\frac{g(t)}{t}=+\infty$. Note that $p^-=3$, $q^-=-\pi^2$ and $r^-=-\pi^4$, we have $\sigma=-2$, and so $\delta=1$. Hence, taking Remark 3.6 into account, since $\sup_{\nu\in]0,1[}\frac{\nu^2}{\int_0^\nu g(\xi)d\xi}=\sup_{\nu\in]0,1[}\frac{\nu^2}{\nu+e^{-\nu}\nu^3}=\frac{e}{1+e}$, by applying Theorem 3.5, for every $\lambda\in]0,\frac{2\pi^2e}{1+e}[$ the problem (3.3) has at least one non-trivial classical solution $u_0\in Y$ such that $\|u_0\|_\infty<1$.

example 3.8. Put $p(x)=1$, $q(x)=\pi^2$, $r(x)=x-\pi$ for all $x\in[0,1]$ and $g(t)=(1+t)e^t$ for every $t\in\mathbb{R}$. Clearly, one has $\sigma=(1-\frac{1}{\pi^3})^{\frac{1}{2}}$. Hence, since

$$\sup_{\nu\in]0,1[}\frac{\nu^2}{\int_0^\nu g(\xi)d\xi}=\sup_{\nu\in]0,1[}\frac{\nu^2}{\nu e^\nu}=\frac{1}{e},$$

from Theorem 3.5, taking Remark 3.6 into account, for every $\lambda\in]0,\frac{2\pi^2(1-\frac{1}{\pi^3})}{e}[$ the problem

$$\begin{gathered}u^{iv}-\pi^2u''+(x-\pi)u=\lambda(1+u)e^u\quad x\in(0,1),\\ u(0)=u(1)=0,\quad u''(0)=u''(1)=0\end{gathered}\tag{3.4}$$

has at least one non-trivial classical solution $u_0\in Y$ such that $\|u_0\|_\infty<1$.

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