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A NON-RESONANCE PROBLEM FOR NON-NEWTONIAN FLUIDS

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ABSTRACT. In this article we study a highly nonlinear problem which describes a non-Newtonian fluid in a specific domain (symmetric channel). This fluid is subjected to pressure of known differences between two parallel plates. We establish the existence and uniqueness of a weak solution. Our solution method is based on a minimization technique when the nonlinearity is asymptotically on the left of the first eigenvalue of the operator k-Laplacian.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with boundary $\partial\Omega = \Gamma = \cup_{i=1}^4 \overline{\Gamma_i}$, where $\Gamma_1 = \{0\}\times]-1,1[$, $\Gamma_2 = \{1\}\times]-1,1[$ and Γ_3, Γ_4 are symmetrical to the x-axis, see Figure (1). In the interior of this domain, a non-Newtonian fluid is subjected to pressures of known differences between the two sides Γ_1 and Γ_2.

We note by $u = (u_1, u_2)^T \in (C^2(\Omega) \cap C^1(\bar{\Omega}))^2$ and $-\vec{\Delta}_k u = (-\Delta_k u_1, -\Delta_k u_2)^T$, where $-\Delta_k u_i = -div(|\nabla u_i|^{k-2}\nabla u_i)$ is the operator k-Laplacian $i = 1, 2$ and $1 < k < \infty$, which is a nonlinear operator, (if $k = 2$, there is the usual Laplacian). Δ_k has been used on Sobolev spaces by several authors we cite for example [3, 4], we extend some results of existence and uniqueness relative to the first eigenvalue of a Stokes problem. Let $p \in L^2(\Omega)$, we note $\vec{g}(x, y, s_1, s_2) = (g_1(x, y, s_1), g_2(x, y, s_2))^T$, where $(x, y)^T \in \Omega$, $(s_1, s_2)^T \in \mathbb{R}^2$, $\vec{g} \in C(\bar{\Omega} \times \mathbb{R}^2, \mathbb{R}^2)$ and $f = (f_1, f_2)^T \in (C(\bar{\Omega}))^2$.

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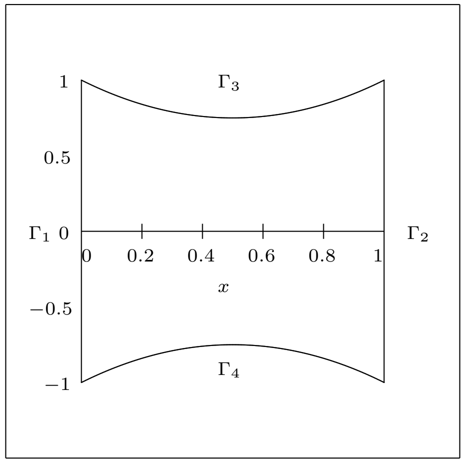

Figure 1. Geometry of channel

For $\alpha \in \mathbb{R}$, we consider the nonlinear Stokes problem

$$\begin{gathered}
-\Delta_k u_1 + \frac{\partial p}{\partial x} = g_1(x,y,u_1) + f_1 \quad \text{in } \Omega, \\
-\Delta_k u_2 + \frac{\partial p}{\partial y} = g_2(x,y,u_2) + f_2 \quad \text{in } \Omega, \\
\operatorname{div} u = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} = 0 \quad \text{in } \Omega, \\
u_1(0,y) = u_1(1,y) \quad \text{on } [-1,1], \\
u_2(0,y) = u_2(1,y) \quad \text{on } [-1,1], \\
\frac{\partial u_1}{\partial x}(0,y) = \frac{\partial u_1}{\partial x}(1,y) \quad \text{on } [-1,1], \\
|\nabla u_2(0,y)|^{k-2}\frac{\partial u_2}{\partial x}(0,y) = |\nabla u_2(1,y)|^{k-2}\frac{\partial u_2}{\partial x}(1,y) \quad \text{on } [-1,1], \\
p(1,y) - p(0,y) = -\alpha \text{on } [-1,1].
\end{gathered} \tag{1.1}$$

We assume also the growth condition:

$$|g_i(x,y,s)| \le c|s|^{k-1} + d(x,y) \quad \forall (x,y)^T \in \Omega,\ \forall s \in \mathbb{R}, \tag{1.2}$$

where $c \in \mathbb{R}$ and $d \in L^{k'}(\Omega)$, with $\frac{1}{k} + \frac{1}{k'} = 1$.

Note that the second member of (1.1) depends on u and since the pressure difference is constant between two parallel plates of the specific domain, we prove that we can associate to (1.1) an energy functional ψ. So a critical point of ψ is a solution of (1.1). We denote by V the closure of $\mathcal{V}$ in the space $(W^{1,k}(\Omega))^2$, where $\mathcal{V} = \{u = (u_1,u_2)^T \in (C^1(\bar{\Omega}))^2 | \operatorname{div} u = 0, u_i(0,y) = u_i(1,y) \text{on } [-1,1]$ for $i = 1,2$ and $u = 0$ on $\Gamma_3 \cup \Gamma_4\}$. We want to extend the work done by Amrouche, Batchi and Batina in the linear case with the Laplacian operator see [1], which showed equivalence between the classical and variational problem, existence and uniqueness of the solution in a linear case where $f = g = 0$. In this paper we introduce the k-Laplacian operator to describe the movement of non-Newtonian fluid with a nonlinear second member, the technique used for the resolution is a minimization and is completely different to that given in [1, 2, 6, 9]. In the case $f = 0$, $g_1(x, y, s_1) = \lambda|s_1|^{k-2}s_1$ and $g_2(x, y, s_1) = \lambda|s_2|^{k-2}s_2$, we have established in [5] that the first eigenfunction λ_1 of (1.1) is well defined, strictly positive and characterized by $\lambda_1^{-1} = \sup\{\int_\Omega |u_1|^k + |u_2|^k; \int_\Omega |\nabla u_1|^k + |\nabla u_2|^k = 1, u \in V\}$.

This article is organized as follows. In Section 2, we prove that u is a weak solution of (1.1) if and only if u satisfies a weak formulation independent of pressure p. In Section 3, we introduce the first eigenvalue of the operator $-\vec{\Delta}_k u + \nabla p$ and as an application, we prove the existence of solution where the primitive of the nonlinear function $\vec{g}$ is asymptotically in the left of the first eigenvalue. In Section 4, we add a condition of monotony for the function $\vec{g}$ and we prove the uniqueness of the solution, then we give an example of such a function $\vec{g}$ which satisfies the conditions. Finally we give in Section 5 a conclusion.

2. Weak formulation of (1.1)

We establish the equivalence between the classical problem and weak formulation of problem which is independent of pressure p, This allows us to find the existence of the weak solution of (1.1) by a new method of minimization.

Definition 2.1. A classical solution of (1.1) is a function $(u, p)^T \in (C^2(\Omega) \cap C^1(\bar{\Omega}))^2 \times L^2(\Omega)$ and $\nabla p \in (C(\overline{\Omega}))^2$ which verify (1.1).

Theorem 2.2. *If $(u, p)^T$ is a classical solution of* (1.1)*, then*

$$\sum_{i=1}^{2} \int_\Omega |\nabla u_i|^{k-2}\nabla u_i.\nabla v_i - \alpha \int_{-1}^{1} v_1(0, y)dy = \int_\Omega \vec{g}(x, y, u).v + \int_\Omega f.v \quad \forall v \in \mathcal{V} \tag{2.1}$$

Proof. If $(u, p)^T$ is a classical solution of (1.1) where $u = (u_1, u_2)^T$, then for $v = (v_1, v_2)^T \in \mathcal{V}$, we multiply the first equation by v_1, the second equation by v_2 of (1.1) and we integrate on Ω, we obtain

$$\int_\Omega -(\Delta_k u_1)v_1 + \int_\Omega -(\Delta_k u_2)v_2 + \int_\Omega (\nabla p).v = \int_\Omega \vec{g}(x, y, u_1, u_2).v + \int_\Omega f.v.$$

According to Green's formula, we have for all $1 \le i \le 2$,

$$\int_\Omega -(\Delta_k u_i)v_i = \int_\Omega |\nabla u_i|^{k-2}\nabla u_i.\nabla v_i - \int_{\partial\Omega} |\nabla u_i|^{k-2}\nabla u_i.\vec{\eta}v_i d\sigma,$$

where $\vec{\eta}$ is the unit outward normal to $\partial\Omega$. On the one hand, we have

$$\begin{aligned}\int_{\partial\Omega} |\nabla u_i|^{k-2}\nabla u_i.\vec{\eta}v_i d\sigma &= \int_{\Gamma_1} |\nabla u_i|^{k-2}\nabla u_i.\vec{\eta}v_i d\sigma + \int_{\Gamma_2} |\nabla u_i|^{k-2}\nabla u_i.\vec{\eta}v_i d\sigma \\ &\quad + \int_{\Gamma_3\cup\Gamma_4} |\nabla u_i|^{k-2}\nabla u_i.\vec{\eta}v_i d\sigma.\end{aligned}$$

As $v \in \mathcal{V}$,

$$\int_{\Gamma_3\cup\Gamma_4} |\nabla u_i|^{k-2}\nabla u_i.\vec{\eta}v_i d\sigma = 0,$$

we have on $\Gamma_1, \vec{\eta} = -(1, 0)^T$ and on $\Gamma_2, \vec{\eta} = (1, 0)^T$, thus

$$\int_{\Gamma_1} |\nabla u_i|^{k-2}\nabla u_i.\vec{\eta}v_i d\sigma = -\int_{-1}^{1} |\nabla u_i(0, y)|^{k-2}\frac{\partial u_i}{\partial x}(0, y)v_i(0, y)dy,$$

and

$$\int_{\Gamma_2} |\nabla u_i|^{k-2}\nabla u_i.\vec{\eta} v_i d\sigma = \int_{-1}^{1} |\nabla u_i(1,y)|^{k-2}\frac{\partial u_i}{\partial x}(1,y)v_i(1,y)dy.$$

As $v \in \mathcal{V}$, we have $v_i(0,y) = v_i(1,y)$, for all $-1 \le y \le 1$, $i = 1,2$. According to (1.1), we have

$$\int_{\partial\Omega} |\nabla u_2|^{k-2}\nabla u_2.\vec{\eta} v_2 = 0. \tag{2.2}$$

On the other hand, as $\frac{\partial u_1}{\partial y}(0,y) = \frac{\partial u_1}{\partial y}(1,y)$, thus $\nabla u_1(0,y) = \nabla u_1(1,y)$, we deduce that

$$\int_{\partial\Omega} |\nabla u_1|^{k-2}\nabla u_1.\vec{\eta} v_1 = 0.$$

Then, by Green's formula and $v \in V$, we have

$$\int_{\Omega} \nabla p.v = \int_{\partial\Omega} pv.\vec{\eta} - \int_{\Omega} p \operatorname{div} v,$$

and

$$\begin{aligned}
\int_{\partial\Omega} pv.\vec{\eta} &= \int_{\Gamma_1} pv.\vec{\eta} + \int_{\Gamma_2} pv.\vec{\eta} + \int_{\Gamma_3\cup\Gamma_4} pv.\vec{\eta} \\
&= -\int_{-1}^{1} p(0,y)v_1(0,y)dy + \int_{-1}^{1} p(1,y)v_1(1,y)dy \\
&= \int_{-1}^{1} (p(1,y) - p(0,y))v_1(0,y)dy \\
&= -\alpha \int_{-1}^{1} v_1(0,y)dy.
\end{aligned}$$

This proves (2.1). □

Now, we study the reciprocal problem; i.e., if u is a weak solution of (1.1) with some regularity, then u is a classical solution of (1.1).

Definition 2.3. A weak solution of (1.1) is a function $u \in V$ satisfying (2.1).

Theorem 2.4. *If u is a weak solution of* (1.1) *with $u \in (C^2(\Omega) \cap C^1(\bar{\Omega}))^2$, then there exists $p \in L^2(\Omega)$ such that $(u,p)^T$ is a classical solution of* (1.1). *Furthermore we have $\nabla p \in (C(\bar{\Omega}))^2$ and $-\alpha = p(1,y) - p(0,y) = \int_0^1 \frac{\partial p}{\partial x}(t,y)dt$.*

Proof. Let $u \in (C^2(\Omega) \cap C^1(\bar{\Omega}))^2$ which satisfies (1.1), by Green's formula, we have

$$\int_{\Omega} (-\vec{\Delta}_k u - \vec{g}(x,y,u_1,u_2) - f).v + \sum_{i=1}^{2} \int_{\partial\Omega} |\nabla u_i|^{k-2}\nabla u_i.\vec{\eta} v_i d\sigma - \alpha \int_{-1}^{1} v_1(0,y)dy = 0 \tag{2.3}$$

for all $v \in \mathcal{V}$. We put $F = \{v \in (\mathfrak{D}(\Omega))^2 | \quad \operatorname{div} v = 0\}$ where $\mathfrak{D}(\Omega)$ is the set of all infinitely differentiable functions with compact support in Ω. (2.3) becomes

$$\int_{\Omega} (-\vec{\Delta}_k u - \vec{g}(x,y,u_1,u_2) - f).v = 0 \quad \forall v \in F.$$

By (1.2), as $u \in (C^2(\Omega) \cap C^1(\bar{\Omega}))^2$ and $f \in (C(\bar{\Omega}))^2$, we have $-\vec{\Delta}_k u - \vec{g}(x,y,u_1,u_2) - f \in (C(\bar{\Omega}))^2 \subset (L^2(\Omega))^2$, according to Rham's theorem see [7, 8], there exists

$p \in L^2(\Omega)$ such that $-\vec{\Delta}_k u + \nabla p = \vec{g}(x, y, u_1, u_2) + f$ in $(C(\bar{\Omega}))^2$. Thus

$$\sum_{i=1}^{2} \int_{\partial\Omega} |\nabla u_i|^{k-2} \nabla u_i.\vec{\eta} v_i d\sigma - \alpha \int_{-1}^{1} v_1(0, y) dy = \int_{\Omega} (\nabla p).v \quad \forall v \in \mathcal{V}, \tag{2.4}$$

where $\nabla p \in (C(\bar{\Omega}))^2$ and $y \mapsto p(1, y) - p(0, y) = \int_{-1}^{1} \frac{\partial p}{\partial x}(t, y) dt \in C^1([-1, 1])$. As $u_i(0, y) = u_i(1, y)$ for all $y \in [-1, 1]$, we have $\frac{\partial u_i}{\partial y}(0, y) = \frac{\partial u_i}{\partial y}(1, y)$. Moreover we know that $\operatorname{div} u = 0$ in Ω and $u \in (C^1(\bar{\Omega}))^2$, we conclude that $\frac{\partial u_1}{\partial x}(0, y) = -\frac{\partial u_2}{\partial y}(0, y) = -\frac{\partial u_2}{\partial y}(1, y)$ for all $y \in [-1, 1]$. Thus $\frac{\partial u_1}{\partial x}(0, y) = \frac{\partial u_1}{\partial x}(1, y)$ and $\nabla u_1(0, y) = \nabla u_1(1, y)$. Hence $\int_{\partial\Omega} |\nabla u_1|^{k-2} \nabla u_1.\vec{\eta} v_1 d\sigma = 0$.

On the other hand, according to (2.4), we have

$$\begin{aligned} \int_{\partial\Omega} |\nabla u_2|^{k-2} \nabla u_2.\vec{\eta} v_2 d\sigma - \alpha \int_{-1}^{1} v_1(0, y) dy &= \int_{\Omega} (\nabla p).v \quad \forall v \in \mathcal{V} \\ &= \int_{\partial\Omega} pv.\vec{\eta} \\ &= \int_{-1}^{1} (p(1, y) - p(0, y)) v_1(0, y) dy. \end{aligned}$$

Therefore,

$$\begin{aligned} &\int_{-1}^{1} -|\nabla u_2(0, y)|^{k-2} \frac{\partial u_2}{\partial x}(0, y) v_2(0, y) dy \\ &+ \int_{-1}^{1} |\nabla u_2(1, y)|^{k-2} \frac{\partial u_2}{\partial x}(1, y) v_2(1, y) dy - \alpha \int_{-1}^{1} v_1(0, y) dy \\ &= \int_{-1}^{1} (p(1, y) - p(0, y)) v_1(0, y) dy. \end{aligned} \tag{2.5}$$

Let $H_{00}^{1/2}(\Gamma_1)$ [1] be the space defined by

$$H_{00}^{1/2}(\Gamma_1) = \{\varphi \in L^2(\Gamma_1); \exists v \in H^1(\Omega), \text{ with } v|_{\Gamma_3 \cup \Gamma_4} = 0, v|_{\Gamma_1 \cup \Gamma_2} = \varphi\}.$$

Let $\mu \in H_{00}^{1/2}(\Gamma_1)$, we put $\nu = (0, \mu_2)^T$ where $\mu_2 = \begin{cases} \mu & \text{on } \Gamma_1 \cup \Gamma_2 \\ 0 & \text{on } \Gamma_3 \cup \Gamma_4. \end{cases}$ It is clear that $\nu \in (H^{1/2}(\Gamma))^2$ and $\int_{\partial\Omega} \nu.\vec{\eta} d\sigma = 0$, so there exists $v \in (H^1(\Omega))^2$ such that $\operatorname{div} v = 0$ in Ω and $v = \nu$ on Γ (see [1]); therefore $v \in V$. According to (2.5), we have for all $\mu \in H_{00}^{1/2}(\Gamma_1)$,

$$\int_{-1}^{1} |\nabla u_2(0, y)|^{k-2} \frac{\partial u_2}{\partial x}(0, y) \mu dy = \int_{-1}^{1} |\nabla u_2(1, y)|^{k-2} \frac{\partial u_2}{\partial x}(1, y) \mu dy,$$

thus

$$|\nabla u_2(0, y)|^{k-2} \frac{\partial u_2}{\partial x}(0, y) = |\nabla u_2(1, y)|^{k-2} \frac{\partial u_2}{\partial x}(1, y).$$

According to (2.5), we have

$$-\alpha \int_{-1}^{1} v_1(0, y) dy = \int_{-1}^{1} (p(1, y) - p(0, y)) v_1(0, y) dy. \tag{2.6}$$

On the other hand, let $\gamma \in H^{1/2}_{00}(\Gamma_1)$. Now we consider $\beta = (\gamma_1, 0)^T$ where $\gamma_1 = \begin{cases} \gamma & \text{on } \Gamma_1 \cup \Gamma_2 \\ 0 & \text{on } \Gamma_3 \cup \Gamma_4. \end{cases}$. We have $\beta \in (H^{1/2}(\Gamma))^2$ and $\int_{\partial\Omega} \beta.\vec{\eta} d\sigma = 0$, so there exists $v \in (H^1(\Omega))^2$ such that $\operatorname{div} v = 0$ in Ω and $v = \beta$ on Γ [1]; therefore, $v \in V$. By (2.6) $-\alpha \int_{-1}^{1} \gamma dy = \int_{-1}^{1} (p(1,y) - p(0,y))\gamma dy$. Finally we prove $p(1,y) - p(0,y) = -\alpha$. $\square$

3. Existence of a solution

Let us introduce the energy functional associated with (2.1), $\psi : V \to \mathbb{R}$:

$$\begin{aligned} \psi(u) = \frac{1}{k}\int_\Omega |\nabla u_1|^k + \frac{1}{k}\int_\Omega |\nabla u_2|^k - \alpha \int_{-1}^{1} u_1(1,y)dy \\ - \int_\Omega F_1(x,y,u_1) - \int_\Omega F_2(x,y,u_2) - \int_\Omega f_1 u_1 - \int_\Omega f_2 u_2, \end{aligned} \tag{3.1}$$

where $F : \Omega \times \mathbb{R}^2 \to \mathbb{R}$; $F(x,y,u) = F_1(x,y,u_1) + F_2(x,y,u_2)$ and $F_i(x,y,s) = \int_0^s g_i(x,y,t)dt$, $i = 1,2$. It is clear that ψ is well defined, C^1 on V and for all $v \in \mathcal{V}$

$$\langle \psi'(u), v\rangle = \sum_{i=1}^{2} \int_\Omega |\nabla u_i|^{k-2} \nabla u_i.\nabla v_i - \alpha \int_{-1}^{1} v_1(0,y)dy - \int_\Omega \vec{g}(x,y,u).v - \int_\Omega f.v. \tag{3.2}$$

We know that a critical point of the function ψ is a weak solution of (1.1) and reciprocally. We assume that the nonlinearity is asymptotically in the left of the first eigenvalue of k-Laplacian; i.e.,

$$F(x,y,s_1,s_2) \le \frac{\lambda}{k}(|s_1|^k + |s_2|^k) + \rho(x,y), \tag{3.3}$$

where $\rho \in L^1(\Omega)$ and $\lambda < \lambda_1$, λ_1 is the first eigenvalue of the problem

$$\begin{gathered} -\Delta_k u_1 + \frac{\partial p}{\partial x} = \lambda |u_1|^{k-2} u_1 \quad \text{in } \Omega, \\ -\Delta_k u_2 + \frac{\partial p}{\partial y} = \lambda |u_2|^{k-2} u_2 \quad \text{in } \Omega, \\ \operatorname{div} u = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} = 0 \quad \text{in } \Omega, \\ u_1(0,y) = u_1(1,y) \quad \text{on } [-1,1], \\ u_2(0,y) = u_2(1,y) \quad \text{on } [-1,1], \\ \frac{\partial u_1}{\partial x}(0,y) = \frac{\partial u_1}{\partial x}(1,y) \quad \text{on } [-1,1], \\ |\nabla u_2(0,y)|^{k-2} \frac{\partial u_2}{\partial x}(0,y) = |\nabla u_2(1,y)|^{k-2} \frac{\partial u_2}{\partial x}(1,y) \quad \text{on } [-1,1], \\ p(1,y) - p(0,y) = 0 \quad \text{on } [-1,1]. \end{gathered} \tag{3.4}$$

In [5], we have proved that the first eigenvalue λ_1 of (3.4) is well defined, strictly positive and characterized by

$$\lambda_1^{-1} = \sup\big\{ \int_\Omega |u_1|^k + |u_2|^k ; \int_\Omega |\nabla u_1|^k + |\nabla u_2|^k = 1, u \in V \big\}. \tag{3.5}$$

So

$$\lambda_1 \int_\Omega |u_1|^k + |u_2|^k \le \int_\Omega |\nabla u_1|^k + |\nabla u_2|^k \quad \forall u \in V. \tag{3.6}$$

Theorem 3.1. *Assume that* (1.2)*and*(3.3) *are satisfied, then there exists* $u \in V$ *such that* $\psi(u) = \inf_{v\in V} \psi(v)$. *Consequently,* u *is the weak solution of* (1.1).

Proof. As ψ is convex and class C^1, it suffices to show that ψ is coercive; i.e., $\psi(u) \to +\infty$ when $\|u\|_{W^{1,k}} \to +\infty$. According to (3.6), the function $u \mapsto (\int_\Omega |\nabla u_1|^k)^{1/k} + (\int_\Omega |\nabla u_2|^k)^{1/k} := \|u\|_V$ define a norm in V. We have successively

$$\psi(u) = \frac{1}{k}\int_\Omega |\nabla u_1|^k + \frac{1}{k}\int_\Omega |\nabla u_2|^k - \alpha\int_{-1}^1 u_1(1,y)dy - \int_\Omega F(x,y,u) - \langle f,u\rangle, \quad (3.7)$$

$$\begin{aligned}\langle f,u\rangle = \int_\Omega f_1u_1 + \int_\Omega f_2u_2 &\le \sum_{i=1}^2 \|f_i\|_{L^{k'}}\|u_i\|_{L^k}\\ &\le c\sum_{i=1}^2 \|\nabla u_i\|_{(L^k)^2}, \quad \text{where } c>0\\ &= c\|u\|_V.\end{aligned}$$

$$\begin{aligned}\lambda\int_{-1}^1 u_1(1,y)dy &\le |\lambda|\int_{\partial\Omega}|u_1(1,y)|dy\\ &\le |\lambda|c'(\int_{\partial\Omega}|u_1(1,y)|^k)^{1/k}dy \quad \text{(Holder's inequality), where } c'>0\\ &\le |\lambda|c'(\int_\Omega|\nabla u_1|^k)^{1/k} \quad (V \to (L^k(\partial\Omega))^2 \quad \text{trace theorem })\\ &= c''\|u\|_V, \quad \text{where } c''>0,\end{aligned}$$

the trace theorem is because $V \subset W^{1,k}_{div}(\Omega)$ and $W^{1,k}_{div}(\Omega) \to (L^k(\partial\Omega))^2$ with continuous injection.

$$\begin{aligned}\int_\Omega F(x,y,u) &\le \frac{\alpha}{k}\int_\Omega |u_1|^k + |u_2|^k + \int_\Omega \rho(x,y)\\ &\le \frac{\widetilde{\alpha}}{\lambda_1 k}\int_\Omega |\nabla u_1|^k + |\nabla u_2|^k + \int_\Omega \rho(x,y),\end{aligned}$$

where $\widetilde{\alpha} := \begin{cases} 0 & \text{if } \alpha < 0\\ \alpha & \text{if } \alpha \ge 0.\end{cases}$ It follows that

$$\psi(u) \ge \frac{1}{k}(1-\frac{\widetilde{\alpha}}{\lambda_1})\int_\Omega |\nabla u_1|^k + |\nabla u_2|^k - c\|u\|_V - c'\|u\|_V - \int_\Omega \rho(x,y).$$

Hence

$$\psi(u) \ge \frac{1}{k}(1-\frac{\widetilde{\alpha}}{\lambda_1})\|u\|_V^k - c''\|u\|_V - \int_\Omega \rho(x,y), \text{ where } c''>0. \quad (3.8)$$

Finally, as $(1-\frac{\widetilde{\alpha}}{\lambda_1}) > 0$, the property is proved. □

4. Uniqueness of the solution

We assume again that the function $\vec{g}$ is decreasing in the following sense:

$$(\vec{g}(x,y,\xi) - \vec{g}(x,y,\xi'), \xi-\xi') \le 0 \quad \text{for all } \xi,\xi' \in \mathbb{R}^2. \quad (4.1)$$

Theorem 4.1. *Problem* (2.1) *has a unique solution.*

Proof. Let u and $\widetilde{u}$ be two solutions of problem (2.1). For all $v \in V$, we have

$$\sum_{i=1}^{2}\{\int_\Omega [(|\nabla u_i|^{k-2}\nabla u_i - |\nabla \widetilde{u}_i|^{k-2}\nabla \widetilde{u}_i).\nabla v_i - (g_i(x,y,u_i) - g_i(x,y,\widetilde{u}_i))v_i]\Big\} = 0. \tag{4.2}$$

In particular for $v = u - \widetilde{u}$, we have

$$\begin{aligned}\sum_{i=1}^{2}\Big\{\int_\Omega &[(|\nabla u_i|^{k-2}\nabla u_i - |\nabla \widetilde{u}_i|^{k-2}\nabla \widetilde{u}_i).\nabla(u_i - \tilde{u}_i)\\ &- (g_i(x,y,u_i) - g_i(x,y,\widetilde{u}_i))(u_i - \tilde{u}_i)]\} = 0.\end{aligned} \tag{4.3}$$

As $(|\xi|^{k-2}\xi - |\xi'|^{k-2}\xi').(\xi - \xi') > 0$ for all $\xi \neq \xi' \in \mathbb{R}^2$ and (4.1), we deduce that

$$\sum_{i=1}^{2}\int_\Omega (|\nabla u_i|^{k-2}\nabla u_i - |\nabla \widetilde{u}_i|^{k-2}\nabla \widetilde{u}_i).\nabla(u_i - \tilde{u}_i) = 0. \tag{4.4}$$

Thus $\nabla u_i = \nabla \tilde{u}_i$, $i = 1, 2$, therefore $u_i = \widetilde{u}_i + \epsilon$, where $\epsilon \in \mathbb{R}$. As $u_i, \tilde{u}_i \in V$, we have $\epsilon = 0$, this completes the proof. $\square$

Example of function $\vec{g}$. We consider $\vec{g}(x,y,s) = (g_1(s_1), g_2(s_2))$ for all $s = (s_1, s_2) \in \mathbb{R}^2$ and $(x,y)^T \in \mathbb{R}^2$, where

$$g_i(s_i) = \begin{cases} \frac{\alpha}{2}\big(\frac{s_i^{k-1}}{1+(s_i^k)}\big) & \text{if } s_i \geq (k-1)^{1/k} \\ \frac{\alpha}{2k}(k-1)^{\frac{k-1}{k}} & \text{if } -(k-1)^{1/k} \leq s_i \leq (k-1)^{1/k} \\ \frac{\alpha}{2}\big(\frac{(-s_i)^{k-2}s_i}{1+(-s_i)^k}\big) + \frac{\alpha}{k}(k-1)^{(k-1)/k} & \text{if } s_i \leq -(k-1)^{1/k}. \end{cases}$$

We have $g_i(x,y,.)$ is a continuous function, so it has a primitive F_i, for $i = 1, 2$.

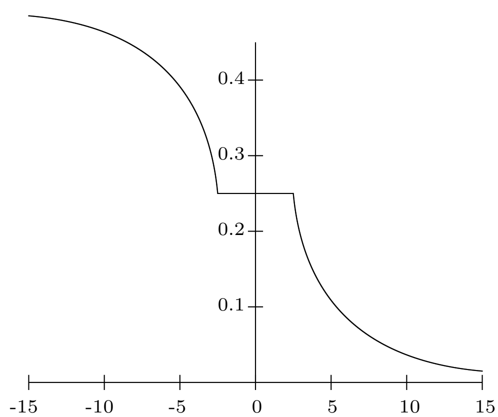

FIGURE 2. Graph of g_i

For $k = 2$ and $\alpha = 1$, Figure (2), we have

$$g_i(s) = \begin{cases} \frac{1}{2}\big(\frac{s}{1+s^2}\big) & \text{if } s \geq 1 \\ \frac{1}{4} & \text{if } -1 \leq s \leq 1 \\ \frac{1}{2}\big(\frac{s}{1+s^2}\big) + \frac{1}{2} & \text{if } s \leq 1. \end{cases}$$

(i) g satisfies (1.2), indeed: If $s_i \geq (k-1)^{1/k}$, then

$$|g_i(s)| = \frac{\alpha}{2}\Big(\frac{|s_i|^{k-1}}{1+(|s_i|^k)}\Big) \leq \frac{\alpha}{2}|s_i|^{k-1}.$$

If $s_i \leq -(k-1)^{1/k}$, then

$$|g_i(s)| = \frac{\alpha}{2}\Big(\frac{|s_i|^{k-1}}{1+(|s_i|^k)}\Big) + c \leq \frac{\alpha}{2}|s_i|^{k-1} + c \leq \frac{\alpha}{2}|s|^{k-1} + c' \text{ for all } s \in \mathbb{R}^2,$$

where $c \in \mathbb{R}$ and $c' = c + \frac{\alpha}{2k}(k-1)^{\frac{k-1}{k}}$.

(ii) We have $F(x,y,s_1,s_2) = F_1(x,y,s_1) + F_2(x,y,s_2)$, where $F_i(x,y,s) = \int_0^s g_i(x,y,t)dt$, i=1,2. So

$$\begin{aligned} F_i(x,y,s) = \int_0^s g_i(t)dt &\leq \int_0^s \frac{\alpha}{2}\Big(\frac{|t|^{k-2}t}{1+|t|^k}\Big)dt + c, \text{ where } c \in \mathbb{R}. \\ &\leq \frac{\alpha}{2k}\ln(1+|s|^k) + c', \text{ where } c' \in \mathbb{R}. \end{aligned}$$

Thus

$$\begin{aligned} F(x,y,s_1,s_2) = F_1(x,y,s_1) + F_2(x,y,s_2) &\leq \frac{\alpha}{2k}\ln(1+|s_1|^k) + \frac{\alpha}{2k}\ln(1+|s_2|^k) + c' \\ &\leq \frac{\alpha}{2k}(|s_1|^k + |s_2|^k) + c' \\ &\leq \frac{\alpha}{k}(s_1^2 + s_1^2)^{\frac{k}{2}} + c', \end{aligned}$$

consequently F satisfies condition (3.3).

(iii) Finally $\vec{g}$ is decreasing. For $s_i \geq (k-1)^{1/k}$,

$$\begin{aligned} g_i'(s_i) &= \frac{\alpha}{2}\Big(\frac{(k-1)s_i^{k-2}(1+(s_i)^k) - s_i^{k-1}(ks_i^{k-1})}{(1+(s_i)^k)^2}\Big) \\ &= \frac{\alpha}{2}\Big(\frac{(ks_i^{k-2} + ks_i^{2k-2} - s_i^{k-2} - s_i^{2k-2} - ks_i^{2k-2}}{(1+(s_i)^k)^2}\Big) \\ &= \frac{\alpha}{2}\Big(\frac{s_i^{k-2}(k-1-s_i^k)}{(1+(s_i)^k)^2}\Big) \leq 0. \end{aligned}$$

For $s_i \leq -(k-1)^{1/k}$,

$$\begin{aligned} g_i'(s) &= \frac{\alpha}{2}\Big[(-(k-2)(-s_i)^{k-3}s_i + (-s_i)^{k-2})(1+(-s_i)^k) \\ &\quad + (-s_i)^{k-2}s_i(k(-s_i)^{k-1})\Big]/(1+(-s_i)^k)^2 \\ &= \frac{\alpha}{2}\Big[-(k-2)(-s_i)^{k-3}s_i - (k-2)(-s_i)^{2k-3}s_i + (-s_i)^{k-2} \\ &\quad + (-s_i)^{2k-2} + k(-s_i)^{2k-3}s_i\Big]/(1+(-s_i)^k)^2 \\ &= \frac{\alpha}{2}\Big[-(k-2)(-s_i)^{k-3}s_i + 2(-s_i)^{2k-3}s_i + (-s_i)^{k-2} \\ &\quad + (-s_i)^{2k-2}\Big]/(1+(-s_i)^k)^2 \\ &= \frac{\alpha}{2}(-s_i)^{k-3}\big[-(k-2)s_i + 2(-s_i)^k s_i + (-s_i) + (-s_i)^{k+1}\big]/(1+(-s_i)^k)^2 \end{aligned}$$

$$
\begin{aligned}
&= \frac{\alpha}{2}(-s_i)^{k-3}\big[-ks_i + s_i + (-s_i)^k(2s_i - s_i)\big]/(1+(-s_i)^k)^2 \\
&= \frac{\alpha}{2}(-s_i)^{k-3}\big[-(k-1)s_i + (-s_i)^k s_i\big]/(1+(-s_i)^k)^2 \\
&= \frac{\alpha}{2}(-s_i)^{k-3}s_i\big[-(k-1) + (-s_i)^k\big]/(1+(-s_i)^k)^2 \le 0.
\end{aligned}
$$

Conclusion. We have shown the existence and uniqueness of a solution by a minimization method. We can also define the other eigenvalues and placed them between two consecutive eigenvalues, in this case we must consider using saddle points.

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