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EXISTENCE AND STABILITY OF PERIODIC SOLUTIONS OF BAM HIGH-ORDER HOPFIELD NEURAL NETWORKS WITH IMPULSES AND DELAYS ON TIME SCALES

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Abstract. By using Mawhins's continuation theorem of coincidence degree theory and constructing some suitable Lyapunov functions, the periodicity and the exponential stability for a class of bidirectional associative memory (BAM) high-order Hopfield neural networks with impulses and delays on time scales are investigated. An example illustrates our results.

1. Introduction

The bidirectional associative memory (BAM) neural network models were introduced by Kosko [9, 10]. It is a special class of recurrent neural networks that can store bipolar vector pairs. The BAM neural network is composed of neurons arranged in two neural fields; i.e., the F_X-field and the F_Y-field. The neurons in one field are fully interconnected to the neurons in the other field, while there is no interconnection among neurons in the same neural field. Through iterations of forward and backward information flows between the two neural fields, it performs a two-way associative search for stored bipolar vector pairs and generalizes the single-field auto-associative Hebbian correlation to a two-field pattern-matched hetero-associative circuits. Therefore, BAM neural networks possesses good application prospects in many fields such as pattern recognition, parallel computation, image and signal processing, optimization automatic control and artificial intelligence. Recently, BAM neural networks have attracted the attention of many scientists (e.g., mathematicians, physicists, computer scientists and so on) and many results for BAM neural networks with or without axonal signal transmission delays including stability and periodic solutions have been obtained [5, 13, 15, 18, 8, 24, 25, 28, 36] and the references quoted therein. For example, in Refs. [21, 22, 23, 29], the authors discussed the problem of the stability of the BAM neural networks with or without delays, and obtained some sufficient conditions to ensure the stability of equilibrium point. Li [12], Ho et al [7], Li and Yang [16], Chen and Cui [4], and Xia et al [30] discussed the existence and exponential stability of the equilibrium point of several classes of impulsive BAM neural networks using different methods,

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such as linear matrix inequality (LMI), Fixed point theorem, Halanay inequality, Lyapunov functional method, M-matrix theory and Topological degree methods, respectively.

On the other hand, due to the fact that high-order Hopfield neural networks (HHNNs) have stronger approximation property, faster convergence rate, greater storage capacity, and higher fault tolerance than lower-order neural networks, high-order neural networks have been the object of intensive analysis by numerous authors in the recent years. We refer the reader to [6, 11, 17, 26, 31, 32, 33, 34, 35]. However, to the best of the author's knowledge, few results have been obtained the periodicity and the exponential stability for a class of BAM high-order Hopfield neural networks with impulses and delays on time scales.

The objective of this paper is to investigate the existence and stability of periodic solutions of BAM HHNNs with impulses and delays on time scales

$$\begin{aligned}
x_i^{\Delta}(t) &= -c_i(t)x_i(t) + \sum_{j=1}^{m} a_{ij}(t)f_j(y_j(t-\tau_{ij}(t))) \\
&\quad + \sum_{j=1}^{m}\sum_{l=1}^{m} b_{ijl}(t)g_j(y_j(t-\sigma_{ijl}(t)))g_l(y_l(t-\upsilon_{ijl}(t))) + I_i(t), \quad t \neq t_k, \\
\Delta x_i(t_k) &= x_i(t_k^+) - x_i(t_k^-) = \alpha_{ik}(x_i(t_k)), \quad i = 1,2,\dots,n,\ k = 1,2,\dots, \\
y_j^{\Delta}(t) &= -d_j(t)y_j(t) + \sum_{i=1}^{n} e_{ji}(t)p_i(x_i(t-\hat{\tau}_{ji}(t))) \\
&\quad + \sum_{i=1}^{n}\sum_{r=1}^{n} h_{jir}(t)q_i(x_i(t-\hat{\sigma}_{jir}(t)))q_r(x_r(t-\hat{\upsilon}_{jir}(t))) + J_j(t), \quad t \neq t_k, \\
\Delta y_j(t_k) &= y_j(t_k^+) - y_j(t_k^-) = \beta_{jk}(y_j(t_k)), \quad j = 1,2,\dots,m,\ k = 1,2,\dots,
\end{aligned} \tag{1.1}$$

where $\mathbb{T}$ is an ω-periodic time scale which has the subspace topology inherited from the standard topology on $\mathbb{R}$. And x_i and y_j are the activations of the i-th neuron in F_X-layer and the jth neuron in F_Y-layer, respectively; $a_i(t) > 0$ and $b_j(t) > 0$ represent the rate with which the ith neuron from F_X-layer and the jth neuron from F_Y-layer will reset their potential to the resting state in isolation when disconnected from the networks and external inputs, respectively; τ_{ij}, σ_{ijl}, υ_{ijl}, and $\hat{\tau}_{ij}$, $\hat{\sigma}_{ijl}$, $\hat{\upsilon}_{ijl}$ represent the axonal signal transmission delays; $I_i(t)$ and $J_j(t)$ are the external inputs on the neurons. Here, $\Delta x_i(t_k) = x_i(t_k^+) - x_i(t_k^-)$ and $\Delta y_j(t_k) = y_j(t_k^+) - y_j(t_k^-)$ are the impulses at moments t_k and $t_1 < t_2 < \cdots$ is a strictly increasing sequence such that $\lim_{k\to\infty} t_k = \infty$.

System (1.1) is supplemented with initial values

$$\begin{gathered}
x_i(t) = \phi_{x_i}(s), \quad s \in [-\theta, 0] \cap \mathbb{T}, \quad \theta = \max\{\tau, \sigma, \upsilon\}, \\
y_j(t) = \phi_{y_j}(s), \quad s \in [-\hat{\theta}, 0] \cap \mathbb{T}, \quad \hat{\theta} = \max\{\hat{\tau}, \hat{\sigma}, \hat{\upsilon}\}, \\
\tau = \max_{1\le i\le n, 1\le j\le m}\big\{\max_{t\in[0,\omega]\cap\mathbb{T}} \tau_{ij}(t)\big\}, \quad \sigma = \max_{1\le i\le n, 1\le j,l\le m}\big\{\max_{t\in[0,\omega]\cap\mathbb{T}} \sigma_{ijl}(t)\big\}, \\
\upsilon = \max_{1\le i\le n, 1\le j,l\le m}\big\{\max_{t\in[0,\omega]\cap\mathbb{T}} \upsilon_{ijl}(t)\big\}, \quad \hat{\tau} = \max_{1\le i\le n, 1\le j\le m}\big\{\max_{t\in[0,\omega]\cap\mathbb{T}} \hat{\tau}_{ij}(t)\big\}, \\
\hat{\sigma} = \max_{1\le i,r\le n, 1\le j\le m}\big\{\max_{t\in[0,\omega]\cap\mathbb{T}} \hat{\sigma}_{jir}(t)\big\}, \quad \hat{\upsilon} = \max_{1\le i,r\le n, 1\le j\le m}\big\{\max_{t\in[0,\omega]\cap\mathbb{T}} \hat{\upsilon}_{jir}(t)\big\},
\end{gathered}$$

where $\phi_{x_i}(\cdot)$ and $\phi_{y_j}(\cdot)$ denote continuous ω-periodic function defined on $[-\theta, 0]\cap\mathbb{T}$ and $[-\hat{\theta}, 0]\cap\mathbb{T}$, respectively.

As usual in the theory of impulsive differential equations, at the points of discontinuity t_k of the solution $t \to (x_1(t), \dots, x_n(t), y_1(t), \dots, y_m(t))^T$, we assume that $(x_1(t_k^+), \dots, x_n(t_k^+), y_1(t_k^+), \dots, y_m(t_k^+))^T$ exists, and $\big(x_1(t_k^-), \dots, x_n(t_k^-), y_1(t_k^-), \dots, y_m(t_k^-)\big)^T = (x_1(t_k), \dots, x_n(t_k), y_1(t_k), \dots, y_m(t_k))^T$. It is clear that, in general,the derivatives $x_i'(t_k)$ and $y_j'(t_k)$ do not exist. On the other hand, according to the first two and the third equalities of (1.1), there exist the limits $x_i'(t_k^{\pm})$ and $y_j'(t_k^{\pm})$. According to the above convention, we assume that $x_i'(t_k) = x_i'(t_k^-)$ and $y_j'(t_k) = y_j'(t_k^-)$.

Throughout this paper, we assume the following.

- (H1) For $i, r = 1, 2, \dots, n$, $j, l = 1, 2, \dots, m$, $c_i(t)$,$d_j(t)$, $a_{ij}(t)$, $e_{ji}(t)$, $b_{ijl}(t)$, $h_{jir}(t)$, $I_i(t)$, $J_j(t)$, $\tau_{ij}(t)$, $\sigma_{ijl}(t)$, $\upsilon_{ijl}(t)$ $\hat{\tau}_{ji}(t)$, $\hat{\sigma}_{jir}(t)$, $\hat{\upsilon}_{jir}(t)$ are positive continuous periodic functions with period $\omega > 0$, and $c_i(t)$ and $d_j(t)$ are regressive. And assume that $t - \tau_{ij}(t)$, $t - \sigma_{ijl}(t)$, $t - \upsilon_{ijl}(t)$ $t - \hat{\tau}_{ji}(t)$, $t - \hat{\sigma}_{jir}(t)$ and $t - \hat{\upsilon}_{jir}(t)$ belong to $\mathbb{T}$ for $t \in \mathbb{T}$.
- (H2) There exist positive constants $M_j, N_j, \hat{M}_i, \hat{N}_i$, such that $|f_j(x)| \le M_j$, $|g_j(x)| \le N_j$, $|p_i(x)| \le \hat{M}_i$, $|q_i(x)| \le \hat{N}_i$, for $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$, $x \in \mathbb{R}$.
- (H3) Functions $f_j(u), g_j(u), p_i(u), q_i(u)$ satisfy the Lipschitz condition; that is, there exist constants L_j, H_j,
  $\hat{L}_i, \hat{H}_i > 0$ such that $|f_j(u_1) - f_j(u_2)| \le L_j|u_1 - u_2|$, $|g_j(u_1) - g_j(u_2)| \le H_j|u_1 - u_2|$, $|p_i(u_1) - p_i(u_2)| \le \hat{L}_i|u_1 - u_2|$, $|q_i(u_1) - q_i(u_2)| \le \hat{H}_i|u_1 - u_2|$ for any $u_1, u_2 \in \mathbb{R}$, for $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$.
- (H4) There exists a positive integer q such that for $k = 1, 2, \dots,$ $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$, $\{t_k, k = 1, 2, \dots\} \cap [0, \omega] = t_1, t_2, \dots, t_q$, $t_{k+q} = t_k + \omega$, $\alpha_{i,k+q}(\cdot) = \alpha_{ik}(\cdot)$, $\beta_{j,k+q}(\cdot) = \beta_{jk}(\cdot)$.

For convenience, we shall use the following notations.

$$\bar{f} = \frac{1}{\omega}\int_{\bar{k}}^{\bar{k}+\omega} f(t)dt, \quad f^+ = \max_{t\in[\bar{k},\bar{k}+\omega]\cap\mathbb{T}} |f(t)|, \quad f^- = \min_{t\in[\bar{k},\bar{k}+\omega]\cap\mathbb{T}} |f(t)|,$$

where $\bar{k} = \min\{[0, +\infty) \cap \mathbb{T}\}$, $f(t)$ is an ω-periodic function.

2. Preliminaries

Some preliminary definitions and theorems on time scales can be found in [2, 3], which are excellent references for the calculus of time scales. We will recall some basic definitions and lemmas which are used in what follows.

Let $\mathbb{T}$ be a nonempty closed subset (time scale) of $\mathbb{R}$. The forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ and the graininess $\mu : \mathbb{T} : \mathbb{T} \to \mathbb{R}^+$ are defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) = \sup\{s \in \mathbb{T} : s < t\}, \quad \mu(t) = \sigma(t) - t.$$

A point $t \in \mathbb{T}$ is called left-dense if $t > \inf\mathbb{T}$ and $\rho(t) = t$, left-scattered if left-scattered if $\rho(t) < t$, right-dense if $t < \sup\mathbb{T}$ and $\sigma(t) = t$, and right-scattered if $\sigma(t) > t$. If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T}\backslash\{m\}$; otherwise $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_k = \mathbb{T}\backslash\{m\}$; otherwise $\mathbb{T}_k = \mathbb{T}$.

A function $f : \mathbb{T} \to \mathbb{R}$ is right-dense continuous provided that it is continuous at right-dense point in $\mathbb{T}$ and its left-side limits exist at left-dense points in $\mathbb{T}$. If f is continuous at each right-dense point and each left-dense point, then f is said to be continuous function on $\mathbb{T}$.

For $y : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$, we define the delta derivative of y(t), $y^{\Delta}(t)$ to be the number (if it exists) with the property that for a given $\varepsilon > 0$, there exists a neighborhood U of t such that

$$|[y(\sigma(t)) - y(s)] - y^{\Delta}(t)[\sigma(t) - s]| < \varepsilon|\sigma(t) - s|, \quad \forall s \in U.$$

If y is continuous, then y is right-dense continuous, and if y is delta differentiable at t, then y is continuous at t. A function $r : \mathbb{T} \to \mathbb{R}$ is called regressive if $1 + \mu(t)r(t) \neq 0$ for all $t \in \mathbb{T}^k$. A function r from $\mathbb{T}$ to $\mathbb{R}$ is positively regressive if $1 + \mu(t)r(t) > 0$ for every $t \in \mathbb{T}$. Denote $\mathfrak{R}^+$ is the set of positively regressive functions from $\mathbb{T}$ to $\mathbb{R}$, and $\mathbb{T}^+ = [0, +\infty) \cap \mathbb{T}$.

If r is regressive function, then the generalized exponential function e_r is defined by

$$e_r(t, s) = \exp\Big\{\int_s^t \xi_{\mu(\tau)}(r(\tau))\Delta\tau\Big\} \quad \text{for} \quad s, t \in \mathbb{T},$$

with the cylinder transformation

$$\xi_h(z) = \begin{cases} \frac{\log(1+hz)}{h}, & \text{if } h \neq 0, \\ z, & \text{if } h = 0. \end{cases}$$

Let $p, q : \mathbb{T} \to \mathbb{R}$ be two regressive functions, we define

$$p \oplus q := p + q + \mu pq, \quad \ominus p := -\frac{p}{1 + \mu p}, \quad p \ominus q := p \oplus (\ominus q).$$

The periodic solution $z^*(t) = (x_1^*(t), \dots, x_n^*(t), y_1^*(t), \dots, y_m^*(t))^T$ of system (1.1) is said to be exponentially stable if there exists a positive constant ϑ such that for every $\varrho \in \mathbb{T}$, there exist $N = N(\varrho) > 1$ such that the solution $z(t) = (x_1(t), \dots, x_n(t), y_1(t), \dots, y_m(t))^T$ of (1.1) satisfies

$$\|z(t) - z^*(t)\| \leq N e_{-\vartheta}(t, \varrho)\Big(\sum_{i=1}^n |\phi_{x_i}(\varrho) - x_i^*(\varrho)| + \sum_{j=1}^m |\phi_{y_j}(\varrho) - y_j^*(\varrho)|\Big),$$

where $\varrho \in [-\max\{\theta, \hat{\theta}\}, 0] \cap \mathbb{T}$.

Lemma 2.1 ([17]). *If $f, g \in C(\mathbb{T}, \mathbb{R})$, and $f(t) \leq g(t)$ on $[\bar{k}, \bar{k} + \omega)$, then*

$$\int_{\bar{k}}^{\bar{k}+\omega} f(t)\Delta t \leq \int_{\bar{k}}^{\bar{k}+\omega} g(t)\Delta t.$$

Lemma 2.2 ([1]). *Let $t_1, t_2 \in [\bar{k}, \bar{k} + \omega] \cap \mathbb{T}$, $t \in \mathbb{T}$. If $f : \mathbb{T} \to \mathbb{R}$ is ω-periodic, then*

$$f(t) \leq f(t_1) + \int_{\bar{k}}^{\bar{k}+\omega} |f^{\Delta}(t)|\Delta t \quad \text{and} \quad f(t) \geq f(t_2) - \int_{\bar{k}}^{\bar{k}+\omega} |f^{\Delta}(t)|\Delta t.$$

Lemma 2.3 ([27]). *Let $a, b \in \mathbb{T}$. For rd-continuous functions $f, g : [a, b] \to \mathbb{R}$, one has*

$$\Big(\int_a^b |f(t)g(t)|\Delta t\Big)^2 \leq \Big(\int_a^b |f(t)|^2\Delta t\Big)\Big(\int_a^b |g(t)|^2\Delta t\Big).$$

Lemma 2.4 ([20]). *Let X and Z be two Banach spaces and let L be a Fredholm mapping of index zero. Let $\Omega \subset X$ be an open bounded set and let $N : \overline{\Omega} \to Z$ be a continuous operator which is L-compact on Ω. Assume that*

(a) *for each $\lambda \in (0, 1)$, $x \in \partial\Omega \cap \operatorname{Dom} L$, $Lx \neq \lambda Nx$;*
(b) *for each $x \in \partial\Omega \cap \ker L$, $QNx \neq 0$;*
(c) $\deg(JNQx, \Omega \cap \ker L, 0) \neq 0$, *where $JQN : \ker L \to \ker L$.*

Then $Lx = Nx$ has at least one solution in $\overline{\Omega} \cap \operatorname{Dom} L$.

Lemma 2.5. *Let $r : \mathbb{T} \to \mathbb{R}$ be right-dense continuous and regressive. Let $a \in \mathbb{T}$ and $y_a \in \mathbb{R}$. The unique solution of the initial value problem*

$$\begin{gathered} y^{\Delta}(t) = r(t)y(t) + h(t), \quad y(a) = y_a, \\ \Delta y(t_k) = y(t_k^+) - y(t_k^-) = \varphi_k(y(t_k)), \quad k = 1, 2, \dots, q, \end{gathered}$$

is

$$y(t) = e_r(t, a)y_a + \int_a^t e_r(t, \sigma(s))h(s)\Delta s + \sum_{k: t_k \in [a,t)_{\mathbb{T}}} e_r(t, t_k)\varphi_k(y(t_k)).$$

Proof. The proof of Lemma 2.5 is similar to that of [14, Lemma 2.7], it is omitted. □

According to (H1)–(H4) and $\bar{k} = \min\{[0, +\infty) \cap \mathbb{T}\}$, for system (1.1), finding the periodic solutions is equivalent to finding those of the boundary-value problem

$$\begin{aligned} x_i^{\Delta}(t) &= -c_i(t)x_i(t) + \sum_{j=1}^{m} a_{ij}(t)f_j(y_j(t - \tau_{ij}(t))) \\ &\quad + \sum_{j=1}^{m}\sum_{l=1}^{m} b_{ijl}(t)g_j(y_j(t - \sigma_{ijl}(t)))g_l(y_l(t - \upsilon_{ijl}(t))) + I_i(t), \\ &\quad t \in [\bar{k}, \bar{k} + \omega],\ t \neq t_k, \\ \Delta x_i(t_k) &= x_i(t_k^+) - x_i(t_k^-) = \alpha_{ik}(x_i(t_k)), x_i(\bar{k}) = x_i(\bar{k} + \omega), \\ &\qquad i = 1, 2, \dots, n, k = 1, 2, \dots, q, \\ y_j^{\Delta}(t) &= -d_j(t)y_j(t) + \sum_{i=1}^{n} e_{ji}(t)p_i(x_i(t - \hat{\tau}_{ji}(t))) \\ &\quad + \sum_{i=1}^{n}\sum_{r=1}^{n} h_{jir}(t)q_i(x_i(t - \hat{\sigma}_{jir}(t)))q_r(x_r(t - \hat{\upsilon}_{jir}(t))) + J_j(t), \\ &\quad t \in [\bar{k}, \bar{k} + \omega],\ t \neq t_k, \\ \Delta y_j(t_k) &= y_j(t_k^+) - y_j(t_k^-) = \beta_{jk}(y_j(t_k)), y_j(\bar{k}) = y_j(\bar{k} + \omega), \\ &\qquad j = 1, 2, \dots, m, k = 1, 2, \dots, q. \end{aligned} \tag{2.1}$$

To apply Lemma 2.4 to system (2.1), we first make the following preparations. For any non-negative integer q, let $t_q < \omega < t_{q+1} = \omega + t_1$ and $C[\bar{k}, \bar{k}+\omega; t_1, t_2, \dots, t_q] = \{z : [\bar{k}, \bar{k} + \omega] \cap \mathbb{T} \to \mathbb{R}^{n+m} | z(t)$ exists for $t \neq t_1, \dots, t_q$; $z(t_k^+)$ and $z(t_k^-)$ exists at $t \neq t_1, \dots, t_q$; and $z(t_k) = z(t_k^-)$, $k = 1, \dots, q\}$. Let

$$\mathbb{X} = \{z \in C[\bar{k}, \bar{k}+\omega; t_1, t_2, \dots, t_q] : z(t+\omega) = z(t),\ t \in \mathbb{T}\}, \quad \mathbb{Z} = \mathbb{X} \times \mathbb{R}^{(n+m)\times(q+1)}$$

be endowed with the norm

$$\|z\| = \sum_{i=1}^{n} \max_{t\in[\bar{k},\bar{k}+\omega]\cap\mathbb{T}} |x_i(t)| + \sum_{j=1}^{m} \max_{t\in[\bar{k},\bar{k}+\omega]\cap\mathbb{T}} |y_j(t)|,$$

for $z(t) = (x_1(t), \dots, x_n(t), y_1(t), \dots, y_m(t))^T$. Then $\mathbb{X}$ and $\mathbb{Z}$ are Banach spaces with the norm $\|\cdot\|$. Let

$$L : \operatorname{Dom} L \cap \mathbb{X} \to \mathbb{Z}, \quad z \mapsto (z^{\Delta}, \Delta z(t_1), \Delta z(t_2), \dots, \Delta z(t_q), 0), \tag{2.2}$$

$$N : \mathbb{X} \to \mathbb{Z}, \quad Nz = \begin{pmatrix} A_1(t) & \Delta x_1(t_1) & \Delta x_1(t_2) & \cdots & \Delta x_1(t_q) & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ A_n(t) & \Delta x_n(t_1) & \Delta x_n(t_1) & \cdots & \Delta x_n(t_q) & 0 \\ B_1(t) & \Delta y_1(t_1) & \Delta y_1(t_2) & \cdots & \Delta y_1(t_q) & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ B_m(t) & \Delta y_n(t_1) & \Delta y_n(t_1) & \cdots & \Delta y_m(t_q) & 0 \end{pmatrix}, \tag{2.3}$$

where $\operatorname{Dom} L = \{z \in C^1[\bar{k}, \bar{k}+\omega; t_1, t_2, \dots, t_q] : z(t+\omega) = z(t)\}$,

$$\begin{aligned} A_i(t) &= -c_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij}(t) f_j(x_j(t-\tau_{ij})) \\ &\quad + \sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}(t) g_j(x_j(t-\sigma_{ijl})) g_l(x_l(t-\upsilon_{ijl})) + I_i(t) \end{aligned}$$

for $i = 1, 2, \dots, n$;

$$\begin{aligned} B_j(t) &= -d_j(t)y_j(t) + \sum_{i=1}^{n} e_{ji}(t) p_i(x_i(t-\hat{\tau}_{ji}(t))) \\ &\quad + \sum_{i=1}^{n}\sum_{r=1}^{n} h_{jir}(t) q_i(x_i(t-\hat{\sigma}_{jir}(t))) q_r(x_r(t-\hat{\upsilon}_{jir}(t))) + J_j(t) \end{aligned}$$

for $j = 1, 2, \dots, m$. Taking $z = (f, C_1, C_2, \dots, C_q, d) \in \operatorname{Im} L \subset \mathbb{Z}$, then

$$\ker L = \{z \in \mathbb{X} : z = h \in \mathbb{R}^{n+m}\},$$

$$\operatorname{Im} L = \Big\{(f, C_1, C_2, \dots, C_q, d) \in \mathbb{Z} : \int_0^{\omega} f(s)ds + \sum_{k=1}^{q} C_k + d = 0\Big\},$$

and $\dim\ker L = \operatorname{codim}\operatorname{Im} L = n+m$. Define the two projectors

$$Pz = \frac{1}{\omega}\int_{\bar{k}}^{\bar{k}+\omega} z(t)\Delta t,$$

$$QNz = Q(f, C_1, \dots, C_q, d) = \Big(\frac{1}{\omega}\Big(\int_{\bar{k}}^{\bar{k}+\omega} f(s)\Delta s + \sum_{k=1}^{q} C_k + d\Big), 0, \dots, 0, 0\Big).$$

It is not difficult to show that P and Q are continuous and satisfy

$$\operatorname{Im} P = \ker L, \quad \operatorname{Im} L = \ker Q = \operatorname{Im}(I-Q).$$

It is easy to see that ImL is closed in $\mathbb{Z}$, which leads to the following lemma.

Lemma 2.6. *Let L and N be defined by* (2.2) *and* (2.3)*, respectively, then L is a Fredholm operator of index zero.*

Lemma 2.7. *Let L and N be defined by* (2.2) *and* (2.3)*, respectively, suppose that Ω is an open bounded subset of* $\operatorname{Dom} L$*, then N is L-compact on $\overline{\Omega}$.*

Proof. Through an easy computation, we can find that the inverse $K_q : \operatorname{Im} L \to \ker P \cap \operatorname{Dom} L$ of L_q has the form

$$(K_q z)(t) = \int_{\bar{k}}^{\bar{k}+\omega} z(s)\Delta s + \sum_{t>t_k} C_k - \frac{1}{\omega}\int_{\bar{k}}^{\bar{k}+\omega}\int_{\bar{k}}^{t} z(s)\Delta s\Delta t - \sum_{k=1}^{q} C_k.$$

Therefore,

$$QNz = \Bigg(\begin{pmatrix} \frac{1}{\omega}\int_0^\omega A_i(t)dt - \frac{1}{\omega}\sum_{k=1}^q \alpha_{ik}(x_i(t_k)) \\ \frac{1}{\omega}\int_0^\omega B_j(t)dt - \frac{1}{\omega}\sum_{k=1}^q \beta_{jk}(y_j(t_k)) \end{pmatrix} \quad 0 \quad \cdots \quad 0\Bigg)_{(n+m)\times(q+1)},$$

and then

$$\begin{aligned} K_q(I-Q)Nz &= \begin{pmatrix} \int_0^t A_i(t)dt - \sum_{t>t_k}\alpha_{ik}(x_i(t_k)) \\ \int_0^t B_j(t)dt - \sum_{t>t_k}\beta_{jk}(y_j(t_k)) \end{pmatrix}_{(n+m)\times 1} \\ &\quad - \begin{pmatrix} \frac{1}{\omega}\int_0^\omega\int_0^t A_i(s)\,ds\,dt \\ \frac{1}{\omega}\int_0^\omega\int_0^t B_j(s)\,ds\,dt \end{pmatrix}_{(n+m)\times 1} \\ &\quad - \Big(\frac{t-\bar{k}}{\omega} - \frac{1}{\omega^2}\int_{\bar{k}}^{\bar{k}+\omega}(t-\bar{k})\Delta t\Big)\begin{pmatrix} \int_0^\omega A_i(t)dt \\ \int_0^\omega B_j(t)dt \end{pmatrix}_{(n+m)\times 1} \\ &\quad - \begin{pmatrix} \sum_{k=1}^q \alpha_{ik}(x_i(t_k)) \\ \sum_{k=1}^q \beta_{jk}(y_j(t_k)) \end{pmatrix}_{(n+m)\times 1}. \end{aligned}$$

Therefore, QN and $K_q(I-Q)N$ are both continuous. Using the Arzela-Ascoli Theorem, it is easy to show that $\overline{K_p(I-Q)N(\overline{\Omega})}$ is relatively compact. Moreover, $QN(\overline{\Omega})$ is bounded. Thus, N is $L-$compact on $\overline{\Omega}$ for any open bounded set $\Omega \subset \mathbb{X}$. The proof is completed. □

3. Existence of periodic solutions

In this section, we study the existence of periodic solution of (1.1) based on Mawhins' continuation theorem.

Theorem 3.1. *Assume that* (H1)–(H4) *hold, then system* (1.1) *has at least one ω-periodic solution.*

Proof. Based on the Lemma 2.6 and Lemma 2.7, what we need to do is just to search for an appropriate open, bounded subset Ω for the application of the continuation theorem. Corresponding to the operator equation $Lz = \lambda Nz$, $\lambda \in (0,1)$, we have

$$
\begin{aligned}
x_i^{\Delta}(t) &= \lambda\Big\{ -c_i(t)x_i(t) + \sum_{j=1}^{m} a_{ij}(t) f_j(y_j(t-\tau_{ij}(t))) \\
&\quad + \sum_{j=1}^{m}\sum_{l=1}^{m} b_{ijl}(t) g_j(y_j(t-\sigma_{ijl}(t))) g_l(y_l(t-\upsilon_{ijl}(t))) + I_i(t)\Big\}, \\
&\quad t \in [\bar{k}, \bar{k}+\omega], t \neq t_k, \\
\Delta x_i(t_k) &= x_i(t_k^+) - x_i(t_k^-) = \alpha_{ik}(x_i(t_k)), \quad x_i(\bar{k}) = x_i(\bar{k}+\omega), \\
&\qquad i = 1,2,\dots,n,\ k = 1,2,\dots,q, \\
y_j^{\Delta}(t) &= \lambda\Big\{ -d_j(t)y_j(t) + \sum_{i=1}^{n} e_{ji}(t) p_i(x_i(t-\hat{\tau}_{ji}(t))) \\
&\quad + \sum_{i=1}^{n}\sum_{r=1}^{n} h_{jir}(t) q_i(x_i(t-\hat{\sigma}_{jir}(t))) q_r(x_r(t-\hat{\upsilon}_{jir}(t))) + J_j(t)\Big\}, \\
&\quad t \in [\bar{k}, \bar{k}+\omega],\ t \neq t_k, \\
\Delta y_j(t_k) &= y_j(t_k^+) - y_j(t_k^-) = \beta_{jk}(y_j(t_k)), \quad y_j(\bar{k}) = y_j(\bar{k}+\omega), \\
&\qquad j = 1,2,\dots,m,\ k = 1,2,\dots,q.
\end{aligned} \tag{3.1}
$$

For the sake of convenience, we define

$$
\|f\|_2 = \Big(\int_{\bar{k}}^{\bar{k}+\omega} |f(t)|^2 \Delta t \Big)^{1/2}, \quad \text{for } f \in (\mathbb{T}, \mathbb{R}).
$$

Suppose that $(x_1(t), x_2(t), \dots, x_n(t), y_1(t), y_2(t), \dots, y_m(t))^T \in X$ is a solution of (3.1) for a certain $\lambda \in (0,1)$. Integrating (3.1) over the interval $[\bar{k}, \bar{k}+\omega]$, we obtain

$$
\begin{aligned}
&\int_{\bar{k}}^{\bar{k}+\omega} A_i(t)\Delta t + \sum_{k=1}^{q} \alpha_{ik}(x_i(t_k)) = 0, \quad i = 1,2,\dots,n, \\
&\int_{\bar{k}}^{\bar{k}+\omega} B_j(t)\Delta t + \sum_{k=1}^{q} \beta_{jk}(y_j(t_k)) = 0, \quad j = 1,2,\dots,m.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\int_{\bar{k}}^{\bar{k}+\omega} c_i(s)x_i(s)\Delta s \\
&= \int_{\bar{k}}^{\bar{k}+\omega} \Big(\sum_{j=1}^{m} a_{ij}(t) f_j(y_j(t-\tau_{ij}(t))) \\
&\quad + \sum_{j=1}^{m}\sum_{l=1}^{m} b_{ijl}(t) g_j(y_j(t-\sigma_{ijl}(t))) g_l(y_l(t-\upsilon_{ijl}(t))) + I_i(t)\Big)\Delta t \\
&\quad + \sum_{k=1}^{q} \alpha_{ik}(x_i(t_k)),
\end{aligned} \tag{3.2}
$$

$$\begin{aligned}
&\int_{\bar{k}}^{\bar{k}+\omega} d_j(s)y_j(s)\Delta s \\
&= \int_{\bar{k}}^{\bar{k}+\omega} \Big(\sum_{i=1}^{n} e_{ji}(t)p_i(x_i(t-\hat{\tau}_{ji}(t))) \\
&\quad + \sum_{i=1}^{n}\sum_{r=1}^{n} h_{jir}(t)q_i(x_i(t-\hat{\sigma}_{jir}(t)))q_r(x_r(t-\hat{\upsilon}_{jir}(t))) + J_j(t)\Big)\Delta t \\
&\quad + \sum_{k=1}^{q} \beta_{jk}(y_j(t_k)),
\end{aligned} \tag{3.3}$$

where $i = 1,2,\dots,n$, $j = 1,2,\dots,m$. Let $\xi_i, \eta_i, \hat{\xi}_j, \hat{\eta}_j (\neq t_k) \in [\bar{k}, \bar{k}+\omega] \cap \mathbb{T}$, $k = 1,2,\dots,q$, such that $x_i(\xi_i) = \inf_{t\in[\bar{k},\bar{k}+\omega]\cap\mathbb{T}} x_i(t)$, $x_i(\eta_i) = \sup_{t\in[\bar{k},\bar{k}+\omega]\cap\mathbb{T}} x_i(t)$, $y_j(\hat{\xi}_j) = \inf_{t\in[\bar{k},\bar{k}+\omega]\cap\mathbb{T}} y_j(t)$, $y_j(\hat{\eta}_j) = \sup_{t\in[\bar{k},\bar{k}+\omega]\cap\mathbb{T}} y_j(t)$, $i = 1,2,\dots,n$, $j = 1,2,\dots,m$. Then by (3.2) and the Lemma 2.1, then we have

$$\begin{aligned}
\omega\bar{c}_i x_i(\xi_i) &\le \int_{\bar{k}}^{\bar{k}+\omega} \Big|\sum_{j=1}^{m} a_{ij}(t)f_j(y_j(t-\tau_{ij}(t))) \\
&\quad + \sum_{j=1}^{m}\sum_{l=1}^{m} b_{ijl}(t)g_j(y_j(t-\sigma_{ijl}(t)))g_l(y_l(t-\upsilon_{ijl}(t))) + I_i(t)\Big|\Delta t \\
&\quad + \sum_{k=1}^{q} |\alpha_{ik}(x_i(t_k))| \\
&\le \int_{\bar{k}}^{\bar{k}+\omega} \sum_{j=1}^{m} |a_{ij}(t)||f_j(y_j(t-\tau_{ij}(t)))|\Delta t \\
&\quad + \int_{\bar{k}}^{\bar{k}+\omega} \sum_{j=1}^{m}\sum_{l=1}^{m} |b_{ijl}(t)||g_j(y_j(t-\sigma_{ijl}(t)))||g_l(y_l(t-\upsilon_{ijl}(t)))|\Delta t \\
&\quad + \int_{\bar{k}}^{\bar{k}+\omega} |I_i(t)|\Delta t + \sum_{k=1}^{q} |\alpha_{ik}(x_i(t_k))| \\
&\le \omega\Big(\sum_{j=1}^{m} a_{ij}^{+}M_j + \sum_{j=1}^{m}\sum_{l=1}^{m} b_{ijl}^{+}N_jN_l + I_i^{+}\Big) + \sum_{k=1}^{q} |\alpha_{ik}(x_i(t_k))|
\end{aligned}$$

for $i = 1,2,\dots,n$. Hence

$$\begin{aligned}
x_i(\xi_i) &\le \frac{1}{\bar{c}_i}\Big(\Big(\sum_{j=1}^{m} a_{ij}^{+}M_j + \sum_{j=1}^{m}\sum_{l=1}^{m} b_{ijl}^{+}N_jN_l + I_i^{+}\Big) + \frac{1}{\omega}\sum_{k=1}^{q} |\alpha_{ik}(x_i(t_k))|\Big) \\
&= B_i, \quad i = 1,2,\dots,n.
\end{aligned} \tag{3.4}$$

Similarly, by (3.3) and Lemma 2.1, we obtain

$$\begin{aligned}
y_j(\hat{\xi}_j) &\le \frac{1}{\bar{d}_j}\Big(\Big(\sum_{i=1}^{n} e_{ji}^{+}\hat{M}_i + \sum_{i=1}^{n}\sum_{r=1}^{n} h_{jir}^{+}\hat{N}_i\hat{N}_r + J_j^{+}\Big) + \frac{1}{\omega}\sum_{k=1}^{q} |\beta_{jk}(y_j(t_k))|\Big) \\
&= \hat{B}_j, \quad j = 1,2,\dots,m.
\end{aligned} \tag{3.5}$$

By (3.2), we have

$$
\begin{aligned}
\omega\bar{c}_i x_i(\eta_i) \ge & -\int_{\bar{k}}^{\bar{k}+\omega} \Big| \sum_{j=1}^{m} a_{ij}(t) f_j(y_j(t-\tau_{ij}(t))) \\
& - \sum_{j=1}^{m}\sum_{l=1}^{m} b_{ijl}(t) g_j(y_j(t-\sigma_{ijl}(t))) g_l(y_l(t-\upsilon_{ijl}(t))) \\
& + I_i(t)\Big|\Delta t - \sum_{k=1}^{q} |\alpha_{ik}(x_i(t_k))| \\
\ge & -\int_{\bar{k}}^{\bar{k}+\omega} \sum_{j=1}^{m} |a_{ij}(t)||f_j(y_j(t-\tau_{ij}(t)))|\Delta t \\
& - \int_{\bar{k}}^{\bar{k}+\omega} \sum_{j=1}^{m}\sum_{l=1}^{m} |b_{ijl}(t)||g_j(y_j(t-\sigma_{ijl}(t)))||g_l(y_l(t-\upsilon_{ijl}(t)))|\Delta t \\
& - \int_{\bar{k}}^{\bar{k}+\omega} |I_i(t)|\Delta t - \sum_{k=1}^{q} |\alpha_{ik}(x_i(t_k))| \\
\ge & -\omega\Big(\sum_{j=1}^{m} a_{ij}^{+} M_j + \sum_{j=1}^{m}\sum_{l=1}^{m} b_{ijl}^{+} N_j N_l + I_i^{+}\Big) - \sum_{k=1}^{q} |\alpha_{ik}(x_i(t_k))|
\end{aligned}
$$

for $i = 1, 2, \dots, n$. Hence

$$
\begin{aligned}
x_i(\eta_i) &\ge -\frac{1}{\bar{c}_i}\Big(\Big(\sum_{j=1}^{m} a_{ij}^{+} M_j + \sum_{j=1}^{m}\sum_{l=1}^{m} b_{ijl}^{+} N_j N_l + I_i^{+}\Big) + \frac{1}{\omega}\sum_{k=1}^{q} |\alpha_{ik}(x_i(t_k))|\Big) \\
&= -B_i, \quad i = 1, 2, \dots, n.
\end{aligned} \tag{3.6}
$$

Similarly, by (3.3), we obtain

$$
\begin{aligned}
y_j(\hat{\eta}_j) &\ge -\frac{1}{\bar{d}_j}\Big(\Big(\sum_{i=1}^{n} e_{ji}^{+} \hat{M}_i + \sum_{i=1}^{n}\sum_{r=1}^{n} h_{jir}^{+} \hat{N}_i \hat{N}_r + J_j^{+}\Big) + \frac{1}{\omega}\sum_{k=1}^{q} |\beta_{jk}(y_j(t_k))|\Big) \\
&= -\hat{B}_j, \quad j = 1, 2, \dots, m.
\end{aligned} \tag{3.7}
$$

Set $t_0 = t_0^{+} = \bar{k}$, $t_{q+1} = \bar{k} + \omega$. From (3.4), (3.6) and Lemma 2.3, we have

$$
\begin{aligned}
& \int_{\bar{k}}^{\bar{k}+\omega} |x_i^{\Delta}(t)|\Delta t \\
&= \sum_{k=1}^{q} \int_{t_{k-1}^{+}}^{t_k} |x^{\Delta}(t)|\Delta t + \sum_{k=1}^{q} |x_i(t_k^{+}) - x_i(t_k)| \\
&\le \int_{\bar{k}}^{\bar{k}+\omega} |c_i(t)||x_i(t)|\Delta t + \int_{\bar{k}}^{\bar{k}+\omega} \sum_{j=1}^{m} |a_{ij}(t)||f_j(y_j(t-\tau_{ij}(t)))|\Delta t \\
&\quad + \int_{\bar{k}}^{\bar{k}+\omega} \sum_{j=1}^{m}\sum_{l=1}^{m} |b_{ijl}(t)||g_j(y_j(t-\sigma_{ijl}(t)))||g_l(y_l(t-\upsilon_{ijl}(t)))|\Delta t \\
&\quad + \int_{\bar{k}}^{\bar{k}+\omega} |I_i(t)|\Delta t + \sum_{k=1}^{q} |e_{ik}(x_i(t_k))|
\end{aligned}
$$

$$
\begin{aligned}
&\le \Big(\int_{\bar{k}}^{\bar{k}+\omega}|c_i(t)|^2\Delta t\Big)^{1/2}\Big(\int_{\bar{k}}^{\bar{k}+\omega}|x_i(t)|^2\Delta t\Big)^{1/2} \\
&\quad+\sum_{j=1}^{m}\Big(\int_{\bar{k}}^{\bar{k}+\omega}|a_{ij}(t)|^2\Delta t\Big)^{1/2}\Big(\int_{\bar{k}}^{\bar{k}+\omega}|f_j(y_j(t-\tau_{ij}(t)))|^2\Delta t\Big)^{1/2} \\
&\quad+\sum_{j=1}^{m}\sum_{l=1}^{m}b_{ijl}^{+}\Big(\int_{\bar{k}}^{\bar{k}+\omega}|g_j(y_j(t-\sigma_{ijl}(t)))|^2\Delta t\Big)^{1/2} \\
&\quad\times\Big(\int_{\bar{k}}^{\bar{k}+\omega}|g_l(y_l(t-\upsilon_{ijl}(t)))|^2\Delta t\Big)^{1/2}+I_i^{+}\omega+\sum_{k=1}^{q}|\alpha_{ik}(x_i(t_k))| \\
&\le \sqrt{\omega}c_i^{+}\|x_i\|_2+\sum_{j=1}^{m}\omega a_{ij}^{+}M_j+\sum_{j=1}^{m}\sum_{l=1}^{m}\omega b_{ijl}^{+}N_jN_l+I_i^{+}\omega+\sum_{k=1}^{q}|\alpha_{ik}(x_i(t_k))|.
\end{aligned}\tag{3.8}
$$

Similarly, from (3.5) and (3.7), and the Lemma 2.3, we obtain

$$
\begin{aligned}
\int_{\bar{k}}^{\bar{k}+\omega}|y_j^{\Delta}(t)|\Delta t &\le \sqrt{\omega}d_j^{+}\|y_j\|_2+\sum_{i=1}^{n}\omega e_{ji}^{+}\hat{M}_i+\sum_{i=1}^{n}\sum_{r=1}^{n}\omega b_{jir}^{+}\hat{N}_i\hat{N}_r \\
&\quad+J_j^{+}\omega+\sum_{k=1}^{q}|\beta_{jk}(y_j(t_k))|.
\end{aligned}\tag{3.9}
$$

From Lemma 2.5 and 3.1, for $i=1,2,\dots,n$, $j=1,2,\dots,m$, we obtain

$$
\begin{aligned}
x_i(t)&=e_{-\lambda c_i(t)}(t,\bar{k})x_i(\bar{k})+\int_{\bar{k}}^{t}\lambda e_{-\lambda c_i(t)}(t,\sigma(s))\Big(\sum_{j=1}^{m}a_{ij}(s)f_j(y_j(s-\tau_{ij}(s))) \\
&\quad+\sum_{j=1}^{m}\sum_{l=1}^{m}b_{ijl}(s)g_j(y_j(s-\sigma_{ijl}(s)))g_l(y_l(s-\upsilon_{ijl}(s)))+I_i(s)\Big)\Delta s \\
&\quad+\sum_{k:t_k\in[\bar{k},t)_{\mathbb{T}}}e_{-\lambda c_i(t)}(t,t_k)\alpha_{ik}(x(t_k)),
\end{aligned}
$$

$$
\begin{aligned}
y_j(t)&=e_{-\lambda d_j(t)}(t,\bar{k})y_j(\bar{k})+\int_{\bar{k}}^{t}\lambda e_{-\lambda d_j(t)}(t,\sigma(s))\Big(\sum_{i=1}^{n}e_{ji}(s)p_i(x_i(s-\hat{\tau}_{ji}(s))) \\
&\quad+\sum_{i=1}^{n}\sum_{r=1}^{n}h_{jir}(s)q_r(x_i(s-\hat{\sigma}_{jir}(s)))q_r(x_r(s-\hat{\upsilon}_{jir}(s)))+J_j(s)\Big)\Delta s \\
&\quad+\sum_{k:t_k\in[\bar{k},t)_{\mathbb{T}}}e_{-\lambda d_j(t)}(t,t_k)\beta_{jk}(y(t_k)).
\end{aligned}
$$

Hence

$$
\begin{aligned}
|x_i(t)|&\le|x_i(\bar{k})|+\sum_{j=1}^{m}\int_{\bar{k}}^{\bar{k}+\omega}|a_{ij}(s)||f_j(y_j(s-\tau_{ij}(s)))|\Delta s \\
&\quad+\sum_{j=1}^{m}\sum_{l=1}^{m}\int_{\bar{k}}^{\bar{k}+\omega}|b_{ijl}(s)||g_j(y_j(s-\sigma_{ijl}(s)))||g_l(y_l(s-\upsilon_{ijl}(s)))|\Delta s \\
&\quad+\int_{\bar{k}}^{\bar{k}+\omega}|I_i(s)|\Delta s+\sum_{k:t_k\in[\bar{k},t)_{\mathbb{T}}}|\alpha_{ik}(x(t_k))|
\end{aligned}
$$

$$
\begin{aligned}
&\le |x_i(\bar{k})| + \sum_{j=1}^{m} \bar{a}_{ij} N_j \omega + \sum_{j=1}^{m}\sum_{l=1}^{m} \bar{b}_{ijl} N_j N_l \omega + \bar{I}_i \omega + \sum_{k=1}^{q} |\alpha_{ik}(x_i(t_k))| \\
&= u_i, \\
|y_j(t)| &\le |y_j(\bar{k})| + \sum_{i=1}^{n} \bar{e}_{ji} \hat{N}_i \omega + \sum_{i=1}^{n}\sum_{r=1}^{n} \bar{h}_{jir} \hat{N}_i \hat{N}_r \omega + \bar{J}_j \omega + \sum_{k=1}^{q} |\beta_{jk}(y_j(t_k))| \\
&= \hat{u}_j,
\end{aligned}
$$

where $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$; that is,

$$
\begin{aligned}
\|x_i\|_2 &= \Big(\int_{\bar{k}}^{\bar{k}+\omega} |x_i(t)|^2 \Delta t \Big)^{1/2} \le u_i \sqrt{\omega}, \\
\|y_j\|_2 &= \Big(\int_{\bar{k}}^{\bar{k}+\omega} |y_j(t)|^2 \Delta t \Big)^{1/2} \le \hat{u}_j \sqrt{\omega}.
\end{aligned} \tag{3.10}
$$

Substituting (3.10) in (3.8) and (3.9), we obtain

$$
\begin{aligned}
\int_{\bar{k}}^{\bar{k}+\omega} |x_i^{\Delta}(t)| \Delta t &\le \omega c_i^{+} u_i + \sum_{j=1}^{m} \omega a_{ij}^{+} M_j + \sum_{j=1}^{m}\sum_{l=1}^{m} \omega b_{ijl}^{+} N_j N_l \\
&\quad + I_i^{+} \omega + \sum_{k=1}^{q} |\alpha_{ik}(x_i(t_k))|, \quad i = 1, 2, \dots, n,
\end{aligned} \tag{3.11}
$$

and

$$
\begin{aligned}
\int_{\bar{k}}^{\bar{k}+\omega} |y_j^{\Delta}(t)| \Delta t &\le \omega d_j^{+} \hat{u}_j + \sum_{i=1}^{n} \omega e_{ji}^{+} \hat{M}_i + \sum_{j=1}^{n}\sum_{r=1}^{n} \omega h_{jir}^{+} \hat{N}_i \hat{N}_r \\
&\quad + J_j^{+} \omega + \sum_{k=1}^{q} |\beta_{jk}(y_j(t_k))|, \quad j = 1, 2, \dots, m.
\end{aligned} \tag{3.12}
$$

From (3.6)-(3.9) and (3.11)-(3.12) and Lemma 2.2, there exist positive constants $\zeta_i, \hat{\zeta}_j$ ($i = 1, 2, \dots, n, j = 1, 2, \dots, m$) such that for $t \in [\bar{k}, \bar{k}+\omega] \cap \mathbb{T}$, $|x_i(t)| \le \zeta_i$, $|y_j(t)| \le \hat{\zeta}_j$, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$. Clearly, ζ_i and $\hat{\zeta}_j$ ($i = 1, 2, \dots, n, j = 1, 2, \dots, m$) are independent of λ. Denote $H^* = \sum_{i=1}^{n} \zeta_i + \sum_{j=1}^{m} \hat{\zeta}_j + C$, where $C > 0$ is taken sufficiently large such that

$$
\begin{aligned}
&\min_{1 \le i \le n, 1 \le j \le m} \{\bar{c}_i, \bar{d}_j\} H^* \\
&> n \max \Big(|\bar{I}_i| + \sum_{j=1}^{m} |\bar{a}_{ij}| M_j + \sum_{j=1}^{m}\sum_{l=1}^{m} \bar{b}_{ijl} M_j N_l - \frac{1}{\omega} \sum_{k=1}^{q} |\alpha_{ik}(x_i(t_k))| \Big) \\
&\quad + m \max \Big(|\bar{J}_j| + \sum_{i=1}^{n} |\bar{e}_{ji}| \hat{M}_i + \sum_{i=1}^{n}\sum_{r=1}^{n} \bar{h}_{jir} \hat{M}_i \hat{N}_r - \frac{1}{\omega} \sum_{k=1}^{q} |\beta_{jk}(y_j(t_k))| \Big).
\end{aligned}
$$

Now we take $\Omega = \{(x_1(t), \dots, x_n(t), y_1(t), \dots, y_m(t))^T : \|(x_1(t), \dots, x_n(t), y_1(t), \dots, y_m(t))^T\| < H^*\}$. Thus (a) of Lemma 2.4 is satisfied. When $(x_1(t), \dots, x_n(t),$

$y_1(t), \dots, y_m(t))^T \in \partial\Omega \cap \mathbb{R}^n$, $(x_1(t), \dots, x_n(t), y_1(t), \dots, y_m(t))^T$ is a constant vector in $\mathbb{R}^n$ with $|x_1| + \dots + |x_n| + |y_1| + \dots + |y_m| = H^*$, then

$$QN\begin{pmatrix}(x_i)_{n\times 1}\\(y_j)_{m\times 1}\end{pmatrix} = \begin{pmatrix} \Big(-\bar{c}_i x_i(t) + \sum_{j=1}^{m} \bar{a}_{ij} f_j(y_j(t-\tau_{ij}(t))) \\ \quad + \sum_{j=1}^{m}\sum_{l=1}^{m} \bar{b}_{ijl} g_j(y_j(t-\sigma_{ijl}(t))) g_l(y_l(t-\upsilon_{ijl}(t))) \\ \quad + \bar{I}_i - \frac{1}{\omega}\sum_{k=1}^{q}\alpha_{ik}(x_i(t_k))\Big)_{n\times 1} \\ \Big(-\bar{d}_j y_j(t) + \sum_{i=1}^{n} \bar{e}_{ji} p_i(x_i(t-\hat{\tau}_{ji}(t))) \\ + \sum_{i=1}^{n}\sum_{r=1}^{n} \bar{h}_{jir} q_i(x_i(t-\hat{\sigma}_{jir}(t))) q_r(x_r(t-\hat{\upsilon}_{jir}(t))) \\ + \bar{J}_j - \frac{1}{\omega}\sum_{k=1}^{q}\beta_{jk}(y_j(t_k))\Big)_{m\times 1} \end{pmatrix}_{(n+m)\times 1}.$$

Therefore,

$$\begin{aligned}
&\Big\| QN\begin{pmatrix}(x_i)_{n\times 1}\\(y_j)_{m\times 1}\end{pmatrix}\Big\| \\
&= \sum_{i=1}^{n}\Big|\bar{c}_i x_i(t) - \sum_{j=1}^{m}\bar{a}_{ij} f_j(y_j(t-\tau_{ij}(t))) \\
&\quad - \sum_{j=1}^{m}\sum_{l=1}^{m}\bar{b}_{ijl} g_j(y_j(t-\sigma_{ijl}(t))) g_l(y_l(t-\upsilon_{ijl}(t))) - \bar{I}_i + \frac{1}{\omega}\sum_{k=1}^{q}\alpha_{ik}(x_i(t_k))\Big| \\
&\quad + \sum_{j=1}^{m}\Big|\bar{d}_j y_j(t) - \sum_{i=1}^{n}\bar{e}_{ji} p_i(x_i(t-\hat{\tau}_{ji}(t))) \\
&\quad - \sum_{i=1}^{n}\sum_{r=1}^{n}\bar{b}_{jir} q_i(x_i(t-\hat{\sigma}_{jir}(t))) q_r(x_r(t-\hat{\upsilon}_{jir}(t))) - \bar{J}_j + \frac{1}{\omega}\sum_{k=1}^{q}\beta_{jk}(y_j(t_k))\Big| \\
&\ge \sum_{i=1}^{n}\bar{c}_i|x_i(t)| + \frac{n}{\omega}\sum_{k=1}^{q}|\alpha_{ik}(x_i(t_k))| - \sum_{i=1}^{n}\sum_{j=1}^{m}|\bar{a}_{ij}|M_j - \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{l=1}^{m}\bar{b}_{ijl}M_jN_l \\
&\quad - \sum_{i=1}^{n}|\bar{I}_i| + \sum_{j=1}^{m}\bar{d}_j|y_j(t)| + \frac{m}{\omega}\sum_{k=1}^{q}|\beta_{jk}(y_j(t_k))| - \sum_{j=1}^{m}\sum_{i=1}^{n}|\bar{e}_{ji}|\hat{M}_i \\
&\quad - \sum_{j=1}^{m}\sum_{i=1}^{n}\sum_{r=1}^{n}\bar{b}_{jir}\hat{M}_i\hat{N}_r - \sum_{j=1}^{m}|\bar{J}_j| \\
&\ge \sum_{i=1}^{n}(\bar{c}_i|x_i(t)|) + \sum_{j=1}^{m}(\bar{d}_j|y_j(t)|) \\
&\quad - \sum_{i=1}^{n}\Big(|\bar{I}_i| + \sum_{j=1}^{m}|\bar{a}_{ij}|M_j + \sum_{j=1}^{m}\sum_{l=1}^{m}\bar{b}_{ijl}M_jN_l - \frac{1}{\omega}\sum_{k=1}^{q}|\alpha_{ik}(x_i(t_k))|\Big)
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{j=1}^{m} \Big(|\bar{J}_j| + \sum_{i=1}^{n} |\bar{e}_{ji}| \hat{M}_i + \sum_{i=1}^{n} \sum_{r=1}^{n} \bar{h}_{jir} \hat{M}_i \hat{N}_r - \frac{1}{\omega} \sum_{k=1}^{q} |\beta_{jk}(y_j(t_k))| \Big) \\
&\geq \min(\bar{c}_i) \sum_{i=1}^{n} |x_i(t)| + \min(\bar{d}_j) \sum_{j=1}^{m} |y_j(t)| \\
&\quad - n \max \Big(|\bar{I}_i| + \sum_{j=1}^{m} |\bar{a}_{ij}| M_j + \sum_{j=1}^{m} \sum_{l=1}^{m} \bar{b}_{ijl} M_j N_l - \frac{1}{\omega} \sum_{k=1}^{q} |\alpha_{ik}(x_i(t_k))| \Big) \\
&\quad - m \max \Big(|\bar{J}_j| + \sum_{i=1}^{n} |\bar{e}_{ji}| \hat{M}_i + \sum_{i=1}^{n} \sum_{r=1}^{n} \bar{h}_{jir} \hat{M}_i \hat{N}_r - \frac{1}{\omega} \sum_{k=1}^{q} |\beta_{jk}(y_j(t_k))| \Big) > 0.
\end{aligned}
$$

Consequently, $QN(x_1, \dots, x_n, y_1, \dots, y_m)^T \neq (0, \dots, 0, 0, \dots, 0)^T$ for $(x_1, \dots, x_n, y_1, \dots, y_m)^T \in \partial\Omega \cap \ker L$. This satisfies condition (b) of Lemma 2.4. Define $\Psi : \ker L \times [0,1] \to X$ by

$$
\begin{aligned}
\Psi(x_1, \dots, x_n, y_1, \dots, y_m, \chi) &= -\chi(x_1, \dots, x_n, y_1, \dots, y_m)^T \\
&\quad + (1-\chi) QN(x_1, \dots, x_n, y_1, \dots, y_m)^T.
\end{aligned}
$$

When $(x_1, \dots, x_n, y_1, \dots, y_m)^T \in \partial\Omega \cap \ker L$, $(x_1, \dots, x_n, y_1, \dots, y_m)^T$ is a constant vector in $\mathbb{R}^{n+m}$ with $\sum_{i=1}^{n} |x_i| + \sum_{j=1}^{m} |y_j| = H^*$, we have

$$
\Psi(x_1, \dots, x_n, y_1, \dots, y_m, \chi) \neq (0, \dots, 0, 0, \dots, 0)^T.
$$

Therefore,

$$
\begin{aligned}
&\deg \Big(QN(x_1(t), \dots, x_n(t), y_1(t), \dots, y_m(t))^T, \Omega \cap \ker L, (0, \dots, 0, 0, \dots, 0)^T \Big) \\
&= \deg \Big(QN(-x_1(t), \dots, -x_n(t), -y_1(t), \dots, -y_m(t))^T, \Omega \cap \ker L, \\
&(0, \dots, 0, 0, \dots, 0)^T \Big) \neq 0.
\end{aligned}
$$

Condition (c) of Lemma 2.4 is also satisfied. Thus, by Lemma 2.4 we obtain that $Lx = Nx$ has at least one solution in $\mathbb{X}$. That is, system (1.1) has at least one ω-periodic solution. The proof is complete. □

4. Global exponential asymptotic stability of periodic solutions

In this section, we will construct some suitable Lyapunov functions to study the global exponential asymptotic stability of the periodic solution of (1.1).

Theorem 4.1. *Assume that* (H1)–(H4) *hold. Suppose further that*

(H5) *there exists $n+m$ positive constants $\varepsilon_i > 0$ and $\varepsilon_{n+j} > 0$, $i = 1, 2 \dots, n$, $j = 1, 2 \dots, m$ such that*

$$
-c_i^- \varepsilon_i + \sum_{j=1}^{m} a_{ij}^+ L_j \varepsilon_{n+j} + \sum_{j=1}^{m} \sum_{l=1}^{m} b_{ijl}^+ (H_j N_l \varepsilon_{n+j} + H_l N_j \varepsilon_{n+l}) < 0, \quad i = 1, 2, \dots, n,
$$

$$
-d_j^- \varepsilon_{n+j} + \sum_{i=1}^{n} e_{ji}^+ \hat{L}_i \varepsilon_i + \sum_{i=1}^{n} \sum_{r=1}^{n} h_{jir}^+ \Big(\hat{H}_i \hat{N}_r \varepsilon_i + \hat{H}_r \hat{N}_i \varepsilon_r \Big) < 0, \quad j = 1, 2, \dots, m;
$$

(H6) *the impulse operators $e_i(x_i(t))$, $i = 1, 2, \dots, n$ satisfy*

$$
\alpha_i(x_i(t_k)) = -\gamma_{ik}(x_i(t_k) - x_i^*(t)), \quad 0 < \gamma_{ik} < 2, \quad i = 1, 2, \dots, n, \quad k \in \mathbb{Z}^+,
$$

$$\beta_j(y_j(t_k)) = -\delta_{jk}(y_j(t_k) - y_j^*(t)), \quad 0 < \delta_{jk} < 2, \quad j = 1, 2, \dots, m, \quad k \in \mathbb{Z}^+.$$

Then the ω-periodic solution of (1.1) *is globally exponentially stable.*

Proof. According to Theorem 3.1, we know that (1.1) has an ω-periodic solution $z^*(t) = (x_1^*(t), \dots, x_n^*(t), y_1^*(t), \dots, y_m^*(t))^T$. Suppose that $x(t) = (x_1(t), \dots, x_n(t), y_1(t), \dots, y_m(t))^T$ is an arbitrary solution of (1.1). Then it follows from system (1.1) that

$$\begin{aligned}
(x_i(t) - x_i^*(t))^\Delta &= -c_i(t)(x_i(t) - x_i^*(t)) \\
&\quad + \sum_{j=1}^m a_{ij}(t)\Big(f_j(y_j(t-\tau_{ij}(t))) - f_j(y_j^*(t-\tau_{ij}(t)))\Big) \\
&\quad + \sum_{j=1}^m \sum_{l=1}^m b_{ijl}(t)\Big(g_j(y_j(t-\sigma_{ijl}(t)))g_l(y_l(t-\upsilon_{ijl}(t))) \\
&\quad - g_j(y_j^*(t-\sigma_{ijl}(t)))g_l(y_l^*(t-\upsilon_{ijl}(t)))\Big),
\end{aligned}$$

$$\begin{aligned}
(y_j(t) - y_j^*(t))^\Delta &= -d_j(t)(y_j(t) - y_j^*(t)) \\
&\quad + \sum_{i=1}^n e_{ji}(t)\Big(p_i(x_i(t-\hat{\tau}_{ji}(t))) - p_i(x_i^*(t-\hat{\tau}_{ji}(t)))\Big) \\
&\quad + \sum_{i=1}^n \sum_{r=1}^n h_{jir}(t)\Big(q_i(x_i(t-\hat{\sigma}_{jir}(t)))q_r(x_r(t-\hat{\upsilon}_{jir}(t))) \\
&\quad - q_i(x_i^*(t-\hat{\sigma}_{jir}(t)))q_r(x_r^*(t-\hat{\upsilon}_{jir}(t)))\Big),
\end{aligned}$$

for $i = 1, 2, \dots, n$, $t > 0$, $t \neq t_k$, $k \in \mathbb{Z}^+$, with initial values given by

$$\begin{gathered}
x_i(t) = \phi_{x_i}(s), \quad s \in [-\theta, 0] \cap \mathbb{T}, \quad i = 1, 2, \dots, n, \\
y_j(t) = \phi_{y_j}(s), \quad s \in [-\hat{\theta}, 0] \cap \mathbb{T}, \quad j = 1, 2, \dots, m,
\end{gathered}$$

where θ and $\hat{\theta}$ are defined as before. By condition (H3), we obtain

$$\begin{aligned}
&[(x_i(t) - x_i^*(t))^\Delta]^+ \\
&\leq -c_i^-|x_i(t) - x_i^*(t)| + \sum_{j=1}^m a_{ij}^+ L_j |y_j(t-\tau_{ij}(t)) - y_j^*(t-\tau_{ij}(t))| \\
&\quad + \sum_{j=1}^m \sum_{l=1}^m b_{ijl}^+\Big(H_j N_l |y_j(t-\sigma_{ijl}(t)) - y_j^*(t-\sigma_{ijl}(t))| \\
&\quad + H_l N_j |y_l(t-\upsilon_{ijl}(t)) - y_l^*(t-\upsilon_{ijl}(t))|\Big),
\end{aligned} \tag{4.1}$$

$$\begin{aligned}
[(y_j(t) - y_j^*(t))^\Delta]^+ &\leq -d_j^-|y_j(t) - y_j^*(t)| + \sum_{i=1}^n e_{ji}^+ \hat{L}_i |x_i(t-\hat{\tau}_{ji}(t)) - x_i^*(t-\hat{\tau}_{ji}(t))| \\
&\quad + \sum_{i=1}^n \sum_{r=1}^n h_{jir}^+\Big(\hat{H}_i \hat{N}_r |x_i(t-\hat{\sigma}_{jir}(t)) - x_i^*(t-\hat{\sigma}_{jir}(t))| \\
&\quad + \hat{H}_r \hat{N}_i |x_r(t-\hat{\upsilon}_{jir}(t)) - x_r^*(t-\hat{\upsilon}_{jir}(t))|\Big),
\end{aligned}$$

for $i=1,2,\dots,n$, $j=1,2,\dots,m$, $t>0$, $t\neq t_k$, $k\in\mathbb{Z}^+$, where $[x_i^\Delta(t)]^+$ and $[y_j^\Delta(t)]^+$ denote the upper right derivative. Also, in view of condition (H6), one has

$$\begin{aligned} x_i(t_k+0)-x_i^*(t_k+0) &= x_i(t_k)+\alpha_i(x_i(t_k))-x_i^*(t_k)-\alpha_i(x_i^*(t_k))\\ &= (1-\gamma_{ik})(x_i(t_k)-x_i^*(t_k)),\\ y_j(t_k+0)-y_j^*(t_k+0) &= y_j(t_k)+\beta_j(y_j(t_k))-y_j^*(t_k)-\beta_i(y_j^*(t_k))\\ &= (1-\beta_{jk})(y_j(t_k)-y_j^*(t_k)), \end{aligned}$$

thus

$$\begin{aligned} |x_i(t_k+0)-x_i^*(t_k+0)| &= |1-\gamma_{ik}||x_i(t_k)-x_i^*(t_k)|\le |x_i(t_k)-x_i^*(t_k)|,\\ |y_j(t_k+0)-y_j^*(t_k+0)| &= |1-\delta_{jk}||y_j(t_k)-y_j^*(t_k)|\le |y_j(t_k)-y_j^*(t_k)|, \end{aligned}$$

for $i=1,2,\dots,n$, $j=1,2,\dots,m$, $k\in\mathbb{Z}^+$. According to condition (H5), it can always find a small enough constant $\eta>0$ satisfying $1-\mu(t)\eta>0$ for all $t\in\mathbb{T}$, namely, $-\eta\in\mathfrak{R}^+$ such that

$$\begin{aligned} &(-c_i^-+\eta)\varepsilon_i+\sum_{j=1}^m a_{ij}^+L_je_\eta(t,t-\tau_{ij}(t))\varepsilon_{n+j}\\ &+\sum_{j=1}^m\sum_{l=1}^m b_{ijl}^+\big(H_jN_l\varepsilon_{n+j}e_\eta(t,t-\sigma_{ijl}(t))+H_lN_j\varepsilon_{n+l}e_\eta(t,t-\upsilon_{ijl}(t))\big)<0 \end{aligned}\tag{4.2}$$

for $i=1,2,\dots,n$;

$$\begin{aligned} &(-d_j^-+\eta)\varepsilon_{n+j}+\sum_{i=1}^n e_{ji}^+\hat{L}_ie_\eta(t,t-\hat{\tau}_{ji}(t))\varepsilon_i\\ &+\sum_{i=1}^n\sum_{r=1}^n h_{jir}^+\big(\hat{H}_i\hat{N}_r\varepsilon_ie_\eta(t,t-\hat{\sigma}_{jir}(t))+\hat{H}_r\hat{N}_i\varepsilon_re_\eta(t,t-\hat{\upsilon}_{jir}(t))\big)<0 \end{aligned}$$

for $j=1,2,\dots,m$.

Now we define a Lyapunov function $V=(\varphi_1,\dots,\varphi_n,\psi_1,\dots,\psi_m)^T$, with

$$\begin{aligned} \varphi_i(t) &= e_\eta(t,\varrho)|x_i(t)-x_i^*(t)|,\quad t\in[-\theta,\infty]\cap\mathbb{T}, i=1,2,\dots,n,\\ \psi_j(t) &= e_\eta(t,\varrho)|y_j(t)-y_j^*(t)|,\quad t\in[-\hat{\theta},\infty]\cap\mathbb{T}, j=1,2,\dots,m, \end{aligned}$$

where $\varrho\in[-\max\{\theta,\hat{\theta}\},0]\cap\mathbb{T}$. In view of (4.1), we obtain

$$\begin{aligned} &[\varphi_i^\Delta(t)]^+\\ &= \eta e_\eta(t,\varrho)|x_i(t)-x_i^*(t)|+e_\eta\sigma(t),\varrho)\operatorname{sign}(x_i(t)-x_i^*(t))\\ &\quad\times\Big(-c_i(t)(x_i(t)-x_i^*(t))+\sum_{j=1}^m a_{ij}(t)\big(f_j(y_j(t-\tau_{ij}(t)))-f_j(y_j^*(t-\tau_{ij}(t)))\big)\\ &\quad+\sum_{j=1}^m\sum_{l=1}^m b_{ijl}(t)\big(g_j(y_j(t-\sigma_{ijl}(t)))g_l(y_l(t-\upsilon_{ijl}(t)))\\ &\quad-g_j(y_j^*(t-\sigma_{ijl}(t)))g_l(y_l^*(t-\upsilon_{ijl}(t)))\big)\Big)\\ &\le \eta e_\eta(t,\varrho)|x_i(t)-x_i^*(t)|+e_\eta(\sigma(t),\varrho)\\ &\quad\times\Big(-c_i^-|x(t)-x^*(t)|+\sum_{j=1}^m a_{ij}^+L_j|y_j(t-\tau_{ij}(t))-y_j^*(t-\tau_{ij}(t))| \end{aligned}\tag{4.3}$$

$$
\begin{aligned}
&+\sum_{j=1}^{m}\sum_{l=1}^{m} b_{ijl}^{+}\big(H_j N_l |y_j(t-\sigma_{ijl}(t)) - y_j^*(t-\sigma_{ijl}(t))| \\
&+ H_l N_j |y_l(t-\upsilon_{ijl}(t)) - y_l^*(t-\upsilon_{ijl}(t))|\big)\Big) \\
\le{}& e_\eta(\sigma(t),\varrho)\Big((-c_i^- + \eta)|x_i(t) - x_i^*(t)| + \sum_{j=1}^{m} a_{ij}^{+} L_j |y_j(t-\tau_{ij}(t)) - y_j^*(t-\tau_{ij}(t))| \\
&+\sum_{j=1}^{m}\sum_{l=1}^{m} b_{ijl}^{+}\big(H_j N_l |y_j(t-\sigma_{ijl}(t)) - y_j^*(t-\sigma_{ijl}(t))| \\
&+ H_l N_j |y_l(t-\upsilon_{ijl}(t)) - y_l^*(t-\upsilon_{ijl}(t))|\big)\Big) \\
\le{}& (1+\mu(t)\eta)\Big((-c_i^- + \eta)\varphi_i(t) + \sum_{j=1}^{m} a_{ij}^{+} L_j e_\eta(t, t-\tau_{ij}(t))\psi_j(t-\tau_{ij}(t)) \\
&+\sum_{j=1}^{m}\sum_{l=1}^{m} b_{ijl}^{+}\big(H_j N_l e_\eta(t, t-\sigma_{ijl}(t))\psi_j(t-\sigma_{ijl}(t)) \\
&+ H_l N_j e_\eta(t, t-\upsilon_{ijl}(t))\psi_l(t-\upsilon_{ijl}(t))\big)\Big),
\end{aligned}
$$

$$
\begin{aligned}
&[\psi_i^\Delta(t)]^+ \\
&\le (1+\mu(t)\eta)\Big((-d_j^- + \eta)\psi_j(t) + \sum_{i=1}^{n} e_{ji}^{+}\hat{L}_i e_\eta(t, t-\hat{\tau}_{ji}(t))\varphi_i(t-\hat{\tau}_{ji}(t)) \\
&\quad+\sum_{i=1}^{n}\sum_{r=1}^{n} h_{jir}^{+}\big(\hat{H}_i\hat{N}_r e_\eta(t, t-\hat{\sigma}_{jir}(t))\varphi_i(t-\hat{\sigma}_{jir}(t)) \\
&\quad+\hat{H}_r\hat{N}_i e_\eta(t, t-\hat{\upsilon}_{jir}(t))\varphi_r(t-\hat{\upsilon}_{jir}(t))\big)\Big),
\end{aligned}
\tag{4.4}
$$

for $i = 1,2,\dots,n$, $j = 1,2,\dots,m$. Also,

$$
\begin{gathered}
\varphi_i(t_k+0) = |1-\gamma_{ik}|\varphi_i(t_k) \le \varphi_i(t_k), \quad i = 1,2,\dots,n,\ k \in \mathbb{Z}^+, \\
\psi_j(t_k+0) = |1-\delta_{jk}|\psi_j(t_k) \le \psi_j(t_k), \quad j = 1,2,\dots,m,\ k \in \mathbb{Z}^+.
\end{gathered}
$$

Let $\varepsilon_M = \max_{1\le i\le n, 1\le j\le m}\{\varepsilon_i, \varepsilon_{n+j}\}$, $\varepsilon_L = \min_{1\le i\le n, 1\le j\le m}\{\varepsilon_i, \varepsilon_{n+j}\}$, $l_0 = (1-\varrho)(\sum_{i=1}^{n}|\phi_{x_i} - x_i^*|_0 + \sum_{j=1}^{m}|\phi_{y_j} - y_j^*|_0)/\varepsilon_L$, where $-\varrho \ge 0$ is a constant, $|\phi_{x_i} - x_i^*|_0 = \sup_{\varrho\in[-\theta,0]\cap\mathbb{T}}|\phi_{x_i}(\varrho) - x_i^*(\varrho)|$, $|\phi_{y_j} - y_j^*|_0 = \sup_{\varrho\in[-\hat{\theta},0]\cap\mathbb{T}}|\phi_{y_j}(\varrho) - y_j^*(\varrho)|$. Then

$$
\begin{gathered}
|\varphi_i(\varrho)| = e_\eta(t,\varrho)|\phi_{x_i}(\varrho) - x_i^*(\varrho)| < \varepsilon_i l_0, \quad \varrho \in [-\theta, 0]\cap\mathbb{T},\ i = 1,2,\dots,n, \\
|\psi_i(\varrho)| = e_\eta(t,\varrho)|\phi_{y_j}(\sigma) - y_j^*(\varrho)| < \varepsilon_{n+j} l_0, \quad \varrho \in [-\hat{\theta}, 0]\cap\mathbb{T},\ j = 1,2,\dots,m.
\end{gathered}
$$

In the following, we will prove that $|\varphi_i(t)| < \varepsilon_i l_0$, $|\psi_j(t)| < \varepsilon_{n+j} l_0$, for $t > 0$, $i = 1,2,\dots,n$, $j = 1,2,\dots,m$. If it is not true, no loss of generality, then there exist some i_0 and t_1 ($t_1 > 0$) such that $|\varphi_{i_0}(t_1)| = \varepsilon_{i_0} l_0$, $[\varphi_{i_0}^\Delta(t_1)]^+ \ge 0$ and $|\varphi_i(t)| < \varepsilon_i l_0$, $t \in [-\theta, t_1]\cap\mathbb{T}$, $|\psi_j(t)| < \varepsilon_{n+j} l_0$, $t \in [-\hat{\theta}, t_1]\cap\mathbb{T}$, for $i = 1,2,\dots,n$, $j = 1,2,\dots,m$. However, from (4.2) and (4.3), we obtain

$$
[\varphi_{i_0}^\Delta(t_1)]^+
$$

$$
\begin{aligned}
&\le (1+\mu(t_1)\eta)\Big((-c_{i_0}^- + \eta)\varepsilon_{i_0} + \sum_{j=1}^m a_{i_0j}^+ L_j e_\eta(t_1, t_1-\tau_{i_0j}(t_1))\varepsilon_j \\
&\quad + \sum_{j=1}^m \sum_{l=1}^m b_{i_0jl}^+ \big(H_j N_l e_\eta(t_1, t_1-\sigma_{i_0jl}(t_1))\varepsilon_j + H_l N_j e_\eta(t_1, t_1-\upsilon_{i_0jl}(t_1))\varepsilon_l\big)\Big) l_0 \\
&< 0,
\end{aligned}
$$

this is a contradiction. So $|\varphi_i(t)| < \varepsilon_i l_0$, $|\psi_j(t)| < \varepsilon_{n+j} l_0$, for $t > 0$, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$. That is,

$$
e_\eta(t,\varrho)|x_i(t) - x_i^*(t)| < \varepsilon_i l_0 \le \frac{\varepsilon_i}{\varepsilon_L}(1-\varrho)\Big(\sum_{i=1}^n |\phi_{x_i} - x_i^*|_0 + \sum_{j=1}^m |\phi_{y_j} - y_j^*|_0\Big)
$$

for $t > 0$, $i = 1, 2, \dots, n$, and

$$
e_\eta(t,\varrho)|y_j(t) - y_j^*(t)| < \varepsilon_{n+j} l_0 \le \frac{\varepsilon_{n+j}}{\varepsilon_L}(1-\varrho)\Big(\sum_{i=1}^n |\phi_{x_i} - x_i^*|_0 + \sum_{j=1}^m |\phi_{y_j} - y_j^*|_0\Big)
$$

for $t > 0$, $j = 1, 2, \dots, m$, which means that

$$
\begin{aligned}
&\sum_{i=1}^n |x_i(t) - x_i^*(t)|_0 + \sum_{j=1}^m |y_j(t) - y_j^*(t)|_0 \\
&\le \frac{\varepsilon_M}{\varepsilon_m} e_{\ominus\eta}(t,\varrho)(1-\varrho)(n+m)\Big(\sum_{i=1}^n |\phi_{x_i}(\varrho) - x_i^*(\varrho)|_0 \\
&\quad + \sum_{j=1}^m |\phi_{y_j}(\varrho) - y_j^*(\varrho)|_0\Big).
\end{aligned} \tag{4.5}
$$

Denote $-\vartheta = \ominus\eta = -\eta/(1+\mu\eta) \in \mathbb{R}$, $N = N(\varrho) = \varepsilon_M/\varepsilon_m(1-\varrho)(n+m) > 1$, in view of (4.5), we have

$$
\|z(t) - z^*(t)\| \le N e_{-\vartheta}(t,\varrho)\Big(\sum_{i=1}^n |\phi_{x_i}(\varrho) - x_i^*(\varrho)|_0 + \sum_{j=1}^m |\phi_{y_j}(\varrho) - y_j^*(\varrho)|_0\Big),
$$

and we can conclude that the ω-periodic solution of (1.1) is globally exponentially stable and this completes the proof. $\square$

5. An illustrative example

In this section, we give an example to illustrate the results in this article. Let $\mathbb{T} = \bigcup_{k=0}^\infty [2k, 2k+1]$. We will apply our main results to the BAM HHNNs with impulses and delays on time scales

$$
\begin{aligned}
x_i^{\Delta}(t) &= -c_i(t)x_i(t) + \sum_{j=1}^{2} a_{ij}(t) f_j(y_j(t-\tau_{ij}(t))) \\
&\quad + \sum_{j=1}^{2}\sum_{l=1}^{2} b_{ijl}(t) g_j(y_j(t-\sigma_{ijl}(t))) g_l(y_l(t-\upsilon_{ijl}(t))) + I_i(t),\ t \neq t_k, \\
\Delta x_i(t_k) &= \alpha_{ik}(x_i(t_k)) = -\gamma_{ik}(x_i(t_k) - x_i^*(t)), \quad i = 1,2,\ k = 1,2,\dots, \\
y_j^{\Delta}(t) &= -d_j(t)y_j(t) + \sum_{i=1}^{2} e_{ji}(t) p_i(x_i(t-\hat{\tau}_{ji}(t))) \\
&\quad + \sum_{i=1}^{2}\sum_{r=1}^{2} h_{jir}(t) q_i(x_i(t-\hat{\sigma}_{jir}(t))) q_r(x_r(t-\hat{\upsilon}_{jir}(t))) + J_j(t),\ t \neq t_k, \\
\Delta y_j(t_k) &= \beta_{jk}(y_j(t_k)) = -\delta_{jk}(y_j(t_k) - y_j^*(t)), \quad j = 1,2, \quad k = 1,2,\dots,
\end{aligned}
\tag{5.1}
$$

where

$$
\begin{gathered}
f_1(x) = \sin\big(\frac{1}{\sqrt{2}}x\big), \quad f_2(x) = \sin\big(\frac{1}{2\sqrt{2}}x\big), \quad g_1(x) = \arctan\big(\frac{1}{\sqrt{2}}x\big), \\
g_2(x) = \arctan\big(\frac{1}{2\sqrt{2}}x\big), \quad p_1(x) = \cos\big(\frac{1}{\sqrt{3}}x\big), \quad p_2(x) = \cos\big(\frac{1}{3\sqrt{3}}x\big), \\
q_1(x) = \operatorname{arccot}\big(\frac{1}{\sqrt{3}}x\big), \quad q_2(x) = \operatorname{arccot}\big(\frac{1}{3\sqrt{3}}x\big).
\end{gathered}
$$

Obviously, $f_j(x), g_j(x) (j = 1,2)$, $p_i(x), q_i(x) (i = 1,2)$ satisfy (H2) and (H3), and

$$
\begin{gathered}
M_1 = M_2 = L_1 = L_2 = H_1 = H_2 = 1, \quad N_1 = N_2 = \frac{\pi}{2}, \\
\hat{M}_1 = \hat{M}_2 = \hat{L}_1 = \hat{L}_2 = \hat{H}_1 = \hat{H}_2 = 1, \quad \hat{N}_1 = \hat{N}_2 = \pi.
\end{gathered}
$$

Take

$$
\begin{gathered}
a_{11}(t) = 1 + \cos(2\pi t), \quad a_{12}(t) = 2 + \cos(2\pi t), \quad a_{21}(t) = 2 + \cos(2\pi t), \\
a_{22}(t) = 3 + \cos(2\pi t), \quad c_1(t) = 20 + 5\sin(2\pi t), \quad c_2(t) = 33 + 16\sin(2\pi t), \\
I_1(t) = 1 + \sin(2\pi t), \quad I_2(t) = 1 + \cos(2\pi t), \quad b_{111}(t) = b_{222}(t) = \frac{1}{4} + \frac{1}{4}\sin(2\pi t), \\
b_{112}(t) = b_{212}(t) = \frac{1}{3} + \frac{1}{3}\cos(2\pi t), \quad b_{121}(t) = b_{221}(t) = \frac{1}{5} + \frac{1}{5}\cos(2\pi t), \\
b_{122}(t) = b_{211}(t) = \frac{1}{6} + \frac{1}{6}\sin(2\pi t), \quad \gamma_{1k} = 1 + \frac{1}{2}\sin(2+k), \\
\gamma_{2k} = 1 + \frac{6}{7}\cos(5+k^2), k \in \mathbb{Z}^+, \quad e_{11}(t) = 1 + \sin(2\pi t), \quad e_{12}(t) = 1 + 2\sin(2\pi t), \\
e_{21}(t) = 1 + 2\cos(2\pi t), \quad e_{22}(t) = 3 + \sin(2\pi t), \quad d_1(t) = 21 + 6\cos(2\pi t), \\
d_2(t) = 31 + 14\cos(2\pi t), \quad J_1(t) = 2 + 3\sin(2\pi t), J_2(t) = 3 + 2\cos(2\pi t), \\
h_{111}(t) = h_{222}(t) = \frac{1}{8} + \frac{1}{8}\sin(2\pi t), \quad h_{112}(t) = h_{212}(t) = \frac{1}{6} + \frac{1}{6}\cos(2\pi t), \\
h_{121}(t) = h_{221}(t) = \frac{1}{10} + \frac{1}{10}\sin(2\pi t), \quad h_{122}(t) = h_{211}(t) = \frac{1}{12} + \frac{1}{12}\cos(2\pi t),
\end{gathered}
$$

$$\delta_{1k} = 1 + \frac{2}{5}\cos(3+k), \quad \delta_{2k} = 1 + \frac{4}{7}\sin(9+k^2), k \in \mathbb{Z}^+.$$

One can verify that (H1) is satisfied, and $\omega = 1$, $c_1^- = 15$, $c_2^- = 17$, $a_{11}^+ = 2$, $a_{12}^+ = 3$, $a_{21}^+ = 3$, $a_{22}^+ = 4$, $b_{111}^+ = b_{222}^+ = 1/2$, $b_{112}^+ = b_{212}^+ = 2/3$, $b_{121}^+ = b_{221}^+ = 2/5$, $b_{122}^+ = b_{211}^+ = 1/3$, $0 < \gamma_{ik} < 2(i = 1, 2)$, $d_1^- = 15$, $d_2^- = 17$, $e_{11}^+ = 2$, $e_{12}^+ = 3$, $e_{21}^+ = 3$, $e_{22}^+ = 4$, $h_{111}^+ = h_{222}^+ = 1/4$, $h_{112}^+ = h_{212}^+ = 1/3$, $h_{121}^+ = h_{221}^+ = 1/5$, $h_{122}^+ = h_{211}^+ = 1/6$, $0 < \delta_{ik} < 2(i = 1, 2)$, so, if we take $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = 1$, we can obtain

$$\begin{aligned}
& - c_1^-\varepsilon_1 + \sum_{j=1}^{2} a_{1j}^+ L_j \varepsilon_{2+j} + \sum_{j=1}^{2}\sum_{l=1}^{2} b_{1jl}^+\big(H_j N_l \varepsilon_{2+j} + H_l N_j \varepsilon_{2+l}\big) \\
& = -15 + 5 + \frac{19}{10}\pi < 0, \\
& - c_2^-\varepsilon_2 + \sum_{j=1}^{2} a_{2j}^+ L_j \varepsilon_{2+j} + \sum_{j=1}^{2}\sum_{l=1}^{2} b_{2jl}^+\big(H_j N_l \varepsilon_{2+j} + H_l N_j \varepsilon_{2+l}\big) \\
& = -17 + 7 + \frac{19}{10}\pi < 0,
\end{aligned}$$

$$-d_3^-\varepsilon_3 + \sum_{i=1}^{2} e_{1i}^+ \hat{L}_i \varepsilon_i + \sum_{i=1}^{2}\sum_{r=1}^{2} h_{1ir}^+\big(\hat{H}_i \hat{N}_r \varepsilon_i + \hat{H}_r \hat{N}_i \varepsilon_r\big) = -15 + 5 + \frac{19}{10}\pi < 0,$$

$$-d_4^-\varepsilon_4 + \sum_{i=1}^{2} e_{2i}^+ \hat{L}_i \varepsilon_i + \sum_{i=1}^{2}\sum_{r=1}^{2} h_{2ir}^+\big(\hat{H}_i \hat{N}_r \varepsilon_i + \hat{H}_r \hat{N}_i \varepsilon_r\big) = -17 + 7 + \frac{19}{10}\pi < 0.$$

Conditions (H5) and (H6) are satisfied. From Theorem 3.1 and 4.1, we know that (5.1) has at least one 1-periodic solution and this solution is exponential stable.

Conclusion. By using the continuation theorem of coincidence degree theory and constructing some suitable Lyapunov functions, sufficient conditions are derived to guarantee the stability and existence of periodic solutions for a class of BAM HHNNs with impulses and delays on time scales. In fact, both continuous and discrete systems, are very important in implementing and applications. But it is troublesome to study the existence and stability of periodic solutions for continuous and discrete systems respectively. Therefore, it is meaningful to study that on time scales which can unify the continuous and discrete situations. The system we study here gives an affirmative exemplum for this problem.

References

[1] M. Bohner, M. Fan, J. Zhang; *Existence of periodic solutions in predator-prey and competition dynamic systems*, Nonlinear Anal. Real World Appl. 7 (2006) 1193-1204.

[2] M. Bohner, A. Peterson; *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhäuser, Boston, Mass, USA, 2001.

[3] M. Bohner, A. Peterson; *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston, Mass, USA, 2003.

[4] J. Chen, B. Cui; *Impulsive effects on global asymptotic stability of delay BAM neural networks*, Chaos Solit. Fract. 38 (2008) 1115-1125.

[5] K. Ding, N. Huang, X. Xu; *Global robust exponential stability of interval BAM neural networks with mixed dalays under uncertainty*, Neural Process. Lett. 25 (2007) 127-141.

[6] A. Dembo, O. Farotimi, T. Kailath; *High-order absolutely stable neural networks*, IEEE Trans. Circuits Syst. 38 (1991) 57-65.

[7] D. Ho, J. Liang, J. Lamc; *Global exponential stability of impulsive high-order BAM neural networks with time-varying delays*, Neural Netw. 19 (2006) 1581-1590.

[8] Z. Huang, Y. Xia; *Global exponential stability of BAM neural networks with transmission delays and nonlinear impulses*, Chaos Solit. Fract. 38 (2008) 489-498.

[9] B. Kosko; *Adaptive bi-directional associative memories*, Appl. Opt. 26 (1987) 4947-4960.

[10] B. Kosko; *Bi-directional associative memories*, IEEE Trans. Systems Man. Cybernet. 18 (1988) 49-60.

[11] E. B. Kosmatopoulos, M. A. Christodoulou; *Structural properties of gradient recurrent high-order neural networks*, IEEE Trans. Circuits Syst. II 42 (1995) 592-603.

[12] Y. Li; *Global exponential stability of BAM neural networks with delays and impulses*, Chaos Solit. Fract. 24 (2005) 279-285.

[13] Y. Li, S. Gao; *Global exponential stability for impulsive BAM neural networks with distributed delays on time scales*, Neural Process. Lett. 31 (2010) 65-91.

[14] Y. Li, M. Hu; *Three positive periodic solutions for a class of higher-dimensional functional differential equations with impulses on time scales*, Adv. Difference Equ. (2009), Article ID 698463, 18 pages, doi:10.1155/2009/698463.

[15] Y. Li, J. Wang; *Ananalysis on the global exponential stability and the existence of periodic solutions for non-autonomous hybrid BAM neural networks with distributed delays and impulses*, Comput. Math. Appl. 56 (2008) 2256-2267.

[16] Y. Li, C. Yang; *Global exponential stability analysis on impulsive BAM neural networks with distributed delays*, J. Math. Anal. Appl. 324 (2006) 1125-1139.

[17] Y. Li, L. Zhao, P. Liu; *Existence and exponential stability of periodic solution of high-order Hopfield neural network with delays on time scales*, Discrete Dyn. Nat. Soc. (2009), Article ID 573534, 18 pages, doi:10.1155/2009/573534.

[18] Y. Liu, Z. Wang, X. Liu; *On global stability of delayed BAM stochastic neural networks with Markovian switching*, Neural Process. Lett. 30 (2009) 19-35.

[19] X. Lou, B. Cui; *Global asymptotic stability of delay BAM neural networks with impulses based on matrix theory*, Appl. Math. Model. 32 (2008) 232-239.

[20] J. Mawhin; *Topological Degree Methods in Nonlinear Boundary Value Problems, vol. 40 of CBMS Regional Conference Series in Mathematics*, American Mathematical Society, Providence, RI, USA, 1979.

[21] J. Park, O. Kwon; *On improved delay-dependent criterion for global stability of bidirectional associative memory neural networks with time-varying delays*, Appl. Math. Comput. 199(2008) 435-446.

[22] J. Park, O. M. Kwon; *A new stability criterion for bidirectional associative memory neural networks of neutral-type*, Appl. Math. Comput. 199 (2008) 716-722.

[23] J. Park, O. Kwon; *Delay-dependent stability criterion for bidirectional associative memory neural networks with interval time-varying delays*, Modern Phys. Lett. B 23 (2009) 35-46.

[24] R. Samidurai, R. Sakthivel, S. Anthoni; *Global asymptotic stability of BAM neural networks with mixed delays and impulses*, Appl. Math. Comput. 212 (2009) 113-119.

[25] Y. Shao, B. Dai; *The existence of exponential periodic attractor of impulsive BAM neural network with periodic coefficients and distributed delays*, Neurocomputing 73 (2010) 3123-3131.

[26] Z. Wang, J. Fang, X. Liu; *Global stability of stochastic high-order neural networks with discrete and distributed delays*, Chaos Solit. Fract. 36 (2008) 388-396.

[27] F.-H. Wong; *Young's inequality and related results on time scales*, Appl. Math. Lett. 18 (2005) 983-988.

[28] R. Wu, C.-C. Yeh, S.-L. Yu, C.-H. Hong; *Exponential convergence of BAM neural networks with time-varying coefficients and distributed delays*, Nonlinear Anal. Real World Appl. 11 (2010) 562-573.

[29] H. Wu, C. Shan; *Stability analysis for periodic solution of BAM neural networks with discontinuous neuron activations and impulses*, Appl. Math. Model. 33(2009) 2564-2574.

[30] Y. Xia, Z. Huang, M. Han; *Existence and global exponential stability of equilibrium for BAM neural networks with impulses*, Chaos Solit. Fract. 37(2008) 588-597.

[31] B. Xiao, H. Meng; *Existence and exponential stability of positive almost periodic solutions for high-order Hopfield neural networks*, Appl. Math. Model. 33 (2009) 532-542.

[32] B. Xu, X. Liu, X. Liao; *Global exponential stability of high order Hopfield type neural networks*, Appl. Math. Comput. 174 (2006) 98-116.

[33] X. Yi, J. Shao, Y. Yu, B. Xiao; *New convergence behavior of high-order Hopfield neural networks with time-varying coefficients*, J. Comput. Appl. Math. 219 (2008) 216-222.

[34] J. Zhang, Z. Gui; *Existence and stability of periodic solutions of high-order Hopfield neural networks with impulses and delays*, J. Comput. Appl. Math. 224 (2009) 602-613.

[35] F. Zhang, Y. Li; *Almost periodic solutions for higher-order Hopfield neural networks without bounded activation functions*, Electron. J. Differ. Equ. 99 (2007) 1-10.

[36] Q. Zhou; *Global exponential stability of BAM neural networks with distributed delays and impulses*, Nonlinear Anal. Real World Appl. 10 (2009) 144-153.

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