Electronic Journal of Differential Equations, Vol. 2012 (2012), No. 47, pp. 1–13.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

WEIGHTED PSEUDO-ALMOST PERIODIC SOLUTIONS FOR SOME NEUTRAL PARTIAL FUNCTIONAL DIFFERENTIAL EQUATIONS

MONDHER DAMAK, KHALIL EZZINBI, LOTFI SOUDEN

Abstract. In this article we study the existence of weighted pseudo almost periodic solutions of an autonomous neutral functional differential equation with Stepanov-Weighted pseudo almost periodic terms in a Banach space. We use the contraction mapping principle to show the existence and the uniqueness of weighted pseudo almost periodic solution of the equation.

1. Introduction

Let $(\mathbb{X}, \|\cdot\|)$ be a Banach space and $p \geq 1$. The concept of pseudo-almost periodicity, was introduced in the early nineties in [17, 18] as a natural generalization of the classical almost periodicity in the sense of Bohr, the existence of pseudo-almost periodic solutions to functional differential equations has been of a great interest to several authors. We refer the reader to [4, 5]. The concept of weighted pseudo-almost periodic functions, which was introduced by Diagana [10], as a natural generalization of the classical pseudo almost periodic functions. We refer to reader [14] [1]. The concept of Stepanov pseudo almost periodic (or S^p-pseudo almost periodic) was introduced and studied, in the recent years in [9]. In recent years Diagana [7] study the existence of pseudo-almost periodic solutions to some nonautonomous differential equations in the case when the semilinear forcing term is both continuous and S^p-pseudo almost periodic for $p > 1$,

$$\frac{d}{dt}u(t) = Au(t) + f(t, u(t)), \quad \text{for } t \in \mathbb{R}$$

Where A is the infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}$, and $f : \mathbb{R}\times\mathbb{X} \to \mathbb{X}$ is Stepanov- pseudo almost periodic functions for $p > 1$. In [8], the author introduced the concept of weighted pseudo-almost periodicity in the sense of Stepanov, also called S^p-weighted pseudo-almost periodicity and study its properties. and present a result on the existence of weighted pseudo-almost periodic solutions to the N-dimensional heat equation with S^p-weighted pseudo almost periodic coefficients of the form

$$u'(t) = Au(t) + f(t, Bu(t))$$

2000 *Mathematics Subject Classification.* 34K40, 35B15, 34C27, 34K14.
Key words and phrases. Neutral equation; semigroup; mild solution;
Stepanov-weighted pseudo almost periodic functions; Banach fixed point theorem.

Submitted November 21, 2011. Published March 23, 2012.

where $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ is a sectorial linear operator on a Banach space $\mathbb{X}$ whose corresponding analytic semigroup $(T(t))_{t\geq 0}$ is hyperbolic and B is an arbitrary linear (possibly unbounded) operator on $\mathbb{X}$, and f is S^p-weighted pseudo almost periodic and jointly continuous function. In [16], the author studied the existence of almost periodic solutions of an autonomous neutral functional differential equation with Stepanov-almost periodic terms in a Banach space of the form

$$\frac{d}{dt}[u(t) - F(t, u(t - g(t)))] = Au(t) + G(t, u(t), u(t - g(t)))$$

for $t \in \mathbb{R}$ and, A is the infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}$, $F : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$, and $G : \mathbb{R} \times \mathbb{X} \times \mathbb{X} \to \mathbb{X}$ are Stepanov almost periodic functions.

In this article, we study the existence of weighted pseudo almost periodic solutions of an autonomous neutral functional differential equation

$$\frac{d}{dt}[u(t) - F(t, u(t - r))] = A[u(t) - F(t, u(t - r))] + G(t, u(t), u(t - r)), \qquad (1.1)$$

for $t \in \mathbb{R}$, Where A is the infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}$, and $F : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$, is Weighted pseudo almost periodic and $G : \mathbb{R} \times \mathbb{X} \times \mathbb{X} \to \mathbb{X}$ is Stepanov-weighted pseudo almost periodic functions.

The rest of this article is organized as follows: In Section 2, we introduce the basic notations and recall the definitions and lemmas. In Section 3, we study the existence of weighted pseudo almost periodic solutions of (1.1). In section 4 we give an example to illustrate our result.

2. Preliminaries

In this section we give some basic results that will be used in the next. In the rest of this paper, $(\mathbb{X}, \|\cdot\|)$ stands for a complex Banach space.

Definition 2.1. A continuous function $f : \mathbb{R} \to \mathbb{X}$ is said to be almost periodic if for every $\epsilon > 0$ there exists a positive number l such that every interval of length l contains a number τ such that

$$\|f(t + \tau) - f(t)\| < \epsilon, \quad \text{for } t \in \mathbb{R}.$$

Let $AP(\mathbb{R}; \mathbb{X})$ be the set of all almost periodic functions from $\mathbb{R}$ to $\mathbb{X}$. Then $(AP(\mathbb{R}; \mathbb{X}), \|\cdot\|_\infty)$ is a Banach space with supremum norm given by

$$\|u\|_\infty = \sup_{t\in\mathbb{R}} \|u(t)\|.$$

Definition 2.2. A continuous function $f : \mathbb{R} \times \mathbb{Y} \to \mathbb{X}$ is said to be almost periodic in t uniformly for $y \in \mathbb{Y}$, if for every $\epsilon > 0$ and any compact subset K of $\mathbb{Y}$, there exists a positive number l such that every interval of length l contains a number τ such that

$$\|f(t + \tau, y) - f(t, y)\| < \epsilon, \quad \text{for } (t, y) \in \mathbb{R} \times K.$$

we denote the set of such functions as $AP(\mathbb{R} \times \mathbb{Y}; \mathbb{X})$.

Lemma 2.3 ([3])**.** *If $f \in AP(\mathbb{R} \times \mathbb{Y}; \mathbb{X})$ and $h \in AP(\mathbb{R}; \mathbb{Y})$, then the function $f(., h(.)) \in AP(\mathbb{R}; \mathbb{X})$.*

Now, let $\mathbb{U}$ be the collection of function (weights) $\rho : \mathbb{R} \to (0, \infty)$, which are locally integrable over $\mathbb{R}$ such that $\rho(x) > 0$ almost everywhere. Set For $T > 0$,

$$\text{meas}(T, \rho) := \int_{-T}^{T} \rho(t)dt,$$

$$\mathbb{U}_\infty ;= \{\rho \in \mathbb{U} : \lim_{T\to\infty} m(T,\rho) = \infty \text{ and } \liminf_{t\in\mathbb{R}} \rho(t) > 0\},$$
$$\mathbb{U}_B := \{\rho \in \mathbb{U}_\infty : \rho \text{ is bounded}\}.$$

Obviously, $\mathbb{U}_B \subset \mathbb{U}_\infty \subset \mathbb{U}$, with strict inclusions.

For each $\rho \in \mathbb{U}_\infty$, define

$$PAP_0(X,\rho) = \{\phi \in BC(\mathbb{R},\mathbb{X}) : \lim_{T\to+\infty} \frac{1}{\text{meas}(T,\rho)} \int_{-T}^{T} \|\phi(s)\|\rho(s)ds = 0\}$$

similarly, $PAP_0(\mathbb{R}\times\mathbb{Y};\mathbb{X},\rho)$ denote the collection of all function, $\phi : \mathbb{R}\times\mathbb{Y} \to \mathbb{X}$, jointly continuous, and $\phi(.,y)$, bounded for each $y \in \mathbb{Y}$, and

$$\lim_{T\to+\infty} \frac{1}{\text{meas}(T,\rho)} \int_{-T}^{T} \|\phi(s,y)\|\rho(s)ds = 0$$

uniformly for any y in any compact subset of $\mathbb{Y}$.

Definition 2.4. Let $\rho \in \mathbb{U}_\infty$. A function $f \in BC(\mathbb{R};\mathbb{X})$ is called weighted pseudo almost periodic or ρ-pseudo almost periodic if it can be expressed as $f = g + h$ where $g \in AP(\mathbb{R};\mathbb{X})$ and $h \in PAP_0(\mathbb{R};\mathbb{X},\rho)$. The collection of such function will be denoted by $PAP(\mathbb{R};\mathbb{X},\rho)$.

Definition 2.5. Let $\rho \in \mathbb{U}_\infty$. A function $f \in BC(\mathbb{R}\times\mathbb{Y};\mathbb{X})$ is called weighted pseudo almost periodic or ρ-pseudo almost periodic if it can be expressed as $f = g + h$ where $g \in AP(\mathbb{R}\times\mathbb{Y};\mathbb{X})$ and $h \in PAP_0(\mathbb{R}\times\mathbb{Y};\mathbb{X},\rho)$. The collection of such function will be denoted by $PAP(\mathbb{R}\times\mathbb{Y};\mathbb{X},\rho)$

Definition 2.6. Let $\rho \in \mathbb{U}_\infty$. A function $f \in BC(\mathbb{R}\times\mathbb{W}\times\mathbb{Y};\mathbb{X})$ is called weighted pseudo almost periodic or ρ-pseudo almost periodic if it can be expressed as $f = g + h$ where $g \in AP(\mathbb{R}\times\mathbb{W}\times\mathbb{Y};\mathbb{X})$ and $h \in PAP_0(\mathbb{R}\times\mathbb{W}\times\mathbb{Y};\mathbb{X},\rho)$. The collection of such function will be denoted by $PAP(\mathbb{R}\times\mathbb{W}\times\mathbb{Y};\mathbb{X},\rho)$

Remark 2.7. (1) The weight $\rho(t) = 1$, is called pseudo-almost periodic functions.
(2) Clearly that $PAP_0(\mathbb{R};\mathbb{X},\rho)$ is a linear subspace of $BC(\mathbb{R},\mathbb{X})$.
(3) Let $\rho \in \mathbb{U}_\infty$, and assume that

$$\limsup_{t\to\infty} \frac{\rho(t+\tau)}{\rho(t)} < \infty, \quad \limsup_{T\to\infty} \frac{\text{meas}(T+\tau,\rho)}{\text{meas}(T,\rho)} < \infty$$

for all $\tau \in \mathbb{R}$. In that case, the space $PAP(\mathbb{R};\mathbb{X},\rho)$ is translation invariant. In this article, all weights $\rho \in \mathbb{U}_\infty$ for which $PAP(\mathbb{R};\mathbb{X},\rho)$ is translation invariant will be denoted $\mathbb{U}_{\text{inv}}$.

Theorem 2.8 ([15])**.** *Fix $\rho \in U_{\text{inv}}$, the decomposition of weighted pseudo almost periodic function $f = g+h$, where $g \in AP(\mathbb{R};\mathbb{X})$ and $h \in PAP_0(\mathbb{R};\mathbb{X},\rho)$, is unique.*

Theorem 2.9 ([12])**.** *Fix $\rho \in U_{\text{inv}}$, then the space $(PAP(\mathbb{R};\mathbb{X},\rho), \|\cdot\|_\infty)$ is a Banach space.*

Lemma 2.10 ([6])**.** *Let $\rho \in U_{\text{inv}}$. If $f \in PAP_0(\mathbb{R};\mathbb{X},\rho)$, then for all $r \in \mathbb{R}$, $f(.-r) \in PAP_0(\mathbb{R};\mathbb{X},\rho)$.*

Lemma 2.11 ([7])**.** *Let $\{f_n\}_{n\in\mathbb{N}} \subset PAP(\mathbb{R};\mathbb{X},\rho)$ be a sequence of functions. If f_n converges uniformly to some f, then $f \in PAP(\mathbb{R};\mathbb{X},\rho)$.*

Theorem 2.12 ([13]). *Let $\rho \in \mathbb{U}_\infty$, $F \in PAP(\mathbb{R}\times\mathbb{Y};\mathbb{X},\rho)$ and $h \in PAP(\mathbb{R};\mathbb{Y},\rho)$. Assume that there exists L such that*

$$\|F(t,x)-F(t,y)\| \le L\|x-y\|$$

for all $t\in\mathbb{R}$ and for each $x,y\in\mathbb{X}$. Then $F(.,h(.))\in PAP(\mathbb{R};\mathbb{X},\rho)$

Definition 2.13. The Bochner transform $f^b(t,s), t\in\mathbb{R}, s\in[0,1]$, of a function $f(t)$ on $\mathbb{R}$, with value in $\mathbb{X}$, is defined by

$$f^b(t,s)=f(t+s).$$

Definition 2.14. Let $1\le p<\infty$. The space $BS^p(\mathbb{R};\mathbb{X})$ of all Stepanov bounded functions, with the exponent p, consists of all measurable functions f on $\mathbb{R}$ with value in $\mathbb{X}$ such that $f^b\in L^\infty(\mathbb{R},L^p(0,1);\mathbb{X})$. This is a Banach space with the norm

$$\|f\|_{S^p}=\|f^b\|_{L^\infty(\mathbb{R},L^p)}=\sup_{t\in\mathbb{R}}\Big(\int_t^{t+1}\|f(s)\|^p ds\Big)^{1/p}<\infty.$$

A function, $f\in L^p_{\rm loc}(\mathbb{R};\mathbb{X})$ is p-Stepanov bounded (S^p-bounded) if $\|f\|_{S^p}<\infty$. It is obvious that $L^p(\mathbb{R};\mathbb{X})\subset BS^p(\mathbb{R};\mathbb{X})\subset L^p_{\rm loc}(\mathbb{R};\mathbb{X})$

Definition 2.15. A function $f\in BS^p(\mathbb{R};\mathbb{X})$ is said to be almost periodic in the sense of Stepanov (S^p-almost periodic) if for every $\epsilon>0$ there exists a positive number l such that every interval of length l contains a number τ such that

$$\sup_{t\in\mathbb{R}}\Big(\int_t^{t+1}\|f(s+\tau)-f(s)\|^p ds\Big)^{1/p}<\epsilon.$$

Let $S^p-AP(\mathbb{R};\mathbb{X})$ be the set of all S^p-almost periodic functions. It is clear that, if f is almost periodic implies f is S^p-almost periodic; that is, $AP(\mathbb{R};\mathbb{X})\subset S^p-AP(\mathbb{R};\mathbb{X})$. Moreover, if $1\le m<p$, then $f(t)$ is S^p-almost periodic implies $f(t)$ is S^m-almost periodic.

Definition 2.16 ([8]). Let $\rho\in\mathbb{U}_\infty$. A function $f\in BS^p$ is said to be a S^p -weighted pseudo almost periodic (or Stepanov- like weighted pseudo almost periodic) if it can be expressed as $f=g+\phi$, where $g^b\in AP(L^p((0,1);\mathbb{X}))$ and $\phi^b\in PAP_0(L^p((0,1);\mathbb{X}),\rho)$. The collection of such functions will be denoted by $PAP(\mathbb{X},\rho,p)$ or $S^p-PAP(\mathbb{X},\rho)$.

Definition 2.17 ([8]). Let $\rho\in\mathbb{U}_\infty$. A function $f:\mathbb{R}\times\mathbb{Y}\to\mathbb{X}$ with $f(.,x)\in L^p_{loc}(\mathbb{R},\mathbb{X})$ for each $x\in\mathbb{X}$ is said to be S^p -weighted pseudo almost periodic if it can be expressed as $f=g+\phi$, where $g^b\in AP(\mathbb{R}\times L^p(0,1));\mathbb{X})$ and $\phi^b\in PAP_0(\mathbb{R}\times L^p((0,1));\mathbb{X},\rho)$. The collection of such functions will be denoted by $PAP(\mathbb{X},\rho,p)$ or $S^p-PAP(\mathbb{X},\rho)$.

Theorem 2.18 ([8]). *Let $\rho\in\mathbb{U}_{\rm inv}$ and let $p\ge1$, if $f\in PAP(\mathbb{X},\rho)$ then $f\in PAP(\mathbb{X},\rho,p)$.*

Theorem 2.19 ([8]). *Let $\rho\in\mathbb{U}_\infty$ and let $f:\mathbb{R}\times\mathbb{Y}\to\mathbb{X}$ be a $S^p-PAP(\mathbb{R}\times\mathbb{Y};\mathbb{X},\rho))$, suppose that F is lipschitz in $y\in\mathbb{Y}$ uniformly in $t\in\mathbb{R}$.*
If $g\in PAP(\mathbb{R};\mathbb{Y},\rho)$ then $f(.,g(.))$ belong to $S^p-PAP(\mathbb{R};\mathbb{X},\rho)$.

It is clear that the space $(PAP(L^p((0,1),\mathbb{X}),\rho),\|\cdot\|_{S^p})$ is a Banach space. In other words, a function $f\in L^p_{\rm loc}(\mathbb{R};\mathbb{X})$ is said to be S^p-weighted pseudo-almost periodic relatively to the weight $\rho\in\mathbb{U}_\infty$, if its Bochner transform $f^b:\mathbb{R}\to L^p((0,1),\mathbb{X})$ is weighted pseudo-almost periodic in the sense that there exist two functions g and

ϕ for $\mathbb{R}$ into $\mathbb{X}$ such that $g^b \in AP(L^p((0,1),\mathbb{X}))$ and $\phi^b \in PAP_0(\mathbb{R}, L^p((0,1);\mathbb{X}),\rho)$, that is $\phi^b \in BC(\mathbb{R},(L^p((0,1);\mathbb{X}))$ and

$$\lim_{T\to+\infty} \frac{1}{\operatorname{meas}(T,\rho)} \int_{-T}^{T} (\int_t^{t+1} \|\phi(s)\|^p ds)^{1/p} \rho(t) dt = 0$$

We define the set $S^p - PAP((\mathbb{R}\times\mathbb{Y};\mathbb{X}),\rho)$ which consists of all functions $f : \mathbb{R}\times\mathbb{Y} \to \mathbb{X}$ such that $f(.,y) \in S^p - PAP((\mathbb{R}\times\mathbb{Y};\mathbb{X},\rho)$ uniformly for each $y \in K$, where K is any compact subset of $\mathbb{Y}$.

We define the set $S^p - PAP((\mathbb{R}\times\mathbb{W}\times\mathbb{Y};\mathbb{X}),\rho)$ which consists of all functions $f : \mathbb{R}\times\mathbb{W}\times\mathbb{Y} \to \mathbb{X}$ such that $f(.,w,y) \in S^p - PAP(\mathbb{R};\mathbb{X},\rho)$ uniformly for each $(w,y) \in K$, where K is any compact subset of $\mathbb{W}\times\mathbb{Y}$.

3. Main results

In this section we prove the existence and uniqueness of weighted pseudo almost periodic mild solution for (1.1). For the rest of this article, we consider the following assumptions.

(H0) Assume that A is the infinitesimal generator of an exponentially stable c_0-semigroup $\{T(t)\}_{t\geq 0}$ acting on $\mathbb{X}$; that is, there exists constants $\omega > 0$ and $M \geq 1$ such that

$$\|T(t)\| \leq Me^{-\omega t} \quad \text{for } t \in \mathbb{R} \tag{3.1}$$

(H1) The function F belong to $PAP(\mathbb{R}\times\mathbb{X};\mathbb{X},\rho)$ satisfy the property that there exists $L_F > 0$ such that

$$\|F(t,u) - F(t,v)\| \leq L_F\|u - v\|$$

for all $t \in \mathbb{R}$ and for each $u, v \in \mathbb{X}$.

(H2) The function G belong $S^p - PAP(\mathbb{R}\times\mathbb{X}\times\mathbb{X};\mathbb{X},\rho)$ and satisfy the followings property that there exists $L_G > 0$ such that

$$\|G(t,x_1,y_1) - G(t,x_2,y_2)\| \leq L_G(\|x_1 - x_2\| + \|y_1 - y_2\|)$$

for all $t \in \mathbb{R}$ and for $(x_1,y_1),(x_2,y_2) \in \mathbb{X}\times\mathbb{X}$.

Definition 3.1. Let $u : \mathbb{R} \to \mathbb{X}$ be an integral solution of (1.1). Then for any $t \geq \sigma$ and any $\sigma \in \mathbb{R}$,

$$\begin{aligned} &u(t) - F(t,u(t-r)) \\ &= T(t-\sigma)[u(\sigma) - F(\sigma,u(\sigma-r))] + \int_\sigma^t T(t-s)G(s,u(s),u(s-r))ds, \quad t \in \mathbb{R}. \end{aligned} \tag{3.2}$$

Theorem 3.2. *Assume that* (H0)–(H2) *hold. Let u be a bounded integral solution of* (1.1) *on $\mathbb{R}$ then, for all $t \in \mathbb{R}$*

$$u(t) = F(t,u(t-r)) + \int_{-\infty}^t T(t-s)G(s,u(s),u(s-r))ds. \tag{3.3}$$

Proof. For any $\sigma \in \mathbb{R}$, we have for all $t > \sigma$,

$$u(t)-F(t,u(t-r)) = T(t-\sigma)[u(\sigma)-F(\sigma,u(\sigma-r))]+\int_\sigma^t T(t-s)G(s,u(s),u(s-r))ds$$

Since u is bounded and F is lipschitz continuous with respect the second argument, then there exists a constant M such that $\|u(t)\| \leq M$ for all $t \in \mathbb{R}$, we have

$$\|T(t-\sigma)[u(\sigma) - F(\sigma, u(\sigma - r))]\| \leq Me^{-\omega(t-\sigma)}(M + 2ML_F + \|F(0, u(-r))\|)$$

and $\lim_{\sigma \to -\infty} T(t-\sigma)[u(\sigma) - F(\sigma, u(\sigma - r))] = 0$. Hence we have

$$u(t) = F(t, u(t-r)) + \int_{-\infty}^{t} T(t-s)G(s, u(s), u(s-r))ds, \quad t \in \mathbb{R}.$$

Conversely if u belongs to $BC(\mathbb{R}, \mathbb{X})$, it is easy to see that the operator $\Gamma u(t) = F(t, u(t-r)) + \int_{-\infty}^{t} T(t-s)G(s, u(s), u(s-r))ds$ defined on $BC(\mathbb{R}, \mathbb{X})$ into itself, if u is given by $u(t) = F(t, u(t-r)) + \int_{-\infty}^{t} T(t-s)G(s, u(s), u(s-r))ds, \quad t \in \mathbb{R}$ then for any $t \geq \sigma$,

$$\begin{aligned}
u(t) &= F(t, u(t-r)) + \int_{-\infty}^{\sigma} T(t-s)G(s, u(s), u(s-r))ds \\
&\quad + \int_{\sigma}^{t} T(t-s)G(s, u(s), u(s-r))ds \\
&= F(t, u(t-r)) + T(t-\sigma)[u(\sigma) - F(\sigma, u(\sigma - r))] \\
&\quad + \int_{\sigma}^{t} T(t-s)G(s, u(s), u(s-r))ds
\end{aligned}$$

□

Theorem 3.3. *Assume that* (H0)–(H2) *hold. If* $(L_F + \frac{2ML_G}{\omega} < 1)$ *then there exists a unique bounded solution of* (1.1) *on* $\mathbb{R}$.

Proof. We consider $\Gamma : BC(\mathbb{R}; \mathbb{X}) \to BC(\mathbb{R}; \mathbb{X})$ defined by, $\Gamma u(t) = F(t, u(t-r)) + \int_{-\infty}^{t} T(t-s)G(s, u(s), u(s-r))ds$. Let $u, v \in BC(\mathbb{R}; \mathbb{X})$. We observed that

$$\begin{aligned}
&\|(\Gamma u)(t) - (\Gamma v)(t)\| \\
&\leq \|F(t, u(t-r)) - F(t - v(t-r))\| \\
&\quad + \int_{-\infty}^{t} \|T(t-s)\| \|G(s, u(s), u(s-r)) - G(s, v(s), v(s-r))\| ds \\
&\leq L_F \|u(t-r) - v(t-r)\| \\
&\quad + ML_G \int_{-\infty}^{t} e^{-\omega(t-s)} (\|u(s) - v(s)\| + \|u - v\|_\infty) ds \\
&\leq L_F \|u - v\|_\infty + 2ML_G \Big(\int_{-\infty}^{t} e^{-\omega(t-s)} ds \Big) \|u - v\|_\infty \\
&\leq \big(L_F + \frac{2M}{\omega} L_G \big) \|u - v\|_\infty.
\end{aligned}$$

Thus

$$\|\Gamma u - \Gamma v)\|_\infty \leq \big(L_F + \frac{2M}{\omega} L_G \big) \|u - v\|_\infty.$$

Thus Γ is a contraction map on $BC(\mathbb{R}; \mathbb{X})$. Therefore, Γ has unique fixed point in $BC(\mathbb{R}; \mathbb{X})$, therefore the equation (1.1) has unique mild solution. □

Theorem 3.4 ([13]). *Let $\rho \in \mathbb{U}_\infty$. If $G \in PAP^p(\mathbb{R}\times\mathbb{W}\times\mathbb{Y};\mathbb{X},\rho)$ satisfies the Lipschitz condition*

$$\|G(t,x_1,y_1)-G(t,x_2,y_2)\|_X \le L_G(\|x_1-x_2\|_W+\|y_1-y_2\|_Y)$$

for all $t\in\mathbb{R}$ and $x_1,x_2\in\mathbb{W}$ and $y_1,y_2\in\mathbb{Y}$. If $h\in PAP(\mathbb{R};\mathbb{Y},\rho)$ and $\phi\in PAP(\mathbb{R};\mathbb{W},\rho)$, then $G(.,\phi(.),h(.))\in PAP^p(\mathbb{R};\mathbb{X},\rho)$.

We define two mappings Γ and Λ by

$$(\Gamma u)(t) = F(t,u(t-r)) + \int_{-\infty}^t T(t-s)G(s,u(s),u(s-r))ds, \tag{3.4}$$

$$(\Lambda f)(t) = \int_{-\infty}^t T(t-s)f(s)ds, \quad t\in\mathbb{R}. \tag{3.5}$$

Proposition 3.5. *Let $\rho\in\mathbb{U}_{\rm inv}$. If $u\in PAP(\mathbb{R};\mathbb{X},\rho)$ and $r\in\mathbb{R}$, then $u(.-r)\in PAP(\mathbb{R};\mathbb{X},\rho)$.*

Proof. We have $u(.)=x(.)+y(.)$, where $x(.)\in AP(\mathbb{R};\mathbb{X})$ and $y(.)\in PAP_0(\mathbb{R};\mathbb{X},\rho)$. It is easy to see that $x(t-r)$ belong to $AP(\mathbb{R};\mathbb{X})$ and from lemma 2.10, we have $y(.-r)\in PAP_0(\mathbb{R};\mathbb{X},\rho)$ $\square$

Lemma 3.6. *Let $\rho\in\mathbb{U}_\infty$. If f is an S^p-weighted pseudo almost periodic function, then the function $(\Lambda f)\in PAP(\mathbb{R};\mathbb{X},\rho)$.*

Proof. Since $f\in S^pPAP(\mathbb{R};\mathbb{X},\rho)$, then $f=g+h$ where $g\in S^pAP(\mathbb{R};\mathbb{X})$ and $h\in S^pPAP_0(\mathbb{R};\mathbb{X},\rho)$. We consider

$$(\Lambda g)(t)=\int_{-\infty}^t T(t-s)g(s)ds, \quad (\Lambda h)(t)=\int_{-\infty}^t T(t-s)h(s)ds, \quad t\in\mathbb{R}.$$

The conjugate of p is denoted by q; that is, $\frac{1}{p}+\frac{1}{q}=1$. We divide the proof onto several steps:

Step 1: • If $p>1$ then $1<q<+\infty$. we prove that $(\Lambda g)(t)\in AP(\mathbb{R};\mathbb{X})$. we consider

$$(\Lambda g)_n(t)=\int_{t-n}^{t-n+1} T(t-s)g(s)ds, \quad n\in\mathbb{N},\ t\in\mathbb{R}.$$

Using the Holder inequality and the estimate 3.1, it follows that

$$\begin{aligned}
\|(\Lambda g)_n(t)\| &= \|\int_{t-n}^{t-n+1} T(t-s)g(s)ds\| \\
&\le \int_{t-n}^{t-n+1} \|T(t-s)\|\|g(s)\|ds \\
&\le M[\int_{t-n}^{t-n+1} e^{-q\omega(t-s)}ds]^{1/q}[\int_{t-n}^{t-n+1}\|g(s)\|^p ds]^{1/p} \\
&\le \frac{M(e^{q\omega}-1)}{\sqrt[q]{q\omega}}e^{-n\omega}\|g\|_{S^p}.
\end{aligned} \tag{3.6}$$

Using the assumption that $\sum_{n=1}^\infty e^{-\omega n}$ is convergent, we then deduce from the well-known Weirstrass theorem that the series $\sum_{n=1}^\infty(\Lambda g)_n(t)$ is uniformly convergent on $\mathbb{R}$, furthermore, $\sum_{n=1}^\infty(\Lambda g)_n(t)=(\Lambda g)(t)$ then $(\Lambda g)(.)$ is continuous and

$$\|(\Lambda g)(t)\| \le \sum_{n=1}^\infty\|(\Lambda g)_n(t)\| \le \frac{M(e^{q\omega}-1)}{\sqrt[q]{q\omega}}\|g\|_{S^p}\sum_{n=1}^\infty e^{-\omega n} \quad \text{for each } t\in\mathbb{R}$$

• If $p = 1$ then $q = \infty$. Using the Holder inequality and the estimate 3.1, we have

$$\begin{aligned} \|(\Lambda g)_n(t)\| &= \| \int_{t-n}^{t-n+1} T(t-s)g(s)ds\| \\ &\le \int_{t-n}^{t-n+1} \|T(t-s)\|\|g(s)\|ds \\ &\le M \sup_{t-n\le s\le t-n+1} e^{-\omega(t-s)} \int_{t-n}^{t-n+1} \|g(s)\|ds \\ &\le Me^{-\omega(n-1)}\|g\|_{S^1}. \end{aligned} \tag{3.7}$$

Using the assumption that $\sum_{n=1}^{\infty} e^{-\omega(n-1)}$ is convergent, we then deduce from the well-known Weirstrass theorem that the series $\sum_{n=1}^{\infty}(\Lambda g)_n(t)$ is uniformly convergent on $\mathbb{R}$, furthermore $\sum_{n=1}^{\infty}(\Lambda g)_n(t) = (\Lambda g)(t)$ then $(\Lambda g)(.)$ is continuous and

$$\|(\Lambda g)(t)\| \le \sum_{n=1}^{\infty} \|(\Lambda g)_n(t)\| \le M\|g\|_{S^1} \sum_{n=1}^{\infty} e^{-\omega(n-1)} \text{ for each } t \in \mathbb{R}$$

Step 2: • If $p > 1$, We prove that $(\Lambda g)(.) \in AP(\mathbb{R};\mathbb{X})$. Since $g \in S^pAP(\mathbb{R};\mathbb{X})$, then for each $\epsilon > 0$ there exists $l(\epsilon) > 0$ such that every interval of length $l(\epsilon)$ contains a τ with the property

$$\sup_{t\in\mathbb{R}} [\int_t^{t+1} \|g(s+\tau) - g(s)\|^p ds^{1/p}] < \epsilon_1\epsilon$$

where $\epsilon_1 = \sqrt[q]{q\omega}(e^{\omega} - 1)/\big(M(e^{q\omega} - 1)\big)$,

$$\begin{aligned} &\|(\Lambda g)_n(t+\tau) - (\Lambda_1 g)_n(t)\| \\ &= \| \int_{t+\tau-n}^{t+\tau-n+1} T(t+\tau-s)g(s)ds - \int_{t-n}^{t-n+1} T(t-s)g(s)ds\| \\ &\le \int_{t-n}^{t-n+1} \|T(t-s)\|\|g(s+\tau) - g(s)\|ds \\ &\le M[\int_{t-n}^{t-n+1} e^{-q\omega(t-s)}ds]^{1/q}[\int_{t-n}^{t-n+1} \|g(s+\tau) - g(s)\|^p ds]^{1/p} \\ &\le \frac{M(e^{q\omega} - 1)}{\sqrt[q]{q\omega}} e^{-n\omega}\epsilon_1\epsilon. \end{aligned} \tag{3.8}$$

Therefore,

$$\sum_{n=1}^{\infty} \|(\Lambda g)_n(t+\tau) - (\Lambda g)_n(t)\| < \epsilon\epsilon_1 \frac{M(e^{q\omega} - 1)}{\sqrt[q]{q\omega}} \sum_{n=1}^{\infty} e^{-n\omega} = \epsilon,$$

hence $\sum_{n=1}^{\infty}(\Lambda g)_n(.) \in AP(\mathbb{R},\mathbb{X})$ for any $n \in \mathbb{N}$ and $(\Lambda g)(.) \in AP(\mathbb{R};\mathbb{X})$.

• If $p = 1$ then $q = \infty$. We prove that $(\Lambda g)(.) \in AP(\mathbb{R};\mathbb{X})$. Since $g \in S^1AP(\mathbb{R};\mathbb{X})$, then for each $\epsilon > 0$ there exists $l(\epsilon) > 0$ such that every interval of length $l(\epsilon)$ contains a τ with the property

$$\sup_{t\in\mathbb{R}} \int_t^{t+1} \|g(s+\tau) - g(s)\|ds < \epsilon_1\epsilon$$

where $\epsilon_1 = (1 - e^{-\omega})/M$,

$$\begin{aligned}
&\|(\Lambda g)_n(t+\tau) - (\Lambda_1 g)_n(t)\| \\
&= \| \int_{t+\tau-n}^{t+\tau-n+1} T(t+\tau-s)g(s)ds - \int_{t-n}^{t-n+1} T(t-s)g(s)ds\| \\
&\le \int_{t-n}^{t-n+1} \|T(t-s)\|\|g(s+\tau)-g(s)\|ds \\
&\le M \sup_{t-n\le s\le t-n+1} e^{\omega(t-s)} \int_{t-n}^{t-n+1} \|g(s+\tau)-g(s)\|ds \\
&\le M e^{-\omega(n-1)}\epsilon_1\epsilon.
\end{aligned} \tag{3.9}$$

Therefore,

$$\sum_{n=1}^{\infty} \|(\Lambda g)_n(t+\tau) - (\Lambda g)_n(t)\| < \epsilon\epsilon_1 M \sum_{n=1}^{\infty} e^{-\omega(n-1)} = \epsilon,$$

hence $\sum_{n=1}^{\infty}(\Lambda g)_n(.) \in AP(\mathbb{R},\mathbb{X})$ for any $n \in \mathbb{N}$ and $(\Lambda g)(.) \in AP(\mathbb{R};\mathbb{X})$.
Step 3: We show that $(\Lambda h)(.) \in PAP_0(\mathbb{R};\mathbb{X},\rho)$.

- If $p > 1$. Let $T > 0$, $h \in S^p PAP_0(\mathbb{R};\mathbb{X},\rho)$ we have that

$$\lim_{T\to+\infty} \frac{1}{\operatorname{meas}(T,\rho)} \int_{-T}^{T} (\int_s^{s+1} \|h(\sigma)\|^p d\sigma)^{1/p} \rho(s)ds = 0 \tag{3.10}$$

First we prove that $\Lambda h \in BC(\mathbb{R};\mathbb{X})$. Indeed it is similar to previous works of Λg. Next we prove that $\Lambda h \in PAP_0(\mathbb{R};\mathbb{X},\rho)$. we consider

$$(\Lambda h)_n(t) = \int_{t-n}^{t-n+1} T(t-s)h(s)ds, \quad n \in \mathbb{N},\ t \in \mathbb{R}.$$

Using the Holder inequality and the estimate 3.1, it follows that

$$\begin{aligned}
\|(\Lambda h)_n(t)\| &= \| \int_{t-n}^{t-n+1} T(t-s)h(s)ds\| \\
&\le \int_{t-n}^{t-n+1} \|T(t-s)\|\|h(s)\|ds \\
&\le M[\int_{t-n}^{t-n+1} e^{-q\omega(t-s)}ds]^{1/q} [\int_{t-n}^{t-n+1} \|h(s)\|^p ds]^{1/p} \\
&\le \frac{M(e^{q\omega}-1)}{\sqrt[q]{q\omega}} e^{-n\omega} \|h\|_{S^p}.
\end{aligned} \tag{3.11}$$

It follows that

$$\begin{aligned}
&\frac{1}{\operatorname{meas}(T,\rho)} \int_{-T}^{T} \|(\Lambda h)_n(t)\|\rho(t)dt \\
&\le \frac{M(e^{q\omega}-1)}{\sqrt[q]{q\omega}} e^{-n\omega} \frac{1}{\operatorname{meas}(T,\rho)} \int_{-T}^{T} \|h(t)\|_{S^p}\rho(t)dt
\end{aligned}$$

and

$$\lim_{T\to+\infty} \frac{1}{\operatorname{meas}(T,\rho)} \int_{-T}^{T} \|(\Lambda h)_n(t)\|\rho(t)dt$$

$$\leq \frac{M(e^{q\omega}-1)}{\sqrt[q]{q\omega}} e^{-n\omega} \cdot \lim_{T\to+\infty} \frac{1}{\operatorname{meas}(T,\rho)} \int_{-T}^{T} \|h(t)\|_{S^p} \rho(t)dt = 0$$

hence $(\Lambda h)_n \in PAP_0(\mathbb{R}, X, \rho)$. On the other hand, using the assumption that $\sum_{n=1}^{\infty} e^{-\omega n}$ is convergent, we then deduce from the well-known Weirstrass theorem that the series $\sum_{n=1}^{\infty} (\Lambda h)_n(t)$ is uniformly convergent on $\mathbb{R}$, furthermore $\sum_{n=1}^{\infty} (\Lambda h)_n(t) = (\Lambda h)(t)$. Consequently $\sum_{n=1}^{\infty} (\Lambda h)_n(t) \in PAP_0(\mathbb{R}, X, \rho)$ and so $(\Lambda h)(t)$ from lemma 2.11.

- If $p = 1$, Let $T > 0$, $h \in S^p PAP_0(\mathbb{R}; \mathbb{X}, \rho)$ we have that

$$\lim_{T\to+\infty} \frac{1}{\operatorname{meas}(T,\rho)} \int_{-T}^{T} \Big(\int_{s}^{s+1} \|h(\sigma)\| d\sigma \Big) \rho(s) ds = 0 \tag{3.12}$$

Using the Holder inequality and the estimate 3.1, it follows that

$$\begin{aligned}
\|(\Lambda h)_n(t)\| &= \| \int_{t-n}^{t-n+1} T(t-s)h(s)ds \| \\
&\leq \int_{t-n}^{t-n+1} \|T(t-s)\| \|h(s)\| ds \\
&\leq M \sup_{t-n\leq s\leq t-n+1} e^{-\omega(t-s)} \int_{t-n}^{t-n+1} \|h(s)\| ds \\
&\leq M e^{-\omega(n-1)} \|h\|_{S^1}.
\end{aligned} \tag{3.13}$$

We have

$$\begin{aligned}
&\lim_{T\to+\infty} \frac{1}{\operatorname{meas}(T,\rho)} \int_{-T}^{T} \|(\Lambda h)_n(t)\| \rho(t) dt \\
&\leq M e^{-\omega(n-1)} \lim_{T\to+\infty} \frac{1}{\operatorname{meas}(T,\rho)} \int_{-T}^{T} \|h(t)\|_{S^1} \rho(t) dt = 0
\end{aligned}$$

hence $(\Lambda h)_n \in PAP_0(\mathbb{R}, X, \rho)$. On the other hand, using the assumption that $\sum_{n=1}^{\infty} e^{-\omega(n-1)}$ is convergent, we then deduce from the well-known Weirstrass theorem that the series $\sum_{n=1}^{\infty} (\Lambda h)_n(t)$ is uniformly convergent on $\mathbb{R}$, furthermore $\sum_{n=1}^{\infty} (\Lambda h)_n(t) = (\Lambda h)(t)$. Consequently $\sum_{n=1}^{\infty} (\Lambda h)_n(t) \in PAP_0(\mathbb{R}, X, \rho)$ and so $(\Lambda h)(t)$ from lemma 2.11. □

Lemma 3.7. *Let $\rho \in \mathbb{U}_{\rm inv}$. The operator Γu is weighted pseudo almost periodic for u is weighted pseudo almost periodic.*

Proof. For $u(t)$ being weighted pseudo almost periodic, from Proposition 3.5, we see that $u(t-r)$ as weighted pseudo almost periodic, and from (H1) and theorem 2.12, it is easy to see that $F(t, u(t-r))$ belong to $PAP(\mathbb{R}; \mathbb{X}, \rho)$. Now we will show that $\int_{-\infty}^{t} T(t-s)G(s, u(s), u(s-r))ds$ beongs to $PAP(\mathbb{R}; \mathbb{X}, \rho)$, indeed from Theorem 3.4 and assumption (H2), it is easy to see that $G(s, u(s), u(s-r))$ belongs to $PAP^p(\mathbb{R}; \mathbb{X}, \rho)$. The proof of lemma is completed using the previous lemma. □

Theorem 3.8. *Let $\rho \in U_{\rm inv}$ and assume that* (H0)–(H2) *hold. If* $(L_F + \frac{2M}{\omega} L_G) < 1$. *Then* (1.1) *has unique weighted pseudo almost periodic mild solution.*

Proof. Let $u, v \in PAP(\mathbb{R}; \mathbb{X}, \rho)$. We observed that

$$\begin{aligned}
&\|(\Gamma u)(t) - (\Gamma v)(t)\| \\
&\leq \|F(t, u(t-r)) - F(t - v(t-r))\|
\end{aligned}$$

$$
\begin{aligned}
&+ \int_{-\infty}^{t} \|T(t-s)\| \|G(s,u(s),u(s-r)) - G(s,v(s),v(s-r))\| ds \\
&\le L_F \|u(t-r) - v(t-r)\| \\
&\quad + ML_G \int_{-\infty}^{t} e^{-\omega(t-s)} (\|u(s)-v(s)\| + \|u-v\|_\infty) ds \\
&\le L_F \|u-v\|_\infty + 2ML_G \Big(\int_{-\infty}^{t} e^{-\omega(t-s)} ds \Big) \|u-v\|_\infty \\
&\le \big(L_F + \frac{2M}{\omega} L_G \big) \|u-v\|_\infty .
\end{aligned}
$$

Thus

$$
\|\Gamma u - \Gamma v)\|_\infty \le \big(L_F + \frac{2M}{\omega} L_G \big) \|u-v\|_\infty .
$$

Then Γ is a contraction map on $PAP(\mathbb{R};\mathbb{X},\rho)$. Therefore, Γ has unique fixed point in $PAP(\mathbb{R};\mathbb{X},\rho)$, that is, there exist unique $u \in PAP(\mathbb{R};\mathbb{X},\rho)$ such that $\Gamma u = u$. Therefore, (1.1) has a unique weighted pseudo almost periodic mild solution. $\square$

4. Application

To illustrate the above results we examine the existence of weighted pseudo almost periodic solution to the differential equation

$$
\begin{gathered}
\frac{d}{dt}[u(t,x) - F(t,u(t-r,x))] \\
= \frac{d^2}{dx^2}[u(t,x) - F(t,u(t-r,x))] + G(t,u(t,x),u(t-r,x)), \quad t \in \mathbb{R},\ x \in [0,\pi] \\
u(t,0) - F(t,u(t-r,0)) = u(t,\pi) - F(t,u(t-r,\pi)) = 0, \quad t \in \mathbb{R}
\end{gathered}
\tag{4.1}
$$

Set $(\mathbb{X}, \|.\|) = (L^2[0,\pi], \|.\|_2)$, and define

$$
D(A) = \{u \in L^2[0,\pi], u'' \in L^2[0,\pi], u((0) = u(\pi) = 0\},
$$
$$
Au = \Delta u = u'' \quad \text{for all} \quad t \in \mathbb{R}
$$

It is well known that A is the infinitesimal generator of an exponentially stable C_0-semigroup $\{T(t)\}_{t\ge 0}$, with $M = \omega = 1$ in (3.1). Let $\rho(t) = 1 + t^2$. It can be easily shown that $\rho \in \mathbb{U}_{\rm inv}$. Let $F : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ defined by

$$
F(t,x) = \sin(t) + \sin(\sqrt{2}t) + \gamma e^{-|t|} \sin(u)
$$

it is checked that F belong to $PAP(\mathbb{R} \times \mathbb{X}, \rho)$ and satisfy

$$
\|F(t,u) - F(t,v)\| \le | \gamma | \, \|u-v\|, \quad \text{for all } t \in \mathbb{R} \text{ and } u,v \in \mathbb{X}
$$

Let $G : \mathbb{R} \times \mathbb{X} \times \mathbb{X} \to \mathbb{X}$ defined by

$$
G(t,u,v) = \cos(t) + \cos(\sqrt{2}t) + \theta(u,v) + \lambda e^{-|t|}
$$

Furthermore, it can be easily checked that $g(t) = \cos(t) + \cos(\sqrt{2}t) + \lambda e^{-|t|}$ belong to $S^p - PAP(\mathbb{R},\rho)$. If we suppose that the function θ satisfying

$$
\|\theta(u,v) - \theta(u',v')\| \le |\beta|(\|u-u'\| + \|v-v'\|) \quad \text{for all } u,v,u',v' \in \mathbb{X}
$$

then there exists $|\beta| > 0$ such that

$$
\|G(t,u,v) - G(t,u',v')\| \le |\beta|(\|u-u'\| + \|v-v'\|) \quad \text{for all} \;\; u,u',v,v' \in \mathbb{X}
$$

Consequently all assumption (H0), (H1) and (H2) are satisfied then by theorem 3.8; we deduce the following result. In conclusion, under the above assumption, if

$$|\gamma| + 2|\beta| < 1,$$

then (4.1) has a unique weighted pseudo almost periodic mild solution on $\mathbb{R}$.

References

[1] M. Adimy, K. Ezzinbi, A. Ouhinou; *Variation of constants formula and almost periodic solutions for some partial functional differential equations with infinite delay* Journal of Mathematical Analysis and Applications, Volume 317, Issue 2, 15 May 2006, Pages 668-689.

[2] E. Ait Dads, P. Cieutat, K. Ezzinbi; *The existence of pseudo-almost periodic solutions for some nonlinear differential equations in Banach spaces* Nonlinear Analysis: Theory, Methods and Applications, Volume 69, Issue 4, 15 August 2008, Pages 1325-1342.

[3] B. Amir and L. Maniar; *Composition of pseudo almost periodic functions and Cauchy problems with operator of non dense domain*, Ann. Math. Blaise Pascal, 6 (1) (1999), 1-11.

[4] R. Benkhalti, H. Bouzahir and K. Ezzinbi; *Existence of a Periodic Solution for Some Partial Functional Differential Equations with Infinite Delay* Journal of Mathematical Analysis and Applications, Volume 256, Issue 1, 1 April 2001, Pages 257-280.

[5] N. Boukli-Hacene and K. Ezzinbi; *Weighted pseudo almost periodic solutions for some partial functional differential equations* Nonlinear Analysis: Theory, Methods and Applications, Volume 71, Issue 9, 1 November 2009, Pages 3612-3621.

[6] X. Chen and X. Hu; *Weighted pseudo almost periodic solutions of neutral functional differential equations* Nonlinear Analysis: RealWorld Applications 12 (2011) 601-610.

[7] T. Diagana; *Stepanov-like pseudo-almost periodicity and its applications to some nonautonomous differential equations* Nonlinear Analysis 69 (2008) 4277-4285.

[8] T. Diagana, G. Mophou and G. M. N'Guérèkata; *Existence of weighted pseudo almost periodic solutions to some classes of differential equations with S^p-weighted pseudo almost periodic coefficients* Nonlinear Analysis 72 (2010) 430-438

[9] T. Diagana; *stepanov -like pseudo almost periodicity and its application to some nonautonomes differential equation*, Commun. Math. Anal.3 (2007)no1 pp 9-18

[10] T. Diagana; *Existence of weighted pseudo almost periodic solutions to some classes of hyperbolic evolution equations.* J. Math. Anal. Appl. 350 (2009) 18-28.

[11] T. Diagana; *Weighted pseudo almost periodic functions and applications* C.R.A.S 343 (10) (2006) 643-646.

[12] T. Diagana; *Existence of weighted pseudo-almost periodic solutions to some classes of nonautonomous partial evolution equations* Nonlinear Analysis 74 (2011) 600-615

[13] T. Diagana; *Weighted pseudo-almost periodic solutions to some differential equations* Nonlinear Analysis 68 (2008) 2250-2260.

[14] K. Ezzinbi, S. Fatajou and G. M. N'guéreèkata; *Massera-type theorem for the existence of C(n)-almost-periodic solutions for partial functional differential equations with infinite delay* Nonlinear Analysis: Theory, Methods and Applications, Volume 69, Issue 4, 15 August 2008, Pages 1413-1424.

[15] J. Liang, T.J. Xiao, J. Zhang; *Decomposition of weighted pseudo almost periodic functions*, Nonlinear Anal. 73 (10) (2010) 3456-3461.

[16] M. Maqbul; *Almost periodic solutions of neutral functional differential equation with Stepanov-Almost periodic terms* Electronic Journal of Differential Equations, Vol. 2011 (2011), No. 72, pp. 1-9.

[17] C. Y. Zhang; *Pseudo almost periodic solutions of some differential equations*, J. Math. Anal. Appl. 151 (1994), 62-76.

[18] C. Y. Zhang; *Pseudo almost periodic solutions of some differential equations II*, J. Math. Anal. Appl. 192 (1995), 543-561.

Mondher Damak

Université de Sfax, Faculté des Sciences Sfax, Route de Soukra, Km 3.5, B.P. 802 3018 Sfax, Tunisie

E-mail address: Mondher.Damak@fss.rnu.tn

Khalil Ezzinbi
Université Cadi Ayyad, Faculté des Sciences Semlalia, Département de Mathématiques, B.P. 2390, Marrakech, Morocco
E-mail address: ezzinbi@ucam.ac.ma

Lotfi Souden
Département de Mathématiques, Faculté Des Sciences De Gafsa. Cité Zarroug 2121, Gafsa, Tunisie
E-mail address: ltfsdn@gmail.com