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INTEGRAL INEQUALITYS FOR PARTIAL DYNAMIC EQUATIONS ON TIME SCALES

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Abstract. The aim of the present paper is to study some basic qualitative properties of solutions of some partial dynamic equations on time scales. A variant of certain fundamental integral inequality with explicit estimates is used to establish our results.

1. Introduction

During past few years many authors have established the time scale analogue of well known dynamic equations used in the development of theory of differential and integral equation see [3, 9, 10, 11, 12, 17, 18, 19]. In [4, 5, 6, 7, 8] authors have obtained some results on multiple integration and partial dynamic equations on time scales. Recently in [13, 14, 15, 16] authors have obtained inequalities on two independent variables on time scales. In the present paper we establish some basic qualitative properties of solutions of some partial dynamic equation on time scales. We use certain fundamental integral inequality with explicit estimates to establish our results. We assume understanding of time scales and its notation. Excellent information about introduction to time scales can be found in [1, 2].

In what follows $\mathbb{R}$ denotes the set of real numbers, $\mathbb{Z}$ the set of integers and $\mathbb{T}$ denotes arbitrary time scales. Let C_{rd} be the set of all rd continuous function. We assume $\mathbb{T}_1$ and $\mathbb{T}_2$ be two time scales and $\Omega = \mathbb{T}_1 \times \mathbb{T}_2$.

In this article, we consider partial dynamic equation of the type

$$u^{\Delta t}(t,x) = f(t,x,u(t,x)) + \int_{s_0}^{s} g(t,x,y,u(t,y))\Delta y + h(t,x), \tag{1.1}$$

which satisfies the initial condition

$$u(t_0,x) = u_0(x), \tag{1.2}$$

for $(t_0,x) \in \Omega$, where $u_0 \in C(I,\mathbb{R})$, $I = [a,b]$ $(a < b)$, $f \in C_{rd}(\Omega \times \mathbb{R}, \mathbb{R}^+)$, $g \in C_{rd}(\Omega \times I \times \mathbb{R}, \mathbb{R}^+)$ and u is unknown function to be found.

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2. Basic Inequality

We will use the following integral inequality.

Lemma 2.1. *Let $w, p \in C_{rd}(\Omega, \mathbb{R}_+)$ and let $c \geq 0$ be a constant*

$$w(t,x) \leq c + \int_{t_0}^{t} [p(s,x)w(s,x) + \int_a^b g(s,x,y)w(s,y)\Delta y]\Delta s, \tag{2.1}$$

for $(t,x) \in \Omega$, then

$$w(t,x) \leq cP(t,x)e_{\int_a^b q(s,x,y)p(s,y)\Delta y}(t,t_0), \tag{2.2}$$

for $(t,x) \in \Omega$, where

$$P(t,x) = e_{p(s,x)}(t_0,t). \tag{2.3}$$

Proof. Define a function

$$m(t,x) = c + \int_{t_0}^{t}\int_a^b q(s,x,y)w(s,y)\Delta y\Delta s. \tag{2.4}$$

Then (2.1) can be restated as

$$w(t,x) \leq m(t,x) + \int_{t_0}^{t} p(s,x)w(s,x)\Delta s, \tag{2.5}$$

$m(t,x)$ is non negative for $(t,x) \in \Omega$ and nondecreasing for t. Now considering (2.5) as a one dimensional integral inequalities in $t \in \mathbb{T}$ for every $x \in \mathbb{T}$ and a suitable application of inequality given in [9, Theorem 3.5], yields

$$m(t,x) \leq c + \int_{t_0}^{t}\int_a^b q(s,x,y)p(s,y)m(s,y)\Delta y\Delta s. \tag{2.6}$$

Let

$$k(s) = \int_a^b q(s,x,y)p(s,y)m(s,y)\Delta y, \tag{2.7}$$

for every $x \in \mathbb{T}$, the inequality (2.6) becomes

$$m(t,x) \leq c + \int_a^b k(s)\Delta s. \tag{2.8}$$

Let

$$z(t) = c + \int_a^b k(s)\Delta s, \tag{2.9}$$

then $z(t_0) = c$ and

$$m(t,x) \leq z(t), \tag{2.10}$$

for $(t,s) \in \Omega$. From (2.9), (2.7) and (2.10), we have

$$\begin{aligned} z^{\Delta}(t) = k(t) &= \int_a^b q(t,x,y)p(t,y)m(t,y)\Delta y \\ &\leq z(t)\int_a^b q(t,x,y)p(t,y)\Delta y. \end{aligned} \tag{2.11}$$

This inequality implies

$$z(t) \leq ce_{\int_a^b q(s,x,y)p(s,y)\Delta s}(t,t_0). \tag{2.12}$$

The required inequality (2.2) follows from (2.12), (2.9) and (2.6). $\square$

3. Main Results

The following theorem provides some estimates on the solution.

Theorem 3.1. *Suppose that the functions f, g, h, u_0 in (1.1) and (1.2) satisfy the conditions*

$$|f(t,x,u) - f(t,x,\overline{u})| \le c(t,x)|u-\overline{u}|, \tag{3.1}$$

$$|g(t,x,y,u) - g(t,x,y,\overline{u})| \le k(t,x,y)|u-\overline{u}|, \tag{3.2}$$

$$d = \sup \big|\phi(t,x) + \int_{t_0}^{t} \Big[f(s,x,t_0) + \int_a^b g(s,x,y,t_0)\Delta y\Big]\Delta s\big| < \infty, \tag{3.3}$$

where $c \in \Omega$, $k \in (\Omega \times \mathbb{R}^n, \mathbb{R}_+)$ and

$$\phi(t,x) = u_0(x) + \int_{t_0}^{t} h(s,x)\Delta s. \tag{3.4}$$

If $u(t,x)$ is any solution of (1.1)-(1.2) then

$$|u(t,x)| \le dC(t,x)e_{\int_a^b k(s,x,y)C(s,y)\Delta y}(t,t_0), \tag{3.5}$$

where

$$C(t,x) = e_{c(s,x)}(t,t_0). \tag{3.6}$$

Proof. Since $u(t,x)$ is a solution of (1.1)-(1.2) and hypotheses, we observe that

$$\begin{aligned}
|u(t,x)| &= \Big|\Big\{\phi(t,x) + \int_{t_0}^{t}\Big[\{f(s,x,u(s,x)) - f(s,x,t_0) + f(s,x,t_0)\}\\
&\quad + \int_a^b \{g(s,x,y,u(s,y)) - g(s,x,y,t_0) + g(s,x,y,t_0)\}\Delta y\Big]\Delta s\Big\}\Big|\\
&\le \Big|\phi(t,x) \int_{t_0}^{t}\Big[f(s,x,t_0) + \int_a^b g(s,x,y,t_0)\Delta y\Big]\Delta s\Big|\\
&\quad + \int_{t_0}^{t}\Big[|f(s,x,u(s,x)) - f(s,x,t_0)|\\
&\quad + \int_a^b |g(s,x,y,u(s,y)) - g(s,x,y,t_0)|\Delta y\Big]\Delta s\\
&\le d + \int_{t_0}^{t}\Big[c(s,x)|u(s,x)| + \int_a^b k(s,x,y)|u(s,y)|\Delta y\Big]\Delta s.
\end{aligned} \tag{3.7}$$

Now an application of Lemma 2.1 to (3.7) yields (3.5). □

Now we give approximation of solutions to (1.1)-(1.2). We obtain conditions under which we estimate errors between true solution and approximate solutions.

Let $u(t,x) \in \Omega$, $u^{\Delta t}(t,x)$ exist on $\mathbb{T}$ and satisfy the inequality

$$\Big|u^{\Delta t}(t,x) - f(t,x,u(t,x)) - \int_a^b g(t,x,y,u(t,y))\Delta y - h(t,x)\Big| \le \epsilon \tag{3.8}$$

for a given constant $\epsilon \ge 0$ where we suppose that (1.2) holds. Then we say that $u(t,x)$ has ϵ-approximate solutions with respect to (1.1).

Theorem 3.2. *Suppose the functions f, g in* (1.1) *satisfy the conditions*

$$|f(t,x,y) - f(t,x,\overline{u})| \le C(t,x)|u - \overline{u}|, \tag{3.9}$$

$$|g(t,x,y,u) - g(t,x,y,\overline{u})| \le K(t,x,y)|u - \overline{u}|, \tag{3.10}$$

Let $u_i(t,x)(i = 1,2)$, $(t,x) \in \Omega$ be respectively ϵ_i approximate solution of (1.1) *with*

$$u_i(t_o, x) = \overline{u}_i(x), \tag{3.11}$$

and let

$$\phi_i(t,x) = \overline{u}_i(x) + \int_{t_0}^{t} h(s,x)\Delta s. \tag{3.12}$$

Suppose that

$$|\phi_1(t,x) - \phi_2(t,x)| \le \delta, \tag{3.13}$$

where $\delta \ge 0$ is a constant and

$$M = \sup_{t \in T}[(\varepsilon_1 + \varepsilon_2)t + \delta] < \infty, \tag{3.14}$$

then

$$|u_1(t,x) - u_2(t,x)| \le MC(t,x)e_{\int_a^b k(s,x,y)C(s,y)\Delta s}(t,t_0), \tag{3.15}$$

where

$$C(t,x) = e_{c(s,x)}(t,t_0). \tag{3.16}$$

Proof. Since $u_i(t,x)$ $(i = 1,2)$, $(t,x) \in \Omega$ are respectively ϵ_i-approximate solutions of (1.1) with (3.8), we have

$$\left|u_i^{\Delta t}(t,x) - f(t,x,u_i(t,x)) - \int_a^b g(t,x,y,u_i(t,y))\Delta y - h(t,x)\right| \le \varepsilon_i. \tag{3.17}$$

By taking $t = s$ in the above inequality and integrating both sides with respect to s from t_0 to t for $t \in \mathbb{T}$, we obtain

$$\begin{aligned}
\varepsilon_i(t - t_0) &\ge \int_{t_0}^{t} \left|u_i^{\Delta s}(s,x) - f(s,x,u_i(s,x)) - \int_a^b g(s,x,y,u_i(s,y))\Delta y - h(s,x)\right|\Delta s \\
&\ge \Big| \int_{t_0}^{t} \{u_i^{\Delta s}(s,x) - f(s,x,u_i(s,x)) \\
&- \int_a^b g(s,x,y,u_i(s,y))\Delta y - h(s,x)\}\Delta s\Big| \\
&= |u_i(t,x) - \phi_i(t,x) \\
&\quad - \int_{t_0}^{t} \Big[f(s,x,u_i(s,x)) + \int_a^b g(s,x,y,u_i(s,y))\Delta y\Big]|\Delta s.
\end{aligned} \tag{3.18}$$

From (3.18) and using elementary inequalities

$$|v - z| \le |v| + |z|, \quad |v - z| \le |v - z|, \tag{3.19}$$

for $v, z \in \mathbb{R}_+$, we have

$$\begin{aligned}
&(\varepsilon_1 + \varepsilon_2)(t - t_0) \\
&\ge \Big|u_1(t,x) - \phi_1(t,x) - \int_{t_0}^{t} \Big[f(s,x,u_1(s,x)) + \int_a^b g(s,x,y,u_1(s,y))\Delta y\Big]\Big|\Delta s
\end{aligned}$$

$$
\begin{aligned}
&+\Big|u_2(t,x)-\phi_2(t,x)-\int_{t_0}^{t}\Big[f(s,x,u_2(s,x))+\int_a^b g(s,x,y,u_2(s,y))\Delta y\Big]\Big|\Delta s\\
&\geq \Big|\Big\{\Big|u_1(t,x)-\phi_1(t,x)-\int_{t_0}^{t}\Big[f(s,x,u_1(s,x))+\int_a^b g(s,x,y,u_1(s,y))\Delta y\Big]\Big|\Delta s\Big\}\\
&\quad-\Big\{\Big|u_2(t,x)-\phi_2(t,x)-\int_{t_0}^{t}\Big[f(s,x,u_2(s,x))+\int_a^b g(s,x,y,u_2(s,y))\Delta y\Big]\Big|\Delta s\Big\}\Big|\\
&\geq |u_1(t,x)-u_2(t,x)|-|\phi_1(t,x)-\phi_2(t,x)|\\
&\quad-\Big|\int_{t_0}^{t}\Big[f(s,x,u_1(s,x))+\int_a^b g(s,x,y,u_1(s,y))\Delta y\Big]\Big|\Delta s\\
&\quad-\int_{t_0}^{t}\Big[f(s,x,u_2(s,x))+\int_a^b g(s,x,y,u_2(s,y))\Delta y\Big]\Delta s\Big|.
\end{aligned}
$$

Let $u(t,x)=|u_1(t,x)-u_2(t,x)|$, $(t,x)\in\Omega$ from the above inequality and using the hypothesis we obtain

$$
\begin{aligned}
u(t,x)&\leq(\varepsilon_1+\varepsilon_2)(t-t_0)+\delta+\int_{t_0}^{t}\Big[c(s,x)u(s,x)+\int_a^b k(s,x,y)u(s,y)\Delta y\Big]\Delta s\\
&\leq M+\int_{t_0}^{t}\Big[c(s,x)u(s,x)+\int_a^b k(s,x,y)u(s,y)\Delta y\Big]\Delta s.
\end{aligned}
$$

Now an application of Lemma 2.1 to the above inequality yields (3.15). □

Remark 3.3. *When $u_1(t,x)$ is a solution of* (1.1) *with $u_1(0,x)=\overline{u}_1(x)$ we obtain $\epsilon_1=0$ and from* (3.15)*, we see that $u_2(t,x)\to u_1(t,x)$ as $\epsilon_2\to\epsilon_1$ and $\delta\to 0$. Furthermore, if we put $\epsilon_1=\epsilon_2=0$, $\overline{u}_1(x)=\overline{u}_2(x)$ in* (3.15)*, then we get the bound which shows the dependency of solutions of* (1.1) *on given initial values.*

Consider (1.1)-(1.2) together with following partial dynamic equation on time scales

$$
v^{\Delta t}(t,x)=\overline{f}(t,x,v(t,x))+\int_a^b \overline{g}(t,x,y,v(t,y))\Delta y+h(t,x) \tag{3.20}
$$

with given initial condition

$$
v(t_0,x)=v_0(x), \tag{3.21}
$$

for $(t,x)\in\Omega$ where $\overline{f}\in C_{rd}(\Omega,\mathbb{R}_+)$, $\overline{g}\in C_{rd}(\Omega\times\mathbb{R}^n,\mathbb{R}_+)$, $h\in C_{rd}(\Omega,\mathbb{R}_+)$.

The following theorem is concerned with the closeness of solutions of (1.1)-(1.2) and (3.20)-(3.21).

Theorem 3.4. *Suppose that the functions f,g in* (1.1)-(1.2) *satisfy the conditions* (3.9)-(3.10) *and that there exists constants $\overline{\varepsilon}_i\geq 0$, $\overline{\delta}_i\geq 0$ $(i=1,2)$ such that*

$$
\begin{gathered}
|f(t,x,u)-\overline{f}(t,x,u)|\leq\overline{\varepsilon}_1, \qquad (3.22)\\
|g(t,x,y,u)-\overline{g}(t,x,y,u)|\leq\overline{\varepsilon}_2, \qquad (3.23)\\
|h(t,x)-\overline{h}(t,x)|\leq\overline{\delta}_1, \qquad (3.24)\\
|u_0(x)-v_0(x)|\leq\overline{\delta}_2, \qquad (3.25)
\end{gathered}
$$

where f,g,h,u_0 and $\overline{f},\overline{g},\overline{h},v_0$ are the functions in (1.1)-(1.2) *and* (3.20)-(3.21) *and*

$$
\overline{M}=\sup_{t\in T}\big[\overline{\delta}_2+[\overline{\delta}_1+\overline{\varepsilon}_1+\overline{\varepsilon}_2(b-a)]t\big]<\infty. \tag{3.26}
$$

Let $u(t,x)$ and $v(t,x)$ be respectively the solutions of (1.1)-(1.2) *and* (3.20)-(3.21) *for $(t,x) \in \Omega$. Then*

$$|u(t,x) - v(t,x)| \le \overline{M} C(t,x) e_{\int_a^b k(s,x,y)C(s,y)\Delta y}(t,t_0), \tag{3.27}$$

for $(t,x) \in \Omega$ where $C(t,x)$ is given by (3.6).

Proof. Let $z(t,x) = |u(t,x) - v(t,x)|$, $(t,x) \in \Omega$. Since $u(t,x), v(t,x)$ are respectively the solutions of (1.1)-(1.2) and (3.20)-(3.21). We have

$$\begin{aligned}
z(t,x) \le & \Big| u_0(x) + \int_{t_0}^t h(s,x)\Delta s - v_0(x) - \int_{t_0}^t \overline{h}(s,x)\Delta s \Big| \\
& + \int_{t_0}^t \Big[|f(s,x,u(s,x)) - f(s,x,v(s,x))| + |f(s,x,v(s,x)) - \overline{f}(s,x,v(s,x))\Big| \\
& + \int_a^b \{ |g(s,x,y,u(s,y)) - g(s,x,y,v(s,y))| \\
& + |g(s,x,y,v(s,y)) - \overline{g}(s,x,y,v(s,y))| \}\Delta y \Big] \Delta s \\
\le & |u_0(x) - v_0(x)| + \int_{t_0}^t |h(s,x) - \overline{h}(s,x)|\Delta s \\
& + \int_{t_0}^t \Big[c(s,x)z(s,x) + \overline{\varepsilon}_1 + \int_a^b \{ k(s,x,y)z(s,y) + \overline{\varepsilon}_2 \}\Delta y \Big] \Delta s \\
\le & [\overline{\delta}_2 + \overline{\delta}_1 t + \overline{\varepsilon}_1 t + \overline{\varepsilon}_2 (b-a)t] \\
& + \int_{t_0}^t \Big[c(s,x)z(s,x) + \int_a^b k(s,x,y)z(s,y)\Delta y \Big] \Delta s \\
\le & M + \int_{t_0}^t \Big[c(s,x)z(s,x) + \int_a^b k(s,x,y)z(s,y)\Delta y \Big] \Delta s.
\end{aligned}$$

Now an application of Lemma 2.1 to the above inequality yields (3.27). □

Remark 3.5. We note that the result given in Theorem 3.2 relates the solutions of (1.1)-(1.2) and (3.20)-(3.21) in the sense that if f, g, h, u_0 are respectively close to f, g, h, v_0 then the solutions of (1.1)-(1.2) and (3.20)-(3.21) are close together.

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