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POSITIVE SOLUTIONS FOR A SYSTEM OF HIGHER ORDER BOUNDARY-VALUE PROBLEMS INVOLVING ALL DERIVATIVES OF ODD ORDERS

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Abstract. In this article we study the existence of positive solutions for the system of higher order boundary-value problems involving all derivatives of odd orders

$$\begin{aligned}
&(-1)^m w^{(2m)}\\
&= f(t, w, w', -w''', \dots, (-1)^{m-1} w^{(2m-1)}, z, z', -z''', \dots, (-1)^{n-1} z^{(2n-1)}),\\
&(-1)^n z^{(2n)}\\
&= g(t, w, w', -w''', \dots, (-1)^{m-1} w^{(2m-1)}, z, z', -z''', \dots, (-1)^{n-1} z^{(2n-1)}),\\
&w^{(2i)}(0) = w^{(2i+1)}(1) = 0 \quad (i = 0, 1, \dots, m-1),\\
&z^{(2j)}(0) = z^{(2j+1)}(1) = 0 \quad (j = 0, 1, \dots, n-1).
\end{aligned}$$

Here $f, g \in C([0,1] \times \mathbb{R}_+^{m+n+2}, \mathbb{R}_+)$ ($\mathbb{R}_+ := [0, +\infty)$). Our hypotheses imposed on the nonlinearities f and g are formulated in terms of two linear functions $h_1(x)$ and $h_2(y)$. We use fixed point index theory to establish our main results based on a priori estimates of positive solutions achieved by utilizing nonnegative matrices.

1. Introduction

In this article we study the existence and multiplicity of positive solutions for the system of higher order boundary-value problems:

$$\begin{aligned}
(-1)^m w^{(2m)} &= f(t, w, w', \dots, (-1)^{m-1} w^{(2m-1)}, z, z', \dots, (-1)^{n-1} z^{(2n-1)}),\\
(-1)^n z^{(2n)} &= g(t, w, w', \dots, (-1)^{m-1} w^{(2m-1)}, z, z', \dots, (-1)^{n-1} z^{(2n-1)}),\\
w^{(2i)}(0) &= w^{(2i+1)}(1) = 0 \quad (i = 0, 1, \dots, m-1),\\
z^{(2j)}(0) &= z^{(2j+1)}(1) = 0 \quad (j = 0, 1, \dots, n-1),
\end{aligned} \tag{1.1}$$

where $m, n \geq 2$, $f \in C([0,1] \times \mathbb{R}_+^{m+n+2}, \mathbb{R}_+)$ and $g \in C([0,1] \times \mathbb{R}_+^{m+n+2}, \mathbb{R}_+)$ ($\mathbb{R}_+ := [0, +\infty)$). By a positive solution of (1.1), we mean a pair of functions $(w, z) \in C^{2m}[0,1] \times C^{2n}[0,1]$ that solve (1.1) and satisfy $w(t) \geq 0$, $z(t) \geq 0$ for all $t \in [0,1]$, with at least one of them positive on $(0,1]$.

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The so-called Lidstone problem

$$\begin{gathered}(-1)^n u^{(2n)} = f(t, u, -u'', \dots, (-1)^{n-1}u^{(2n-2)}),\\ u^{(2i)}(0) = u^{(2i)}(1) = 0, \quad (i = 0, 1, \dots, n-1),\end{gathered} \tag{1.2}$$

has been extensively studied in recent years; see [17, 10, 13, 14, 15, 16] and the references cited therein. The existence of positive solutions for systems of nonlinear differential equations have been studied by many authors; see for instance, [3, 5, 9, 1], to cite a few. In [19], the author studied the existence of positive solutions of the system

$$\begin{gathered}(-1)^m u^{(2m)} = f_1(t, u, -u'', \dots, (-1)^{m-1}u^{(2m-2)}, v, -v'', \dots, (-1)^{n-1}v^{(2n-2)}),\\ (-1)^n v^{(2n)} = f_2(t, u, -u'', \dots, (-1)^{m-1}u^{(2m-2)}, v, -v'', \dots, (-1)^{n-1}v^{(2n-2)}),\\ \alpha_0 u^{(2i)}(0) - \beta_0 u^{(2i+1)}(0) = \alpha_1 u^{(2i)}(1) + \beta_1 u^{(2i+1)}(1) = 0 \quad (i = 0, 1, \dots, m-1),\\ \alpha_0 v^{(2j)}(0) - \beta_0 v^{(2j+1)}(0) = \alpha_1 v^{(2j)}(1) + \beta_1 v^{(2j+1)}(1) = 0 \quad (j = 0, 1, \dots, n-1),\end{gathered}$$

where $m, n \geq 1$ and $f_1, f_2 \in C([0,1] \times \mathbb{R}_+^{m+n}, \mathbb{R}_+)$. The main results obtained in [19] are presented in terms of nonnegative matrices and the author used the method of order reduction to overcome the difficulty arising from high order derivatives. Furthermore, in [6], Kang et al., using the fixed point theorem of cone expansion and compression type due to Krasnosel'skill, established some simple criteria for the existence, multiplicity and nonexistence of positive solutions of the following systems of singular boundary value problems with integral boundary value conditions:

$$\begin{gathered}(-1)^p u^{(2p)} = \lambda a_1(t) f(t, u, -u'', \dots, (-1)^{p-1}u^{2p-2}, v, -v'', \dots, (-1)^{q-1}v^{2q-2}),\\ (-1)^q v^{(2q)} = \mu a_2(t) g(t, u, -u'', \dots, (-1)^{p-1}u^{2p-2}, v, -v'', \dots, (-1)^{q-1}v^{2q-2}),\\ a_i u^{(2i)}(0) - b_i u^{(2i+1)}(0) = \int_0^1 m_i(s) u^{(2i)}(s) ds, \quad 0 \leq i \leq p-1,\\ c_i u^{(2i)}(1) - d_i u^{(2i+1)}(1) = \int_0^1 n_i(s) u^{(2i)}(s) ds, \quad 0 \leq i \leq p-1,\\ \alpha_j v^{(2j)}(0) - \beta_j v^{(2j+1)}(0) = \int_0^1 \varphi_j(s) v^{(2j)}(s) ds, \quad 0 \leq j \leq q-1,\\ \gamma_j v^{(2j)}(1) - \delta_j v^{(2j+1)}(1) = \int_0^1 \psi_j(s) v^{(2j)}(s) ds, \quad 0 \leq j \leq q-1,\end{gathered}$$

where $0 < t < 1, a_i \in ((0,1), [0, +\infty))$, $a_i(t)$ are allowed to be singular at $t = 0$ or $t = 1$, $i = 1, 2$.

Anand et al. [11] addressed the question of the existence of at least three symmetric positive solutions for the system of dynamical equations on symmetric times scales

$$\begin{gathered}(-1)^n y_1^{(\Delta\nabla)^n} = f_1(t, y_1, y_2), \quad t \in [a, b]_{\mathbb{T}},\\ (-1)^m y_2^{(\Delta\nabla)^m} = f_2(t, y_1, y_2), \quad t \in [a, b]_{\mathbb{T}},\end{gathered}$$

subject to the two-point boundary conditions

$$\begin{gathered}y_1^{(\Delta\nabla)^i}(a) = 0 = y_1^{(\Delta\nabla)^i}(b), \quad i = 0, 1, 2, \dots, n-1,\\ y_2^{(\Delta\nabla)^j}(a) = 0 = y_2^{(\Delta\nabla)^j}(b), \quad j = 0, 1, 2, \dots, m-1,\end{gathered}$$

where $f_i : [a,b]_{\mathbb{T}} \times \mathbb{R}^2 \to [0,\infty)$ are continuous and $f_i(t,y_1,y_2) = f_i(a+b-t,y_1,y_2)$ for $i = 1,2$, $a \in \mathbb{T}_k$, $b \in \mathbb{T}^k$ for a time scale $\mathbb{T}$, and $\sigma(a) < \rho(b)$. The main tool in [11] is the Avery fixed point theorem, a generalization of the Leggett-Williams fixed point theorem.

Yang et al. [18] studied the existence, multiplicity, and uniqueness of positive solutions for the boundary value problem

$$\begin{gathered}(-1)^n u^{(2n)} = f(t,u,u',\dots,(-1)^{n-1}u^{(2n-1)}),\\ u^{(2i)}(0) = u^{(2i+1)}(1) = 0 (j = 0,1,\dots,n-1),\end{gathered} \tag{1.3}$$

where $n \geq 2$, and $f \in C([0,1] \times \mathbb{R}_+^{n+1}, \mathbb{R}_+)$. The main results obtained in [18] are presented in terms of a linear function associated with the nonlinearity f in (1.3). They also apply their main results to establish the existence, multiplicity, and uniqueness of positive symmetric solutions for a Lidstone problem involving an open question posed by Eloe in 2000.

However, the existence problem of positive solutions for systems, like (1.1), has not been extensively studied yet. Our main difficulty here arises from the presence of all derivatives of all odd orders in the nonlinearities f and g in (1.1). To overcome this difficulty, as in [19], we first use the method of order reduction to transform (1.1) into an equivalent system of integro-differential equations, then prove the existence and multiplicity of positive solutions for the resultant equivalent system, thereby establishing our main results for (1.1). Our main features are threefold. Firstly, the nonlinear functions f and g contain all derivatives of odd orders. Secondly, nonnegative matrices are used to obtain the priori estimates of positive solutions. Finally, the orders $2m$ and $2n$ in (1.1) may be different. Such problems can be found in applied sciences; see [8].

This paper is organized as follows. Section 2 contains some preliminary results, including some basic facts recalled from [18]. Our main results, namely Theorems 3.4–3.6, are stated and proved in Section 3. Finally, three examples that illustrate our main results are presented in Section 4.

2. Preliminaries

Let

$$E := C^1([0,1],\mathbb{R}), \|u\| := \max\{\|u\|_0, \|u'\|_0\},$$

where $\|u\|_0 = \max\{|u(t)| : t \in [0,1]\}$. Furthermore, put

$$P := \{u \in E : u(t) \geq 0, u'(t) \geq 0, \forall t \in [0,1]\}.$$

Clearly, $(E, \|\cdot\|)$ is a real Banach space and P is a cone in E. Let

$$k(t,s) := \min\{t,s\}, (Tu)(t) := \int_0^1 k(t,s)u(s)ds.$$

Now let $u := (-1)^{m-1}w^{(2m-2)}, v := (-1)^{n-1}z^{(2n-2)}$. Then (1.1) is equivalent to the system of integro-differential equations

$$\begin{gathered}-u'' = f(t,T^{m-1}u,(T^{m-1}u)',\dots,(Tu)',u',T^{n-1}v,(T^{n-1}v)',\dots,(Tv)',v'),\\ -v'' = g(t,T^{m-1}u,(T^{m-1}u)',\dots,(Tu)',u',T^{n-1}v,(T^{n-1}v)',\dots,(Tv)',v'),\\ u(0) = u'(1) = 0,\\ v(0) = v'(1) = 0.\end{gathered} \tag{2.1}$$

Furthermore, the above problem is equivalent to

$$\begin{aligned} u(t) = \int_0^1 & k(t,s) f(s, (T^{m-1}u)(s), (T^{m-1}u)'(s), \dots, (Tu)'(s), u'(s), \\ & (T^{n-1}v)(s), (T^{n-1}v)'(s), \dots, (Tv)'(s), v'(s))ds, \\ v(t) = \int_0^1 & k(t,s) g(s, (T^{m-1}u)(s), (T^{m-1}u)'(s), \dots, (Tu)'(s), u'(s), \\ & (T^{n-1}v)(s), (T^{n-1}v)'(s), \dots, (Tv)'(s), v'(s))ds, \end{aligned} \tag{2.2}$$

Define the operators $A_i : P \times P \to P$ $(i = 1, 2)$ and $A : P \times P \to P \times P$ by

$$\begin{aligned} A_1(u,v)(t) := \int_0^1 & k(t,s) f(s, (T^{m-1}u)(s), (T^{m-1}u)'(s), \dots, (Tu)'(s), u'(s), \\ & (T^{n-1}v)(s), (T^{n-1}v)'(s), \dots, (Tv)'(s), v'(s))ds, \\ A_2(u,v)(t) := \int_0^1 & k(t,s) g(s, (T^{m-1}u)(s), (T^{m-1}u)'(s), \dots, (Tu)'(s), u'(s), \\ & (T^{n-1}v)(s), (T^{n-1}v)'(s), \dots, (Tv)'(s), v'(s))ds, \end{aligned}$$

$$A(u,v)(t) := (A_1(u,v), A_2(u,v)).$$

Now $f \in C([0,1] \times \mathbb{R}_+^{m+n+2}, \mathbb{R}_+)$ and $g \in C([0,1] \times \mathbb{R}_+^{m+n+2}, \mathbb{R}_+)$ imply that A_i and A are completely continuous operators. In our setting, the existence of positive solutions for (1.1) is equivalent to that of positive fixed points of $A : P \times P \to P \times P$. Let

$$\begin{aligned} G_1(u,v)(t) := f(t, & (T^{m-1}u)(t), (T^{m-1}u)'(t), \dots, (Tu)'(t), u'(t), (T^{n-1}v)(t), \\ & (T^{n-1}v)'(t), \dots, (Tv)'(t), v'(t)) \end{aligned} \tag{2.3}$$

$$\begin{aligned} G_2(u,v)(t) := g(t, & (T^{m-1}u)(t), (T^{m-1}u)'(t), \dots, (Tu)'(t), u'(t), (T^{n-1}v)(t), \\ & (T^{n-1}v)'(t), \dots, (Tv)'(t), v'(t)) \end{aligned} \tag{2.4}$$

Then $G_i : P \times P \to P$ $(i = 1, 2)$ is a continuous, bounded operator, and

$$A_i(u,v)(t) = \int_0^1 k(t,s) G_i(u,v)(s) ds, i = 1, 2.$$

Lemma 2.1 ([18, Lemma 2.2])**.** *Let $q \in P$. then*

$$\int_0^1 ((T^{n-1}q)(t) + 2\sum_{i=0}^{n-1} (T^{n-1-i}q)'(t)) te^t dt = \int_0^1 (q(t) + 2q'(t)) te^t dt.$$

Lemma 2.2 ([18, Lemma 2.3])**.** *If $q \in P \bigcap C^2[0,1]$, $q(0) = q'(1) = 0$, then*

$$\int_0^1 (-q''(t)) te^t dt = \int_0^1 (q(t) + 2q'(t)) te^t \, dt \,.$$

Lemma 2.3 ([18, Lemma 2.4])**.** *If $q \in P$, $q(0) = 0$, then*

$$q(1) \le \int_0^1 (q(t) + 2q'(t)) te^t \, dt \,.$$

Lemma 2.4 ([4]). *Let E be a real Banach space and P a cone on E. Suppose that $\Omega \subset E$ is a bounded open set and that $T : \overline{\Omega} \bigcap P \to P$ is a completely continuous operator. If there exists $\omega_0 \in P\backslash\{0\}$ such that*

$$\omega - T\omega \neq \lambda\omega_0, \forall\lambda \geq 0, \omega \in \partial\Omega \cap P,$$

then $i(T, \Omega \bigcap P, P) = 0$, where i indicates the fixed point index on P.

Lemma 2.5 ([4]). *Let E be a real Banach space and P a cone on E. Suppose that $\Omega \subset E$ is a bounded open set with $0 \in \Omega$ and that $T : \overline{\Omega} \cap P \to P$ is a completely continuous operator. If*

$$\omega - \lambda T\omega \neq 0, \forall\lambda \in [0, 1], \omega \in \partial\Omega \cap P,$$

then $i(T, \Omega \cap P, P) = 1$.

3. Existence of positive solutions for (1.1)

A real matrix B is said to be nonnegative if all elements of B are nonnegative.

For the sake of simplicity, we denote by $x := (x_1, \dots, x_{m+1}) \in \mathbb{R}_+^{m+1}$, $y := (y_1, \dots, y_{n+1}) \in \mathbb{R}_+^{n+1}$. Let

$$h_1(x) := x_1 + 2\sum_{i=2}^{m+1} x_i, \quad h_2(y) := y_1 + 2\sum_{i=2}^{n+1} y_i, \quad x \in \mathbb{R}_+^{m+1},\ y \in \mathbb{R}_+^{n+1}.$$

We now list our hypotheses on f and g.

(F1) $f, g \in C([0, 1] \times \mathbb{R}_+^{m+n+2}, \mathbb{R}_+)$.

(F2) There are four nonnegative constants a_1, a_2, b_1, b_2, and a real number $c > 0$ such that

$$f(t, x, y) \geq a_1h_1(x) + b_1h_2(y) - c, g(t, x, y) \geq a_2h_1(x) + b_2h_2(y) - c,$$

for all $(t, x, y) \in [0, 1] \times \mathbb{R}_+^{m+n+2}$ and the matrix $B_1 := \begin{pmatrix} a_1 - 1 & b_1 \\ a_2 & b_2 - 1 \end{pmatrix}$ is invertible with B_1^{-1} nonnegative.

(F3) For every $N > 0$, there exist two functions $\Phi_N, \Psi_N \in C(\mathbb{R}_+, \mathbb{R}_+)$ such that

$$f(t, x, y) \leq \Phi_N(x_{m+1} + y_{n+1}), g(t, x, y) \leq \Psi_N(x_{m+1} + y_{n+1})$$

for all $(x_1, \dots, x_m) \in \underbrace{[0, N] \times \dots \times [0, N]}_{m}$, $(y_1, \dots, y_n) \in \underbrace{[0, N] \times \dots \times [0, N]}_{n}$ and $x_{m+1}, y_{n+1} \geq 0$, and

$$\int_0^{\infty} \frac{\tau d\tau}{\Phi_N(\tau) + \Psi_N(\tau) + \delta} = \infty$$

for all $\delta > 0$.

(F4) There are four nonnegative constants c_1, c_2, d_1, d_2 and a positive constant r such that

$$f(t, x, y) \leq c_1h_1(x) + d_1h_2(y), \quad g(t, x, y) \leq c_2h_1(x) + d_2h_2(y)$$

for all $(t, x, y) \in [0, 1] \times ([0, r])^{m+n+2}$ and $B_2 := \begin{pmatrix} 1 - c_1 & -d_1 \\ -c_2 & 1 - d_2 \end{pmatrix}$ is invertible with B_2^{-1} nonnegative.

(F5) There are four nonnegative constants l_1, l_2, m_1, m_2 and a postive constant c such that

$$f(t,x,y) \le l_1h_1(x) + m_1h_2(y) + c, g(t,x,y) \le l_2h_1(x) + m_2h_2(y) + c,$$

for all $(t,x,y) \in [0,1] \times \mathbb{R}_+^{m+n+2}$ and $B_3 := \begin{pmatrix} 1-l_1 & -m_1 \\ -l_2 & 1-m_2 \end{pmatrix}$ is invertible with B_3^{-1} nonnegative.

(F6) There are four nonnegative constants p_1, p_2, q_1, q_2 and a positive constant r such that

$$f(t,x,y) \ge p_1h_1(x) + q_1h_2(y), g(t,x,y) \ge p_2h_1(x) + q_2h_2(y),$$

for all $(t,x,y) \in [0,1] \times ([0,r])^{m+n+2}$ and $B_4 := \begin{pmatrix} p_1-1 & q_1 \\ p_2 & q_2-1 \end{pmatrix}$ is invertible with B_4^{-1} nonnegative.

(F7) $f(t,x,y)$ and $g(t,x,y)$ are increasing in x and y,and there is a constant $\Lambda > 0$ such that

$$\int_0^1 f(s, \underbrace{\Lambda, \dots, \Lambda}_{m+n+2})ds < \Lambda, \int_0^1 g(s, \underbrace{\Lambda, \dots, \Lambda}_{m+n+2})ds < \Lambda.$$

Remark 3.1 ([19, Remark 2])**.** Let $l_{ij}(i,j = 1,2)$ be four nonnegative constants. Then it is easy to see that the matrix $B := \begin{pmatrix} l_{11}-1 & l_{12} \\ l_{21} & l_{22}-1 \end{pmatrix}$ is invertible with B^{-1} nonnegative if and only if one of the following two conditions is satisfied:

(1) $l_{11} > 1$, $l_{22} > 1$, $l_{12} = l_{21} = 0$.
(2) $l_{11} \le 1$, $l_{22} \le 1$, $\det B = (1-l_{11})(1-l_{22}) - l_{12}l_{21} < 0$.

Remark 3.2 ([19, Remark 3])**.** Let $l_{ij}(i,j = 1,2)$ be four nonnegative constants. Then it is easy to see that the matrix $D := \begin{pmatrix} 1-l_{11} & -l_{12} \\ -l_{21} & 1-l_{22} \end{pmatrix}$ is invertible with D^{-1} nonnegative if and only if $l_{11} < 1, l_{22} < 1, \det D = (1-l_{11})(1-l_{22}) - l_{12}l_{21} > 0$.

Remark 3.3. $f(t,x,y)$ is said to be increasing in x and y if $f(t,x,y) \le f(t,x',y')$ holds for every pair $(x,y),(x',y') \in \mathbb{R}_+^{m+n+2}$ with $(x,y) \le (x',y')$, where the partial ordering $\le$ in $\mathbb{R}_+^{m+n+2}$ is understood componentwise.

We have the following comments about the functions f and g.

(1) Condition (F3) is of Berstein-Nagumo type;
(2) f and g grow superlinearly both at $+\infty$ and at 0 if (F2) and (F4) hold;
(3) f and g grow sublinearly both at $+\infty$ and at 0 if (F5) and (F6) hold.

We adopt the convention in the sequel that $n_1, n_2, \dots$ stand for different positive constants. G_1 and G_2 are defined by (2.3) and (2.4).

Theorem 3.4. *If* (F1)–(F4) *hold, then* (1.1) *has at least one positive solution.*

Proof. It suffices to prove that (2.2) has at least one positive solution. We claim that the set

$$\mathcal{M}_1 := \{(u,v) \in P \times P : (u,v) = A(u,v) + \lambda(\varphi,\varphi), \lambda \ge 0\}$$

is bounded, where $\varphi(t) := te^{-t}$. Indeed, if $(u_0,v_0) \in \mathcal{M}_1$, then there exist a constant $\lambda_0 \ge 0$ such that $(u_0,v_0) = A(u_0,v_0) + \lambda_0(\varphi,\varphi)$, which can be written in the form

$$-u_0''(t) = G_1(u_0, v_0)(t) + \lambda_0(2-t)e^{-t}, -v_0''(t) = G_2(u_0, v_0)(t) + \lambda_0(2-t)e^{-t}.$$

By (F2), we have

$$\begin{aligned}-u_0''(t) \geq a_1((T^{m-1}u_0)(t) + 2\sum_{i=0}^{m-1}(T^{m-i-1}u_0)'(t)) + b_1((T^{n-1}v_0)(t)\\ + 2\sum_{i=0}^{n-1}(T^{n-i-1}v_0)'(t)) - c,\end{aligned}$$

$$\begin{aligned}-v_0''(t) \geq a_2((T^{m-1}u_0)(t) + 2\sum_{i=0}^{m-1}(T^{m-i-1}u_0)'(t)) + b_2((T^{n-1}v_0)(t)\\ + 2\sum_{i=0}^{n-1}(T^{n-i-1}v_0)'(t)) - c.\end{aligned}$$

Multiply by $\psi(t) := te^t$ on both sides of the last two inequalities and integrate over $[0,1]$, and use Lemmas 2.1 and 2.2 to obtain

$$\int_0^1 (u_0(t)+2u_0'(t))te^t dt \geq a_1 \int_0^1 (u_0(t)+2u_0'(t))te^t dt + b_1 \int_0^1 (v_0(t)+2v_0'(t))te^t dt - c$$

and

$$\int_0^1 (v_0(t)+2v_0'(t))te^t dt \geq a_2 \int_0^1 (u_0(t)+2u_0'(t))te^t dt + b_2 \int_0^1 (v_0(t)+2v_0'(t))te^t dt - c.$$

The above two inequalities can be written as

$$\begin{aligned}\begin{pmatrix} a_1-1 & b_1 \\ a_2 & b_2-1 \end{pmatrix} \begin{pmatrix} \int_0^1 (u_0(t)+2u_0'(t))te^t dt \\ \int_0^1 (v_0(t)+2v_0'(t))te^t dt \end{pmatrix} = B_1 \begin{pmatrix} \int_0^1 (u_0(t)+2u_0'(t))te^t dt \\ \int_0^1 (v_0(t)+2v_0'(t))te^t dt \end{pmatrix}\\ \leq \begin{pmatrix} c \\ c \end{pmatrix}.\end{aligned}$$

Now (F2) implies

$$\begin{pmatrix} \int_0^1 (u_0(t)+2u_0'(t))te^t dt \\ \int_0^1 (v_0(t)+2v_0'(t))te^t dt \end{pmatrix} \leq B_1^{-1} \begin{pmatrix} c \\ c \end{pmatrix} := \begin{pmatrix} N_1 \\ N_2 \end{pmatrix}.$$

Let $N := \max\{N_1, N_2\} > 0$. Then we have

$$\int_0^1 (u_0(t) + 2u_0'(t))te^t dt \leq N, \quad \int_0^1 (v_0(t) + 2v_0'(t))te^t dt \leq N, \quad \forall (u_0, v_0) \in \mathcal{M}_1.$$

Now Lemma 2.3 implies

$$\|u_0\|_0 = u_0(1) \leq N, \|v_0\|_0 = v_0(1) \leq N, \quad \forall (u_0, v_0) \in \mathcal{M}_1. \tag{3.1}$$

Furthermore, this estimate leads to

$$\|T^{m-1}u_0\|_0 = (T^{m-1}u_0)(1) \leq N, \|T^{n-1}v_0\|_0 = (T^{n-1}v_0)(1) \leq N,$$

for all $(u_0, v_0) \in \mathcal{M}_1$ and

$$\|(T^{m-i-1}u_0)'\|_0 = (T^{m-i-1}u_0)'(0) = \int_0^1 (T^{m-i-2}u_0)(t)dt \leq N,$$

$$\|(T^{n-j-1}v_0)'\|_0 = (T^{n-j-1}v_0)'(0) = \int_0^1 (T^{n-j-2}v_0)(t)dt \le N, \forall (u_0, v_0) \in \mathcal{M}_1,$$

$i = 0, \dots, m-2$, $j = 0, \dots, n-2$. Let

$$\mathbb{H} := \{\mu \ge 0 : \text{ there exists } (u,v) \in P \times P, \text{ such that } (u,v) = A(u,v) + \mu(\varphi, \varphi)\}.$$

Now (3.1) implies that $\mu_0 := \sup \mathbb{H} < +\infty$. By (F3), there are two functions $\Phi_N, \Psi_N \in C(\mathbb{R}_+, \mathbb{R}_+)$ such that

$$G_1(u,v)(t) \le \Phi_N(u'(t) + v'(t)), \quad G_2(u,v)(t) \le \Psi_N(u'(t) + v'(t)),$$

for all $(u, v) \in \mathcal{M}_1$. Hence we obtain

$$\begin{aligned} -u''(t) &= G_1(u,v)(t) + \mu(2-t)e^{-t} \\ &\le \Phi_N(u'(t) + v'(t)) + \mu(2-t)e^{-t} \\ &\le \Phi_N(u'(t) + v'(t)) + \mu(2-t)e^{-t} \\ &\le \Phi_N(u'(t) + v'(t)) + 2\mu_0, \end{aligned}$$

$$\begin{aligned} -v''(t) &= G_2(u,v)(t) + \mu(2-t)e^{-t} \\ &\le \Psi_N(u'(t) + v'(t)) + \mu(2-t)e^{-t} \\ &\le \Psi_N(u'(t) + v'(t)) + \mu(2-t)e^{-t} \\ &\le \Psi_N(u'(t) + v'(t)) + 2\mu_0, \end{aligned}$$

so that

$$-u''(t) - v''(t) \le \Phi_N(u'(t) + v'(t)) + \Psi_N(u'(t) + v'(t)) + 4\mu_0$$

for all $(u, v) \in \mathcal{M}_1$, $\mu \in \mathbb{H}$, and

$$\int_0^{u'(0)+v'(0)} \frac{\tau d\tau}{\Phi_N(\tau) + \Psi_N(\tau) + 4\mu_0} \le \int_0^1 u'(t) + v'(t)dt = u(1) + v(1) \le 2N$$

for all $(u, v) \in \mathcal{M}_1$. By (F3) again, there exists a constant $N_1 > 0$ such that

$$\|u' + v'\|_0 = u'(0) + v'(0) \le N_1, \quad \forall (u,v) \in \mathcal{M}_1.$$

This means that $\mathcal{M}_1$ is bounded. Taking $R > \sup\{\|(u,v)\| : (u,v) \in \mathcal{M}_1\}$, we have

$$(u,v) \ne A(u,v) + \lambda(\varphi, \varphi), \quad \forall (u,v) \in \partial\Omega_R \cap (P \times P), \lambda \ge 0.$$

Now Lemma 2.4 yields

$$i(A, \Omega_R \cap (P \times P), P \times P) = 0. \tag{3.2}$$

Let

$$\mathcal{M}_2 := \{(u,v) \in \overline{\Omega}_r \cap (P \times P) : (u,v) = \lambda A(u,v), \lambda \in [0,1]\}.$$

Now we want to prove that $\mathcal{M}_2 = \{0\}$. Indeed, if $(u, v) \in \mathcal{M}_2$, then there is $\lambda \in [0, 1]$ such that

$$(u(t), v(t)) = (\lambda \int_0^1 k(t,s)G_1(u,v)(s)ds, \lambda \int_0^1 k(t,s)G_2(u,v)(s)ds)$$

which can be written in the form

$$-u''(t) = \lambda G_1(u,v)(t), \quad -v''(t) = \lambda G_2(u,v)(t).$$

By (F4), we have

$$\begin{aligned}-u''(t) &\le c_1((T^{m-1}u)(t) + 2\sum_{i=0}^{m-1}(T^{m-i-1}u)'(t)) + d_1((T^{n-1}v)(t)\\ &\quad + 2\sum_{i=0}^{n-1}(T^{n-i-1}v)'(t)),\\ -v''(t) &\le c_2((T^{m-1}u)(t) + 2\sum_{i=0}^{m-1}(T^{m-i-1}u)'(t)) + d_2((T^{n-1}v)(t)\\ &\quad + 2\sum_{i=0}^{n-1}(T^{n-i-1}v)'(t)).\end{aligned}$$

Multiply by $\psi(t) := te^t$ on both sides of the above and integrate over $[0,1]$, and use Lemmas 2.1 and 2.2 to obtain

$$\begin{aligned}\begin{pmatrix}1-c_1 & -d_1\\ -c_2 & 1-d_2\end{pmatrix}\begin{pmatrix}\int_0^1(u(t)+2u'(t))te^tdt\\ \int_0^1(v(t)+2v'(t))te^tdt\end{pmatrix} &= B_2\begin{pmatrix}\int_0^1(u(t)+2u'(t))te^tdt\\ \int_0^1(v(t)+2v'(t))te^tdt\end{pmatrix}\\ &\le \begin{pmatrix}0\\0\end{pmatrix}.\end{aligned}$$

(F4) again implies

$$\begin{pmatrix}\int_0^1(u(t)+2u'(t))te^tdt\\ \int_0^1(v(t)+2v'(t))te^tdt\end{pmatrix} \le B_2^{-1}\begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}.$$

Consequently,

$$\int_0^1(u(t)+2u'(t))te^tdt = \int_0^1(v(t)+2v'(t))te^tdt = 0$$

and whence $u \equiv 0$, $v \equiv 0$, as required. Thus we have

$$(u,v) \ne \lambda A(u,v), \quad \forall (u,v) \in \partial\Omega_r \cap (P\times P),\ \lambda \in [0,1].$$

Now Lemma 2.5 yields

$$i(A, \Omega_r \cap (P\times P), P\times P) = 1.$$

This together with (3.2) implies

$$i(A, (\Omega_R\backslash\overline{\Omega}_r)\cap(P\times P), P\times P) = 0 - 1 = -1.$$

Therefore, A has at least one fixed point on $(\Omega_R\backslash\overline{\Omega}_r)\cap(P\times P)$ and (2.2) has at least one positive solution (u,v), and thus (1.1) has at least one positive solution $(w,z) = (T^{m-1}u, T^{n-1}v)$. This completes the proof. □

Theorem 3.5. *If* (F1), (F5), (F6) *hold, then* (1.1) *has at least one positive solution.*

Proof. Let

$$\mathcal{M}_3 := \{(u,v) \in P\times P : (u,v) = \lambda A(u,v), \lambda \in [0,1]\}.$$

We now assert that $\mathcal{M}_3$ is bounded. Indeed, if $(u,v) \in \mathcal{M}_3$, then there is $\lambda \in [0,1]$ such that

$$u(t) = \lambda\int_0^1 k(t,s)G_1(u,v)(s)ds, v(t) = \lambda\int_0^1 k(t,s)G_2(u,v)(s)ds,$$

which can be written in the form

$$-u''(t) = \lambda G_1(u,v)(t), \quad -v''(t) = \lambda G_2(u,v)(t).$$

By (F5), we have

$$\begin{aligned} -u''(t) \le\ & l_1((T^{m-1}u)(t) + 2\sum_{i=0}^{m-1}(T^{m-i-1}u)'(t)) + m_1((T^{n-1}v)(t) \\ & + 2\sum_{i=0}^{n-1}(T^{n-i-1}v)'(t)) + c \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} -v''(t) \le\ & l_2((T^{m-1}u)(t) + 2\sum_{i=0}^{m-1}(T^{m-i-1}u)'(t)) + m_2((T^{n-1}v)(t) \\ & + 2\sum_{i=0}^{n-1}(T^{n-i-1}v)'(t)) + c. \end{aligned} \tag{3.4}$$

Multiply by $\psi(t) := te^t$ on both sides of the above two inequalities and integrate over $[0,1]$, and then use Lemmas 2.1 and 2.2 to obtain

$$\int_0^1 (u(t)+2u'(t))te^t dt \le l_1 \int_0^1 (u(t)+2u'(t))te^t dt + m_1 \int_0^1 (v(t)+2v'(t))te^t dt + c$$

and

$$\int_0^1 (v(t)+2v'(t))te^t dt \le l_2 \int_0^1 (u(t)+2u'(t))te^t dt + m_2 \int_0^1 (v(t)+2v'(t))te^t dt + c,$$

which can be written in the form

$$\begin{aligned} \begin{pmatrix} 1-l_1 & -m_1 \\ -l_2 & 1-m_2 \end{pmatrix} \begin{pmatrix} \int_0^1 (u(t)+2u'(t))te^t dt \\ \int_0^1 (v(t)+2v'(t))te^t dt \end{pmatrix} &= B_3 \begin{pmatrix} \int_0^1 (u(t)+2u'(t))te^t dt \\ \int_0^1 (v(t)+2v'(t))te^t dt \end{pmatrix} \\ &\le \begin{pmatrix} c \\ c \end{pmatrix}. \end{aligned}$$

(F5) again implies

$$\begin{pmatrix} \int_0^1 (u(t)+2u'(t))te^t dt \\ \int_0^1 (v(t)+2v'(t))te^t dt \end{pmatrix} \le B_3^{-1} \begin{pmatrix} c \\ c \end{pmatrix} := \begin{pmatrix} n_3 \\ n_4 \end{pmatrix}.$$

Let $N = \max\{n_3, n_4\} > 0$. Then we have

$$\int_0^1 (u(t)+2u'(t))te^t dt \le N, \int_0^1 (v(t)+2v'(t))te^t dt \le N, \forall (u,v) \in \mathcal{M}_3.$$

By Lemma 2.3, we obtain

$$\begin{aligned} \|u\|_0 = u(1) &\le \int_0^1 (u(t)+2u'(t))te^t dt \le N, \\ \|v\|_0 = v(1) &\le \int_0^1 (v(t)+2u'(t))te^t dt \le N, \end{aligned}$$

for all $(u,v) \in \mathcal{M}_3$. Furthermore, those estimates lead to

$$\|T^{m-1}u\|_0 = (T^{m-1})(1) \le N, \quad \|T^{n-1}u\|_0 = (T^{n-1})(1) \le N,$$

for all $(u,v) \in \mathcal{M}_3$ and

$$\|(T^{m-i-1}u)'\|_0 = (T^{m-i-1}u)'(0) = \int_0^1 (T^{m-i-2}u)(t)dt \le N,$$
$$\|(T^{n-j-1}v)'\|_0 = (T^{n-j-1}v)'(0) = \int_0^1 (T^{n-j-2}v)(t)dt \le N$$

for all $(u,v) \in \mathcal{M}_3$ and $i = 0, \dots, m-2$, $j = 0, \dots, n-2$. By (3.3) and (3.4), we have

$$-u''(t) \le (l_1(2m-1) + m_1(2n-1))N + 2l_1u'(t) + 2m_1v'(t) + c,$$
$$-v''(t) \le (l_2(2m-1) + m_2(2n-1))N + 2l_2u'(t) + 2m_2v'(t) + c,$$

for all $(u,v) \in \mathcal{M}_3$. So we have

$$\begin{aligned}-(u''(t) + v''(t)) &\le (l_1(2m-1) + m_1(2n-1) + l_2(2m-1) + m_2(2n-1))N \\ &\quad + 2(l_1 + l_2)u'(t) + 2(m_1 + m_2)v'(t) + 2c.\end{aligned}$$

Noticing $u'(1) = v'(1) = 0$ and letting

$$N_2 := (l_1(2m-1) + m_1(2n-1) + l_2(2m-1) + m_2(2n-1))N + 2c$$

and $L := 2(l_1 + l_2 + m_1 + m_2) + 1$, we obtain

$$u'(t) + v'(t) \le \frac{N_2}{L}(e^{L-Lt} - 1),$$

so that

$$\|u' + v'\|_0 = u'(0) + v'(0) \le \frac{N_2}{L}(e^L - 1).$$

This proves the boundedness of $\mathcal{M}_3$. Taking $R > \sup\{\|(u,v)\| : (u,v) \in \mathcal{M}_3\}$, we have

$$(u,v) \ne \lambda A(u,v), \quad \forall (u,v) \in \partial\Omega_R \cap (P \times P), \lambda \in [0,1].$$

Now Lemma 2.5 yields

$$i(A, \Omega_R \cap (P \times P), P \times P) = 1. \tag{3.5}$$

Let

$$\mathcal{M}_4 := \{(u,v) \in \overline{\Omega}_r \cap (P \times P) : (u,v) = A(u,v) + \lambda(\varphi,\varphi), \lambda \ge 0\}$$

where $\varphi(t) := te^{-t}$. We want to prove that $\mathcal{M}_4 \subset \{0\}$. Indeed, if $(u,v) \in \mathcal{M}_4$, then there is $\lambda \ge 0$ such that

$$u(t) = \int_0^1 k(t,s)G_1(u,v)(s)ds + \lambda\varphi(t), \quad v(t) = \int_0^1 k(t,s)G_2(u,v)(s)ds + \lambda\varphi(t),$$

which can be written in the form

$$-u''(t) = G_1(u,v)(t) + \lambda(2-t)e^{-t}, \quad -v''(t) = G_2(u,v)(t) + \lambda(2-t)e^{-t}.$$

By (F6), we have

$$\begin{aligned}&-u''(t) \\ &\ge p_1\big((T^{m-1}u)(t) + 2\sum_{i=0}^{m-1}(T^{m-i-1}u)'(t)\big) + q_1\big((T^{n-1}v)(t) + 2\sum_{i=0}^{n-1}(T^{n-i-1}v)'(t)\big), \\ &-v''(t)\end{aligned}$$

$$\geq p_2\big((T^{m-1}u)(t)+2\sum_{i=0}^{m-1}(T^{m-i-1}u)'(t)\big)+q_2\big((T^{n-1}v)(t)+2\sum_{i=0}^{n-1}(T^{n-i-1}v)'(t)\big).$$

Multiply by $\psi(t):=te^t$ on both sides of the above two inequalities and integrate over $[0,1]$ and use Lemmas 2.1 and 2.2 to obtain

$$\int_0^1(u(t)+2u'(t))te^tdt\geq p_1\int_0^1(u(t)+2u'(t))te^tdt+q_1\int_0^1(v(t)+2v'(t))te^tdt$$

and

$$\int_0^1(v(t)+2v'(t))te^tdt\geq p_2\int_0^1(u(t)+2u'(t))te^tdt+q_2\int_0^1(v(t)+2v'(t))te^tdt,$$

which can be written in the form

$$\begin{aligned}\begin{pmatrix}p_1-1 & q_1\\ p_2 & q_2-1\end{pmatrix}\begin{pmatrix}\int_0^1(u(t)+2u'(t))te^tdt\\ \int_0^1(v(t)+2v'(t))te^tdt\end{pmatrix}&=B_4\begin{pmatrix}\int_0^1(u(t)+2u'(t))te^tdt\\ \int_0^1(v(t)+2v'(t))te^tdt\end{pmatrix}\\ &\leq\begin{pmatrix}0\\0\end{pmatrix}.\end{aligned}$$

Now (F6) implies

$$\begin{pmatrix}\int_0^1(u(t)+2u'(t))te^tdt\\ \int_0^1(v(t)+2v'(t))te^tdt\end{pmatrix}\leq B_4^{-1}\begin{pmatrix}0\\0\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}.$$

Therefore, $\int_0^1(u(t)+2u'(t))te^tdt=\int_0^1(v(t)+2v'(t))te^tdt=0$ and $u\equiv 0$, $v\equiv 0$. This implies $\mathcal{M}_4\subset\{0\}$, as required. As a result of this, we obtain

$$(u,v)\neq A(u,v)+\lambda(\varphi,\varphi),\quad \forall(u,v)\in\partial\Omega_r\cap(P\times P),\ \lambda\geq 0.$$

Now Lemma 2.4 yields

$$i(A,\Omega_r\cap(P\times P),P\times P)=0. \tag{3.6}$$

Combining (3.5) and (3.6) gives

$$i(A,(\Omega_R\backslash\overline{\Omega}_r)\cap(P\times P),P\times P)=1.$$

Hence A has at least one fixed point on $(\Omega_R\backslash\overline{\Omega}_r)\cap(P\times P)$. Thus (1.1) has at least one positive solution $(w,z)=(T^{m-1}u,T^{n-1}v)$. This complete the proof. □

Theorem 3.6. *If* (F1)–(F3), (F6), (F7) *hold, then* (1.1) *has at least two positive solutions.*

Proof. By (F7), the following inequalities

$$f(t,x,y)\leq f(t,\underbrace{\Lambda,\dots,\Lambda}_{m+n+2})<\Lambda,\quad g(t,x,y)\leq g(t,\underbrace{\Lambda,\dots,\Lambda}_{m+n+2})<\Lambda,$$

hold for all $t\in[0,1]$ and all $(x,y)\in\underbrace{[0,\Lambda]\times\dots\times[0,\Lambda]}_{m+n+2}$. Consequently, we have for all $(u,v)\in\partial\Omega_\Lambda\cap(P\times P)$,

$$\begin{aligned}\|A_1(u,v)\|_0=A_1(u,v)(1)&=\int_0^1 sG_1(u,v)(s)ds\leq\int_0^1 G_1(u,v)(s)ds\\ &\leq\int_0^1 f(s,\Lambda,\dots,\Lambda)ds<\Lambda=\|(u,v)\|,\end{aligned}$$

$$
\begin{aligned}
\|A_2(u,v)\|_0 = A_2(u,v)(1) &= \int_0^1 sG_2(u,v)(s)ds \le \int_0^1 G_2(u,v)(s)ds \\
&\le \int_0^1 g(s,\Lambda,\dots,\Lambda)ds < \Lambda = \|(u,v)\|,
\end{aligned}
$$

$$
\begin{aligned}
\|(A_1(u,v))'\|_0 = (A_1(u,v))'(0) &= \int_0^1 G_1(u,v)(s)ds \\
&\le \int_0^1 f(s,\Lambda,\dots,\Lambda)ds < \Lambda = \|(u,v)\|,
\end{aligned}
$$

$$
\begin{aligned}
\|(A_2(u,v))'\|_0 = (A_2(u,v))'(0) &= \int_0^1 G_2(u,v)(s)ds \\
&\le \int_0^1 g(s,\Lambda,\dots,\Lambda)ds < \Lambda = \|(u,v)\|.
\end{aligned}
$$

The preceding inequalities imply $\|A(u,v)\| = \|(A_1(u,v), A_2(u,v))\| < \Lambda = \|(u,v)\|$, and thus

$$
(u,v) \ne \lambda A(u,v), \quad \forall (u,v) \in \partial\Omega_\Lambda \cap (P \times P),\ 0 \le \lambda \le 1.
$$

Now Lemma 2.5 yields

$$
i(A, \Omega_\Lambda \cap (P \times P), P \times P) = 1. \tag{3.7}
$$

By (F2), (F3) and (F6), we know that (3.2) and (3.6) hold. Note we can choose $R > \Lambda > r$ in (3.2) and (3.6) (see the proofs of Theorems 3.4 and 3.5). Combining (3.2), (3.6) and (3.7), we obtain

$$
i(A, (\Omega_R \backslash \overline{\Omega}_\Lambda) \cap (P \times P), P \times P) = 0 - 1 = -1,
$$

and

$$
i(A, (\Omega_\Lambda \backslash \overline{\Omega}_r) \cap (P \times P), P \times P) = 1 - 0 = 1.
$$

Therefore, A has at least two fixed points, with one on $(\Omega_R \backslash \overline{\Omega}_\Lambda) \cap (P \times P)$ and the other on $(\Omega_\Lambda \backslash \overline{\Omega}_r) \cap (P \times P)$. Hence (1.1) has at least two positive solutions. □

4. Examples

In this section we present three examples that illustrate our main results.

Example 4.1. Suppose $(\xi_{ij})_{2\times(m+1)}$ and $(\eta_{ij})_{2\times(n+1)}$ be two positive matrices and $1 < \alpha_i \le 2 (i = 1, 2.)$. Let

$$
\begin{aligned}
f(t,x,y) &:= \Big(\sum_{j=1}^{m+1} \xi_{1j} x_j + \sum_{j=1}^{n+1} \eta_{1j} y_j \Big)^{\alpha_1} \quad t \in [0,1],\ x \in \mathbb{R}_+^{m+1},\ y \in \mathbb{R}_+^{n+1}, \\
g(t,x,y) &:= \Big(\sum_{j=1}^{m+1} \xi_{2j} x_j + \sum_{j=1}^{n+1} \eta_{2j} y_j \Big)^{\alpha_2} \quad t \in [0,1],\ x \in \mathbb{R}_+^{m+1},\ y \in \mathbb{R}_+^{n+1}.
\end{aligned}
$$

Now (F1)–(F4) hold. By Theorem 3.4, (1.1) has at least one positive solution.

Example 4.2. Suppose $(\xi'_{ij})_{2\times(m+1)}$ and $(\eta'_{ij})_{2\times(n+1)}$ be two positive matrices and $0<\alpha_i<1 (i=3,4.)$. Let

$$f(t,x,y) := \Big(\sum_{j=1}^{m+1}\xi'_{1j}x_j + \sum_{j=1}^{n+1}\eta'_{1j}y_j\Big)^{\alpha_3} \quad t\in[0,1],\ x\in\mathbb{R}_+^{m+1},\ y\in\mathbb{R}_+^{n+1},$$

$$g(t,x,y) := \Big(\sum_{j=1}^{m+1}\xi'_{2j}x_j + \sum_{j=1}^{n+1}\eta'_{2j}y_j\Big)^{\alpha_4} \quad t\in[0,1],\ x\in\mathbb{R}_+^{m+1},\ y\in\mathbb{R}_+^{n+1}.$$

Now (F1), (F5) and (F6) are satisfied. By Theorem 3.5, (1.1) has at least one positive solution.

Example 4.3. Suppose $(\xi_{ij})_{2\times(m+1)}$, $(\xi'_{ij})_{2\times(m+1)}$, $(\eta_{ij})_{2\times(n+1)}$ and $(\eta'_{ij})_{2\times(n+1)}$ be four positive matrices, $1<\beta_i\le 2 (i=1,2)$, $0<\gamma_i<1$ $(i=1,2)$. Let

$$f(t,x,y) := \Big(\sum_{j=1}^{m+1}\xi_{1j}x_j + \sum_{j=1}^{n+1}\eta_{1j}y_j\Big)^{\beta_1} + \Big(\sum_{j=1}^{m+1}\xi'_{1j}x_j + \sum_{j=1}^{n+1}\eta'_{1j}y_j\Big)^{\gamma_1}$$
$$t\in[0,1], x\in\mathbb{R}_+^{m+1}, y\in\mathbb{R}_+^{n+1},$$
$$g(t,x,y) := \Big(\sum_{j=1}^{m+1}\xi_{2j}x_j + \sum_{j=1}^{n+1}\eta_{2j}y_j\Big)^{\beta_2} + \Big(\sum_{j=1}^{m+1}\xi'_{2j}x_j + \sum_{j=1}^{n+1}\eta'_{2j}y_j\Big)^{\gamma_2}$$
$$t\in[0,1], x\in\mathbb{R}_+^{m+1}, y\in\mathbb{R}_+^{n+1}.$$

Now (F1)-(F3), (F6) and (F7) are satisfied. By Theorem 3.6, (1.1) has at least two positive solutions.

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