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SELF-SIMILAR DECAY TO THE MARGINALLY STABLE GROUND STATE IN A MODEL FOR FILM FLOW OVER INCLINED WAVY BOTTOMS

TOBIAS HÄCKER, GUIDO SCHNEIDER, HANNES UECKER

Abstract. The integral boundary layer system (IBL) with spatially periodic coefficients arises as a long wave approximation for the flow of a viscous incompressible fluid down a wavy inclined plane. The Nusselt-like stationary solution of the IBL is linearly at best marginally stable; i.e., it has essential spectrum at least up to the imaginary axis. Nevertheless, in this stable case we show that localized perturbations of the ground state decay in a self-similar way. The proof uses the renormalization group method in Bloch variables and the fact that in the stable case the Burgers equation is the amplitude equation for long waves of small amplitude in the IBL. It is the first time that such a proof is given for a quasilinear PDE with spatially periodic coefficients.

1. Introduction

The gravity driven free surface flow of a viscous incompressible fluid down an inclined plate plays an important role in heat exchanging devices. Numerous applications are found in coating processes ranging from the production of compact discs to photographic industries. For a flat bottom, the inclined film problem has been extensively studied experimentally, numerically, and analytically; see [7] for a review. In particular, it is well known that for a given film height the underlying Navier-Stokes equations possess a stationary solution with a parabolic velocity profile and a flat surface. Denoting the inclination angle by α, this so-called Nusselt solution is spectrally stable if the Reynolds number R is below the critical value $\mathrm{R}_{\mathrm{crit}} = 5/6 \cot\alpha$, and unstable to long waves for $\mathrm{R} > \mathrm{R}_{\mathrm{crit}}$, cf. [2, 24]. Nonlinear diffusive stability in the sense of the present paper in the spectrally stable case was shown in [21], while for $\mathrm{R} > \mathrm{R}_{\mathrm{crit}}$ surface waves are generated, which pass through a number of secondary instabilities until turbulence occurs at high Reynolds numbers; see [6], for instance.

In many applications the bottom is not perfectly flat but rather has a wavy profile. This may be due to natural irregularities or by design, for example in cooling processes. Thus, it is of interest to study the impact of an undulated bottom on the film flow. However, to study the stability of stationary solutions, the Navier-Stokes equations in combination with the free surface are hard to handle and

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thus there has been much effort to derive simpler model equations. Starting from the 2D Navier-Stokes equations in curvilinear coordinates, in [12] we derived a 2-dimensional system with periodic coefficients for the film thickness $F = F(t,x) \in \mathbb{R}$ and the local flow rate $Q = Q(t,x) := \int_0^{F(t,x)} U(t,x,z)\,\mathrm{d}z$, where U is the velocity in direction parallel to the bottom. In [12] this system is called weighted residual integral boundary layer system, here IBL in short, and may be written as

$$\partial_t F = -\frac{1}{1+\kappa F}\partial_x Q, \tag{1.1}$$

$$\begin{aligned}\partial_t Q = \frac{5}{2\mathrm{R}}\Big(\frac{\sin(\alpha-\theta)}{\sin\alpha}F - \frac{Q}{F^2} - \frac{\cos(\alpha-\theta)}{\sin\alpha}\partial_x F\, F - \frac{3}{8}\frac{\sin(\alpha-\theta)}{\sin\alpha}\partial_x\theta\, F^2\Big)\\ + \frac{5}{6}\mathrm{W}(\partial_x^3 F - \partial_x\kappa)F - \frac{17}{7}\frac{Q}{F}\partial_x Q + \frac{9}{7}\frac{Q^2}{F^2}\partial_x F - \frac{1}{210}\mathrm{R}(\partial_x Q)^2 Q\\ + \frac{1}{\mathrm{R}}\Big(\frac{9}{2}\partial_x^2 Q + \frac{45}{16}\kappa\frac{Q}{F} + 4\frac{Q}{F^2}(\partial_x F)^2 - 6\frac{Q}{F}\partial_x^2 F - \frac{9}{2}\frac{1}{F}\partial_x Q\,\partial_x F\Big).\end{aligned} \tag{1.2}$$

Here $t \geq 0$ denotes time, $x \in \mathbb{R}$ corresponds to arclength along the bottom, and we simplified notation of the IBL used in [12, (31),(32)] by redefining the spatial variable X, the temporal variable T, and the curvature K used in [12, (31),(32)] via

$$x := \frac{1}{\delta}X, \quad t := \frac{1}{\delta}T, \quad \kappa := \delta\zeta K, \tag{1.3}$$

where $\delta > 0$ is a dimensionless wave number, $\zeta \geq 0$ describes the bottom waviness, and $\kappa = \kappa(x)$ is the curvature of the bottom which is periodic with period $\gamma > 0$. For the surface tension effects here we replaced the inverse Bond number $\mathrm{B_i}$ from [12] by the Weber number W, defined by $\mathrm{W} := 3\delta^{-2}\mathrm{B_i}\mathrm{R}^{-1}$. Finally, $\alpha > 0$ is the mean inclination angle such that $\alpha - \theta$, with $\theta = \theta(x)$ is the γ-periodic local inclination angle, and R is the Reynolds number which measures the ratio between inertia and viscous forces.

Remark 1.1. (a) From the non-dimensionalization and derivation in [12] we have that $F \approx 1$ and $1 + \kappa F \approx 1$ and thus the denominators in (1.1), (1.2) are bounded from below by, e.g., 1/2.

(b) In [12] we also considered a regularized version (rIBL) of (1.1), (1.2), mainly to correct some unphysical behaviour of (1.1), (1.2) for $\mathrm{R} \gg \mathrm{R_{crit}}$. Here we are interested in $\mathrm{R} \leq \mathrm{R_{crit}}$ where the difference between (1.1), (1.2) and the rIBL is very small. In particular, the two versions only differ by terms which for $\mathrm{R} < \mathrm{R_{crit}}$ are asymptotically irrelevant. Therefore we stick to the slightly simpler version (1.1), (1.2), but nevertheless (1.1), (1.2) is a quasilinear parabolic system with spatially periodic coefficients.

Numerical simulations for (1.1), (1.2) showed very good agreement with data available from experiment and full Navier-Stokes simulations. In particular, (1.1), (1.2) can be used to approximate stationary solutions of the original Navier-Stokes systems, even with eddies, see [12]. Moreover, from linear stability analysis one can again find a critical Reynolds number $\mathrm{R_{crit}}$ beyond which the free surface of stationary solutions undergoes a long wave instability [23], and again the numerical stability results from [12] for the IBL agree very well with [23].

Thus, here we use the IBL as a model problem to study nonlinear stability of Nusselt-like stationary γ-periodic solutions (f_s, q_s) in the spectrally stable case. For stationary solutions q_s is constant, and it turns out that we always have families of stationary solutions which can be parametrized by q_s. Therefore, the stability of any spectrally stable (f_s, q_s) is nontrivial since linearizations around such (f_s, q_s) always have essential spectrum up to the imaginary axis. Thus, we cannot conclude stability from the linearization alone but have to take into account the nonlinearity.

If we restrict to spatially localized perturbations, dissipative systems often show dynamics which are similar to those of linear diffusion equations. To be more precise, denoting the solution by $v(t, x)$, the rescaled solution $\sqrt{t}v(t, \sqrt{t}x)$ converges towards a Gaussian limit. In this case, the nonlinearity is called asymptotically irrelevant. However, if the nonlinearity has an advection term $\partial_x(v^2)$, then it becomes relevant and the resulting non-Gaussian limit of the rescaled solution is determined by the Burgers equation, see [5], for instance.

Here we show a similar result for the IBL, namely that localized perturbations of spectrally stable stationary solutions $(f_s, q_s)^\top$ decay in a universal manner, which is determined by the Burgers equation. The proof relies on renormalization group (RG) methods [5] for nonlinear parabolic PDEs, which have been used for systems like the Ginzburg-Landau equation, see [3, 8, 4, 9], or pattern forming systems, see [16, 17, 19, 10, 18]. Also for film flow over flat inclines RG methods were used to show nonlinear stability of spectrally stable stationary solutions, namely in [20] for an IBL and in [21] for the full Navier-Stokes system.

Mathematically, (1.1), (1.2) can be classified as a quasilinear second order parabolic system. Besides the quasilinearity, which makes the local existence theory difficult, we have the following issues. First, in contrast to the Nusselt solution over flat bottoms, over wavy bottoms the stationary solutions are not known in closed form. Second, Fourier analysis, which is an essential tool in the stability proofs for flat inclines, has to be replaced by Bloch wave analysis. This was used in [22] to prove nonlinear stability for a semilinear model problem, namely a spatially periodic Kuramoto-Shivashinsky equation.

Notation. For $m, r \in \mathbb{R}$ the weighted Sobolev spaces $H^r(m)$ are defined as

$$H^r(m) := \{v : \mathbb{R} \to \mathbb{C} \mid \|v\|_{H^r(m)} = \|\varrho^m v\|_{H^r} < \infty\} \text{ with } \varrho(x) = (1+x^2)^{1/2}. \quad (1.4)$$

Fourier transform $\mathcal{F}$ is defined by

$$\mathcal{F}v(k) = \frac{1}{2\pi}\int_{\mathbb{R}} v(x)\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}x, \quad v(x) = \mathcal{F}^{-1}\hat{v}(x) = \int_{\mathbb{R}} \hat{v}(k)\mathrm{e}^{\mathrm{i}kx}\mathrm{d}k, \quad (1.5)$$

and is an isomorphism between $H^r(m)$ and $H^m(r)$.

Our main result now reads as follows, where for notational convenience we take initial conditions for (1.1), (1.2) at $t = 1$, and where the spectral stability assumptions will be discussed below in Assumption 2.3.

Theorem 1.2. *Let $p \in (0, 1/2)$, $3 < r < 4$, and let $(f_s, q_s)^\top$ be a spectrally stable stationary solution of the IBL* (1.1), (1.2)*, cf. Assumption 2.3 below. Then there exist constants $C_1, C_2 > 0$ such that the following holds. If $\|f_0\|_{H^r(2)} + \|q_0\|_{H^{r-1}(2)} \le C_1$, then there exists a unique global solution $(F, Q)^\top = (f_s, q_s)^\top + (f, q)^\top$ of the IBL* (1.1), (1.2) *with $(f, q)^\top\big|_{t=1} = (f_0, q_0)^\top$ and*

$$\sup_{x\in\mathbb{R}} \Big|(f, q)^\top - t^{-1/2} f_{z_0}(t^{-1/2}(x + c_1 t))\Phi^1(0, x)\Big| \le C_2 t^{-1+p/2} \quad (1.6)$$

for $t \in [1,\infty)$. *Here,* $\Phi^1(0,\cdot) = (\mathrm{d}f_s/\mathrm{d}q_s, 1)^\top$ *is the critical eigenfunction of the linearization of* (1.1), (1.2) *around* $(f_s, q_s)^\top$, *and*

$$f_{z_0}(y) = \frac{\sqrt{c_2}}{d} \frac{z_0 \,\mathrm{erf}'(y/(2\sqrt{c_2}))}{4 + 2z_0 \left(1 + \mathrm{erf}(y/(2\sqrt{c_2}))\right)}, \tag{1.7}$$

denotes the non-Gaussian profile determined by the Burgers equation, where $c_1 < 0$, $c_2 > 0$ *and* $d < 0$ *are likewise determined by the linearization around* $(f_s, q_s)^\top$, *while* $z_0 > -1$ *can be given explicitly in terms of the excess mass* $\int_{\mathbb{R}} f_0 \,\mathrm{d}x$, *see* (4.31).

The behaviour of the function $v_z(t,x) := t^{-1/2} f_z(t^{-1/2}x)$ is shown in Fig. 1.

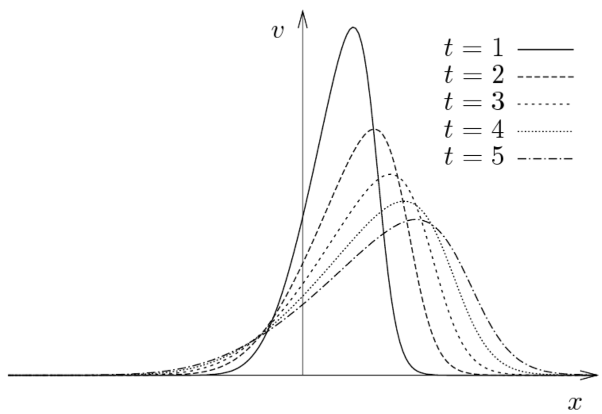
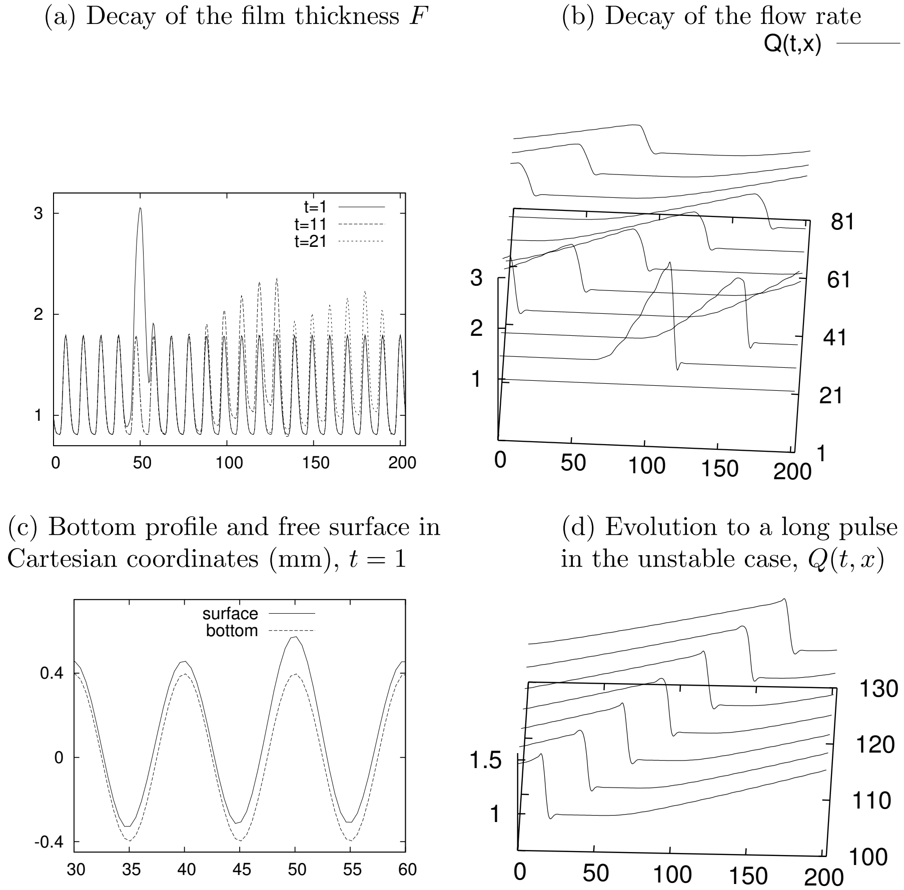


FIGURE 1. Sketch of self-similar decay of the amplitude in a co-moving frame in (1.6)

Figure 2 shows numerical simulations of (1.1), (1.2) in the stable case (a)-(c) and the unstable case (d), with periodic boundary conditions. This is also intended to relate (1.1), (1.2) to the underlying physics. In (a)-(c) we used a sinusoidal bottom with amplitude $a = 0.4$mm and wavelength λ=10mm (bottom profile $\hat{b}(\hat{x}) = a\cos(\frac{2\pi}{\lambda}\hat{x})$). The mean film thickness is $h \approx 0.06$mm, inclination angle $\alpha = 60°$, and the fluid parameters correspond to water, which yields $\delta \approx 0.037$, $\zeta = 0.25$, $\mathrm{B_i} \approx 3.25$ and $\mathrm{R} = 0.6$. The initial condition is $F = f_s + 2/\cosh((x-50)/5)$, $Q = q_s \equiv 1$. Although R is larger than the critical Reynolds number over flat bottom, which is $\mathrm{R_{crit}} \approx 0.48$, the stationary solution is stable and the perturbation decays in the self-similar way predicted by (1.6). (a) shows F at times as indicated, while (b) shows the evolution of Q. In the latter we directly see the envelope $t^{-1/2} f_{z_0}(t^{-1/2}(x + c_1 t))$ since $\Phi^1_2(0,\cdot) \equiv 1$, while $\Phi^1_1(0,x) = \frac{\mathrm{d}f_s}{\mathrm{d}q_s}(x)$ is γ-periodic. To illustrate the physical situation, panel (c) shows the bottom contour and the free surface at initial time $t = 1$ in dimensional (mm) cartesian coordinates, between the 4^{th} and 6^{th} bottom wave. Finally, panel (d) shows Q (for large time) after we increased α to 90°. Here $(f_s, q_s)^\top$ has become unstable: the perturbation does not decay to 0, but instead evolves into a long pulse. We expect that this situation can be described by a generalized KS equation, see, e.g., [7, 15] for the situation over flat bottoms, and [22] for a model problem for wavy bottoms.



(a) Decay of the film thickness F

(b) Decay of the flow rate

(c) Bottom profile and free surface in Cartesian coordinates (mm), $t = 1$

(d) Evolution to a long pulse in the unstable case, $Q(t, x)$

Figure 2. Numerical simulations of (1.1), (1.2)

The plan of this article is as follows. First we make precise the assumptions on spectral stability of $(f_s, q_s)^\top$, review basics of the RG method and of Bloch transform, and formally derive the Burgers equation from (1.1), (1.2). Then, using maximal regularity results we first prove local existence for (1.1), (1.2) and then use the RG method to prove Theorem 1.2. The RG method is worked out here for the first time for a realistic quasilinear fluid dynamical system with spatially periodic coefficients in which the renormalized solution converges to a non-Gaussian limit. We expect that the analysis is useful for a number of similar problems, for instance the full Navier-Stokes film flow problem over wavy bottom, and other parabolic systems with spatially periodic coefficients and a nonlinearity with lowest order terms of convective type.

2. Background and result

2.1. **Stationary solutions.** From (1.1), for γ-periodic stationary solutions $(F, Q)^\top = (f_s, q_s)^\top$, we immediately obtain that $\partial_x q_s \equiv 0$. Plugging $\partial_t q_s = \partial_x q_s =$

0 into (1.2) and multiplying it by f_s^2, we obtain

$$\begin{aligned} 0 = \frac{5}{2\mathrm{R}}\Big(\frac{\sin(\alpha-\theta)}{\sin\alpha}f_s^3 - q_s - \frac{\cos(\alpha-\theta)}{\sin\alpha}\partial_x f_s\, f_s^3 - \frac{3}{8}\frac{\sin(\alpha-\theta)}{\sin\alpha}\partial_x\theta\, f_s^4\Big) \\ + \frac{5}{6}\mathrm{W}(\partial_x^3 f_s - \partial_x\kappa)f_s^3 + \frac{9}{7}q_s^2\,\partial_x f_s + \frac{1}{\mathrm{R}}\Big(\frac{45}{16}\kappa q_s f_s + 4q_s(\partial_x f_s)^2 - 6q_s\,\partial_x^2 f_s\, f_s\Big). \end{aligned} \tag{2.1}$$

If the bottom waviness ζ is zero, the coefficients κ and θ vanish and we have the well known Nusselt solution $f_s = f_N$ with constant film thickness $f_N = q_s^{1/3}$. Thus, one possibility to obtain solutions of (2.1) is to continue (f_N, q_s) for $\zeta > 0$ using the implicit function theorem. Since x is measured in curvilinear coordinates, the periodicity γ of x depends on the bottom waviness ζ, and in order to apply the implicit function theorem in a function space with fixed periodicity we temporarily replace x by $k_0 x$, where we set $k_0 = 2\pi/\gamma$. This yields

$$\begin{aligned} 0 = \frac{5}{2\mathrm{R}}\Big(\frac{\sin(\alpha-\theta)}{\sin\alpha}f_s^3 - q_s - \frac{\cos(\alpha-\theta)}{\sin\alpha}k_0\partial_x f_s\, f_s^3 - \frac{3}{8}\frac{\sin(\alpha-\theta)}{\sin\alpha}k_0\partial_x\theta\, f_s^4\Big) \\ + \frac{5}{6}\mathrm{W}(k_0^3\partial_x^3 f_s - k_0\partial_x\kappa)f_s^3 + \frac{9}{7}k_0 q_s^2\,\partial_x f_s \\ + \frac{1}{\mathrm{R}}\Big(\frac{45}{16}\kappa q_s f_s + 4k_0^2 q_s(\partial_x f_s)^2 - 6k_0^2 q_s\,\partial_x^2 f_s\, f_s\Big). \end{aligned} \tag{2.2}$$

To solve this equation we fix the parameters $\alpha, \delta, \mathrm{R}, \mathrm{W}$ and the flow rate q_s. For $\zeta \geq 0$, we write (2.2) as $S(f_s, \zeta) = 0$. Assuming that the bottom contour is in $H^s_{\rm per}(0, 2\pi)$ with $s \geq 3$, we obtain $\partial_x\kappa \in H^{s-3}_{\rm per}(0, 2\pi)$, and thus,

$$S \in C^1\left(H^s_{\rm per}(0,2\pi) \times U, H^{s-3}_{\rm per}(0,2\pi)\right)$$

with $U \subset \mathbb{R}_0^+$. For $A_0 := \partial_f S(f_N, 0),\ H^s_{\rm per}(0,2\pi) \to H^{s-3}_{\rm per}(0,2\pi)$ we have

$$A_0 = \frac{15}{2\mathrm{R}}q_s^{2/3} + \Big(\frac{9}{7}q_s^2 - \frac{5}{2\mathrm{R}}\cot(\alpha)\, q_s\Big)k_0\partial_x - \frac{6}{\mathrm{R}}k_0^2 q_s^{4/3}\partial_x^2 + \frac{5}{6}k_0^3\mathrm{W} q_s\partial_x^3,$$

and the eigenfunctions of this constant coefficient linear differential operator are e^{ikx}, $k \in \mathbb{Z}$. The real part of the eigenvalue ω_k is given by

$$\mathrm{Re}\,\omega_k = \frac{15}{2\mathrm{R}}q_s^{2/3} + \frac{6}{\mathrm{R}}k_0^2 q_s^{4/3}k^2;$$

i.e., the spectrum is bounded away from zero. Therefore, A_0 is an isomorphism between $H^s_{\rm per}(0,2\pi)$ and $H^{s-3}_{\rm per}(0,2\pi)$, and the implicit function theorem yields that for each ζ small enough the equation $S(f_s,\zeta) = 0$ has a unique solution $f_s(\zeta) \in H^s_{\rm per}(0,2\pi)$ which depends continuously on ζ. Altogether, for each constant flow rate $q_s > 0$ and for small bottom waviness ζ there exists a unique stationary solution of the IBL (1.1), (1.2).

Remark 2.1. The implicit function theorem yields the existence of f_s for small values of ζ. This can be extended until a bifurcation occurs, but it is not clear for which parameters the stationary solution f_s for fixed q_s is unique. However, numerically this was the case in our simulations in [12] up to moderate R much larger than the critical Reynolds number, beyond which the branch of Nusselt-like solutions becomes unstable. Thus, it is mainly this branch that we have in mind here. However, we shall prove a general nonlinear stability result for all spectrally stable (f_s, q_s). Thus, instead of discussing the existence and spectral properties of stationary solutions in more detail, we simply postulate the pertinent properties in Assumptions 2.2 and 2.3.

Assumption 2.2. For fixed $\alpha, \mathrm{R}, \mathrm{W} > 0$ and $\kappa \in H^{s-3}_{per}(0,\gamma)$, $s \geq 3$, the IBL (1.1), (1.2) has a family of γ-periodic stationary solutions $(f_s, q_s)^\top$ with

$$f_s \in H^s_{\mathrm{per}}(0,\gamma), \quad q_s = \text{const.}, \tag{2.3}$$

which can be parametrized by the flow rate $q_s \in (q_{s,\min}, q_{s,\max})$, where $q_{s,\min,\max}$ may depend on the branch considered.

2.2. **Perturbation of stationary solutions.** Let $(f_s, q_s)^\top$ be a fixed stationary solution of the IBL (1.1), (1.2). Then the perturbation $(f,q)^\top := (F - f_s, Q - q_s)^\top$ satisfies

$$\partial_t f = -\frac{1}{1+\kappa(f_s+f)}\partial_x q \tag{2.4}$$

and

$$\begin{aligned}
\partial_t q = \frac{5}{2\mathrm{R}}\Big(&\frac{\sin(\alpha-\theta)}{\sin\alpha} f + \frac{-f_s^2 q + 2 f_s q_s f + q_s f^2}{f_s^2 (f_s+f)^2} - \frac{\cos(\alpha-\theta)}{\sin\alpha}(\partial_x f_s\, f + f_s\,\partial_x f + \partial_x f\, f) \\
&- \frac{3}{8}\frac{\sin(\alpha-\theta)}{\sin\alpha}\partial_x\theta(2 f_s f + f^2)\Big) + \frac{5}{6}\mathrm{W} f_s\,\partial_x^3 f + \frac{5}{6}\mathrm{W}(\partial_x^3 f_s + \partial_x^3 f - \partial_x\kappa) f \\
&- \frac{17}{7}\frac{q_s+q}{f_s+f}\partial_x q + \frac{9}{7}\frac{(q_s+q)^2}{(f_s+f)^2}\partial_x f + \frac{9}{7}\frac{2 f_s^2 q_s q - 2 f_s q_s^2 f + f_s^2 q^2 - q_s^2 f^2}{f_s^2 (f_s+f)^2}\partial_x f_s \\
&+ \frac{1}{\mathrm{R}}\Big(\frac{9}{2}\partial_x^2 q + \frac{45}{16}\kappa\frac{f_s q - q_s f}{(f_s+f) f_s} + 4\frac{q_s+q}{(f_s+f)^2}(2\partial_x f_s\,\partial_x f + (\partial_x f)^2) \\
&+ 4\frac{f_s^2 q - 2 f_s q_s f - q_s f^2}{f_s^2 (f_s+f)^2}(\partial_x f_s)^2 - 6\frac{q_s+q}{f_s+f}\partial_x^2 f - 6\frac{f_s q - q_s f}{(f_s+f) f_s}\partial_x^2 f_s \\
&- \frac{9}{2}\frac{\partial_x q(\partial_x f_s + \partial_x f)}{f_s+f}\Big) - \frac{1}{210}\mathrm{R}(\partial_x q)^2 (q_s+q).
\end{aligned} \tag{2.5}$$

The denominators in (2.5) are bounded from below since F is of order 1, cf. Remark 1.1. The linearization of (2.4), (2.5) around $(f,q)^\top = 0$ reads

$$\partial_t \begin{pmatrix} f \\ q \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{1+\kappa f_s}\partial_x \\ \tilde a_{10} + \tilde a_{11}\partial_x + \tilde a_{12}\partial_x^2 + \tilde a_{13}\partial_x^3 & a_{20} + a_{21}\partial_x + a_{22}\partial_x^2 \end{pmatrix}\begin{pmatrix} f \\ q \end{pmatrix}, \tag{2.6}$$

with the γ-periodic coefficients

$$\begin{aligned}
\tilde a_{10} = \frac{5}{2\mathrm{R}}&\left(\frac{\sin(\alpha-\theta)}{\sin\alpha} + 2\frac{q_s}{f_s^3} - \frac{\cos(\alpha-\theta)}{\sin\alpha}\partial_x f_s - \frac{3}{4}\frac{\sin(\alpha-\theta)}{\sin\alpha}\partial_x\theta\, f_s\right) \\
&+ \frac{5}{6}\mathrm{W}(\partial_x^3 f_s - \partial_x\kappa) - \frac{18}{7}\frac{\partial_x f_s\, q_s^2}{f_s^3} - \frac{45}{16\mathrm{R}}\kappa\frac{q_s}{f_s^2} - 8\frac{1}{\mathrm{R}}\frac{(\partial_x f_s)^2 q_s}{f_s^3} + 6\frac{1}{\mathrm{R}}\frac{\partial_x^2 f_s\, q_s}{f_s^2},
\end{aligned} \tag{2.7}$$

$$\tilde a_{11} = -\frac{5}{2}\frac{1}{\mathrm{R}}\frac{\cos(\alpha-\theta)}{\sin\alpha} f_s + \frac{9}{7}\frac{q_s^2}{f_s^2} + 8\frac{1}{\mathrm{R}}\frac{\partial_x f_s\, q_s}{f_s^2}, \quad \tilde a_{12} = -6\frac{1}{\mathrm{R}}\frac{q_s}{f_s}, \quad \tilde a_{13} = \frac{5}{6}\mathrm{W} f_s, \tag{2.8}$$

$$a_{20} = -\frac{5}{2\mathrm{R}}\frac{1}{f_s^2} + \frac{18}{7}\frac{\partial_x f_s\, q_s}{f_s^2} + \frac{45}{16\mathrm{R}}\kappa\frac{1}{f_s} + 4\frac{1}{\mathrm{R}}\frac{(\partial_x f_s)^2}{f_s^2} - 6\frac{1}{\mathrm{R}}\frac{\partial_x^2 f_s}{f_s}, \tag{2.9}$$

$$a_{21} = -\frac{17}{7}\frac{q_s}{f_s} - \frac{9}{2\mathrm{R}}\frac{\partial_x f_s}{f_s}, \quad a_{22} = \frac{9}{2\mathrm{R}}. \tag{2.10}$$

By the transformation

$$H := F + \frac{1}{2}\kappa F^2 \tag{2.11}$$

the nonlinear equation (1.1) becomes linear, namely $\partial_t H = -\partial_x Q$. The transformation (2.11) is one-to-one if the film thickness F is of order 1, and we can express F by H as

$$F = \frac{-1 + \sqrt{1 + 2\kappa H}}{\kappa} = H - \frac{1}{2}\kappa H^2 + \mathcal{O}(H^3). \tag{2.12}$$

The family of stationary solutions $(f_s, q_s)^\top$ from Assumption 2.2 is transformed into $(h_s, q_s)^\top$, where $h_s = f_s + \frac{1}{2}\kappa f_s^2$. Setting $h := H - h_s$ we obtain

$$\begin{aligned} f = F - f_s &= \frac{-1 + \sqrt{1 + 2\kappa H}}{\kappa} - \frac{-1 + \sqrt{1 + 2\kappa h_s}}{\kappa} \\ &= \frac{1}{\kappa}\Big(\sqrt{1 + 2\kappa(h_s + h)} - \sqrt{1 + 2\kappa h_s}\Big) \\ &= \frac{1}{(1 + 2\kappa h_s)^{1/2}} h - \frac{\kappa}{2(1 + 2\kappa h_s)^{3/2}} h^2 + \mathcal{O}(h^3) \\ &= \frac{1}{1 + \kappa f_s} h - \frac{\kappa}{2(1 + \kappa f_s)^3} h^2 + \mathcal{O}(h^3), \end{aligned} \tag{2.13}$$

while the inverse transformation is given by

$$h = (1 + \kappa f_s) f + \frac{1}{2}\kappa f^2. \tag{2.14}$$

For the time derivative of the perturbation's total mass

$$M = \int_{\mathbb{R}} \int_{f_s}^{f_s + f} (1 + \kappa z)\,\mathrm{d}z\,\mathrm{d}x = \int_{\mathbb{R}} f(1 + \kappa(f_s + f/2))\,\mathrm{d}x \tag{2.15}$$

we obtain

$$\partial_t M = \int_{\mathbb{R}} \partial_t f(1 + \kappa(f_s + f))\,\mathrm{d}x = -\int_{\mathbb{R}} \partial_x q\,\mathrm{d}x = 0. \tag{2.16}$$

Thus, the total mass of perturbations is conserved. This simply reads $\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} h\,\mathrm{d}x = 0$, and the IBL (2.4), (2.5) is equivalent to solving $\partial_t h = -\partial_x q$ together with (2.5), where f must be replaced everywhere according to (2.13). For the linear terms we write in short

$$A(\partial_x)\begin{pmatrix} h \\ q \end{pmatrix} := \begin{pmatrix} 0 & -\partial_x \\ a_{10} + a_{11}\partial_x + a_{12}\partial_x^2 + a_{13}\partial_x^3 & a_{20} + a_{21}\partial_x + a_{22}\partial_x^2 \end{pmatrix}\begin{pmatrix} h \\ q \end{pmatrix}, \tag{2.17}$$

where $a_{10} = \tilde{a}_{10}\beta + \tilde{a}_{11}\partial_x\beta + \tilde{a}_{12}\partial_x^2\beta + \tilde{a}_{13}\partial_x^3\beta$, $a_{11} = \tilde{a}_{11}\beta + 2\tilde{a}_{12}\partial_x\beta + 3\tilde{a}_{13}\partial_x^2\beta$, $a_{12} = \tilde{a}_{12}\beta + 3\tilde{a}_{13}\partial_x\beta$, $a_{13} = \tilde{a}_{13}\beta$, with $\beta(x) := \frac{1}{1 + \kappa(x) f_s(x)}$. Since all fractions in (2.5) are finite for small perturbations with $\|f\|_{L^\infty} < \|f_s\|_{L^\infty}/2$, they can be expanded in powers of f, and thus, in powers of h. Hence we can write the transformed IBL as

$$\partial_t \begin{pmatrix} h \\ q \end{pmatrix} = A(\partial_x)\begin{pmatrix} h \\ q \end{pmatrix} + N(h, q), \tag{2.18}$$

where N contains the nonlinear terms. The first component of N vanishes, since the equation for $\partial_t h$ is linear. We look for a solution $(h, q)^\top$ of (2.18) with $(h(t), q(t))^\top \in H^r(2) \times H^{r-1}(2)$ for fixed t and $r \geq 3$ in order to avoid Sobolev spaces with negative orders. Due to the weight we will achieve C^1-regularity with respect to the wave number ℓ in Bloch space, which is necessary to expand the critical mode in terms of ℓ in Section 4.3.

2.3. **Bloch transform.** Considering a bottom with fixed wavelength γ and setting $k_0 := 2\pi/\gamma$, we define for $v \in H^r(m)$ the Bloch transform $\mathcal{J}v$ as

$$\mathcal{J}v(\ell, x) = \tilde{v}(\ell, x) := \sum_{j\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}jk_0x}\hat{v}(k_0j + \ell). \tag{2.19}$$

From (2.19) we have that $\mathcal{J}v(\ell, x + \gamma) = \mathcal{J}v(\ell, x)$, and that Bloch transform is an isomorphism between the weighted Sobolev space $H^r(m)$ and the Bloch space $B(m, r)$ defined by

$$\begin{gathered} B(m, r) = H^m((-k_0/2, k_0/2), H^r_{\mathrm{per}}(0, \gamma)), \\ \|\tilde{v}\|_{B(m,r)} := \Big(\sum_{j\le m}\int_{I_{k_0}} \|\partial_\ell^j \tilde{v}(\ell, \cdot)\|^2_{H^r(I_\gamma)}\,\mathrm{d}\ell\Big)^{1/2}, \end{gathered} \tag{2.20}$$

where $I_\delta := (-\delta/2, \delta/2)$. The inverse Bloch transform is given by

$$v(x) = \int_{I_{k_0}} \mathrm{e}^{\mathrm{i}\ell x}\mathcal{J}v(\ell, x)\,\mathrm{d}\ell. \tag{2.21}$$

We collect some useful properties of Bloch transform. For a real-valued function v, we have

$$\mathcal{J}v(-\ell, x) = \overline{\mathcal{J}v(\ell, x)}. \tag{2.22}$$

If $a : \mathbb{R} \to \mathbb{R}$ is γ-periodic, then

$$\mathcal{J}(av)(\ell, x) = a(x)\mathcal{J}v(\ell, x). \tag{2.23}$$

Thus, Bloch transform is invariant under multiplication with γ-periodic coefficients. So far, functions in Bloch space are only defined for $\ell \in (-k_0/2, k_0/2]$. In order to transform products uv with $u, v \in H^r(m)$ we extend the domain of $\tilde{v} \in B(r, m)$ corresponding to (2.19); i.e.,

$$\tilde{v}(\ell + k_0, x) = \sum_{j\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}jk_0x}\hat{v}(k_0j + \ell + k_0) = \mathrm{e}^{-\mathrm{i}k_0x}\sum_{j\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}jk_0x}\hat{v}(k_0j + \ell) = \mathrm{e}^{-\mathrm{i}k_0x}\tilde{v}(\ell, x).$$

Then, multiplication in x-space corresponds to convolution in Bloch space; i.e.,

$$\mathcal{J}(uv)(\ell, x) = \int_{-k_0/2}^{k_0/2} \mathcal{J}u(l - k, x)\mathcal{J}v(k, x)\,\mathrm{d}k =: (\mathcal{J}u *_1 \mathcal{J}v)(\ell, x). \tag{2.24}$$

Therefore we adapt the definition of $B(m, r)$ in (2.20) to

$$B(m, r) := \Big\{\tilde{v} \mid \tilde{v}\big|_{\ell\in I_{k_0}} \in H^m(I_{k_0}, H^r_{\mathrm{per}}(0, \gamma)) \text{ and } \tilde{v}(\ell + k_0, x) = \mathrm{e}^{-\mathrm{i}k_0x}\tilde{v}(\ell, x)\Big\}. \tag{2.25}$$

The notation $*_1$ in (2.24) becomes clear in (4.23), where we define a more general convolution operator. If there is no ambiguity we omit the subscript in the following and write $\mathcal{J}u * \mathcal{J}v$. Due to the extension in (2.25) convolution becomes commutative. From (2.21) we obtain

$$\partial_x v(x) = \int_{-k_0/2}^{k_0/2} \mathrm{e}^{\mathrm{i}\ell x}(\partial_x + \mathrm{i}\ell)\mathcal{J}v(\ell, x)\,\mathrm{d}\ell; \tag{2.26}$$

i.e., ∂_x in x-space corresponds to the operator $(\partial_x + \mathrm{i}\ell)$ in Bloch space. Thus, setting $\tilde{h} := \mathcal{J}h$ and $\tilde{q} := \mathcal{J}q$ the IBL (2.18) is equivalent to

$$\partial_t \begin{pmatrix}\tilde{h}\\ \tilde{q}\end{pmatrix} = A(\partial_x + \mathrm{i}\ell)\begin{pmatrix}\tilde{h}\\ \tilde{q}\end{pmatrix} + \tilde{N}(\tilde{h}, \tilde{q}) \tag{2.27}$$

in Bloch space, where

$$\tilde{N}(\tilde{h},\tilde{q}) := \mathcal{J}N(\mathcal{J}^{-1}\tilde{h},\mathcal{J}^{-1}\tilde{q}). \tag{2.28}$$

Since Bloch transform is an isomorphism between $H^r(2)$ and $B(2,r)$, we look for a solution $(\tilde{h},\tilde{q})^\top$ of (2.27) with $(\tilde{h}(t),\tilde{q}(t))^\top \in B(2,r)\times B(2,r-1)$ for fixed t and $r \geq 3$.

2.4. **Spectral situation and mode filters. Spectral situation.**

By (2.11) the family of stationary solutions $(f_s,q_s)^\top$ from Assumption 2.2 is transformed into a family of stationary solutions $(h_s,q_s)^\top$ of the IBL for $(H,Q)^\top$, which we write in short as

$$\partial_t \begin{pmatrix} H \\ Q \end{pmatrix} = \mathcal{G}(H,Q) = \begin{pmatrix} \mathcal{G}_1(H,Q) \\ \mathcal{G}_2(H,Q) \end{pmatrix}. \tag{2.29}$$

Since the γ-periodic stationary solutions are parametrized by the x-independent flow rates q_s, we have

$$\mathcal{G}(h_s(q_s),q_s) = 0 \quad \text{for all } q_s \in (q_{s,\min},q_{s,\max}), \tag{2.30}$$

and differentiating with respect to q_s gives

$$0 = \frac{\mathrm{d}}{\mathrm{d}q_s}\mathcal{G}(h_s(q_s),q_s) = \begin{pmatrix} \frac{\partial \mathcal{G}_1}{\partial H}(h_s(q_s),q_s) & \frac{\partial \mathcal{G}_1}{\partial Q}(h_s(q_s),q_s) \\ \frac{\partial \mathcal{G}_2}{\partial H}(h_s(q_s),q_s) & \frac{\partial \mathcal{G}_2}{\partial Q}(h_s(q_s),q_s) \end{pmatrix} \begin{pmatrix} \frac{\mathrm{d}h_s}{\mathrm{d}q_s}(q_s) \\ 1 \end{pmatrix}. \tag{2.31}$$

The linear differential operator on the right-hand side of (2.31) also occurs in the linearization of the IBL (2.29) around a stationary solution: Choosing in the following q_s fixed, the perturbation $(h,q)^\top = (H-h_s,Q-q_s)^\top$ satisfies $\partial_t(h,q)^\top = \mathcal{G}(h_s+h,q_s+q)$. Thus, the linearization around $(h,q)^\top = 0$ reads

$$\partial_t \begin{pmatrix} h \\ q \end{pmatrix} = \frac{\partial \mathcal{G}}{\partial(H,Q)}(h_s,q_s)\begin{pmatrix} h \\ q \end{pmatrix}, \tag{2.32}$$

which we have already expressed in (2.17) with the help of the differential operator $A(\partial_x)$. Therefore, combining (2.31) and (2.32) gives

$$A(\partial_x)\begin{pmatrix} \frac{\mathrm{d}h_s}{\mathrm{d}q_s}(q_s) \\ 1 \end{pmatrix} = 0 \quad \text{for all } q_s \in (0,q_{s,\max}). \tag{2.33}$$

Transferring the IBL to Bloch space, we know from (2.27) that the linear operator in the evolution equation for $(\tilde{h},\tilde{q})^\top$ is given by $A(\partial_x+\mathrm{i}\ell)$. Corresponding to (2.33), $(\mathrm{d}h_s/\mathrm{d}q_s,1)^\top \in H^s_{\mathrm{per}}(0,\gamma)\times H^{s-1}_{\mathrm{per}}(0,\gamma)$ is an eigenfunction of $A(\partial_x+\mathrm{i}\ell)$ to the eigenvalue $\lambda_1(0)=0$ for $\ell=0$. Thus, in Bloch space the linearization of the IBL around a stationary solution has always a zero eigenvalue. This property corresponds to the free surface in the underlying physical problem. Furthermore, for fixed $\ell \in (-k_0/2,k_0/2)$ the differential operator $A(\partial_x+\mathrm{i}\ell) : H^s_{\mathrm{per}}(0,\gamma)\times H^{s-1}_{\mathrm{per}}(0,\gamma) \to H^{s-2}_{\mathrm{per}}(0,\gamma)\times H^{s-3}_{\mathrm{per}}(0,\gamma)$ is elliptic, and thus we obtain countable many curves of eigenvalues λ_n with $\operatorname{Re}\lambda_n(\ell) \to -\infty$ for $n\to\infty$. Like for the stationary solutions, instead of calculating the spectrum of $A(\partial_x+\mathrm{i}\ell)$, we state an assumption based on the properties derived above. A typical spectrum is then sketched in Figure 3.

Assumption 2.3 (Spectral stability)**.** Let s be the bottom regularity from Assumption 2.2. We assume that $A(\partial_x+\mathrm{i}\cdot)$ with $A(\partial_x+\mathrm{i}\ell) : H^s_{\mathrm{per}}(0,\gamma)\times H^{s-1}_{\mathrm{per}}(0,\gamma) \to H^{s-2}_{\mathrm{per}}(0,\gamma)\times H^{s-3}_{\mathrm{per}}(0,\gamma)$ has countable many curves of eigenvalues $\lambda_n : (-k_0/2,k_0/2) \to \mathbb{C}$, $n\in\mathbb{N}$, with eigenfunctions $\ell \mapsto \phi^n(\ell,\cdot) \in H^s_{\mathrm{per}}(0,\gamma)\times H^{s-1}_{\mathrm{per}}(0,\gamma)$ and

(i) $\lambda_1(\ell) = c_1\mathrm{i}\ell - c_2\ell^2 + \mathcal{O}(\ell^3)$ with $c_1\in\mathbb{R}$, $\operatorname{Re}c_2>0$,

(ii) $\operatorname{Re}\lambda_1(\ell) < -\tilde{c}_2\ell^2$ for $|\ell| \le 4r_\chi$ and a $\tilde{c}_2 < c_2$,
(iii) $\operatorname{Re}\lambda_1(\ell) < -\sigma_0 < 0$ for $|\ell| > 4r_\chi$ and $\operatorname{Re}\lambda_1(\ell) > -\sigma_0$ for $|\ell| < 4r_\chi$,
(iv) $\operatorname{Re}\lambda_n(\ell) < -\sigma_0$ for all $n \ge 2$, $\ell \in (-k_0/2, k_0/2)$.

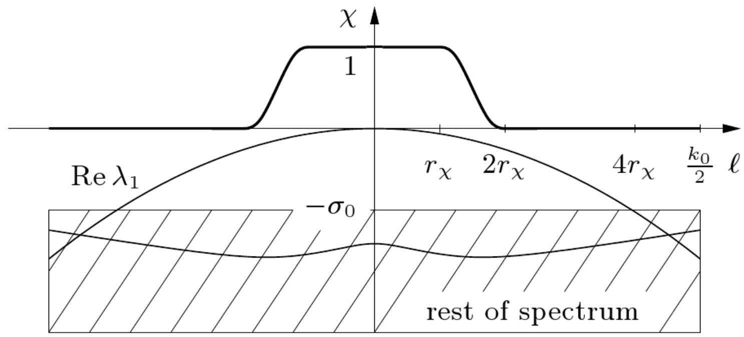


FIGURE 3. Sketch of the spectral situation and the cut-off function χ

Eigenfunctions. The relation between the eigenvalues and eigenvectors of the two versions of the IBL, namely system (2.4), (2.5) for $(f,q)^\top$ and system (2.18) for $(h,q)^\top$, is as follows. Let us denote the linearized (f,q)-system (2.6) by $\partial_t(f,q)^\top = \hat{A}(\partial_x)(f,q)^\top$. Since $h = \frac{1}{\beta}f + \mathcal{O}(f^2)$, see (2.17), we have

$$A(\partial_x)\begin{pmatrix} h \\ q \end{pmatrix} = \begin{pmatrix} 1/\beta & 0 \\ 0 & 1 \end{pmatrix}\hat{A}(\partial_x)\begin{pmatrix} \beta h \\ q \end{pmatrix},$$

where $\beta(x) = 1/(1+\kappa(x)f_s(x))$, see (2.17). Thus, for each eigenvalue λ_n of $A(\partial_x+\mathrm{i}\ell)$ we obtain

$$\lambda_n\phi^n = A(\partial_x+\mathrm{i}\ell)\phi^n = \begin{pmatrix} 1/\beta & 0 \\ 0 & 1 \end{pmatrix}\hat{A}(\partial_x+\mathrm{i}\ell)\begin{pmatrix} \beta\phi_1^n \\ \phi_2^n \end{pmatrix};$$

i.e.,

$$\lambda_n\begin{pmatrix} \beta\phi_1^n \\ \phi_2^n \end{pmatrix} = \hat{A}(\partial_x+\mathrm{i}\ell)\begin{pmatrix} \beta\phi_1^n \\ \phi_2^n \end{pmatrix}.$$

Therefore, in Bloch space the two systems for $(f,q)^\top$ and $(h,q)^\top$ have exactly the same eigenvalues, where the eigenvectors of the (f,q)-system are given by

$$\Phi^n := \begin{pmatrix} \beta & 0 \\ 0 & 1 \end{pmatrix}\phi^n. \tag{2.34}$$

In particular, the critical eigenfunctions read

$$\phi^1(\ell,\cdot) = \begin{pmatrix} \frac{\mathrm{d}f_s}{\mathrm{d}q_s} \\ 1 \end{pmatrix} + \mathcal{O}(\ell), \quad \Phi^1(\ell,\cdot) = \begin{pmatrix} \frac{1}{1+\kappa f_s}\frac{\mathrm{d}h_s}{\mathrm{d}q_s} \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{\mathrm{d}f_s}{\mathrm{d}q_s} \\ 1 \end{pmatrix} + \mathcal{O}(\ell). \tag{2.35}$$

This property is used in the proof of Theorem 1.2, where the universal decay behavior for the (h,q)-system is transferred back to the original (f,q)-system.

Since the IBL (2.27) in Bloch space has a zero eigenvalue, we have to split $(\tilde{h},\tilde{q})^\top$ into its stable part and into a multiple of the critical eigenvector ϕ^1. On the linear level, the critical curve $\lambda_1(\ell) = c_1\mathrm{i}\ell - c_2\ell^2 + \mathcal{O}(\ell^3)$ for the mode $\phi^1(\ell,\cdot)$ corresponds to $\partial_t v = (c_1\partial_x + c_2\partial_x^2)v$, which is the linear diffusion equation in the comoving frame $y = x + c_1 t$. However, going into this comoving frame in (2.18) leads to a time dependent differential operator, which would make the subsequent analysis more complicated. Therefore, we introduce the rotated variable $\tilde{w}$ by

$$\tilde{w}(t,\ell,x) = \begin{pmatrix}\tilde{w}_1\\ \tilde{w}_2\end{pmatrix}(t,\ell,x) := \mathrm{e}^{-c_1\mathrm{i}\ell t}\begin{pmatrix}\tilde{h}\\ \tilde{q}\end{pmatrix}(t,\ell,x), \tag{2.36}$$

which satisfies

$$\partial_t\tilde{w}(t,\ell,x) = \tilde{A}(\ell)\tilde{w}(t,\ell,x) + \tilde{N}(\tilde{w})(t,\ell,x) \tag{2.37}$$

with

$$\begin{aligned}\tilde{A}(\ell) &:= -c_1\mathrm{i}\ell + A(\partial_x + \mathrm{i}\ell)\\ &= \begin{pmatrix} -c_1\mathrm{i}\ell & -(\partial_x+\mathrm{i}\ell)\\ a_{10}+a_{11}(\partial_x+\mathrm{i}\ell)+a_{12}(\partial_x+\mathrm{i}\ell)^2+a_{13}(\partial_x+\mathrm{i}\ell)^3 & (a_{20}-c_1\mathrm{i}\ell)+a_{21}(\partial_x+\mathrm{i}\ell)+a_{22}(\partial_x+\mathrm{i}\ell)^2\end{pmatrix}.\end{aligned} \tag{2.38}$$

The nonlinearity $\tilde{N}$ is exactly the same as for the $(\tilde{h},\tilde{q})$-system in (2.27) since $(\tilde{v}_i * \tilde{v}_j)(\ell) = \int_{-k_0/2}^{k_0/2}\tilde{w}_i(\ell-k)\mathrm{e}^{c_1\mathrm{i}(\ell-k)t}\tilde{w}_j(k)\mathrm{e}^{c_1\mathrm{i}kt}\,\mathrm{d}k = \mathrm{e}^{c_1\mathrm{i}\ell t}(\tilde{w}_i * \tilde{w}_j)(\ell)$ for $\tilde{v} := \mathrm{e}^{c_1\mathrm{i}\ell t}\tilde{w}$ and $i,j\in\{1,2\}$. Clearly, $\tilde{A}$ has the same eigenfunctions ϕ^n as $A(\partial_x+\mathrm{i}\ell)$ with eigenvalues $\mu_n(\ell) = \lambda_n(\ell) - c_1\mathrm{i}\ell$. In particular, for the critical eigenvalue we obtain

$$\mu_1(\ell) = -c_2\ell^2 + \mathcal{O}(\ell^3). \tag{2.39}$$

Mode filters. We introduce mode filters to extract the critical mode ϕ^1. Let $\chi:\mathbb{R}\to[0,1]$ be a smooth cut-off function with $\chi(\ell)=1$ for $|\ell|\le r_\chi$ and $\chi(\ell)=0$ for $|\ell|\ge 2r_\chi$, see Figure 3. Due to Assumption 2.3 the curve of critical eigenvalues μ_1 is isolated from the rest of the spectrum for $|\ell|<4r_\chi$. Thus, denoting the scalar product in $L^2(0,\gamma)$ by $\langle\cdot,\cdot\rangle$; i.e.,

$$\langle u,v\rangle := \int_0^\gamma u\cdot\bar{v}\,\mathrm{d}x,$$

where the "$\cdot$" stands for the standard scalar product in $\mathbb{R}^2$, we can define the critical mode filter $\tilde{E}_c$ by

$$(\tilde{E}_c\tilde{w})(\ell,x) := \chi(\ell)\big\langle\tilde{w}(\ell,\cdot),\psi^1(\ell,\cdot)\big\rangle\phi^1(\ell,x). \tag{2.40}$$

Here $\psi^1(\ell,\cdot)$ is an eigenfunction of the $L^2(0,\gamma)$-adjoint operator $\tilde{A}^*(\ell)$ to the eigenvalue $\bar{\mu}_1(\ell)$. The $L^2(0,\gamma)$-adjoint operator of a differential operator $L = a(x)(\partial_x + \mathrm{i}\ell)$ with a γ-periodic coefficient a is given by $L^*v = -(\partial_x+\mathrm{i}\ell)(\bar{a}v)$. Thus, for the critical eigenfunction we obtain $\psi^1(0,x) = (c_0,0)^\top$; i.e.,

$$\psi^1(\ell,x) = c_0\begin{pmatrix}1\\0\end{pmatrix} + \mathcal{O}(\ell), \tag{2.41}$$

and we choose ψ^1 such that $\langle\phi^1(\ell,\cdot),\psi^1(\ell,\cdot)\rangle = 1$ for all $\ell\in(-4r_\chi,4r_\chi)$. Additionally to $\tilde{E}_c$, we define the scalar mode filter $\tilde{E}_c^*$ and the stable mode filter $\tilde{E}_s$ by

$$(\tilde{E}_c^*\tilde{w})(\ell) := \chi(\ell)\langle\tilde{w}(\ell,\cdot),\psi^1(\ell,\cdot)\rangle, \quad \tilde{E}_s := \mathrm{Id}-\tilde{E}_c. \tag{2.42}$$

Moreover, we define auxiliary mode filters

$$(\tilde{E}_c^h\tilde{w})(\ell,x) := \chi(\ell/2)\langle\tilde{w}(\ell,\cdot),\psi^1(\ell,\cdot)\rangle\phi^1(\ell,x), \quad \tilde{E}_s^h := \mathrm{Id}-\tilde{E}_c^h \tag{2.43}$$

such that $\tilde{E}_c^h \tilde{E}_c = \tilde{E}_c$ and $\tilde{E}_s^h \tilde{E}_s = \tilde{E}_s$, which is used to substitute for missing projection properties of $\tilde{E}_c$ and $\tilde{E}_s$. Setting $\tilde{\alpha}(t,\ell) := (\tilde{E}_c^* \tilde{w}(t))(\ell)$, $\tilde{w}_s(t,\ell,x) := (\tilde{E}_s \tilde{w}(t))(\ell,x)$, we obtain the splitting

$$\tilde{w}(t,\ell,x) = \tilde{\alpha}(t,\ell)\phi^1(\ell,x) + \tilde{w}_s(t,\ell,x) \tag{2.44}$$

into the critical mode $\tilde{\alpha}\phi^1$ and the stable component $\tilde{w}_s$.

Remark 2.4. The idea of this splitting is that due to the spectral properties of $\tilde{E}_s^h \tilde{A}(\ell)$, w_s is linearly exponentially damped. Thus, we expect the dynamics of (2.37) to be governed by the dynamics of the critical mode $\tilde{\alpha}\phi^1$.

Altogether, after applying mode filters, the IBL in Bloch space reads

$$\partial_t \tilde{\alpha}(t,\ell) = \mu_1(\ell)\tilde{\alpha}(t,\ell) + \tilde{B}_c(\tilde{\alpha}(t))(\ell) + \tilde{H}_c(\tilde{\alpha}(t), \tilde{w}_s(t))(\ell), \tag{2.45}$$

$$\partial_t \tilde{w}_s(t,\ell,x) = \tilde{A}_s(\ell)\tilde{w}_s(t,\ell,x) + \tilde{H}_s(\tilde{\alpha}(t), \tilde{w}_s(t))(\ell,x), \tag{2.46}$$

where

$$\tilde{B}_c(\tilde{\alpha})(\ell) := \mathrm{i}d\ell\chi(\ell)(\tilde{\alpha}^{*2})(\ell), \tag{2.47}$$

$$\tilde{H}_c(\tilde{\alpha}, \tilde{w}_s)(\ell) := \tilde{E}_c^* \Big(\tilde{N}(\tilde{\alpha}\phi^1 + \tilde{w}_s)\Big)(\ell) - \mathrm{i}d\ell\chi(\ell)\tilde{\alpha}^{*2}(\ell), \tag{2.48}$$

$$\tilde{A}_s(\ell) := \tilde{E}_s^h \tilde{A}(\ell), \quad \tilde{H}_s(\tilde{\alpha}, \tilde{w}_s)(\ell,x) := \tilde{E}_s\Big(\tilde{N}(\tilde{\alpha}\phi^1 + \tilde{w}_s)\Big)(\ell,x), \tag{2.49}$$

with d specified subsequently in (2.64). Below we will see that cubic terms as well as those involving $\tilde{w}_s$ are asymptotically irrelevant. Thus, the only dangerous terms are the quadratic ones in $\tilde{N}(\tilde{\alpha}\phi^1)$, which are not damped by the decay of $\tilde{w}_s$. In the formal derivation in §2.6 we will see that these terms have the "derivative-like" structure $\mathrm{i}d\ell\chi(\ell)\tilde{\alpha}^{*2}$ with $d \in \mathbb{R}$, which leads to a Burgers-like decay. There also occur terms of the order of $\mathcal{O}(\ell^2)\tilde{\alpha}^{*2}$, but as they turn out to be irrelevant due to the additional factor ℓ, we put them into $\tilde{H}_c$ and denote by $\tilde{B}_c$ the term $\mathrm{i}d\ell\chi(\ell)\tilde{\alpha}^{*2}$, which is the only relevant one.

Function spaces. It remains to choose appropriate function spaces for $\tilde{\alpha}$ and $\tilde{w}_s$. For fixed t we have $(h,q)^\top \in H^r(2) \times H^{r-1}(2)$ if and only if $\tilde{w} \in B(2,r) \times B(2,r-1)$; i.e., both $\tilde{\alpha}\phi^1$ and $\tilde{w}_s \in B(2,r) \times B(2,r-1)$.

Thus, in a first step we assume that $\tilde{\alpha}\phi^1 \in B(2,r) \times B(2,r-1)$. In the following let the bottom profile be at least in $H^r_{\mathrm{per}}(0,\gamma)$, such that due to Assumption 2.3 we have $\phi^1(\ell) \in H^r_{\mathrm{per}}(0,\gamma) \times H^{r-1}_{\mathrm{per}}(0,\gamma)$ for fixed ℓ. Since the critical eigenvalue $\mu_1(\ell)$ is isolated from the rest of the spectrum for $|\ell| < 4r_\chi$, the eigenfunction ϕ^1 is smooth with respect to ℓ in this interval. In particular, we have $\phi^1 \in H^2((-2r_\chi, 2r_\chi), H^r_{\mathrm{per}}(0,\gamma) \times H^{r-1}_{\mathrm{per}}(0,\gamma))$. Since the same is true for the adjoint eigenfunction ψ^1, the definition of the critical mode filter in (2.42) leads to

$$\tilde{\alpha} \in H^2(\mathbb{R}), \quad \operatorname{supp}\tilde{\alpha} \in [-2r_\chi, 2r_\chi]. \tag{2.50}$$

Next, we conversely assume that $\tilde{\alpha} \in H^2(\mathbb{R})$ with $\operatorname{supp}\tilde{\alpha} \in [-2r_\chi, 2r_\chi]$, and $\tilde{w}_s \in B(2,r) \times B(2,r-1)$. It immediately follows that $\tilde{\alpha}\phi^1$ is in $H^2(I_{k_0}, H^r_{\mathrm{per}}(0,\gamma) \times H^{r-1}_{\mathrm{per}}(0,\gamma))$, but not in $B(2,r) \times B(2,r-1)$ since the extension property from (2.25) is missing, which is required to calculate convolutions. However, since $\tilde{\alpha}$ has compact support, this is not needed. On the one hand, in convolutions like

$$\int_{-k_0/2}^{k_0/2} \tilde{\alpha}(\ell-k)\phi^1(\ell-k)\tilde{v}(k)\,\mathrm{d}k = \int_{-k_0/2}^{k_0/2} \tilde{\alpha}(k)\phi^1(k)\tilde{v}(\ell-k)\,\mathrm{d}k,$$

with $\tilde{v} \in B(2,r) \times B(2,r-1)$, we can use the extension property of $\tilde{v}$ such that $\tilde{\alpha}\phi^1$ must only be evaluated for $\ell \in I_{k_0}$. On the other hand, for convolutions

$$\int_{-k_0/2}^{k_0/2} \tilde{\alpha}(\ell-k)\phi^1(\ell-k)\tilde{\alpha}(k)\phi^1(k)\,\mathrm{d}k = \int_{-2r_\chi}^{2r_\chi} \tilde{\alpha}(\ell-k)\phi^1(\ell-k)\tilde{\alpha}(k)\phi^1(k)\,\mathrm{d}k$$

we have to extend $\tilde{\alpha}\phi^1$ to $|\ell| \le k_0/2+2r_\chi$ by $(\tilde{\alpha}\phi^1)(\ell+k_0) = \mathrm{e}^{-\mathrm{i}k_0 x}(\tilde{\alpha}\phi^1)(\ell)$. Thus, $\tilde{\alpha}\phi^1$ is extended with values of $(\tilde{\alpha}\phi^1)(\ell)$, $\ell \in (-1/2, -1/2+2r_\chi] \cup (1/2-2r_\chi, 1/2]$, where $\tilde{\alpha}$ and hence $\tilde{\alpha}\phi^1$ is zero. Thus, there is no difference if we extend $\tilde{\alpha}\phi^1$ according to the extension rule in (2.25) or if we use $\tilde{\alpha} \in H^2(\mathbb{R})$ with compact support. If necessary, we must replace r_χ in Assumption 2.3 by a smaller value depending on the final degree of the nonlinearity since each convolution enlarges the support of $(\tilde{\alpha}\phi^1)^{*j}$. Altogether, we obtain the equivalence

$$\begin{aligned} \tilde{w} \in B(2,r)\times B(2,r-1) \Leftrightarrow\ & \tilde{\alpha} \in H^2(\mathbb{R}),\ \operatorname{supp}\tilde{\alpha} \in [-2r_\chi, 2r_\chi], \\ & \text{and } \tilde{w}_s \in B(2,r)\times B(2,r-1). \end{aligned} \tag{2.51}$$

Moreover, since $\tilde{\alpha}$ is independent of x we obtain

$$\begin{aligned} \|\tilde{\alpha}\|^2_{B(2,r)} &= \sum_{j\le 2} \int_{-k_0/2}^{k_0/2} \|\partial_\ell^j \tilde{\alpha}(\ell)\|^2_{H^r(I_\gamma)}\,\mathrm{d}\ell \\ &= \sum_{j\le 2} \int_{-k_0/2}^{k_0/2} \gamma^2 |\partial_\ell^j \tilde{\alpha}(\ell)|^2\,\mathrm{d}\ell = \gamma^2 \|\tilde{\alpha}\|^2_{H^2(I_{k_0})} \end{aligned}$$

for all $r \ge 0$. Therefore, and since it does not matter how $\tilde{\alpha}$ is extended to $|\ell| > k_0/2$, $\tilde{\alpha} \in H^2(\mathbb{R})$ in (2.51) can be substituted by $\tilde{\alpha} \in B(2,r)$. Thus, we look for a solution $(\tilde{\alpha}, \tilde{w}_s)$ of (2.45), (2.46) with $\tilde{\alpha}(t) \in B(2,r)$ and $\tilde{w}(t) \in B(2,r)\times B(2,r-1)$ for fixed t and $r \ge 3$.

2.5. **Self-similar decay in the viscous Burgers equation.** The idea behind the splitting of $\tilde{w}$ into $\tilde{\alpha}$ and $\tilde{w}_s$ is that $\tilde{\alpha}$ will fulfill a perturbed Burgers equation while $\tilde{w}$ is linearly exponentially damped. Here we collect some basic facts about the dynamics of the Burgers equation, mainly from [5], see also [20, 21] for more details.

By the Cole-Hopf transformation $\eta(t,\xi) = \exp\Big(\frac{d}{c_2}\int_{-\infty}^{\sqrt{c_2}\xi} v(t,x)\mathrm{d}x\Big)$, the viscous Burgers equation

$$\partial_t v = c_2 \partial_x^2 v + d\partial_x(v^2),\ x \in \mathbb{R},\ t \ge 0 \tag{2.52}$$

is transformed into the linear diffusion equation $\partial_t \eta = \partial_\xi^2 \eta$. The inverse transformation is given by

$$v(t,x) = \frac{\sqrt{c_2}}{d}\frac{\partial_\xi \eta(t, x/\sqrt{c_2})}{\eta(t, x/\sqrt{c_2})}.$$

By construction, we have $\lim_{\xi\to-\infty}\eta(t,\xi) = 1$ for all $t \ge 0$. Setting $\lim_{\xi\to\infty}\eta(0,\xi) = 1+z$ for the initial condition, it is well known that

$$\eta(t,\xi) = 1 + \frac{z}{2}\Big(1 + \operatorname{erf}\big(\frac{\xi}{2\sqrt{t}}\big)\Big) \text{ with } \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \mathrm{e}^{-y^2}\mathrm{d}y$$

is an exact solution of the linear diffusion equation. Thus, for every $z > -1$ there exists a self-similar solution of the Burgers equation (2.52) given by

$$v_z(t,x) := t^{-1/2} f_z(t^{-1/2}x) \text{ with } f_z(y) = \frac{\sqrt{c_2}}{d}\frac{z \operatorname{erf}'(y/(2\sqrt{c_2}))}{4 + 2z\big(1 + \operatorname{erf}(y/(2\sqrt{c_2}))\big)}, \tag{2.53}$$

where $\ln(z+1) = \frac{d}{c_2}\int_{\mathbb{R}} v_z(t,x)\mathrm{d}x$.

Moreover, if we consider an arbitrary initial condition $\eta\big|_{t=0} = \eta_0 \in L^\infty$ with the boundary conditions $\lim_{\xi\to-\infty}\eta_0(\xi) = 1$ and $\lim_{\xi\to\infty}\eta_0(\xi) = 1+z$, the solution can be written as

$$\eta(t,\xi) = \frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}} \mathrm{e}^{-(\xi-y)^2/(4t)}\eta_0(y)\mathrm{d}y.$$

If we assume that η_0 decays sufficiently fast to 1 for $\xi\to\pm\infty$, we have $\varphi_0 := \partial_\xi\eta_0 \in L^1$, and $\varphi := \partial_\xi\eta$ satisfies the linear diffusion equation with the localized initial condition $\varphi(0,\xi) = \varphi_0(\xi)$. Then $\sup_{\xi\in\mathbb{R}}|\varphi(t,\xi) - \sqrt{\pi/t}\,\hat\varphi_0(0)\mathrm{e}^{-\xi^2/(4t)}| \le Ct^{-1}$, which, by integration with respect to ξ, yields

$$\sup_{\xi\in\mathbb{R}}\Big|\eta(t,\xi) - 1 - \frac{z}{2}\Big(1+\mathrm{erf}\,\big(\frac{\xi}{2\sqrt{t}}\big)\Big)\Big| \le Ct^{-1/2}.$$

Therefore, the renormalized solution of the Burgers equation (2.52) with initial condition $v\big|_{t=0} = v_0 \in L^1$ satisfies

$$\sup_{x\in\mathbb{R}}|t^{1/2}v(t,t^{1/2}x) - f_z(x)| \le Ct^{-1/2}, \tag{2.54}$$

where $\ln(z+1) = \frac{d}{c_2}\int_{\mathbb{R}} v_0(x)\mathrm{d}x$. Thus, solutions of the Burgers equation to localized initial conditions converge to a non-Gaussian profile, see Fig. 1. This behaviour is stable under suitable perturbations of the Burgers equation, cf., e.g., [21, Theorem 1.5].

Lemma 2.5. *Let $p \in (0,1/2)$ and $h(v,\partial_x v,\partial_x^2 v) = v^{q_1}(\partial_x v)^{q_2}(\partial_x^2 v)^{q_3}$ with $d_h = q_1+2q_2+3q_3 > 3$, $q_j\in\mathbb{N}_0$, and $q_3\le 1$. Then there exist $C_1, C_2 > 0$ such that the following holds. If $\|v_0\|_{H^2(2)} \le C_1$, then the perturbed Burgers equation*

$$\partial_t v = c_2\partial_x^2 v + d\partial_x(v^2) + h(v,\partial_x v,\partial_x^2 v)$$

with $c_2 > 0$, $d \ne 0$ has a unique solution v with $v\big|_{t=1} = v_0$. For a $z > -1$ it satisfies

$$\|\sqrt{t}v(t,\sqrt{t}x) - f_z(x)\|_{H^2(2)} \le C_2 t^{-1/2+p} \tag{2.55}$$

for all $t\ge 1$, where f_z is the non–Gaussian profile from (2.53).

In particular, nonlinearities h with degree $d_h > 3$, or more general nonlinearities (not necessarily monomials) such that (2.55) holds, are called asymptotically irrelevant.

2.6. **Derivation of the Burgers equation. Splitting of the nonlinearity.** To distinguish relevant from asymptotically irrelevant terms we split the nonlinearity N from (2.18) into $N = B + G$, where the second component of $B(h,q)$ contains all quadratic terms without a factor $\partial_x q$. The terms in B turn out to have a "derivative-like" structure and hence lead to a Burgers-like decay, see Remark 2.6 below. For all other terms, which we collect in $G(h,q)$, we later show that they are irrelevant. By construction,

$$\begin{aligned} B(h,q) &= \begin{pmatrix} 0 \\ \mathcal{B}_2(h,q)\end{pmatrix} \\ &= \begin{pmatrix} 0 \\ b_{00}h^2 + b_{01}h\,\partial_x h + b_{02}h\,\partial_x^2 h + b_{03}h\,\partial_x^3 h + b_{11}(\partial_x h)^2 + b_1 hq + b_2\partial_x h\,q + b_3\partial_x^2 h\,q + b_4 q^2\end{pmatrix}, \end{aligned} \tag{2.56}$$

where again all coefficients are γ-periodic in x and depend on the stationary solution (h_s, q_s).

Since the equation for $\partial_t h$ is linear, also in $G(h,q)$ the first component vanishes. The terms in the second component of G can be characterized as follows:

(i) Terms in $B_2(h,q)$, multiplied by h^j, $j \geq 1$.
(ii) $(\partial_x q)^2$, $h^j \,\partial_x q\, q$, $h^{j+1}\,\partial_x q$, $h^j\,\partial_x h\,\partial_x q$ with $j \geq 0$.
(iii) $(\partial_x q)^2 q$, $h^j\,\partial_x h\, q^2$, $h^j(\partial_x h)^2 q$ with $j \geq 0$.

The terms in (i) are due to the expansions $1/(f_s+f) = \sum_{j\geq 0} c_j f^j$ and $f = \sum_{j\geq 1} \tilde{c}_j h^j$. They are at least cubic and contain the quasilinear terms $h^j\,\partial_x^3 h$, $j \geq 2$. The terms in (ii) are the quadratic ones in (2.5) which contain a factor $\partial_x q$. Except of the first one, they also occur multiplied by powers of h due to the denominator $1/(f_s+f)$. Finally, the terms in (iii) originate from the terms in (2.5) having a cubic numerator. Altogether, we can write the IBL (2.18) for $(h,q)^\top$ as

$$\partial_t \begin{pmatrix} h \\ q \end{pmatrix} = A(\partial_x) \begin{pmatrix} h \\ q \end{pmatrix} + B(h,q) + G(h,q). \tag{2.57}$$

Setting $\tilde{B}(\tilde{h},\tilde{q}) = \mathcal{J} B(\mathcal{J}^{-1}\tilde{h}, \mathcal{J}^{-1}\tilde{q})$ and $\tilde{G}(\tilde{h},\tilde{q}) = \mathcal{J} G(\mathcal{J}^{-1}\tilde{h}, \mathcal{J}^{-1}\tilde{q})$, this corresponds to

$$\partial_t \tilde{w}(t,\ell,x) = \tilde{A}(\ell)\tilde{w}(t,\ell,x) + \tilde{B}(\tilde{w})(t,\ell,x) + \tilde{G}(\tilde{w})(t,\ell,x) \tag{2.58}$$

in Bloch space, cf. (2.37).

Remark 2.6. Heuristically, the reason for splitting the nonlinearity into B and G is the following. To project the nonlinearity $\tilde{N}$ onto the critical eigenfunction we take the scalar product of $\tilde{N}(\ell,\cdot)$ with the eigenvector of the adjoint linear operator $\tilde{A}^*(\ell,\cdot)$, which, by (2.41), reads $\psi^1(\ell) = (c_0, 0)^\top + \mathcal{O}(\ell)$. Thus, since the equation for $\partial_t h$ is linear, the critical component of the nonlinearity obtains an additional factor ℓ in Bloch space, which increases its degree by 1. This is the reason why terms like h^2 turn out to have the same degree as the nonlinearity $\partial_x(v^2)$ in the Burgers equation. As the IBL has non-constant coefficients, a ∂_x in x-space, which corresponds to $(\partial_x + \mathrm{i}\ell)$ in Bloch space, does not automatically increase the degree. Therefore, also terms like $h\,\partial_x^3 h$, which at first view appear to be irrelevant, make an contribution to the relevant terms. On the other hand, since the q-component of the critical eigenvector ϕ^1 is independent of x at wave number $\ell = 0$, a factor $\partial_x q$ leads to a further factor ℓ after projecting it onto the critical eigenvector, and thus to an asymptotically irrelevant term. That is why quadratic terms with a factor $\partial_x q$ are assigned to G. These considerations are made rigorous in §4.

Formal derivation of the Burgers equation. Following Remarks 2.4 and 2.6 we formally derive the Burgers equation for $\tilde{\alpha}$ by ignoring $\tilde{w}_s$ as well as the nonlinearity $\tilde{G}$. Thus, setting $\tilde{w} = (\tilde{w}_1, \tilde{w}_2)^\top = \tilde{\alpha}\phi^1$, (2.45) becomes

$$\partial_t \tilde{\alpha}(t,\ell) = \mu_1(\ell)\tilde{\alpha}(t,\ell) + \tilde{E}_c^*(\tilde{B}(\tilde{\alpha}(t)\phi^1))(\ell). \tag{2.59}$$

Since the equation for $\partial_t h$ is linear, the nonlinearity reads

$$\tilde{E}_c^*(\tilde{B}(\tilde{\alpha}\phi^1))(\ell) = \chi(\ell) \int_0^\gamma \tilde{B}_2(\tilde{\alpha}\phi^1)(\ell,x)\,\bar{\psi}_2^1(\ell,x)\,\mathrm{d}x,$$

where (2.56) yields

$$\begin{aligned}\tilde{B}_2(\tilde{w}) &= b_{00}\tilde{w}_1^{*2} + b_{01}\tilde{w}_1 * [(\partial_x + \mathrm{i}\ell)\tilde{w}_1] + b_{02}\tilde{w}_1 * [(\partial_x + \mathrm{i}\ell)^2\tilde{w}_1] \\ &\quad + b_{03}\tilde{w}_1 * [(\partial_x + \mathrm{i}\ell)^3\tilde{w}_1] + b_{11}[(\partial_x + \mathrm{i}\ell)\tilde{w}_1]^{*2} + b_1\tilde{w}_1 * \tilde{w}_2 \\ &\quad + b_2[(\partial_x + \mathrm{i}\ell)\tilde{w}_1] * \tilde{w}_2 + b_3[(\partial_x + \mathrm{i}\ell)^2\tilde{w}_1] * \tilde{w}_2 + b_4\tilde{w}_2^{*2}.\end{aligned} \tag{2.60}$$

We study in detail only the first term of $\tilde{B}_2(\tilde{w})$ and show afterwards that all other terms can be treated the same way. We have

$$\begin{aligned}&\int_0^\gamma b_{00}(x)\tilde{w}_1^{*2}(\ell,x)\bar{\psi}_2^1(\ell,x)\,\mathrm{d}x \\ &= \int_0^\gamma b_{00}(x)\int_{-k_0/2}^{k_0/2}\tilde{\alpha}(\ell-k)\phi_1^1(\ell-k,x)\tilde{\alpha}(k)\phi_1^1(k,x)\,\mathrm{d}k\,\bar{\psi}_2^1(\ell,x)\,\mathrm{d}x \\ &= \int_{-k_0/2}^{k_0/2}\tilde{\alpha}(\ell-k)\tilde{\alpha}(k)\int_0^\gamma b_{00}(x)\phi_1^1(\ell-k,x)\phi_1^1(k,x)\bar{\psi}_2^1(\ell,x)\,\mathrm{d}x\,\mathrm{d}k \\ &=: \int_{-k_0/2}^{k_0/2}\tilde{\alpha}(\ell-k)\tilde{\alpha}(k)K(\ell,\ell-k,k)\,\mathrm{d}k.\end{aligned} \tag{2.61}$$

Before we expand the kernel $K(\ell,\ell-k,k)$ in terms of ℓ, we state the following useful properties.

Lemma 2.7. *The adjoint eigenfunction ψ^1 satisfies $\psi_2^1(0,x) = 0$ and $\partial_\ell\psi_2^1(0,x) \in \mathrm{i}\mathbb{R}$.*

Proof. The first property immediately follows from (2.41). Differentiating the eigenvalue equation $\tilde{A}^*(\ell)\psi^1(\ell,x) = \bar{\mu}_1(\ell)\psi^1(\ell,x)$ with respect to ℓ gives

$$\partial_\ell\tilde{A}^*(\ell)\psi^1(\ell,x) + \tilde{A}^*(\ell)\partial_\ell\psi^1(\ell,x) = \partial_\ell\bar{\mu}_1(\ell)\psi^1(\ell,x) + \bar{\mu}_1(\ell)\partial_\ell\psi^1(\ell,x);$$

i.e., from the locally parabolic shape of $\bar{\mu}_1(\ell) = -c_2\ell^2 + \mathcal{O}(\ell^3)$ it follows that

$$\partial_\ell\tilde{A}^*(0)\psi^1(0,x) + \tilde{A}^*(0)\partial_\ell\psi^1(0,x) = 0.$$

Since $\partial_\ell\tilde{A}^*(0)\psi^1(0,x) = (\mathrm{i}c_0c_1, \mathrm{i}c_0)^\top$, cf. (2.38), we obtain $\tilde{A}^*(0)\partial_\ell\psi^1(0,x) \in \mathrm{i}\mathbb{R}^2$. As all coefficients of $\tilde{A}^*(0)$ are real, we obtain $\partial_\ell\psi^1(0,x) \in \mathrm{i}\mathbb{R}^2 + \ker\tilde{A}^*(0) = \mathrm{i}\mathbb{R}^2 + \mathbb{C}\psi^1(0,x)$ and thus $\partial_\ell\psi_2^1(0,x) \in \mathrm{i}\mathbb{R}$. □

Returning to (2.61) we can write the integral kernel as

$$K(\ell,\ell-k,k) = \partial_1K(\mathbf{0})\ell + \partial_2K(\mathbf{0})(\ell-k) + \partial_3K(\mathbf{0})k + \mathcal{O}(|\ell|^2 + |\ell-k|^2 + |k|^2),$$

since $K(\mathbf{0}) = 0$ due to $\psi_2^1(0,x) = 0$. For the same reason we obtain $\partial_2K(\mathbf{0}) = \partial_3K(\mathbf{0}) = 0$, while the first term reads

$$\partial_1K(\mathbf{0})\ell = \int_0^\gamma b_{00}(x)\phi_1^1(0,x)\phi_1^1(0,x)\partial_\ell\bar{\psi}_2^1(0,x)\,\mathrm{d}x\,\ell =: \mathrm{i}K_1\ell \in \mathrm{i}\mathbb{R}, \tag{2.62}$$

since $\phi^1(0,x) \in \mathbb{R}^2$, see (2.35). Altogether, we have

$$K(\ell,\ell-k,k) = \mathrm{i}K_1\ell + \mathcal{O}(|\ell|^2 + |\ell-k|^2 + |k|^2). \tag{2.63}$$

Therefore,

$$\int_0^\gamma b_{00}(x)\tilde{w}_1^{*2}(\ell,x)\bar{\psi}_2^1(\ell,x)\,\mathrm{d}x$$

$$
\begin{aligned}
&= \int_{-k_0/2}^{k_0/2} \tilde{\alpha}(\ell-k)\tilde{\alpha}(k)\left(\mathrm{i}K_1\ell + \mathcal{O}(|\ell|^2+|\ell-k|^2+|k|^2)\right) \mathrm{d}k \\
&= \mathrm{i}K_1\ell\tilde{\alpha}^{*2}(\ell) + \mathcal{O}(\ell^2)\tilde{\alpha}^{*2}(\ell).
\end{aligned}
$$

For the remaining terms in $\tilde{B}_2(\tilde{w})$ the same considerations lead to

$$
\int_0^\gamma \tilde{B}_2(\tilde{\alpha}\phi^1)(\ell,x)\bar{\psi}_2^1(\ell,x)\,\mathrm{d}x = \int_{-k_0/2}^{k_0/2} \tilde{\alpha}(\ell-k)\tilde{\alpha}(k)K(\ell,\ell-k,k)\,\mathrm{d}k,
$$

where the integral kernel K is given as sum of terms of the type

$$
\int_0^\gamma b_{ij}(x)\phi_{j_1}^1(\ell-k,x)(\mathrm{i}k)^{n_1}\partial_x^{n_2}\phi_{j_2}^1(k,x)\bar{\psi}_2^1(\ell,x)\,\mathrm{d}x.
$$

Thus, there exists a $d\in\mathbb{R}$ such that we can write

$$
K(\ell,\ell-k,k) = \mathrm{i}d\ell + \mathcal{O}(|\ell|^2+|\ell-k|^2+|k|^2), \tag{2.64}
$$

which yields

$$
\tilde{E}_c^*(\tilde{B}(\tilde{\alpha}\phi^1))(\ell) = \mathrm{i}d\ell\chi(\ell)\tilde{\alpha}^{*2}(\ell) + \mathcal{O}(\ell^2)\tilde{\alpha}^{*2}(\ell). \tag{2.65}
$$

Altogether (2.59) leads to

$$
\partial_t\tilde{\alpha}(t,\ell) = -c_2\ell^2\tilde{\alpha}(t,\ell) + \mathrm{i}d\ell\chi(\ell)\tilde{\alpha}^{*2}(t,\ell) + \mathcal{O}(\ell^3)\tilde{\alpha}(t,\ell) + \mathcal{O}(\ell^2)\tilde{\alpha}^{*2}(t,\ell). \tag{2.66}
$$

Since ∂_x in x-space corresponds to $\mathrm{i}\ell$ in Fourier space and since a derivative increases the degree of irrelevance by one, this reminds us strongly of an asymptotically irrelevant perturbation of the Fourier transformed Burgers equation $\partial_t\hat{v} = -c_2k^2\hat{v} + \mathrm{i}dk\hat{v}^{*2}$. This formally explains why in the main Theorem 1.2 the comoving non-Gaussian profile $t^{-1/2}f_{z_0}(t^{-1/2}(x+c_1t))\Phi^1(0,x)$ governs the asymptotics of the IBL at lowest order.

2.7. **The result.** §2.5 about the Burgers equation and the formal calculations in §2.6 motivate the formulation of the following theorem about nonlinear stability of stationary solutions of the IBL.

Theorem 2.8. *Let $p\in(0,1/2)$, $3<r<4$, and let $(f_s,q_s)^\top$ be a spectrally stable stationary solution of the IBL (1.1), (1.2); i.e., Assumption 2.3 is fulfilled. Then there exist constants $C_1,C_2>0$ such that the following holds. If $\|h_0\|_{H^r(2)} + \|q_0\|_{H^{r-1}(2)} \le C_1$, then there exists a unique global solution (h,q) of the transformed IBL (2.18) with $(h,q)\big|_{t=1} = (h_0,q_0)$ and*

$$
\sup_{x\in\mathbb{R}}\Big|(h,q)^\top - t^{-1/2}f_{z_0}(t^{-1/2}(x+c_1t))\phi^1(0,x)\Big| \le C_2t^{-1+p/2} \tag{2.67}
$$

for $t\in[1,\infty)$. Here, $z_0>-1$, f_{z_0} denotes the non-Gaussian profile from (2.53), and $\phi^1(0,\cdot) = (\mathrm{d}h_s/\mathrm{d}q_s,1)^\top$ is an eigenfunction to the critical eigenvalue $\lambda_1(\ell) = c_1\mathrm{i}\ell - c_2\ell^2 + \mathcal{O}(\ell^3)$ from Assumption 2.3.

Theorem 2.8 follows from the subsequent Theorem 4.2 about nonlinear stability in rescaled Bloch spaces. To transfer Theorem 2.8 to the original (F,Q)-system (1.1), (1.2) we note that (2.67) yields $\sup_{x\in\mathbb{R}} h^2 \le Ct^{-1}$. Since due to (2.13) the transformation for the film thickness reads $f = \beta h + \mathcal{O}(h^2)$, we can write

$$
\begin{aligned}
&|(f,q)^\top - t^{-1/2}f_{z_0}(t^{-1/2}(x+c_1t))\Phi^1(0,x)| \\
&= \left|\begin{pmatrix}\beta & 0\\ 0 & 1\end{pmatrix}\Big((h,q)^\top - t^{-1/2}f_{z_0}(t^{-1/2}(x+c_1t))\phi^1(0,x)\Big)\right| + \mathcal{O}(h^2),
\end{aligned}
$$

where Φ^1 is the eigenvector corresponding to the critical eigenvalue λ_1 in the (f,q)-system, see (2.34). This yields Theorem 1.2.

3. Existence of a local solution

For the proof of Theorem 2.8 we use the RG method [5] for (2.45), (2.46). The main steps consist in a proof of local existence using maximal regularity methods, and in a careful estimate of the nonlinear terms. The local existence and uniqueness of solutions is carried out via resolvent estimates in x-space, while the RG method is set up in Bloch space.

3.1. **Function spaces depending on time and space.** In the following, we always assume that X is a Hilbert space and $t_0, t_1 \in \mathbb{R} \cup \{-\infty, \infty\}$. If not stated otherwise, H^r stands for $H^r(\mathbb{R})$.

Definition 3.1. $L^2((t_0,t_1),X)$ denotes the space of (strongly) measurable functions u with values in X such that the norm $\|u\|_{L^2((t_0,t_1),X)} := \Big(\int_{t_0}^{t_1} \|u(t)\|_X^2 \,\mathrm{d}t\Big)^{1/2}$ is finite. For $m \in \mathbb{N}$ we write

$$H^m((t_0,t_1),X) := \{u \mid \partial_t^j u \in L^2((t_0,t_1),X) \text{ for } 0 \le j \le m\},$$
$$\|u\|_{H^m((t_0,t_1),X)} := \Big(\sum_{j=0}^m \|\partial_t^j u\|^2_{L^2((t_0,t_1),X)}\Big)^{1/2}.$$

In the special case $(t_0,t_1) = \mathbb{R}$ and $X = H^r(\mathbb{R}), r \in \mathbb{R}^+$ we find the equivalent norm

$$\|u\|_{H^m(\mathbb{R},H^r)} \sim \Big(\int_{\mathbb{R}}\int_{\mathbb{R}} (1+\tau^2)^m (1+k^2)^r |\mathcal{F}_{tx} u(\tau,k)|^2 \,\mathrm{d}\tau\,\mathrm{d}k\Big)^{1/2}, \tag{3.1}$$

where $\mathcal{F}_{tx}u$ denotes the Fourier transform of u with respect to time and space. Obviously, this definition can be extended to all $m \in \mathbb{R}^+$.

Lemma 3.2. *Let $s \ge 0$. Then, we have*

(i) $u \in H^s(\mathbb{R},X) \Leftrightarrow (1+\tau^2)^{\frac{s}{2}} \mathcal{F}_t u \in L^2(\mathbb{R},X)$.

(ii) *$H^s((t_0,t_1),X)$ coincides with the space of restrictions to (t_0,t_1) of the elements in $H^s(\mathbb{R},X)$. Extension and restriction are both continuous operators.*

For a proof see [13, p. 58 and Theorem 9.1]. By (i) we see that (3.1) is an equivalent norm in $H^m(\mathbb{R},H^r)$ also for non-integer values of m. The next lemma shows that the regularity of space and time derivatives is the same as in the scalar valued case. Here we denote by $[s]$ the integer part of s.

Lemma 3.3. *Let $u \in H^s((t_0,t_1),H^r)$, $j \le [s]$, $l \le [r]$. Then*

$$\partial_x^l \partial_t^j u \in H^{s-j}((t_0,t_1),H^{r-l}).$$

Proof. We assume $(t_0,t_1) = \mathbb{R}$. Then by (3.1) we have

$$\begin{aligned} \|\partial_x^l\partial_t^j u\|^2_{H^{s-j}(\mathbb{R},H^{r-l})} &\le C \int_{\mathbb{R}}\int_{\mathbb{R}} (1+\tau^2)^{s-j}(1+k^2)^{r-l} |\mathcal{F}_{tx}[\partial_x^l\partial_t^j u](\tau,k)|^2 \,\mathrm{d}\tau\,\mathrm{d}k \\ &= C \int_{\mathbb{R}}\int_{\mathbb{R}} (1+\tau^2)^{s-j}(1+k^2)^{r-l} |k|^{2l}|\tau|^{2j} |\mathcal{F}_{tx} u(\tau,k)|^2 \,\mathrm{d}\tau\,\mathrm{d}k \\ &\le C\|u\|_{H^s(\mathbb{R},H^r)}. \end{aligned}$$

□

As we will see in Lemma 3.11 below, for $u \in H^s((t_0,\infty),L^2)$, $s > 1/2$, there exist traces $\partial_t^j u(t_0,\cdot) \in L^2(\mathbb{R})$ for all $j \in \mathbb{N}_0$ with $j < s - 1/2$. Thus, we can define the following subspace of $H^s((t_0,\infty),H^r)$.

Definition 3.4.

$$H_0^s((t_0,\infty),H^r) := \{u \in H^s((t_0,\infty),H^r) \mid \|\partial_t^j u(t_0,\cdot)\|_{L^2}{=}0 \text{ for } j < s-\frac{1}{2}, j{\in}\mathbb{N}_0\}.$$

By the following lemma these functions can be extended by zero for $t \le 0$.

Lemma 3.5. *Let $s \ge 0$ be not a half integer, $t_0 \in \mathbb{R}$, $u \in H_0^s((t_0,\infty),H^r)$, and*

$$u_0(t,\cdot) := \begin{cases} u(t,\cdot) & \text{for } t > t_0, \\ 0 & \text{for } t \le t_0. \end{cases}$$

Then $u \mapsto u_0$ is a continuous mapping from $H_0^s((t_0,\infty),H^r)$ into $H^s(\mathbb{R},H^r)$; i.e., there exist $C_1, C_2 > 0$ such that

$$C_1\|u\|_{H^s((t_0,\infty),H^r)} \le \|u_0\|_{H^s(\mathbb{R},H^r)} \le C_2\|u\|_{H^s((t_0,\infty),H^r)}.$$

For a proof see [13], in particular Theorem 11.4. Next we characterize functions u in $H_0^s((0,\infty),H^r)$. Since they can be extended by zero for $t \le 0$ we can apply Fourier transform in time. The problem is that without making further demands on the regularity of $\mathcal{F}_t u$ we can not guarantee that the inverse Fourier transform is again in H_0^s. The following two lemmas show conditions based upon the Paley-Wiener Theorem which ensure that the inverse Fourier transform maps back to functions vanishing on the negative time axis. As u is only defined for $t > 0$ and $\mathcal{F}_t u$ must be treated as function on $\tau \in \mathbb{C}$ it is common to replace Fourier transform in time by Laplace transform:

$$\mathcal{L}u(\tau) := \frac{1}{2\pi}\int_0^\infty u(t)\mathrm{e}^{-t\tau}\,\mathrm{d}t. \tag{3.2}$$

The relation to Fourier transform is

$$\mathcal{L}u(\tau_1 + \mathrm{i}\tau_2, x) = \frac{1}{2\pi}\int_{\mathbb{R}} u_0(t,x)\mathrm{e}^{-t\tau_1}\mathrm{e}^{-\mathrm{i}t\tau_2}\,\mathrm{d}t = \mathcal{F}_t[\mathrm{e}^{-\cdot\tau_1}u_0(\cdot,x)](\tau_2).$$

Lemma 3.6. *Let $s \ge 0$ be not a half integer, $r \ge 0$. If $u \in H_0^s((0,\infty),H^r)$, then the Laplace transform $\mathcal{L}u$ satisfies*

(i) *$\tau \mapsto \mathcal{L}u(\tau,x)$ is holomorphic in the half-plane $\mathrm{Re}\,\tau > 0$ for almost every $x \in \mathbb{R}$.*

(ii) $\sup_{\tau_1>0}\int_{\mathbb{R}}|\mathcal{L}u(\tau_1+\mathrm{i}\tau_2,x)|^2\,\mathrm{d}\tau_2 < \infty$ *for almost every $x \in \mathbb{R}$.*

(iii) $\|u\|_{H^s((0,\infty),H^r)} \sim \Big(\int_{\mathbb{R}}(1+\tau_2^2)^s\|\mathcal{L}u(\mathrm{i}\tau_2,\cdot)\|_{H^r}^2\,\mathrm{d}\tau_2\Big)^{1/2}$.

Proof. Since $u \in H_0^s((0,\infty),H^r)$, we have $u \in L^2((0,\infty),L^2)$. Thus

$$\int_{\mathbb{R}}\int_0^\infty |u(t,x)|^2\,\mathrm{d}t\,\mathrm{d}x = \int_0^\infty\int_{\mathbb{R}}|u(t,x)|^2\,\mathrm{d}x\,\mathrm{d}t < \infty$$

which yields $\int_0^\infty |u(t,x)|^2\,\mathrm{d}t < \infty$ for almost every $x \in \mathbb{R}$. Applying the Paley-Wiener Theorem, see [25] for instance, gives the first property. Now let $\tau_1 > 0$. Parseval's identity implies

$$\int_{\mathbb{R}}|\mathcal{L}u(\tau_1+\mathrm{i}\tau_2,x)|^2\,\mathrm{d}\tau_2$$

$$
\begin{aligned}
&= \int_{\mathbb{R}} |\mathcal{F}_t[\mathrm{e}^{-\cdot\tau_1} u_0(\cdot, x)](\tau_2)|^2 \, \mathrm{d}\tau_2 \\
&\leq C \int_{\mathbb{R}} |\mathrm{e}^{-t\tau_1} u_0(t,x)|^2 \, \mathrm{d}t \leq C \int_0^\infty |u(t,x)|^2 \, \mathrm{d}t < \infty \text{ for almost every } x \in \mathbb{R}
\end{aligned}
$$

independently of τ_1. This shows the second property. The third property follows with the help of Lemma 3.5 and Lemma 3.2; i.e.,

$$
\begin{aligned}
\int_{\mathbb{R}} (1+\tau_2^2)^s \|\mathcal{L}u(\mathrm{i}\tau_2, \cdot)\|_{H^r}^2 \, \mathrm{d}\tau_2 &= \int_{\mathbb{R}} (1+\tau_2^2)^s \|\mathcal{F}_t u_0(\tau_2, \cdot)\|_{H^r}^2 \, \mathrm{d}\tau_2 \\
&\sim \|u_0\|_{H^s(\mathbb{R}, H^r)}^2 \sim \|u\|_{H^s((0,\infty), H^r)}^2 .
\end{aligned}
$$

□

The following lemma shows that regularity in x is preserved under Laplace transform.

Lemma 3.7. *Let $r \geq 0$. If $u \in L^2((0,\infty), H^r)$, then*

$$
\sup_{\tau_1 \geq 0} \int_{\mathbb{R}} \|\mathcal{L}u(\tau_1 + \mathrm{i}\tau_2, \cdot)\|_{H^r}^2 \, \mathrm{d}\tau_2 \leq C \|u\|_{L^2((0,\infty), H^r)}^2 .
$$

In particular, $\mathcal{L}u(\tau, \cdot) \in H^r$ for almost every τ with $\operatorname{Re}\tau \geq 0$.

Proof. Let $\tau_1 \geq 0$. Then

$$
\begin{aligned}
\int_{\mathbb{R}} \|\mathcal{L}u(\tau_1 + \mathrm{i}\tau_2, \cdot)\|_{H^r}^2 \, \mathrm{d}\tau_2 &= \int_{\mathbb{R}} \|\mathcal{F}_t[\mathrm{e}^{-t\tau_1} u_0(t, \cdot)](\tau_2)\|_{H^r}^2 \, \mathrm{d}\tau_2 \\
&\leq C \int_{\mathbb{R}} \|\mathrm{e}^{-t\tau_1} u_0(t, \cdot)\|_{H^r}^2 \, \mathrm{d}t \\
&= C \int_0^\infty e^{-2t\tau_1} \|u(t, \cdot)\|_{H^r}^2 \, \mathrm{d}t \leq C \|u\|_{L^2((0,\infty), H^r)}^2
\end{aligned}
$$

independently of τ_1. □

Lemma 3.6 has the following inverse.

Lemma 3.8. *Let $s \geq 0$ be not a half integer, $r \geq 0$, and assume $f : \mathbb{C} \times \mathbb{R} \to \mathbb{C}$ fulfills the following conditions.*

- (i) *$f(\tau, x)$ is holomorphic in the half-plane $\operatorname{Re}\tau > 0$ for almost every $x \in \mathbb{R}$.*
- (ii) $\sup_{\tau_1 > 0} \int_{\mathbb{R}} |f(\tau_1 + \mathrm{i}\tau_2, x)|^2 \, \mathrm{d}\tau_2 < \infty$ *for almost every $x \in \mathbb{R}$.*
- (iii) $\int_{\mathbb{R}} (1+\tau_2^2)^s \|f(\mathrm{i}\tau_2, \cdot)\|_{H^r}^2 \, \mathrm{d}\tau_2 < \infty$.

Then the inverse Fourier transform $g(t,x) = \int_{\mathbb{R}} f(i\tau_2, x) \mathrm{e}^{\mathrm{i}t\tau_2} \, \mathrm{d}\tau_2$ satisfies

- (iv) $g\big|_{\mathbb{R}^+ \times \mathbb{R}} \in H_0^s((0,\infty), H^r)$ *and* $\mathcal{L}g = f$.

Proof. Due to (i) and (ii), for almost every $x \in \mathbb{R}$ we can apply the Paley-Wiener Theorem and obtain $g(t,x) = 0$ for $t < 0, \mathcal{L}g(\tau, x) = f(\tau, x)$, see [25], for instance. It remains to prove that $g\big|_{\mathbb{R}^+ \times \mathbb{R}} \in H_0^s((0,\infty), H^r)$. Due to Lemma 3.2 we have

$$
\begin{aligned}
\|g\|_{H^s(\mathbb{R}, H^r)}^2 &\leq C \int_{\mathbb{R}} \|(1+\tau_2^2)^{\frac{s}{2}} \mathcal{F}_t g(\tau_2, \cdot)\|_{H^r}^2 \, \mathrm{d}\tau_2 = C \int_{\mathbb{R}} (1+\tau_2^2)^s \|\mathcal{L}g(\mathrm{i}\tau_2, \cdot)\|_{H^r}^2 \, \mathrm{d}\tau_2 \\
&= C \int_{\mathbb{R}} (1+\tau_2^2)^s \|f(\mathrm{i}\tau_2, \cdot)\|_{H^r}^2 \, \mathrm{d}\tau_2 < \infty,
\end{aligned}
$$

thus $g \in H^s(\mathbb{R}, H^r)$. Let $j \in \mathbb{N}$ such that $s - j > 1/2$. Lemma 3.3 gives $\partial_t^j g \in H^{s-j}(\mathbb{R}, H^r)$, and due to standard Sobolev embedding we have $\partial_t^j g \in C(\mathbb{R}, H^r)$. Since $g(t,x) = 0$ for $t < 0$ and almost every $x \in \mathbb{R}$ we achieve $\|\partial_t^j g(0,\cdot)\|_{L^2} = 0$. This shows the first property. □

To prove local existence for the IBL (2.18) we use the following Sobolev spaces.

Definition 3.9. Let $H^{r,s}((t_0,t_1)) := L^2((t_0,t_1), H^r(\mathbb{R})) \cap H^s((t_0,t_1), L^2(\mathbb{R}))$, with norm

$$\|u\|_{H^{r,s}((t_0,t_1))} := \Big(\int_{t_0}^{t_1} \|u(t,\cdot)\|_{H^r}^2 \,\mathrm{d}t + \|u\|_{H^s((t_0,t_1),L^2)}^2 \Big)^{1/2}.$$

Applying (3.1) we can state an equivalent norm for the case $(t_0,t_1) = \mathbb{R}$, namely

$$\|u\|_{H^{r,s}(\mathbb{R})} \sim \Big(\int_{\mathbb{R}} \int_{\mathbb{R}} \big((1+\tau^2)^{\frac{s}{2}} + (1+k^2)^{\frac{r}{2}} \big)^2 |\mathcal{F}_{tx} u(\tau,k)|^2 \,\mathrm{d}\tau \,\mathrm{d}k \Big)^{1/2}. \tag{3.3}$$

Functions in $H^{r,s}(\mathbb{R})$ also belong to "intermediate spaces" with intermediate regularities in time and space.

Lemma 3.10. *Let $r,s \geq 0$, $\vartheta \in (0,1)$. Then $H^{r,s}(\mathbb{R})$ is continuously embedded into $H^{\vartheta s}(\mathbb{R}, H^{(1-\vartheta)r})$.*

Proof. For $u \in H^{r,s}(\mathbb{R})$, Lemma 3.2 yields

$$\|u\|_{H^{\vartheta s}(\mathbb{R},H^{(1-\vartheta)r})}^2 \leq C \int_{\mathbb{R}} \int_{\mathbb{R}} (1+\tau^2)^{\vartheta s} (1+k^2)^{(1-\vartheta)r} |\mathcal{F}_{tx} u(\tau,k)|^2 \,\mathrm{d}\tau \,\mathrm{d}k.$$

By Young's inequality, $(1+\tau^2)^{\vartheta s}(1+k^2)^{(1-\vartheta)r} \leq \vartheta(1+\tau^2)^s + (1-\vartheta)(1+k^2)^r$ we obtain $\|u\|_{H^{\vartheta s}(\mathbb{R},H^{(1-\vartheta)r})}^2 \leq C\|u\|_{H^{r,s}(\mathbb{R})}^2$. □

Later we need estimates of the $H^{\frac{r}{s}(s-1/2)}$-norm for fixed times. Lemma 3.10 particularly yields that $H^{r,s}(\mathbb{R})$ is continuously embedded into $H^{1/2}(\mathbb{R}, H^{\frac{r}{s}(s-1/2)})$, but by standard Sobolev embedding theory this does not allow any conclusion for fixed t. However, by interpolation theory the following trace theorem can be shown.

Lemma 3.11. *Let $u \in H^{r,s}((t_0,t_1))$, $r \geq 0$, $s > 1/2$. Then for all integers $j < s - \frac{1}{2}$ there exists the trace*

$$\partial_t^j u(t_0,\cdot) \in H^{p_j}(\mathbb{R}), \; p_j = \frac{r}{s}\big(s - j - \frac{1}{2}\big).$$

The mappings $H^{r,s}((t_0,t_1)) \to H^{p_j}(\mathbb{R}) : u \mapsto \partial_t^j u$ are continuous. Furthermore, the mapping $u \mapsto (\partial_t^j u(t_0,\cdot))_{0\leq j<s-\frac{1}{2}}$ from $H^{r,s}((t_0,t_1))$ into $\prod_{0\leq j<s-\frac{1}{2}} H^{p_j}$ is surjective.

A proof can be found in [14, Theorem 4.2.1]. For the surjectivity of the trace operator see [13, Theorem 4.4.2] with $X = H^r, Y = L^2$. Since the trace operator is continuous, we have the following corollary, see [13, Theorem 1.3.1] and the proof of [13, Theorem 1.4.2].

Corollary 3.12. *Let $u \in H^{r,s}((t_0,t_1))$, $r \geq 0$, $s > 1/2$. Then there exists a $C > 0$ such that*

$$\sup_{t\in[t_0,t_1]} \|u(t,\cdot)\|_{H^{r-1}} < C\|u\|_{H^{r,s}((t_0,t_1))}.$$

It turns out, that the IBL is a second-order parabolic evolution system, and therefore the spaces $H^{r,s}$ always occur with $s = r/2$ and usually consist of functions defined only for $t \geq 0$. Hence we set

Definition 3.13. $K^r((t_0, t_1)) := H^{r, \frac{r}{2}}((t_0, t_1))$, and

$$K_0^r((t_0, t_1)) := \{u \in K^r((t_0, t_1)) \mid \|\partial_t^j u(t_0, \cdot)\|_{L^2} = 0 \text{ for } j \in \mathbb{N}_0 \text{ with } 2j < r - 1\}.$$

Thus we have $K_0^r((0, \infty)) = H_0^{\frac{r}{2}}((0, \infty), L^2) \cap L^2((0, \infty), H^r)$ and with the help of Lemma 3.6 and Lemma 3.8 we can characterize $K_0^r((0, \infty))$ in Fourier space.

Theorem 3.14. *Let $r \geq 0$, $(r+1)/2 \notin \mathbb{N}$. Then $u \in K_0^r((0, \infty))$ if and only if the Laplace transform $\mathcal{L}u$ fulfills*

(i) *$\mathcal{L}u(\tau, x)$ is holomorphic in the half-plane $\operatorname{Re} \tau > 0$ for almost every $x \in \mathbb{R}$.*

(ii) *$\sup_{\tau_1 > 0} \int_{\mathbb{R}} |\mathcal{L}u(\tau_1 + \mathrm{i}\tau_2, x)|^2 \, \mathrm{d}\tau_2 < \infty$ for almost every $x \in \mathbb{R}$.*

(iii) $\Big(\int_{\mathbb{R}} (\|\mathcal{L}u(\mathrm{i}\tau, \cdot)\|_{H^r}^2 + |\tau|^r \|\mathcal{L}u(\mathrm{i}\tau, \cdot)\|_{L^2}^2) \, \mathrm{d}\tau\Big)^{1/2} < \infty.$

The left-hand side in (iii) *defines a norm equivalent to $\| \cdot \|_{K^r((0,\infty))}$.*

Proof. Due to Lemmas 3.6 and 3.8 it remains to show the equivalence of norms. Since $r/2$ is not a half integer we have

$$\|u\|_{K^r((0,\infty))}^2 \sim \int_{\mathbb{R}} \|\mathcal{L}u(\mathrm{i}\tau, \cdot)\|_{H^r}^2 \, \mathrm{d}\tau + \int_{\mathbb{R}} (1 + \tau^2)^{\frac{r}{2}} \|\mathcal{L}u(\mathrm{i}\tau, \cdot)\|_{L^2}^2 \, \mathrm{d}\tau.$$

Now using $(1 + \tau^2)^{\frac{r}{2}} \sim 1 + |\tau|^r$ yields the result. □

Next, we collect some useful properties of the K^r-spaces, concerning derivatives and nonlinear interaction.

Lemma 3.15. *Let $r>0$ and $l, j \in \mathbb{N}$ with $l+2j \leq r$. If $u \in K^r((t_0, t_1))$ then $\partial_x^l \partial_t^j u \in K^{r-l-2j}((t_0, t_1))$.*

Proof. Applying Lemma 3.10 with $\vartheta = 1 - \frac{l}{r}$ and $\vartheta = \frac{2j}{r}$, respectively, we obtain $u \in H^{\frac{r-l}{2}}((t_0, t_1), H^l) \cap H^j((t_0, t_1), H^{r-2j})$. By Lemma 3.3 it follows that $\partial_x^l \partial_t^j u \in H^{\frac{r-l}{2} - j}((t_0, t_1), L^2) \cap L^2((t_0, t_1), H^{r-l-2j})$. □

Lemma 3.16. *Let $r > 3/2$, $r \geq s \geq 0$. If $u \in K^r((t_0, t_1))$ and $v \in K^s((t_0, t_1))$, then $uv \in K^s((t_0, t_1))$ and there exists a $C > 0$ such that*

$$\|uv\|_{K^s((t_0,t_1))} \leq C \|u\|_{K^r((t_0,t_1))} \|v\|_{K^s((t_0,t_1))}. \tag{3.4}$$

If $u \in K_0^r$ or $v \in K_0^s$, then $uv \in K_0^s$.

A proof of (3.4) can be found in [1, Lemma 5.1] while the second statement is obvious. We need function spaces with weights in the spatial variable, namely $H^s((t_0, t_1), H^r(n))$ where $H^r(n)$ is the weighted Sobolev space introduced in (1.4); i.e., $\|u\|_{H^r(n)} = \|\varrho^n u\|_{H^r}$ with $\varrho(x) := (1+x^2)^{1/2}$. A natural description equivalent to Definition 3.1 is given by the following obvious lemma.

Lemma 3.17. *Let $s, r \geq 0$, $n \in \mathbb{N}$. Then $u \in H^s((t_0, t_1), H^r(n)) \Leftrightarrow \varrho^n u \in H^s((t_0, t_1), H^r)$.*

Definition 3.18. For $r, s, n \geq 0$ let

$$H^{r,s}((t_0,t_1),n) := L^2((t_0,t_1),H^r(n)) \cap H^s((t_0,t_1),L^2(n)),$$

$$K^r((t_0,t_1),n) := H^{r,\frac{r}{2}}((t_0,t_1),n).$$

Remark 3.19. Due to Lemma 3.17, Theorem 3.14 also holds for $u \in K_0^r((0,\infty),n)$ if we replace H^r by $H^r(n)$ and L^2 by $L^2(n)$ in property (iii). The same is true for Lemmas 3.5, 3.7, 3.11, 3.15, 3.16 if we replace the respective Sobolev spaces by weighted ones.

3.2. **Existence of a local solution.** Taking into account that the space regularity of h should be taken higher than that of q, we introduce the vector-valued function spaces

$$\mathcal{H}^r(m) := H^r(m) \times H^{r-1}(m), \tag{3.5}$$

$$\mathcal{K}^{r+1}((t_0,t_1),m) := K^{r+1}((t_0,t_1),m) \times K^r((t_0,t_1),m). \tag{3.6}$$

To prove that spectrally stable stationary solutions of the IBL are nonlinearly stable we first need local existence in a given time interval (t_0, t_1).

Theorem 3.20 (Local existence)**.** *Let $3 < r < 4$ and fix some $t_0 < t_1$. Then there exist $C_1, C_2 > 0$ such that the following holds. If $(h_0, q_0)^\top \in \mathcal{H}^r(2) = H^r(2) \times H^{r-1}(2)$ satisfies*

$$\rho := \|(h_0,q_0)^\top\|_{\mathcal{H}^r(2)} \leq C_1,$$

then there exists a unique local solution

$$(h,q)^\top \in \mathcal{K}^{r+1}((t_0,t_1),2) = K^{r+1}((t_0,t_1),2) \times K^r((t_0,t_1),2)$$

of IBL (2.18) *with*

$$\|(h,q)^\top\|_{\mathcal{K}^{r+1}((t_0,t_1),2)} \leq C_2\rho \tag{3.7}$$

and $(h,q)^\top|_{t=t_0} = (h_0,q_0)^\top$. Moreover, for $t_0 < \tilde{t}_0 < t_1$ and any $m \in \mathbb{N}$ we have $(h,q)^\top \in \mathcal{K}^{r+m}((\tilde{t}_0,t_1),2)$, and there exists $C_3 = C_3(\tilde{t}_0,m)$ such that

$$\|(h,q)^\top\|_{\mathcal{K}^{r+m}((\tilde{t}_0,t_1),2)} \leq C_3\rho. \tag{3.8}$$

To prove this theorem we need to apply maximal regularity results based on Laplace transform. First we solve the linearized problem with inhomogeneous right-hand side and zero initial condition. This requires resolvent estimates for the linear operator A. Due to the periodic coefficients, these cannot be shown by applying Fourier transform in space. Instead, we have to test in x-space with appropriate test functions. This is carried out in detail in §3.3. The higher regularity in the time interval $[\tilde{t}_0, t_1]$ then follows from a bootstrapping argument in §3.4; i.e., since (3.7) yields $(h(\tilde{t}), q(\tilde{t}))^\top \in \mathcal{H}^{r+1}$ for almost every $\tilde{t} \in (t_0,t_1)$, we can start again at $t = \tilde{t}$. This gives $(h,q)^\top \in \mathcal{K}^{r+2}$, and iterating this argument shows (3.8).

3.3. **Resolvent estimates.** The resolvent equation is obtained by Laplace transform with respect to time of the linear inhomogeneous equation $(\partial_t - A(\partial_x))u = g$ and reads $(\lambda - A(\partial_x))u = g$ with $g \in H^{r-1} \times H^{r-2}$ and the linear operator A from (2.17); i.e.,

$$\lambda\begin{pmatrix} h \\ q \end{pmatrix} - \begin{pmatrix} 0 & -\partial_x \\ a_{10} + a_{11}\partial_x + a_{12}\partial_x^2 + a_{13}\partial_x^3 & a_{20} + a_{21}\partial_x + a_{22}\partial_x^2 \end{pmatrix}\begin{pmatrix} h \\ q \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}. \tag{3.9}$$

Theorem 3.21. *Let $r \geq 2$. Then there exist $C, a > 0$ such that for all $g = (g_1, g_2)^\top \in H^{r-1} \times H^{r-2}$ and all λ with $\operatorname{Re}\lambda \geq a$ the resolvent equation $(\lambda - A(\partial_x))(h,q)^\top = g$ has a unique solution, which satisfies*

$$\begin{aligned} &\|h\|_{H^{r+1}} + |\lambda|^{(r+1)/2}\|h\|_{L^2} + \|q\|_{H^r} + |\lambda|^{r/2}\|q\|_{L^2} \\ &\leq C\Big(\|g_1\|_{H^{r-1}} + |\lambda|^{(r-1)/2}\|g_1\|_{L^2} + \|g_2\|_{H^{r-2}} + |\lambda|^{(r-2)/2}\|g_2\|_{L^2}\Big). \end{aligned} \tag{3.10}$$

For the proof we give separate estimates for q and h, and moreover first restrict to $r = 2$.

Estimates for q. From the first equation in (3.9) we obtain

$$h = \frac{-\partial_x q + g_1}{\lambda}, \tag{3.11}$$

and plugging this into the second equation yields

$$(\lambda + a_0)q + a_1\partial_x q + a_2\partial_x^2 q + a_3\partial_x^3 q + a_4\partial_x^4 q = g_2 + \frac{1}{\lambda}(a_{10} + a_{11}\partial_x + a_{12}\partial_x^2 + a_{13}\partial_x^3)g_1. \tag{3.12}$$

The γ-periodic coefficients are given by

$$a_0 = -a_{20}, \quad a_1 = -a_{21} + \frac{a_{10}}{\lambda}, \quad a_2 = -a_{22} + \frac{a_{11}}{\lambda}, \quad a_3 = \frac{a_{12}}{\lambda}, \quad a_4 = \frac{a_{13}}{\lambda}. \tag{3.13}$$

To solve (3.12) we define on $H^2 \times H^2$ the bilinear form

$$\begin{aligned} b(q,\varphi) := \int_{\mathbb{R}} \Big(&(\lambda + a_0)q\bar\varphi + a_1\partial_x q\,\bar\varphi + a_2\partial_x^2 q\,\bar\varphi - \partial_x a_3\,\partial_x^2 q\,\bar\varphi - a_3\partial_x^2 q\,\partial_x\bar\varphi \\ &+ \partial_x^2 a_4\,\partial_x^2 q\,\bar\varphi + 2\partial_x a_4\,\partial_x^2 q\,\partial_x\bar\varphi + a_4\partial_x^2 q\,\partial_x^2\bar\varphi\Big)\,\mathrm{d}x. \end{aligned}$$

Using integration by parts, $q \in H^2$ is a weak solution of (3.12) if and only if

$$b(q,\varphi) = \int_{\mathbb{R}} \Big(g_2 + \frac{1}{\lambda}(a_{10} + a_{11}\partial_x + a_{12}\partial_x^2 + a_{13}\partial_x^3)g_1\Big)\bar\varphi\,\mathrm{d}x \tag{3.14}$$

for all $\varphi \in H^2$. To prove the existence of a unique weak solution we apply the Lemma of Lax-Milgram. Therefore, we have to show that the bilinear form b is continuous and elliptic. Since all coefficients of b are in L^∞, the continuity is obvious. To verify the ellipticity of b we have to estimate $b(q,q)$, which reads

$$\begin{aligned} b(q,q) &= \int_{\mathbb{R}} (\lambda + a_0)|q|^2\,\mathrm{d}x + \int_{\mathbb{R}} a_1\partial_x q\,\bar q\,\mathrm{d}x - \int_{\mathbb{R}} a_2|\partial_x q|^2\,\mathrm{d}x - \int_{\mathbb{R}} \partial_x a_2\,\partial_x q\,\bar q\,\mathrm{d}x \\ &\quad - \int_{\mathbb{R}} \partial_x a_3\,\partial_x^2 q\,\bar q\,\mathrm{d}x - \int_{\mathbb{R}} a_3\partial_x^2 q\,\partial_x\bar q\,\mathrm{d}x + \int_{\mathbb{R}} \partial_x^2 a_4\,\partial_x^2 q\,\bar q\,\mathrm{d}x \\ &\quad + 2\int_{\mathbb{R}} \partial_x a_4\,\partial_x^2 q\,\partial_x\bar q\,\mathrm{d}x + \int_{\mathbb{R}} a_4|\partial_x^2 q|^2\,\mathrm{d}x \\ &= \int_{\mathbb{R}} (\lambda + a_0)|q|^2\,\mathrm{d}x - \int_{\mathbb{R}} a_2|\partial_x q|^2\,\mathrm{d}x + \int_{\mathbb{R}} a_4|\partial_x^2 q|^2\,\mathrm{d}x \\ &\quad + \int_{\mathbb{R}} (a_1 - \partial_x a_2)\,\partial_x q\,\bar q\,\mathrm{d}x + \int_{\mathbb{R}} \big(\partial_x^2 a_4 - \partial_x a_3\big)\,\partial_x^2 q\,\bar q\,\mathrm{d}x \\ &\quad + \int_{\mathbb{R}} (2\partial_x a_4 - a_3)\,\partial_x^2 q\,\partial_x\bar q\,\mathrm{d}x. \end{aligned} \tag{3.15}$$

We begin with estimating the real part of the first three integrals on the right-hand side of (3.15), related to the H^2-norm of q, and which for $\operatorname{Re}\lambda$ large enough also absorb the mixed terms. For $\operatorname{Re}\lambda \geq \|a_{20}\|_{L^\infty}$ we have

$$\operatorname{Re}\int_{\mathbb{R}} (\lambda + a_0)|q|^2 \, \mathrm{d}x \geq (\operatorname{Re}\lambda - \|a_{20}\|_{L^\infty})\|q\|_{L^2}^2. \tag{3.16}$$

Since $\operatorname{Re}(1/\lambda) = \operatorname{Re}\lambda/|\lambda|^2$ we obtain from (2.8) $-\operatorname{Re} a_2 \geq \frac{9}{2\mathrm{R}} - \frac{\operatorname{Re}\lambda}{|\lambda|^2}\|a_{11}\|_{L^\infty}$. For $\operatorname{Re}\lambda \geq (4\mathrm{R}/9)\|a_{11}\|_{L^\infty}$ the second integral in (3.15) can be estimated by

$$-\operatorname{Re}\int_{\mathbb{R}} a_2|\partial_x q|^2 \, \mathrm{d}x \geq \frac{9}{4\mathrm{R}}\|\partial_x q\|_{L^2}^2. \tag{3.17}$$

By (2.8) we obtain $a_4 = \frac{a_{13}}{\lambda} = \frac{5\mathrm{W}}{6\lambda}\frac{f_s}{1+\kappa f_s}$, and since κ is small we have $C_1 := \min_{x\in[0,\gamma]} a_{13}(x) > 0$, and therefore $\operatorname{Re} a_4 \geq C_1 \operatorname{Re}\lambda/|\lambda|^2$. Thus,

$$\operatorname{Re}\int_{\mathbb{R}} a_4|\partial_x^2 q|^2 \, \mathrm{d}x \geq C_1\frac{\operatorname{Re}\lambda}{|\lambda|^2}\|\partial_x^2 q\|_{L^2}^2. \tag{3.18}$$

Next, we estimate the mixed terms in (3.15) by applying Young's inequality $ab \leq \frac{\epsilon}{2}a^2 + \frac{1}{2\epsilon}b^2$ for $\epsilon > 0$. Looking at (3.16)-(3.18) we find that in case $|\lambda| \to \infty$ the inequalities for $\|q\|_{L^2}^2$, $\|\partial_x q\|_{L^2}^2$, and $\|\partial_x^2 q\|_{L^2}^2$ get worse the more derivatives we have. Thus, we have to choose ϵ with care such that the mixed terms can be absorbed by (3.16)-(3.18) without losing the positive coefficients. Therefore, we start with the integral containing the highest derivatives; i.e.,

$$\Big|\int_{\mathbb{R}} (2\partial_x a_4 - a_3)\, \partial_x^2 q\, \partial_x \bar{q} \, \mathrm{d}x\Big| \leq \frac{1}{2\epsilon}\int_{\mathbb{R}} |2\partial_x a_4 - a_3|^2|\partial_x^2 q|^2 \, \mathrm{d}x + \frac{\epsilon}{2}\int_{\mathbb{R}} |\partial_x q|^2 \, \mathrm{d}x.$$

Choosing $\epsilon = 9/(8\mathrm{R})$, we obtain

$$\frac{1}{2\epsilon}|2\partial_x a_4 - a_3|^2 \leq \frac{1}{|\lambda|^2}\frac{4\mathrm{R}}{9}\|2\partial_x a_{13} - a_{12}\|_{L^\infty}^2 =: C_2\frac{1}{|\lambda|^2},$$

and thus,

$$\begin{aligned}\Big|\int_{\mathbb{R}} (2\partial_x a_4 - a_3)\, \partial_x^2 q\, \partial_x \bar{q} \, \mathrm{d}x\Big| &\leq C_2\frac{1}{|\lambda|^2}\|\partial_x^2 q\|_{L^2}^2 + \frac{9}{16\mathrm{R}}\|\partial_x q\|_{L^2}^2 \\ &\leq C_1\frac{\operatorname{Re}\lambda}{4|\lambda|^2}\|\partial_x^2 q\|_{L^2}^2 + \frac{9}{16\mathrm{R}}\|\partial_x q\|_{L^2}^2\end{aligned}$$

for $\operatorname{Re}\lambda \geq 4C_2/C_1$. Analogously, for $\operatorname{Re}\lambda \geq \|\partial_x^2 a_{13} - \partial_x a_{12}\|_{L^\infty}^2/(C_1\|a_{20}\|_{L^\infty}) =: C_3/C_1$ we obtain

$$\Big|\int_{\mathbb{R}} \big(\partial_x^2 a_4 - \partial_x a_3\big)\, \partial_x^2 q\, \bar{q} \, \mathrm{d}x\Big| \leq C_1\frac{\operatorname{Re}\lambda}{4|\lambda|^2}\|\partial_x^2 q\|_{L^2}^2 + \|a_{20}\|_{L^\infty}\|q\|_{L^2}^2. \tag{3.19}$$

Finally, for the fourth integral in (3.15) we use the inequality

$$\begin{aligned}\|a_1 - \partial_x a_2\|_{L^\infty} &\leq \|-a_{21} + \partial_x a_{22}\|_{L^\infty} + \frac{1}{|\lambda|}\|a_{10} - \partial_x a_{11}\|_{L^\infty} \\ &\leq \|-a_{21} + \partial_x a_{22}\|_{L^\infty} + \frac{1}{\|a_{20}\|_{L^\infty}}\|a_{10} - \partial_x a_{11}\|_{L^\infty} =: \sqrt{\frac{9}{4\mathrm{R}}C_4}\end{aligned}$$

for $\operatorname{Re}\lambda \geq \|a_{20}\|_{L^\infty}$. Using again $\epsilon = 9/(8\mathrm{R})$ in Young's inequality, we obtain

$$\Big|\int_{\mathbb{R}} (a_1 - \partial_x a_2)\, \partial_x q\, \bar{q} \, \mathrm{d}x\Big| \leq \frac{1}{2\epsilon}\|a_1 - \partial_x a_2\|_{L^\infty}^2\|q\|_{L^2}^2 + \frac{\epsilon}{2}\|\partial_x q\|_{L^2}^2$$

$$\leq C_4\|q\|_{L^2}^2 + \frac{9}{16\mathrm{R}}\|\partial_x q\|_{L^2}^2.$$

Altogether, we have

$$\operatorname{Re} b(q,q) \geq (\operatorname{Re}\lambda - 2\|a_{20}\|_{L^\infty} - C_4)\|q\|_{L^2}^2 + \frac{9}{8\mathrm{R}}\|\partial_x q\|_{L^2}^2 + \frac{C_1 \operatorname{Re}\lambda}{2|\lambda|^2}\|\partial_x^2 q\|_{L^2}^2 \tag{3.20}$$

for all λ with

$$\operatorname{Re}\lambda \geq \max\big\{2\|a_{20}\|_{L^\infty} + C_4, \frac{4\mathrm{R}}{9}\|a_{11}\|_{L^\infty}, \frac{4C_2}{C_1}, \frac{C_3}{C_1}\big\}.$$

This shows the ellipticity of b; i.e., there exist $a, C > 0$ such that for all λ with $\operatorname{Re}\lambda \geq a$ we have

$$C \operatorname{Re} b(q,q) \geq (\operatorname{Re}\lambda - a/2)\|q\|_{L^2}^2 + \|\partial_x q\|_{L^2}^2 + \frac{\operatorname{Re}\lambda}{|\lambda|^2}\|\partial_x^2 q\|_{L^2}^2.$$

Thus, by the Lax-Milgram Lemma, there exists a unique weak solution $q \in H^2$ of (3.12) if $\operatorname{Re}\lambda \geq a$. Furthermore, from the weak formulation (3.14) we obtain the estimate

$$(\operatorname{Re}\lambda - a/2)\|q\|_{L^2}^2 + \|\partial_x q\|_{L^2}^2 + \frac{\operatorname{Re}\lambda}{|\lambda|^2}\|\partial_x^2 q\|_{L^2}^2 \leq C\|g_2\|_{L^2}\|q\|_{L^2} + C\frac{\operatorname{Re}\lambda}{|\lambda|^2}\|g_1\|_{H^1}\|q\|_{H^2}.$$

To estimate the H^2-norm of q, we can use $\operatorname{Re}\lambda - a/2 \geq a/2$. Thus, the coefficient in front of $\|q\|_{L^2}^2$ can be estimated from below independently of λ. However, the coefficient of $\|\partial_x^2 q\|_{L^2}^2$ converges to zero for $|\lambda| \to \infty$. Therefore, it is necessary to test the resolvent equation (3.12) not only with q itself, but also with $\partial_x^2 q$. However, since g_1 is only in H^1, on the right-hand side of the weak formulation (3.14) there occurs the integral $\int_{\mathbb{R}} \partial_x g_1\, \partial_x^4 \bar{q}\, \mathrm{d}x$, for instance. This can only be estimated with the help of $\|q\|_{H^4}$, which is not helpful for estimating $\|q\|_{H^2}$. Therefore, we split q into $q = q_0 + \tilde{q}$, where the two components are supposed to fulfill

$$(\lambda + a_0)q_0 + (a_1\partial_x + a_2\partial_x^2 + a_3\partial_x^3 + a_4\partial_x^4)q_0 = \frac{1}{\lambda}(a_{10} + a_{11}\partial_x + a_{12}\partial_x^2 + a_{13}\partial_x^3)g_1, \tag{3.21}$$

$$(\lambda + a_0)\tilde{q} + (a_1\partial_x + a_2\partial_x^2 + a_3\partial_x^3 + a_4\partial_x^4)\tilde{q} = g_2. \tag{3.22}$$

Since the right-hand side of (3.21) has a leading factor $1/\lambda$, it is sufficient to test with q_0. In (3.22), the right-hand side is in L^2, thus it can be tested with $\partial_x^2\tilde{q}$, which leads to an estimate of $\|\tilde{q}\|_{H^2}$ independent of λ.

We begin with estimating q_0. By the considerations above we find a unique weak solution q_0 of (3.21) with

$$(\operatorname{Re}\lambda - a/2)\|q_0\|_{L^2}^2 + \|\partial_x q_0\|_{L^2}^2 + \frac{\operatorname{Re}\lambda}{|\lambda|^2}\|\partial_x^2 q_0\|_{L^2}^2 \leq C\frac{\operatorname{Re}\lambda}{|\lambda|^2}\|g_1\|_{H^1}\|q_0\|_{H^2}$$

for $\operatorname{Re}\lambda \geq a$. Since $(\operatorname{Re}\lambda - a/2)/\operatorname{Re}\lambda \geq 1/2$ and since $|\lambda|^2/\operatorname{Re}\lambda \geq |\lambda|$, we obtain

$$|\lambda|^2\|q_0\|_{L^2}^2 + |\lambda|\|\partial_x q_0\|_{L^2}^2 + \|\partial_x^2 q_0\|_{L^2}^2 \leq C\|g_1\|_{H^1}\|q_0\|_{H^2}. \tag{3.23}$$

As $|\lambda| \geq a$ is bounded from below, it follows that

$$\|q_0\|_{H^2} \leq C\|g_1\|_{H^1}. \tag{3.24}$$

In particular, together with (3.23) this leads to $|\lambda|^2\|q_0\|_{L^2}^2 \leq C\|g_1\|_{H^1}\|q_0\|_{H^2} \leq C\|g_1\|_{H^1}^2$; i.e.,

$$|\lambda|\|q_0\|_{L^2} \leq C\|g_1\|_{H^1}. \tag{3.25}$$

Next, we look for the corresponding estimates for $\tilde q$. Exactly as for q_0, by testing (3.22) with $\tilde q$ and taking the real part we obtain

$$(\operatorname{Re}\lambda - a/2)\|\tilde q\|_{L^2}^2 + \|\partial_x \tilde q\|_{L^2}^2 + \frac{\operatorname{Re}\lambda}{|\lambda|^2}\|\partial_x^2\tilde q\|_{L^2}^2 \le C\|g_2\|_{L^2}\|\tilde q\|_{L^2} \tag{3.26}$$

for $\operatorname{Re}\lambda \ge a$. Testing (3.22) with $\partial_x^2\tilde q$; i.e., $b(\tilde q, \partial_x^2\tilde q) = \int_{\mathbb{R}} g_2\, \partial_x^2\bar{\tilde q}\,\mathrm{d}x$, and integration by parts leads to

$$\begin{aligned} &- b(\tilde q, \partial_x^2\tilde q)\\ &= \int_{\mathbb{R}} (\lambda + a_0)|\partial_x\tilde q|^2\,\mathrm{d}x + \int_{\mathbb{R}}(-a_2+\partial_x a_3 - \partial_x^2 a_4)|\partial_x^2\tilde q|^2\,\mathrm{d}x + \int_{\mathbb{R}} a_4|\partial_x^3\tilde q|^2\,\mathrm{d}x\\ &\quad + \int_{\mathbb{R}} \partial_x a_0\,\tilde q\,\partial_x\bar{\tilde q}\,\mathrm{d}x - \int_{\mathbb{R}} a_1\partial_x\tilde q\,\partial_x^2\bar{\tilde q}\,\mathrm{d}x + \int_{\mathbb{R}}(a_3 - \partial_x a_4)\partial_x^2\tilde q\,\partial_x^3\bar{\tilde q}\,\mathrm{d}x. \end{aligned} \tag{3.27}$$

Since both a_3 and a_4 have a leading $1/\lambda$ and since $-a_2 > 0$ is bounded from below, (3.27) can be estimated similarly to (3.15) by applying Young's inequality. An exception is the integral $\int_{\mathbb{R}} \partial_x a_0\,\tilde q\,\partial_x\bar{\tilde q}\,\mathrm{d}x$, which cannot be absorbed by the first three integrals, and therefore

$$\begin{aligned} &(\operatorname{Re}\lambda - a/2)\|\partial_x\tilde q\|_{L^2}^2 + \|\partial_x^2\tilde q\|_{L^2}^2 + \frac{\operatorname{Re}\lambda}{|\lambda|^2}\|\partial_x^3\tilde q\|_{L^2}^2\\ &\le C\|g_2\|_{L^2}\|\partial_x^2\tilde q\|_{L^2} + C\int_{\mathbb{R}}(|\tilde q|^2 + |\partial_x\tilde q|^2)\,\mathrm{d}x \le C\|g_2\|_{L^2}\|\partial_x^2\tilde q\|_{L^2} + C\|g_2\|_{L^2}\|\tilde q\|_{L^2}. \end{aligned} \tag{3.28}$$

Here, we used (3.26) in the second estimate. Combining (3.26) and (3.28) yields the resolvent estimate

$$\|\tilde q\|_{H^2} \le C\|g_2\|_{L^2} \tag{3.29}$$

for $\operatorname{Re}\lambda \ge a$.

Remark 3.22. To test (3.22) with $\partial_x^2\tilde q$ we actually have to test with smooth functions which are dense in H^2 and then extend the resulting resolvent estimates continuously to the respective Sobolev spaces.

Finally, to estimate $|\lambda|\|\tilde q\|_{L^2}$, we also have to estimate the imaginary part of $b(\tilde q, \tilde q)$. Using $\operatorname{Im}(1/\lambda) = -\operatorname{Im}\lambda/|\lambda|^2$ we obtain from (3.15) and (3.22),

$$\begin{aligned} &\Big|\int_{\mathbb{R}} g_2\bar{\tilde q}\,\mathrm{d}x\Big| \ge \operatorname{Im} b(\tilde q, \tilde q)\\ &\ge (\operatorname{Im}\lambda - \|a_{20}\|_{L^\infty})\|\tilde q\|_{L^2}^2 + \frac{\operatorname{Im}\lambda}{|\lambda|^2}\int_{\mathbb{R}} a_{11}|\partial_x\tilde q|^2\,\mathrm{d}x - \frac{\operatorname{Im}\lambda}{|\lambda|^2}\int_{\mathbb{R}} a_{13}|\partial_x^2\tilde q|^2\,\mathrm{d}x\\ &\quad + \operatorname{Im}\int_{\mathbb{R}}(a_1 - \partial_x a_2)\,\partial_x\tilde q\,\bar{\tilde q}\,\mathrm{d}x + \operatorname{Im}\int_{\mathbb{R}}\frac{1}{\lambda}\left(\partial_x^2 a_{13} - \partial_x a_{12}\right)\partial_x^2\tilde q\,\bar{\tilde q}\,\mathrm{d}x\\ &\quad + \operatorname{Im}\int_{\mathbb{R}}\frac{1}{\lambda}\,(2\partial_x a_{13} - a_{12})\,\partial_x^2\tilde q\,\partial_x\bar{\tilde q}\,\mathrm{d}x. \end{aligned} \tag{3.30}$$

This estimate is less powerful than the one for the real part, since a_{11} and a_{13} have an undefined or even the wrong sign. However, since the coefficients a_{11}, a_{12}, and a_{13} have a leading $\operatorname{Im}(1/\lambda)$, it allows for $\operatorname{Im}\lambda \ge a$ the inequality

$$\begin{aligned} &(\operatorname{Im}\lambda - a/2)\|\tilde q\|_{L^2}^2\\ &\le C\frac{1}{|\lambda|}(\|\partial_x\tilde q\|_{L^2}^2 + \|\partial_x^2\tilde q\|_{L^2}^2) + \|a_1 - \partial_x a_2\|_{L^\infty}\int_{\mathbb{R}}|\partial_x\tilde q||\tilde q|\,\mathrm{d}x + \Big|\int_{\mathbb{R}} g_2\bar{\tilde q}\,\mathrm{d}x\Big| \end{aligned}$$

$$\begin{aligned}&\le C\frac{1}{|\lambda|}(\|\partial_x\tilde q\|_{L^2}^2+\|\partial_x^2\tilde q\|_{L^2}^2)+\frac{\operatorname{Im}\lambda}{8}\int_{\mathbb R}|\tilde q|^2\,\mathrm dx+\frac{2}{\operatorname{Im}\lambda}\|a_1-\partial_x a_2\|_{L^\infty}^2\int_{\mathbb R}|\partial_x\tilde q|^2\,\mathrm dx\\&\quad+\frac{\operatorname{Im}\lambda}{8}\int_{\mathbb R}|\tilde q|^2\,\mathrm dx+\frac{2}{\operatorname{Im}\lambda}\int_{\mathbb R}|g_2|^2\,\mathrm dx\\&\le C\frac{1}{\operatorname{Im}\lambda}(\|\partial_x\tilde q\|_{L^2}^2+\|\partial_x^2\tilde q\|_{L^2}^2)+\frac{\operatorname{Im}\lambda}{4}\|\tilde q\|_{L^2}^2+\frac{2}{\operatorname{Im}\lambda}\|g_2\|_{L^2}^2,\end{aligned}$$

where we used Young's inequality twice. Thus,

$$\big(\frac34\operatorname{Im}\lambda-a/2\big)\|\tilde q\|_{L^2}^2\le C\frac{1}{\operatorname{Im}\lambda}(\|\partial_x\tilde q\|_{L^2}^2+\|\partial_x^2\tilde q\|_{L^2}^2)+\frac{2}{\operatorname{Im}\lambda}\|g_2\|_{L^2}^2.$$

Since $(3/4)\operatorname{Im}\lambda=((3/4)\operatorname{Im}\lambda-a/2)+a/2\le((3/4)\operatorname{Im}\lambda-a/2)+\operatorname{Im}\lambda/2$, we have $\operatorname{Im}\lambda\le 4((3/4)\operatorname{Im}\lambda-a/2)$, and as we have already estimated $\|\tilde q\|_{H^2}$ in (3.29), we obtain

$$\operatorname{Im}\lambda\,\|\tilde q\|_{L^2}^2\le C\frac{1}{\operatorname{Im}\lambda}(\|\partial_x\tilde q\|_{L^2}^2+\|\partial_x^2\tilde q\|_{L^2}^2+\|g_2\|_{L^2}^2)\le C\frac{1}{\operatorname{Im}\lambda}\|g_2\|_{L^2}^2.$$

Considering $-\operatorname{Im}b(\tilde q,\tilde q)$ in (3.30) gives the same estimate for $-\operatorname{Im}\lambda$ instead of $\operatorname{Im}\lambda$. Thus, for $|\operatorname{Im}\lambda|>a$ we have

$$|\operatorname{Im}\lambda|\,\|\tilde q\|_{L^2}\le C\|g_2\|_{L^2}.$$

By (3.26), the same estimate is true if we replace $|\operatorname{Im}\lambda|$ by $\operatorname{Re}\lambda$, since $\operatorname{Re}\lambda\le(\operatorname{Re}\lambda-a/2)+\operatorname{Re}\lambda/2$, which implies $\operatorname{Re}\lambda\le 2(\operatorname{Re}\lambda-a/2)$. Altogether, for $\operatorname{Re}\lambda>a$ we obtain

$$|\lambda|\|\tilde q\|_{L^2}\le C\|g_2\|_{L^2}. \tag{3.31}$$

Combining (3.24), (3.25), (3.29), and (3.31) yields for $q=q_0+\tilde q$ the resolvent estimate

$$|\lambda|\|q\|_{L^2}+\|q\|_{H^2}\le C\left(\|g_1\|_{H^1}+\|g_2\|_{L^2}\right). \tag{3.32}$$

Estimates for h. It remains to estimate the L^2- and the H^3-norm of h. Identity (3.11) leads to

$$\|h\|_{L^2}\le\frac{1}{|\lambda|}\left(\|\partial_x q\|_{L^2}+\|g_1\|_{L^2}\right),\quad\|\partial_x h\|_{L^2}\le\frac{1}{|\lambda|}\left(\|\partial_x^2 q\|_{L^2}+\|\partial_x g_1\|_{L^2}\right), \tag{3.33}$$

thus, by applying (3.32) we obtain

$$\|h\|_{L^2}+\|\partial_x h\|_{L^2}\le C\left(\|g_1\|_{H^1}+\|g_2\|_{L^2}\right). \tag{3.34}$$

From the second equation in the resolvent equation (3.9) we obtain

$$\begin{aligned}&a_{12}\partial_x^2h+a_{13}\partial_x^3h\\&=-g_2+(\lambda-a_{20})q-a_{21}\partial_xq-a_{22}\partial_x^2q-a_{10}h-a_{11}\partial_xh\\&=-g_2+(\lambda-a_{20})q-a_{21}\partial_xq-a_{22}\partial_x^2q+\frac1\lambda a_{10}(\partial_xq-g_1)+\frac1\lambda a_{11}(\partial_x^2q-\partial_xg_1).\end{aligned} \tag{3.35}$$

Since all coefficients a_{ij} are real, testing (3.35) with $(-\partial_x^2 h + \partial_x^3 h)$ yields for the left-hand side

$$\begin{aligned}
&\operatorname{Re}\int_{\mathbb{R}} (a_{12}\partial_x^2 h + a_{13}\partial_x^3 h)(-\partial_x^2\bar{h} + \partial_x^3\bar{h})\,\mathrm{d}x \\
&= -\int_{\mathbb{R}} a_{12}|\partial_x^2 h|^2\,\mathrm{d}x + \int_{\mathbb{R}} a_{13}|\partial_x^3 h|^2\,\mathrm{d}x + \operatorname{Re}\int_{\mathbb{R}} a_{12}\partial_x^2 h\,\partial_x^3\bar{h}\,\mathrm{d}x \\
&\quad - \operatorname{Re}\int_{\mathbb{R}} a_{13}\partial_x^3 h\,\partial_x^2\bar{h}\,\mathrm{d}x \\
&= -\int_{\mathbb{R}} a_{12}|\partial_x^2 h|^2\,\mathrm{d}x + \int_{\mathbb{R}} a_{13}|\partial_x^3 h|^2\,\mathrm{d}x + \operatorname{Re}\int_{\mathbb{R}} (a_{12} - a_{13})\partial_x^2 h\,\partial_x^3\bar{h}\,\mathrm{d}x.
\end{aligned} \tag{3.36}$$

Integration by parts leads to

$$\int_{\mathbb{R}} (a_{12} - a_{13})\partial_x^2 h\,\partial_x^3\bar{h}\,\mathrm{d}x = -\int_{\mathbb{R}} (\partial_x a_{12} - \partial_x a_{13})|\partial_x^2 h|^2\,\mathrm{d}x - \int_{\mathbb{R}} (a_{12} - a_{13})\partial_x^3 h\,\partial_x^2\bar{h}\,\mathrm{d}x,$$

and thus,

$$\operatorname{Re}\int_{\mathbb{R}} (a_{12} - a_{13})\partial_x^2 h\,\partial_x^3\bar{h}\,\mathrm{d}x = -\frac{1}{2}\int_{\mathbb{R}} (\partial_x a_{12} - \partial_x a_{13})|\partial_x^2 h|^2\,\mathrm{d}x.$$

Due to the definitions of a_{12}, a_{13} in (2.17), there exist positive constants $C_1, \tilde{C}_1$ with $\min_{x\in[0,\gamma]} a_{13} = C_1$, $\min_{x\in[0,\gamma]}(-a_{12}) = 2\tilde{C}_1$. Hence, (3.36) can be estimated by

$$\begin{aligned}
&\operatorname{Re}\int_{\mathbb{R}} (a_{12}\partial_x^2 h + a_{13}\partial_x^3 h)(-\partial_x^2\bar{h} + \partial_x^3\bar{h})\,\mathrm{d}x \\
&\geq 2\tilde{C}_1\|\partial_x^2 h\|_{L^2}^2 + C_1\|\partial_x^3 h\|_{L^2}^2 - \frac{1}{2}\left(\|\partial_x a_{12}\|_{L^\infty} + \|\partial_x a_{13}\|_{L^\infty}\right)\|\partial_x^2 h\|_{L^2}^2.
\end{aligned} \tag{3.37}$$

On closer inspection of (2.17) we find that due to the additional x-derivative, the coefficients $\partial_x a_{12}, \partial_x a_{13}$ are of the order of $\mathcal{O}(\varepsilon)$, where ε is proportional to the bottom waviness. Hence, without loss of generality we may assume that $\frac{1}{2}\left(\|\partial_x a_{12}\|_{L^\infty} + \|\partial_x a_{13}\|_{L^\infty}\right) < \tilde{C}_1$. Then (3.37) reads

$$\operatorname{Re}\int_{\mathbb{R}} (a_{12}\partial_x^2 h + a_{13}\partial_x^3 h)(-\partial_x^2\bar{h} + \partial_x^3\bar{h})\,\mathrm{d}x \geq \tilde{C}_1\|\partial_x^2 h\|_{L^2}^2 + C_1\|\partial_x^3 h\|_{L^2}^2. \tag{3.38}$$

Thus, testing (3.35) with $(-\partial_x^2 h + \partial_x^3 h)$ yields

$$\begin{aligned}
&\tilde{C}_1\|\partial_x^2 h\|_{L^2}^2 + C_1\|\partial_x^3 h\|_{L^2}^2 \\
&\leq \Big|\int_{\mathbb{R}} \Big(-g_2 - \frac{1}{\lambda}a_{10}g_1 - \frac{1}{\lambda}a_{11}\partial_x g_1 + (\lambda - a_{20})q + \big(\frac{1}{\lambda}a_{10} - a_{21}\big)\partial_x q \\
&\quad + \big(\frac{1}{\lambda}a_{11} - a_{22}\big)\partial_x^2 q\Big)(-\partial_x^2\bar{h} + \partial_x^3\bar{h})\,\mathrm{d}x\Big| \\
&\leq C\left(\|g_2\|_{L^2} + \|g_1\|_{H^1} + |\lambda|\|q\|_{L^2} + \|q\|_{H^2}\right)\left(\|\partial_x^2 h\|_{L^2} + \|\partial_x^3 h\|_{L^2}\right),
\end{aligned}$$

together with (3.32) we obtain $\|\partial_x^2 h\|_{L^2} + \|\partial_x^3 h\|_{L^2} \leq C\left(\|g_1\|_{H^1} + \|g_2\|_{L^2}\right)$. Combination with (3.34) yields

$$\|h\|_{H^3} \leq C\left(\|g_1\|_{H^1} + \|g_2\|_{L^2}\right), \tag{3.39}$$

and (3.33) finally implies

$$|\lambda|^{3/2}\|h\|_{L^2} \leq C|\lambda|^{1/2}(\|\partial_x q\|_{L^2} + \|g_1\|_{L^2}). \tag{3.40}$$

Since integration by parts gives

$$|\lambda|\|\partial_x q\|_{L^2}^2 = -|\lambda| \int_{\mathbb{R}} q\, \partial_x^2 \bar{q}\, \mathrm{d}x \le \frac{1}{2}|\lambda|^2\|q\|_{L^2}^2 + \frac{1}{2}\|\partial_x^2 q\|_{L^2}^2,$$

we have $|\lambda|^{1/2}\|\partial_x q\|_{L^2} \le C(|\lambda|\|q\|_{L^2} + \|\partial_x^2 q\|_{L^2})$, and (3.40) reads

$$|\lambda|^{3/2}\|h\|_{L^2} \le C\left(|\lambda|^{1/2}\|g_1\|_{L^2} + \|g_1\|_{H^1} + \|g_2\|_{L^2}\right). \tag{3.41}$$

This proves Theorem 3.21 for $r = 2$. For $r \ge 3$, $r \in \mathbb{N}$, the proof works the same way as above by testing with the respective derivatives of h, q. For non-integer values of r, the resolvent estimate follows by interpolation theory, see [13], for instance.

Analytic semigroup. With a few additional expenses the proof of Theorem 3.21 allows to show that the linear operator A is sectorial; i.e., there exists a $\vartheta \in (0, \pi/2)$ such that a slightly modified resolvent estimate can be extended to the sector $S_{a,\vartheta} := \{\lambda \mid 0 \le |\arg(a - \lambda)| \le \vartheta + \pi/2\}$ covering the half-plane $\operatorname{Re}\lambda \ge a$. Setting $X := \mathcal{H}^1 = H^1 \times L^2$, the domain of A is $\mathcal{D}(A) = \mathcal{H}^3 = H^3 \times H^2 \subset X$. Moreover, let $g \in \mathcal{D}(A)$ and $\operatorname{Re}\lambda' \ge a$. By Theorem 3.21 there exists a unique solution of the resolvent equation $(\lambda' - A)u = g$, and from (3.32), (3.33) it follows the estimate

$$\|(\lambda' - A)^{-1} g\|_X \le \frac{M}{|\lambda'|}\|g\|_X \tag{3.42}$$

for a $M > 0$ independent of λ' and g. It remains to extend this estimate to the sector $S_{a,\vartheta}$ by a perturbation argument, which we recall for completeness in the following. Let $\lambda \in S_{a,\vartheta}$ with $\operatorname{Re}\lambda < a$, where ϑ is specified later. Setting $\lambda' := a + \mathrm{i}(\operatorname{Im}\lambda)$ yields

$$\lambda - A = \lambda' - A + \lambda - \lambda' = (\lambda' - A)\left(\operatorname{Id} + (\lambda' - A)^{-1}(\lambda - \lambda')\right). \tag{3.43}$$

Choosing $\vartheta \in (0, \pi/2)$ small enough we can always ensure that

$$\frac{|\lambda - \lambda'|}{|\lambda'|} \le \frac{|\lambda - \lambda'|}{|\operatorname{Im}\lambda|} \le \tan\vartheta < \frac{1}{M}.$$

Hence,

$$\|(\lambda' - A)^{-1}(\lambda - \lambda')\|_{\mathscr{L}(X,X)} \le \frac{M}{|\lambda'|}|\lambda - \lambda'| < 1,$$

and the Neumann series

$$\left(\operatorname{Id} + (\lambda' - A)^{-1}(\lambda - \lambda')\right)^{-1} = \sum_{j=0}^{\infty}\left(-(\lambda' - A)^{-1}(\lambda - \lambda')\right)^j$$

converges in $\mathscr{L}(X, X)$. By (3.43), there exists the inverse

$$(\lambda - A)^{-1} = \left(\operatorname{Id} + (\lambda' - A)^{-1}(\lambda - \lambda')\right)^{-1}(\lambda' - A)^{-1} \tag{3.44}$$

with

$$\|(\lambda - A)^{-1} g\|_X \le C\|(\lambda' - A)^{-1} g\|_X \le C\frac{M}{|\lambda'|}\|g\|_X \le C\frac{M}{|\lambda|}\|g\|_X. \tag{3.45}$$

Lemma 3.23. *Let $X = H^1 \times L^2$. Then the operator $A : \mathcal{D}(A) \to X$ from (2.17) is sectorial; i.e., there exist $M, a > 0$ and $\vartheta \in (0, \pi/2)$ such that the sector $S_{a,\vartheta} = \{\lambda \mid 0 \le |\arg(a - \lambda)| \le \vartheta + \pi/2\}$ is part of the resolvent set and*

$$\|(\lambda - A)^{-1} g\|_X \le \frac{M}{|\lambda|}\|g\|_X$$

for all $\lambda \in S_{a,\vartheta}$ and $g \in X$.

Thus, the linear operator A generates an analytic semigroup, see [11], for instance.

Weighted Sobolev spaces. Since we will need some decay rate in x, which corresponds to some regularity with respect to the wave number in Bloch space, we transfer the result of Theorem 3.21 to the case of weighted Sobolev spaces.

Theorem 3.24 (Resolvent estimate in weighted Sobolev spaces)**.** *Theorem 3.21 also holds for the weighted spaces $H^r(2)$; i.e.,* (3.10) *becomes*

$$\begin{aligned} &\|h\|_{H^{r+1}(2)} + |\lambda|^{(r+1)/2}\|h\|_{L^2(2)} + \|q\|_{H^r(2)} + |\lambda|^{r/2}\|q\|_{L^2(2)} \\ &\le C\Big(\|g_1\|_{H^{r-1}(2)} + |\lambda|^{(r-1)/2}\|g_1\|_{L^2(2)} + \|g_2\|_{H^{r-2}(2)} + |\lambda|^{(r-2)/2}\|g_2\|_{L^2(2)}\Big). \end{aligned}$$

Proof. In contrast to the proof of (3.10), we have to multiply the test functions by $\varrho(x) = (1+x^2)^{1/2}$ before testing. Differentiating the weight leads to additional terms in the estimates of the weak formulation. However, since derivatives of ϱ are of lower order, the additional mixed terms can be controlled by the terms in which ϱ occurs without a derivative if we choose a and C larger than in Theorem 3.21. Details for a related problem can be found in [21, Appendix A.2], for instance. □

3.4. **Maximal regularity.** With the resolvent estimate from Theorem 3.24 we are now able to prove Theorem 3.20 concerning local existence. For this purpose we fix some times $t_0 < t_1$ and denote again by A the linear operator from (2.17). Furthermore, let $r > 2$ be not an integer such that both $(r+1)/2$ and $r/2$ are not half integers in order to apply Theorem 3.14 and use Laplace transform in time.

The linear inhomogeneous problem. We begin with the linear inhomogeneous equation

$$Mu := (\partial_t - A)u = g, \quad u\big|_{t=t_0} = 0, \tag{3.46}$$

where $g \in \mathcal{K}_0^{r-1}((t_0,t_1),2) := K_0^{r-1}((t_0,t_1),2) \times K_0^{r-2}((t_0,t_1),2)$. Due to Lemma 3.2 we can identify g with its extension to $[t_0,\infty)$. Thus, without loss of generality, we can write $g \in \mathcal{K}_0^{r-1}((t_0,\infty),2)$. For a $\sigma_1 > 0$ chosen below we set

$$U(t,x) := \mathrm{e}^{-\sigma_1 t}u(t+t_0,x), \quad G(t,x) := \mathrm{e}^{-\sigma_1 t}g(t+t_0,x). \tag{3.47}$$

Then $G \in \mathcal{K}_0^{r-1}((0,\infty),2)$, and (3.46) is equivalent to solving

$$(\partial_t + \sigma_1 - A)U = G, \quad U\big|_{t=0} = 0. \tag{3.48}$$

Since $U\big|_{t=0} = 0$, the Laplace transform of U satisfies

$$\mathcal{L}(\partial_t v)(\tau) = \frac{1}{2\pi}\int_0^\infty \partial_t v(t)\mathrm{e}^{-t\tau}\,\mathrm{d}t = \tau\mathcal{L}v(\tau),$$

and (3.48) becomes

$$(\tau + \sigma_1 - A)\mathcal{L}U(\tau,x) = \mathcal{L}G(\tau,x). \tag{3.49}$$

From Lemma 3.7 and Remark 3.19 it follows that $\mathcal{L}G \in H^{r-1}(2) \times H^{r-2}(2)$ for almost every τ with $\operatorname{Re}\tau \ge 0$. Thus, according to Theorem 3.24, for almost every τ with $\operatorname{Re}\tau + \sigma_1 \ge a$ there exists a unique solution of the resolvent equation (3.49).

Choosing $\sigma_1 \geq a$ and setting $U = (U_1, U_2)^\top$, $G = (G_1, G_2)^\top$, we achieve the estimate

$$\begin{aligned}
&\|\mathcal{L}U_1\|_{H^{r+1}(2)} + |\tau|^{(r+1)/2}\|\mathcal{L}U_1\|_{L^2(2)} + \|\mathcal{L}U_2\|_{H^r(2)} + |\tau|^{r/2}\|\mathcal{L}U_2\|_{L^2(2)} \\
&\leq C\Big(\|\mathcal{L}G_1\|_{H^{r-1}(2)} + |\tau|^{(r-1)/2}\|\mathcal{L}G_1\|_{L^2(2)} \\
&\quad + \|\mathcal{L}G_2\|_{H^{r-2}(2)} + |\tau|^{(r-2)/2}\|\mathcal{L}G_2\|_{L^2(2)}\Big)
\end{aligned} \tag{3.50}$$

for almost every τ with $\operatorname{Re}\tau \geq 0$. To apply Theorem 3.14, which yields $U \in \mathcal{K}_0^{r+1}((0,\infty), 2)$, we additionally have to show for $j \in \{1, 2\}$ that

(i) $\tau \mapsto \mathcal{L}U_j(\tau, x)$ is holomorphic in the half-plane $\operatorname{Re}\tau > 0$ for almost every $x \in \mathbb{R}$,

(ii) $\sup_{\tau_1>0} \int_{\mathbb{R}} |\mathcal{L}U_j(\tau_1 + \mathrm{i}\tau_2, x)|^2 \,\mathrm{d}\tau_2 < \infty$ for almost every $x \in \mathbb{R}$.

Property (ii) immediately follows from the corresponding estimate for $\mathcal{L}G$; i.e.,

$$\begin{aligned}
&\sup_{\tau_1>0} \int_{\mathbb{R}} \big(|\mathcal{L}U_1(\tau_1 + \mathrm{i}\tau_2, x)|^2 + |\mathcal{L}U_2(\tau_1 + \mathrm{i}\tau_2, x)|^2\big) \,\mathrm{d}\tau_2 \\
&\leq C \sup_{\tau_1>0} \int_{\mathbb{R}} \Big(\|\mathcal{L}U_1(\tau_1 + \mathrm{i}\tau_2, \cdot)\|^2_{H^2(2)} + \|\mathcal{L}U_2(\tau_1 + \mathrm{i}\tau_2, \cdot)\|^2_{H^2(2)}\Big) \,\mathrm{d}\tau_2 \\
&\leq C \sup_{\tau_1>0} \int_{\mathbb{R}} \Big(\|\mathcal{L}G_1(\tau_1 + \mathrm{i}\tau_2, \cdot)\|^2_{L^2(2)} + \|\mathcal{L}G_2(\tau_1 + \mathrm{i}\tau_2, \cdot)\|^2_{L^2(2)}\Big) \,\mathrm{d}\tau_2 < \infty
\end{aligned}$$

due to Lemma 3.7. In order to show that $\mathcal{L}U$ is holomorphic, we set $\tau = \tau_r + \mathrm{i}\tau_i$ and $\mathcal{L}U = \mathcal{U}_r + \mathrm{i}\mathcal{U}_i$, thus the resolvent equation (3.49) reads

$$(\tau_r + \sigma_1 + \mathrm{i}\tau_i - A)(\mathcal{U}_r + \mathrm{i}\mathcal{U}_i) = \mathcal{L}G.$$

Differentiating with respect to τ_r and τ_i and using on the right-hand side that $\mathcal{L}G$ is holomorphic, we obtain

$$(\tau_r + \sigma_1 + \mathrm{i}\tau_i - A)\left[\partial_{\tau_r}\mathcal{U}_r - \partial_{\tau_i}\mathcal{U}_i + \mathrm{i}\left(\partial_{\tau_r}\mathcal{U}_i + \partial_{\tau_i}\mathcal{U}_r\right)\right] = 0 \tag{3.51}$$

for $\tau_r > 0$. Due to Theorem 3.24, there exists a unique solution of (3.51), given by

$$\partial_{\tau_r}\mathcal{U}_r - \partial_{\tau_i}\mathcal{U}_i + \mathrm{i}\left(\partial_{\tau_r}\mathcal{U}_i + \partial_{\tau_i}\mathcal{U}_r\right) = 0. \tag{3.52}$$

Thus, $\mathcal{L}U$ fulfills the Cauchy-Riemann differential equations for $\operatorname{Re}\tau > 0$. Transferring the results back to u, g proves the following lemma.

Lemma 3.25. *Let $r > 2$ be not an integer, and fix some $t_1 > t_0$. Then there exists a $C > 0$ such that for $g \in \mathcal{K}_0^{r-1}((t_0, t_1), 2)$ there exists a unique solution $u \in \mathcal{K}_0^{r+1}((t_0, t_1), 2)$ of*

$$Mu = (\partial_t - A)u = g, \quad u\big|_{t=t_0} = 0$$

with $\|u\|_{\mathcal{K}_0^{r+1}((t_0,t_1),2)} \leq C\|g\|_{\mathcal{K}_0^{r-1}((t_0,t_1),2)}$.

The nonlinear problem. To prove Theorem 3.20 we look for a solution $u = (h, q)^\top \in \mathcal{K}^{r+1}((t_0, t_1), 2)$ of the nonlinear problem (2.18); i.e.,

$$\partial_t \begin{pmatrix} h \\ q \end{pmatrix} = A \begin{pmatrix} h \\ q \end{pmatrix} + N(h, q). \tag{3.53}$$

As initial condition we take

$$u\big|_{t=t_0} = u_0 = (h_0, q_0)^\top \in \mathcal{H}^r(2) = H^r(2) \times H^{r-1}(2). \tag{3.54}$$

Since the nonlinearity N contains a third derivative of h, we restrict our calculations to the case $r \geq 3$. According to Lemma 3.11 there exists a function $v \in \mathcal{K}^{r+1}((t_0,t_1),2)$ with $v\big|_{t=t_0} = u_0$. Setting $u = v + w$ the initial value problem (3.53), (3.54) is equivalent to $\partial_t v + \partial_t w = Av + Aw + N(v+w)$ satisfied by $w \in \mathcal{K}^{r+1}((t_0,t_1),2)$ with homogeneous initial condition; i.e.,

$$Mw = N(v+w) - Mv, \quad w\big|_{t=t_0} = 0. \tag{3.55}$$

In the next step we assume that $w \in \mathcal{K}_0^{r+1}((t_0,t_1),2)$. To invert the operator M on the left-hand side of (3.55), we have to show that $N(v+w) - Mv \in \mathcal{K}_0^{r-1}((t_0,t_1),2)$. As Lemma 3.25 does not work for integers, we take $r > 3$ in the following. The highest derivatives occurring in the nonlinearity $N(v+w)$ are $\partial_x^3 h$ and $\partial_x q$. Since $v, w \in \mathcal{K}^{r+1}((t_0,t_1),2) = K^{r+1}((t_0,t_1),2) \times K^r((t_0,t_1),2)$, Lemma 3.15 yields $\partial_x^3 h \in K^{r-2}((t_0,t_1),2)$ and $\partial_x q \in K^{r-1}((t_0,t_1),2)$. Thus, we can apply Lemma 3.16 which gives $N(v+w) \in \mathcal{K}^{r-1}((t_0,t_1),2)$. Due to Lemma 3.15 the same is true for Mv, hence

$$N(v+w) - Mv \in \mathcal{K}^{r-1}((t_0,t_1),2). \tag{3.56}$$

According to Definition 3.13 it remains to show that

$$\partial_t^j \left(N(v+w) - Mv \right) \big|_{t=t_0} = 0 \quad \text{for all } j < (r-2)/2. \tag{3.57}$$

Restricting the regularity to $3 < r < 4$, we have to check (3.57) only for $j = 0$. Since $v \in K^{r+1}((t_0,t_1),2) \times K^r((t_0,t_1),2)$, Lemma 3.11 additionally allows to choose $\partial_t^j v\big|_{t=t_0}$ for $j < (r-1)/2$ arbitrarily. Hence we set $\partial_t v\big|_{t=t_0} = Au_0 + N(u_0)$, which yields $\left(N(v+w) - Mv \right)\big|_{t=t_0} = N(u_0) - \partial_t v\big|_{t=t_0} + Au_0 = 0$, thus, $N(v+w) - Mv \in \mathcal{K}_0^{r-1}((t_0,t_1),2)$ if $w \in \mathcal{K}_0^{r+1}((t_0,t_1),2)$. Therefore, we can apply Lemma 3.25 and write (3.55) as fixed point equation, namely

$$w = M_0^{-1} \left(N(v+w) - Mv \right), \tag{3.58}$$

where we denote the solution operator of Lemma 3.25 by M_0^{-1}. The choice of v is not unique, but by applying a cut-off function in time we can always ensure that $\|v\|_{\mathcal{K}^{r+1}((t_0,t_1),2)} < C\|u_0\|_{\mathcal{H}^r(2)}$ for a fixed $C > 0$. Setting $\|u_0\|_{\mathcal{H}^r(2)} = \epsilon^2 < 1$ and assuming $\|w\|_{\mathcal{K}^{r+1}((t_0,t_1),2)} \leq \epsilon$ we obtain

$$\begin{aligned}
&\|M_0^{-1} \left(N(v+w) - Mv \right)\|_{\mathcal{K}^{r+1}((t_0,t_1),2)} \\
&\leq C\|N(v+w) - Mv\|_{\mathcal{K}^{r-1}((t_0,t_1),2)} \\
&\leq C \left(\|w\|^2_{\mathcal{K}^{r+1}((t_0,t_1),2)} + \|v\|_{\mathcal{K}^{r+1}((t_0,t_1),2)} \right) \\
&\leq C\epsilon^2 = C\|u_0\|_{\mathcal{H}^r(2)} < \epsilon
\end{aligned} \tag{3.59}$$

for $\epsilon > 0$ small enough. Therefore, the right-hand side of (3.58) maps a small ball in $\mathcal{K}_0^{r+1}((t_0,t_1),2)$ into itself if the initial condition u_0 is small enough. For w_1 and w_2 in this ball, additionally we have

$$\begin{aligned}
&\|M_0^{-1} \left(N(v+w_1) - Mv \right) - M_0^{-1} \left(N(v+w_2) - Mv \right)\|_{\mathcal{K}^{r+1}((t_0,t_1),2)} \\
&\leq C\|N(v+w_1) - N(v+w_2)\|_{\mathcal{K}^{r-1}((t_0,t_1),2)} \leq \frac{1}{2}\|w_1 - w_2\|_{\mathcal{K}^{r+1}((t_0,t_1),2)},
\end{aligned} \tag{3.60}$$

since N is at least quadratic and pure v-terms drop out.

Thus, for a sufficiently small initial condition the right-hand side of (3.58) defines a contraction in $\mathcal{K}_0^{r+1}((t_0,t_1),2)$, and the contraction mapping theorem yields the existence of a $w \in \mathcal{K}_0^{r+1}((t_0,t_1),2)$ satisfying (3.55). Since from (3.59) we obtain

$\|w\|_{\mathcal{K}^{r+1}((t_0,t_1),2)} \le C\|u_0\|_{\mathcal{H}^r(2)}$, there exists a solution $u = v + w$ of (3.53), (3.54) with

$$\|u\|_{\mathcal{K}^{r+1}((t_0,t_1),2)} \le C\|u_0\|_{\mathcal{H}^r(2)}.$$

Uniqueness. To show uniqueness of u suppose there are two solutions u_1, u_2. Then the difference $u_1 - u_2$ fulfills

$$M(u_1 - u_2) = N(u_1) - N(u_2), \quad (u_1 - u_2)\big|_{t=t_0} = 0. \tag{3.61}$$

Since $N(u_1)\big|_{t=t_0} = N(u_2)\big|_{t=t_0}$, we have $N(u_1) - N(u_2) \in \mathcal{K}_0^{r-1}((t_0,t_1),2)$. Thus, we can write $u_1 - u_2 = M_0^{-1}(N(u_1) - N(u_2))$, and similarly to (3.60) we obtain

$$\|u_1-u_2\|_{\mathcal{K}^{r+1}((t_0,t_1),2)} \le C\|N(u_1)-N(u_2)\|_{\mathcal{K}^{r-1}((t_0,t_1),2)} \le \frac{1}{2}\|u_1-u_2\|_{\mathcal{K}^{r+1}((t_0,t_1),2)},$$

if the initial condition u_0 is small enough. Thus, $u_1 = u_2$.

Higher regularity. The higher regularity in the time interval $t \in [\tilde{t}_0, t_1]$ for $t_0 < \tilde{t}_0 < t_1$ follows from a bootstrapping argument, which we sketch next. As $u \in L^2((t_0,t_1),\mathcal{H}^{r+1}(2))$, there exists a $\tilde{t} \in \left[\frac{t_0+\tilde{t}_0}{2}, \tilde{t}_0\right]$ with

$$\|u(\tilde{t},\cdot)\|_{\mathcal{H}^{r+1}(2)} \le \frac{2}{\tilde{t}_0 - t_0}\|u\|_{L^2((t_0,\tilde{t}_0),\mathcal{H}^{r+1}(2))},$$

since otherwise we had

$$\int_{\frac{t_0+\tilde{t}_0}{2}}^{\tilde{t}_0} \|u(\tilde{t},\cdot)\|^2_{\mathcal{H}^{r+1}(2)}\,\mathrm{d}\tilde{t} > \|u\|^2_{L^2((t_0,\tilde{t}_0),\mathcal{H}^{r+1}(2))}.$$

Starting again at $t = \tilde{t}$ yields $u \in \mathcal{K}^{r+2}((\tilde{t},t_1),2)$ with

$$\|u\|_{\mathcal{K}^{r+2}((\tilde{t},t_1),2)} \le C\|u(\tilde{t},\cdot)\|_{\mathcal{H}^{r+1}(2)} \le C\|u\|_{\mathcal{K}^{r+1}((t_0,t_1),2)} \le C\|u_0\|_{\mathcal{H}^r(2)}.$$

In particular, we have

$$\|u\|_{\mathcal{K}^{r+2}((\tilde{t}_0,t_1),2)} \le C\|u_0\|_{\mathcal{H}^r(2)}. \tag{3.62}$$

Iterating this procedure for $m \in \mathbb{N}$ yields $\|u\|_{\mathcal{K}^{r+m}((\tilde{t}_0,t_1),2)} \le C\|u_0\|_{\mathcal{H}^r(2)}$, and hence the second assertion in Theorem 3.20.

Remark 3.26. In (3.57) we had to choose $r < 4$ in order to achieve $(N(v+w) - Mv) \in \mathcal{K}_0^{r-1}((t_0,t_1),2)$, hence it is not obvious why the bootstrapping argument can be applied to initial conditions with higher regularity. However, considering the two components of $N(v+w) - Mv$ separately, condition (3.57) can be substituted by

$$\partial_t^{j_1}(\partial_t v_1 + \partial_x v_2)\big|_{t=t_0} = 0 \quad \text{for all } j_1 < (r-2)/2, \tag{3.63}$$

$$\partial_t^{j_2}(N_2(v+w) - \partial_t v_2 + A_2 v)\big|_{t=t_0} = 0 \quad \text{for all } j_2 < (r-3)/2, \tag{3.64}$$

where $A_2 v$ denotes the second component of Av. According to Lemma 3.11, we can choose the time derivatives

$$\partial_t^{j_1} v_1\big|_{t=t_0}, \quad \partial_t^{j_2} v_2\big|_{t=t_0}$$

arbitrarily for all $j_1 < r/2$ and $j_2 < (r-1)/2$. The essential property is that the regularities of $v \in \mathcal{K}^{r+1}$ and $N(v+w) - Mv \in \mathcal{K}^{r-1}$ differ by two, such that there is always one degree of freedom left in the choice of $v\big|_{t=t_0}$ to fulfill (3.63), (3.64). For instance, if $5 < r < 6$, we additionally have to fulfill (3.63) for $j_1 = 1$ and (3.64) for $j_2 = 1$, which is no problem since we can arbitrarily choose $\partial_t^2 v_1$ and $\partial_t^2 v_2$ at $t = t_0$. Thus, the restriction to $r < 4$ above is only for notational convenience.

4. Renormalization

To make the formal calculations in §2.6 rigorous and hence prove Theorem 2.8 we establish a renormalization process as in [5, 22]. Additional to iterating the application of the local existence and uniqueness theorem, the key issue is to extract the leading order behavior formally described by the Burgers equation (2.66). Therefore we now consider the IBL in Bloch space which is split in (2.45) for the linearly diffusive mode $\tilde{\alpha}\phi^1$ and in (2.46) for the linearly exponentially damped remainder. Here rescaled Bloch spaces with different weights in ℓ turn out to be useful.

4.1. **Basic setup.** For $m \in \mathbb{N}_0$, $r, b \geq 0$, and $L > 0$ we set

$$\begin{aligned} B_L(m,r,b) &:= H^m(I_{Lk_0}, H^r_{\text{per}}(I_\gamma)),\ \|\tilde{v}\|_{B_L(m,r,b)} \\ &:= \Big(\sum_{j\leq m}\int_{I_{Lk_0}} (1+\ell^2)^b \|\partial_\ell^j \tilde{v}(\ell,\cdot)\|^2_{H^r(I_\gamma)}\,\mathrm{d}\ell\Big)^{1/2}, \end{aligned} \tag{4.1}$$

where again $I_\delta = (-\delta/2, \delta/2)$. Note that the spaces adhere to the fixed choice of periodicity $\gamma = 2\pi/k_0$. Let $B(m,r,b) := B_1(m,r,b)$. Regarding the original Bloch spaces from §2.3 we have $B(m,r) = B(m,r,0)$. At first view, the introduction of weights in the ℓ-variable seems dispensable since all norms $\|\cdot\|_{B_L(m,r,b_1)}$ and $\|\cdot\|_{B_L(m,r,b_2)}$ are equivalent due to the compact support in ℓ. But as constants depend on L this step is crucial in §4.2 to control nonlinear interaction without losing powers of L^{-1}. For $L > 0$ we define the renormalization operator $\mathcal{R}_{1/L}$ by

$$\mathcal{R}_{1/L} : B(m,r,b) \to B_L(m,r,b), \quad \mathcal{R}_{1/L}\tilde{v}(\ell,x) := \tilde{v}(\ell/L, x). \tag{4.2}$$

Note that only ℓ is rescaled, and thus there is no matching rescaling in x-space. For $L \geq 1$ we have

$$L^{\frac{1-2m}{2}}\|\tilde{v}\|_{B(m,r,b)} \leq \|\mathcal{R}_{1/L}\tilde{v}\|_{B_L(m,r,b)} \leq L^{\frac{1+2b}{2}}\|\tilde{v}\|_{B(m,r,b)} \leq CL^{\frac{1+2b}{2}}\|\tilde{v}\|_{B(m,r,0)}. \tag{4.3}$$

We will mainly need the second inequality for $b = 2$, which yields an additional factor $L^{5/2}$ in the estimates.

For a fixed $p \in (0, 1/2)$ we introduce the renormalized variables

$$\alpha_n(t,\ell) := \mathcal{R}_{L^{-n}}\tilde{\alpha}(L^{2n}t,\ell), \quad w_n(t,\ell,x) := L^{(1-p)n}\mathcal{R}_{L^{-n}}\tilde{w}_s(L^{2n}t,\ell,x). \tag{4.4}$$

Since we suppose the stable component to decay like t^{-1} and since time is scaled by L^{2n}, we blow up w_n by multiplying it with $L^{(1-p)n}$. The factor L^{pn} is needed later to control some constants. From the IBL (2.45), (2.46) in Bloch space we obtain

$$\begin{aligned} \partial_t\alpha_n(t,\ell) &= L^{2n}\mu_1(L^{-n}\ell)\alpha_n(t,\ell) + L^{2n}B^c_n(\alpha_n(t))(\ell) \\ &\quad + L^{2n}H^c_n(\alpha_n(t), L^{-(1-p)n}w_n(t))(\ell), \end{aligned} \tag{4.5}$$

$$\partial_t w_n(t,\ell,x) = L^{2n}\tilde{A}_s(L^{-n}\ell)w_n(t,\ell,x) + L^{(3-p)n}H^s_n(\alpha_n(t), L^{-(1-p)n}w_n(t))(\ell,x), \tag{4.6}$$

where

$$\begin{aligned} B^c_n(\alpha_n) &:= \mathcal{R}_{L^{-n}}\tilde{B}_c(\mathcal{R}_{L^n}\alpha_n), \quad H^c_n(\alpha_n,w_n) := \mathcal{R}_{L^{-n}}\tilde{H}_c(\mathcal{R}_{L^n}\alpha_n, \mathcal{R}_{L^n}w_n), \\ H^s_n(\alpha_n,w_n) &:= \mathcal{R}_{L^{-n}}\tilde{H}_s(\mathcal{R}_{L^n}\alpha_n, \mathcal{R}_{L^n}w_n). \end{aligned} \tag{4.7}$$

Thus, solving (2.45), (2.46) with the initial condition $(\tilde{\alpha}, \tilde{w}_s)\big|_{t=1} = (\alpha_0, w_0)$ is equivalent to iterating the following renormalization process: For $n \in \mathbb{N}$ solve (4.5), (4.6) for $t \in [1/L^2, 1]$ with the initial condition

$$\alpha_n(1/L^2, \ell) = \alpha_{n-1}(1, \ell/L), \quad w_n(1/L^2, \ell, x) = L^{1-p} w_{n-1}(1, \ell/L, x). \tag{4.8}$$

We take $(\alpha_n, w_n) \in X_{L^n}(2, r, b)$, where

$$\begin{aligned} X_{L^n}(m, r, b) &:= B_{L^n}(m, r, b) \times \mathcal{B}_{L^n}(m, r, b), \\ \mathcal{B}_{L^n}(m, r, b) &:= B_{L^n}(m, r, b) \times B_{L^n}(m, r-1, b), \end{aligned} \tag{4.9}$$

with $r \geq 3$ and $b > 0$ to be chosen later. Note that the value of r does not play any role in the critical component $\tilde{\alpha}$ since $\tilde{\alpha}$ is independent of x. We introduce

$$\tilde{K}_L^r((t_0, t_1), m, b) := H^{r/2}((t_0, t_1), B_L(m, 0, b)) \cap L^2((t_0, t_1), B_L(m, r, b)). \tag{4.10}$$

Finally, let

$$\tilde{\mathcal{K}}_L^{r+1}((t_0, t_1), m, b) := \tilde{K}_L^{r+1}((t_0, t_1), m, b) \times \tilde{K}_L^r((t_0, t_1), m, b), \tag{4.11}$$

$$\mathcal{X}_L^{r+1}((1/L^2, 1), 2, 2) := \tilde{K}_L^{r+1}((1/L^2, 1), 2, 2) \times \tilde{\mathcal{K}}_L^{r+1}((1/L^2, 1), 2, 2). \tag{4.12}$$

Then, due to Theorem 3.20 concerning local existence in the original system, we expect local solutions of (4.5), (4.6) in the space $\mathcal{X}_{L^n}^{r+1}((1/L^2, 1), 2, 2)$.

Theorem 4.1 (Local existence in the renormalized system)**.** *Let $3 < r < 4$. There exist $L_0 > 1$ and $C_1, C_2 > 0$ such that for all $L > L_0$ the following holds. Let*

$$\rho_{n-1} := \|(\alpha_{n-1}(1), w_{n-1}(1))\|_{X_{L^{n-1}}(2,r,2)} \leq C_1 L^{-5/2}.$$

Then there exists a unique local solution $(\alpha_n, w_n) \in \mathcal{X}_{L^n}^{r+1}((1/L^2, 1), 2, 2)$ of (4.5), (4.6) with

$$\|(\alpha_n, w_n)\|_{\mathcal{X}_{L^n}^{r+1}((1/L^2,1),2,2)} \leq C_2 L^{5/2} \rho_{n-1}. \tag{4.13}$$

Moreover, for any $m \in \mathbb{N}$ we have $(\alpha_n, w_n) \in \mathcal{X}_{L^n}^{r+m}((1/2, 1), 2, 2)$ and there exists a $C_3 > 0$ such that

$$\|(\alpha_n, w_n)\|_{\mathcal{X}_{L^n}^{r+m}((1/2,1),2,2)} \leq C_3 L^{5/2} \rho_{n-1}.$$

Proof. The proof can be adapted from Theorem 3.20. The crucial point is to have C_2, C_3 independent of n, which depends on suitable resolvent estimates of the linear parts $L^{2n}\mu_1(L^{-n}\ell)$ and $L^{2n}\tilde{A}_s(L^{-n}\ell)$, and on estimates for the nonlinearities. The latter is worked out in detail in §4.2 in a slightly different form suitable to obtain more detailed asymptotics. Thus, here we only sketch the main ideas. First, we consider the linear inhomogeneous system

$$\partial_t \alpha_n - L^{2n}\mu_1(L^{-n}\ell)\alpha_n = g_{n,1}, \tag{4.14}$$

$$\partial_t w_n - L^{2n}\tilde{A}_s(L^{-n}\ell) w_n = g_{n,2}. \tag{4.15}$$

Since (4.14) is independent of x no smoothing properties are needed, and hence, as well as (4.5), it can be solved by the variation-of-constants formula. For (4.15) we find resolvent estimates for $\lambda - L^{2n}\tilde{A}_s(L^{-n}\ell)$ which correspond to Theorem 3.24 transferred to Bloch space. Since we are in the stable part, we can choose $a = 0$ independently of n, which yields

$$\|(\alpha_n, w_n)\|_{\mathcal{X}_{L^n}^{r+1}((1/L^2,1),2,2)} \leq C_2 \|(g_{n,1}, g_{n,2})\|_{\mathcal{X}_{L^n}^{r-1}((1/L^2,1),2,2)} \tag{4.16}$$

with C_2 independent of n. Note that (4.16) could be improved by choosing $a = -L^{2n}\sigma_0/2$, but to show the local existence result (4.13), $a = 0$ is sufficient here.

The estimates for the nonlinearities, see §4.2, together with Banach's fixed point theorem and (4.3) then yield the first result, while the higher regularity follows as in Theorem 3.20 by a bootstrapping argument. □

The local existence Theorem 4.1 turns out to be a fundamental step in the proof of the following nonlinear stability result.

Theorem 4.2. *Let $p \in (0, 1/2)$ and $3 < r < 4$. In the spectrally stable case, cf. Assumption 2.3, there exist $C_1, C_2 > 0$ such that the following holds. If $\|\alpha_0\|_{B(2,r,2)} + \|w_0\|_{\mathcal{B}(2,r,2)} \leq C_1$, then there exists a unique global solution $\tilde{w} = \tilde{\alpha}\phi^1 + \tilde{w}_s$ of the IBL (2.45), (2.46) in Bloch space with $(\tilde{\alpha}, \tilde{w}_s)\big|_{t=1} = (\alpha_0, w_0)$. Moreover, we have*

$$\big\|(\ell, x) \mapsto \Big(\tilde{w}(t, t^{-1/2}\ell, x) - \chi(t^{-1/2}\ell)\hat{f}_{z_0}(\ell)\phi^1(0, x)\Big)\,\|_{\mathcal{B}_{\sqrt{t}}(2,r,1)} \leq C_2 t^{-(1-p)/2}, \tag{4.17}$$

where $\hat{f}_{z_0}$ is the Fourier transformed profile from (2.53), ϕ^1 is the eigenvector to the critical eigenvalue λ_1, see (2.35), and $\ln(z_0 + 1) = 2\pi \frac{d}{c_2}\alpha_0(0)$ with d from (2.64).

Theorem 4.2 is proved in §4.2 and §4.3 by an iteration scheme for the renormalized system. Here, we translate (4.17) back to x-space in order to show Theorem 2.8.

Proof of Theorem 2.8. We have

$$\begin{aligned}(h, q)^\top(t, x) &= \int_{-1/2k_0}^{1/2k_0} \mathrm{e}^{\mathrm{i}\ell x}(\tilde{h}, \tilde{q})^\top(t, \ell, x)\,\mathrm{d}\ell = \int_{-1/2k_0}^{1/2k_0} \mathrm{e}^{\mathrm{i}\ell(x+c_1 t)}\tilde{w}(t, \ell, x)\,\mathrm{d}\ell \\ &= t^{-1/2}\int_{-1/2k_0\sqrt{t}}^{1/2k_0\sqrt{t}} \mathrm{e}^{\mathrm{i}\ell t^{-1/2}(x+c_1 t)}\tilde{w}(t, t^{-1/2}\ell, x)\,\mathrm{d}\ell.\end{aligned}$$

With the inverse Fourier transform $f_{z_0}(t^{-1/2}(x + c_1 t)) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\ell t^{-1/2}(x+c_1 t)}\hat{f}_{z_0}(\ell)\,\mathrm{d}\ell$ we obtain

$$\begin{aligned}&(h, q)^\top(t, x) - t^{-1/2}f_{z_0}(t^{-1/2}(x + c_1 t))\phi^1(0, x) \\ &= t^{-1/2}\int_{-1/2k_0\sqrt{t}}^{1/2k_0\sqrt{t}} \mathrm{e}^{\mathrm{i}\ell t^{-1/2}(x+c_1 t)}\Big(\tilde{w}(t, t^{-1/2}\ell, x) - \chi(t^{-1/2}\ell)\hat{f}_{z_0}(\ell)\phi^1(0, x)\Big)\,\mathrm{d}\ell \\ &\quad - t^{-1/2}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\ell t^{-1/2}(x+c_1 t)}\hat{f}_{z_0}(\ell)\Big(1 - \chi(t^{-1/2}\ell)\Big)\,\mathrm{d}\ell\,\phi^1(0, x).\end{aligned} \tag{4.18}$$

The first integral on the right-hand side of (4.18) can be estimated by

$$\begin{aligned}&t^{-1/2}\sup_{x\in\mathbb{R}}\Big|\int_{-1/2k_0\sqrt{t}}^{1/2k_0\sqrt{t}} \mathrm{e}^{\mathrm{i}\ell t^{-1/2}(x+c_1 t)}\Big(\tilde{w}(t, t^{-1/2}\ell, x) - \chi(t^{-1/2}\ell)\hat{f}_{z_0}(\ell)\phi^1(0, x)\Big)\,\mathrm{d}\ell\Big| \\ &\leq t^{-1/2}\int_{-1/2k_0\sqrt{t}}^{1/2k_0\sqrt{t}} \sup_{x\in[0,\gamma]}|\tilde{w}(t, t^{-1/2}\ell, x) - \chi(t^{-1/2}\ell)\hat{f}_{z_0}(\ell)\phi^1(0, x)|\mathrm{d}\ell \\ &\leq Ct^{-1+p/2}.\end{aligned}$$

Since f_{z_0} is analytic, the Fourier transform $\hat{f}_{z_0}$ is exponentially decaying. Thus, the second integral in (4.18) can be estimated as

$$t^{-1/2}\sup_{x\in\mathbb{R}}\Big|\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\ell t^{-1/2}(x+c_1 t)}\hat{f}_{z_0}(\ell)\Big(1 - \chi(t^{-1/2}\ell)\Big)\,\mathrm{d}\ell\,\phi^1(0, x)\Big| \leq Ct^{-1}.$$

Altogether, this proves Theorem 2.8. Thus, it remains to prove Theorem 4.2.

4.2. **Estimates.** We write the solution of the renormalized IBL (4.5), (4.6) in Bloch space with the help of the variation-of-constants formula; i.e.,

$$\begin{aligned}
&\alpha_n(t,\ell)\\
&= e^{(t-L^{-2})L^{2n}\mu_1(L^{-n}\ell)}\alpha_{n-1}(1,L^{-1}\ell)\\
&\quad + L^{2n}\int_{L^{-2}}^{t} e^{(t-s)L^{2n}\mu_1(L^{-n}\ell)}\Big(B_n^c(\alpha_n(s))(\ell)+H_n^c(\alpha_n(s),L^{-(1-p)n}w_n(s))(\ell)\Big)\,ds,
\end{aligned} \tag{4.19}$$

$$\begin{aligned}
&w_n(t,\ell,x)\\
&= L^{1-p}e^{(t-L^{-2})L^{2n}\tilde{A}_s(L^{-n}\ell)}w_{n-1}(1,L^{-1}\ell,x)\\
&\quad + L^{(3-p)n}\int_{L^{-2}}^{t} e^{(t-s)L^{2n}\tilde{A}_s(L^{-n}\ell)}H_n^s(\alpha_n(s),L^{-(1-p)n}w_n(s))(\ell,x)\,ds,
\end{aligned} \tag{4.20}$$

where $e^{tL^{2n}\tilde{A}_s(L^{-n}\ell)}$ stands for the analytic semigroup generated by $L^{2n}\tilde{A}_s(L^{-n}\ell)$, cf. Lemma 3.23, which clearly transfers from $\tilde{A}_s(\ell)$ to $L^{2n}\tilde{A}_s(L^{-n}\ell)$. To prove existence of a solution for $t\in[1/L^2,1]$ we need estimates of the linear semigroups and the nonlinearities. These are shown in Lemma 4.3, Lemma 4.8, and Lemma 4.9, respectively.

Lemma 4.3. *For $0\le b_1\le b_2$, $0\le j\le 2$ there exists a $C>0$ such that for $\alpha\in B_{L^n}(2,r,b_1)$ independent of x we have*

$$\|e^{tL^{2n}\mu_1(L^{-n}\cdot)}\alpha\|_{B_{L^n}(2,r,b_2)}\le C\max\{1,t^{-(b_2-b_1)/2}\}\|\alpha\|_{B_{L^n}(2,r,b_1)} \tag{4.21}$$

in the critical part. The stable part is linearly exponentially damped; i.e., there exists a $\sigma_1>0$ such that for $u\in B_{L^n}(2,r-j,b)\times B_{L^n}(2,r-1-j,b)$ we have

$$\begin{aligned}
&\|e^{tL^{2n}\tilde{A}_s(L^{-n}\cdot)}u\|_{B_{L^n}(2,r,b)\times B_{L^n}(2,r-1,b)}\\
&\le Ce^{-\sigma_1 tL^{2n}}\max\{1,(L^{2n}t)^{-j/2}\}\|u\|_{B_{L^n}(2,r-j,b)\times B_{L^n}(2,r-1-j,b)}.
\end{aligned} \tag{4.22}$$

Proof. Estimate (4.21) follows from the locally parabolic shape of $L^{2n}\mu_1(L^{-n}\ell)=-c_2\ell^2+\mathcal{O}(L^{-n}\ell^3)$ around $\ell=0$, see Assumption 2.3. Since $\operatorname{Re}\mu_n(\ell)=\operatorname{Re}\lambda_n(\ell)<-\sigma_0$ for all $n\ge 2$ and $\ell\in(-k_0/2,k_0/2)$ and since $\operatorname{Re}\mu_1(\ell)<-\sigma_0$ for all $|\ell|>4r_\chi$, the real part of the spectrum of $\tilde{A}_s$ is bounded from above by $-\sigma_0$. Thus, we have inequality (4.22) for a $\sigma_1<\sigma_0$, which avoids the treatment of Jordan blocks. □

To control the integral in (4.20) we use the following lemma.

Lemma 4.4. *There exists a $C>0$ such that for $t_0\in[1/L^2,1]$ and $0\le j\le 1$ we have $\int_{t_0}^1 e^{-\sigma_1(1-s)L^{2n}}(1-s)^{-j/2}ds\le CL^{-(2-j)n}$. For $0\le j\le 2$ we obtain $\int_{1/L^2}^{1/2} e^{-\sigma_1(1-s)L^{2n}}(1-s)^{-j/2}ds\le CL^{-2n}$.*

The next lemma exploits the role of leading ℓ's in the critical part of the nonlinearity, cf. [16, Lemma 14]. Remember again that $k_0:=2\pi/\gamma$, where γ denotes the bottom periodicity.

Lemma 4.5. *Let $\beta\in C^2([-k_0/2,k_0/2),C^2((0,\gamma),\mathbb{C}))$ with $\|\beta(\ell,\cdot)\|_{C^2((0,\gamma),\mathbb{C})}\le C\ell^{b_2-b_1}$ for a $b_1\in[0,b_2]$. Then there exists a $C>0$ such that for all $L>1$ we* *have*

$$\|(\mathcal{R}_{1/L}\beta)u\|_{B_L(2,r,b_1)} \leq CL^{-(b_2-b_1)}\|\beta\|_{C^2([-k_0/2,k_0/2),C^2((0,\gamma),\mathbb{C}))}\|u\|_{B_L(2,r,b_2)}.$$

The idea of the lemma is as follows. If the nonlinearity in (4.5) exhibits, e.g., a leading $(\ell/L^n)^j$, $j>0$, we can extract the factor $\ell^{b_2-b_1}/L^{(b_2-b_1)n}$, $b_2-b_1 \leq j$. While in (4.19) the factor $\ell^{b_2-b_1}$ can be balanced by the linear semigroup in Lemma 4.3, the factor $L^{-(b_2-b_1)n}$ increases the degree of irrelevance. However, according to the term $t^{-(b_2-b_1)/2}$ in (4.21) this is only possible as long as b_2-b_1 is bounded from above by, e.g., $2-p$.

As products in x-space correspond to convolutions in Bloch space, the nonlinearities in (4.7) produce terms of the type $\mathcal{R}_{L^{-1}}(\mathcal{R}_L u * \mathcal{R}_L v)$. Thus, we define an adapted convolution operator $*_L$ for $u, v \in B_L(m,r,b)$ as follows:

$$u *_L v := \int_{-Lk_0/2}^{Lk_0/2} u(\ell-m)v(m)\mathrm{d}m = L\mathcal{R}_{L^{-1}}\left((\mathcal{R}_L u) * (\mathcal{R}_L v)\right). \tag{4.23}$$

To estimate convolutions we use the following lemma, based on standard Sobolev embeddings.

Lemma 4.6. *Let $b_2 > 1/2$, $b_2 \geq b_1 \geq 0$. There exists a $C>0$ such that, for all $L \geq 1$,*

$$\|u *_L v\|_{B_L(2,r,b_1)} \leq C\|u\|_{B_L(2,r,b_1)}\|v\|_{B_L(2,r,b_2)}.$$

Remark 4.7. Before estimating the nonlinearities in detail we want to summarize the different effects in a descriptive way. By combining (4.23) and Lemma 4.6, each convolution produces a factor L^{-n}. Due to the rescaling in (4.4), each factor w_n gives a further $L^{-(1-p)n}$. In the critical component, a factor $\mathcal{R}_{L^{-n}}(\ell^{b_2-b_1})$ in the renormalized nonlinearity leads to an additional factor $L^{-(b_2-b_1)n}$ as long as $b_2-b_1 \leq 2-p$.

Lemma 4.8. *Let $r \geq 3$. For $p \in (0,1/2)$ there exists a $C>0$ such that for all $(\alpha_n, w_n) \in X_{L^n}(2,r,2) = B_{L^n}(2,r,2) \times \mathcal{B}_{L^n}(2,r,2)$ with $(\alpha_n, w_n)_{X_{L^n}(2,r,2)} \leq 1$ we have*

$$L^{2n}H_n^c(\alpha_n, L^{-(1-p)n}w_n) = s_1 + s_2 + s_3$$

with

$$\|s_1\|_{B_{L^n}(2,r,p)} \leq CL^{-(1-p)n}\|\alpha_n\|^2_{B_{L^n}(2,r,2)}, \tag{4.24}$$

$$\|s_2\|_{B_{L^n}(2,r,1)} \leq CL^{-(1-p)n}\|\alpha_n\|_{B_{L^n}(2,r,2)}\|w_n\|_{\mathcal{B}_{L^n}(2,r,2)}, \tag{4.25}$$

$$\|s_3\|_{B_{L^n}(2,r,1)} \leq CL^{-2(1-p)n}\|w_n\|^2_{\mathcal{B}_{L^n}(2,r,2)}. \tag{4.26}$$

Proof. By construction we have

$$\begin{aligned}
&H_n^c(\alpha_n, L^{-(1-p)n}w_n)\\
&= \mathcal{R}_{L^{-n}}\tilde{H}_c(\mathcal{R}_{L^n}\alpha_n, L^{-(1-p)n}\mathcal{R}_{L^n}w_n)\\
&= \mathcal{R}_{L^{-n}}\tilde{E}_c^*\left(\tilde{B}((\mathcal{R}_{L^n}\alpha_n)\phi^1 + L^{-(1-p)n}\mathcal{R}_{L^n}w_n)\right) - \mathcal{R}_{L^{-n}}\left(\tilde{B}_c(\mathcal{R}_{L^n}\alpha_n)\right)\\
&\quad + \mathcal{R}_{L^{-n}}\tilde{E}_c^*\left(\tilde{G}((\mathcal{R}_{L^n}\alpha_n)\phi^1 + L^{-(1-p)n}\mathcal{R}_{L^n}w_n)\right),
\end{aligned} \tag{4.27}$$

see (2.45) and (2.58). We start with the estimates of the first two terms on the right-hand side of (4.27). Since they are quadratic, we split them according to the multiplicities of α_n and w_n.

1. Terms quadratic in α_n. In the formal derivation of the Burgers equation in §2.6 we obtained

$$\tilde{E}_c^*\left(\tilde{B}((\mathcal{R}_{L^n}\alpha_n)\phi^1)\right)(\ell) - \tilde{B}_c(\mathcal{R}_{L^n}\alpha_n)(\ell) = \beta(\ell)(\mathcal{R}_{L^n}\alpha_n)^{*2}(\ell)$$

with $\beta(\ell) = \mathcal{O}(\ell^2)$, cf. (2.65). Therefore, we have to estimate

$$\mathcal{R}_{L^{-n}}(\beta(\mathcal{R}_{L^n}\alpha_n)^{*2}) = (\mathcal{R}_{L^{-n}}\beta)\mathcal{R}_{L^{-n}}((\mathcal{R}_{L^n}\alpha_n)^{*2}).$$

Applying (4.23), Lemma 4.5, and Lemma 4.6 we obtain

$$\begin{aligned}\|\mathcal{R}_{L^{-n}}(\beta(\mathcal{R}_{L^n}\alpha_n)^{*2})\|_{B_{L^n}(2,r,p)} &= L^{-n}\|(\mathcal{R}_{L^{-n}}\beta)(\alpha_n *_{L^n} \alpha_n)\|_{B_{L^n}(2,r,p)}\\ &\le CL^{-(3-p)n}\|\alpha_n\|^2_{B_{L^n}(2,r,2)}.\end{aligned}$$

Thus, the terms considered in this part can be assigned to s_1. Note that here we only used $\beta(\ell) = \mathcal{O}(\ell^{2-p})$ instead of $\beta(\ell) = \mathcal{O}(\ell^2)$, since otherwise the missing weight in ℓ could not be balanced by the linear semigroup in (4.21).

2. Mixed terms in α_n, w_n. The terms in

$$\tilde{E}_c^*\left(\tilde{B}(\tilde{\alpha}\phi^1 + \tilde{w}_s)\right)(\ell) = \chi(\ell)\int_0^\gamma B_2(\tilde{\alpha}(\ell)\phi^1(\ell) + \tilde{w}_s(\ell))(x)\bar{\psi}_2^1(\ell,x)\mathrm{d}x$$

which contain both $\tilde{\alpha}$ and $\tilde{w}_s$ are all of the type

$$N(\tilde{\alpha},\tilde{w}_s)(\ell) = \chi(\ell)\int_0^\gamma \left[(\tilde{\alpha}(\cdot)(\partial_x+\mathrm{i}\cdot)^{k_c}\phi_i^1(\cdot,x)) * ((\partial_x+\mathrm{i}\cdot)^{k_s}\tilde{w}_{s,j}(\cdot,x))\right](\ell)\bar{\psi}_2^1(\ell,x)\mathrm{d}x$$

with $k_c, k_s \in \{1,2,3\}$ and $i,j \in \{1,2\}$. Applying (4.23) yields

$$\begin{aligned}&\mathcal{R}_{L^{-n}}N(\mathcal{R}_{L^n}\alpha_n, L^{-(1-p)n}\mathcal{R}_{L^n}w_n)\\
&= L^{-(1-p)n}\mathcal{R}_{L^{-n}}\Big(\chi\int_0^\gamma \left[((\mathcal{R}_{L^n}\alpha_n)(\partial_x+\mathrm{i}\cdot)^{k_c}\phi_i^1) * ((\partial_x+\mathrm{i}\cdot)^{k_s}(\mathcal{R}_{L^n}w_{n,j}))\right]\bar{\psi}_2^1\mathrm{d}x\Big)\\
&= L^{-(1-p)n}\mathcal{R}_{L^{-n}}\Big(\chi\int_0^\gamma \Big[\mathcal{R}_{L^n}\Big(\alpha_n(\partial_x+\mathrm{i}\frac{\cdot}{L^n})^{k_c}\mathcal{R}_{L^{-n}}\phi_i^1\Big)\\
&\quad * \mathcal{R}_{L^n}\Big((\partial_x+\mathrm{i}\frac{\cdot}{L^n})^{k_s}w_{n,j}\Big)\Big]\bar{\psi}_2^1\mathrm{d}x\Big)\\
&= L^{-(2-p)n}\int_0^\gamma \Big[\Big(\alpha_n(\partial_x+\mathrm{i}\frac{\cdot}{L^n})^{k_c}\mathcal{R}_{L^{-n}}\phi_i^1\Big) *_{L^n}\Big((\partial_x+\mathrm{i}\frac{\cdot}{L^n})^{k_s}w_{n,j}\Big)\Big]\mathcal{R}_{L^{-n}}(\chi\bar{\psi}_2^1)\mathrm{d}x,\end{aligned}$$

where $\chi(\ell)\bar{\psi}_2^1(\ell) = \mathcal{O}(\ell)$, cf. (2.41). If $u \in B_{L^n}(2,r,1)$ is independent of x, we have $\|u\|_{B_{L^n}(2,r,1)} = \|u\|_{B_{L^n}(2,0,1)}$. Thus, we obtain

$$\begin{aligned}&\|\mathcal{R}_{L^{-n}}N(\mathcal{R}_{L^n}\alpha_n, L^{-(1-p)n}\mathcal{R}_{L^n}w_n)\|_{B_{L^n}(2,r,1)}\\
&\le L^{-(2-p)n}\int_0^\gamma \Big\|\Big(\Big(\alpha_n(\partial_x+\mathrm{i}\frac{\cdot}{L^n})^{k_c}\mathcal{R}_{L^{-n}}\phi_i^1\Big) *_{L^n}\Big((\partial_x+\mathrm{i}\frac{\cdot}{L^n})^{k_s}w_{n,j}\Big)\Big)\Big\|_{B_{L^n}(2,0,1)}\mathrm{d}x\\
&\le CL^{-(3-p)n}\|\alpha_n(\partial_x+\mathrm{i}\frac{\cdot}{L^n})^{k_c}\mathcal{R}_{L^{-n}}\phi_i^1\|_{B_{L^n}(2,0,2)}\|(\partial_x+\mathrm{i}\frac{\cdot}{L^n})^{k_s}w_{n,j}\|_{B_{L^n}(2,0,2)}\\
&\le CL^{-(3-p)n}\|\alpha_n\|_{B_{L^n}(2,r,2)}\|w_n\|_{\mathcal{B}_{L^n}(2,r,2)}.\end{aligned}$$

Therefore, the mixed terms in α_n, w_n can be assigned to s_2.

3. Terms quadratic in w_n. The estimates for the terms in $\tilde{E}_c^*\big(\tilde{B}(L^{-(1-p)n}\mathcal{R}_{L^n}w_n)\big)$ are the same as for the mixed terms, except that we have an additional factor $L^{-(1-p)n}$ due to the scaling of w_n, which yields the estimate for s_3 in (4.26).

It remains to estimate the third term in (4.27). We have

$$\tilde{E}_c^* \left(\tilde{G}((\mathcal{R}_{L^n}\alpha_n)\phi^1)\right)(\ell) = \mathcal{O}(\ell^2)(\mathcal{R}_{L^n}\alpha_n)^{*2}(\ell) + \sum_{j\geq 3}\mathcal{O}(\ell)(\mathcal{R}_{L^n}\alpha_n)^{*j}(\ell), \quad (4.28)$$

where the $\mathcal{O}(\ell^2)$-terms are due to the quadratic terms in the IBL with a factor $\partial_x q$. Since $(\alpha_n, w_n)_{X_{L^n}(2,r,2)} \leq 1$ we have $\|\alpha_n\|^j_{B_{L^n}(2,r,2)} \leq \|\alpha_n\|^2_{B_{L^n}(2,r,2)}$ for $j \geq 3$, thus the terms in (4.28) belong to s_1 and can be estimated as stated in (4.24). By the same considerations, all other terms in $\tilde{E}_c^*\big(\tilde{G}((\mathcal{R}_{L^n}\alpha_n)\phi^1 + L^{-(1-p)n}\mathcal{R}_{L^n}w_n)\big)$ are absorbed by s_2 and s_3 and can be estimated as specified in (4.24) and (4.26). $\square$

To prove estimates for the stable part of the nonlinearity, which in contrast to the critical part depends on x, we have to split H_n^s according to the different regularities in space. Moreover, lowering the weight in ℓ is not useful in this case. On the one hand, this is because we do not gain an additional factor ℓ by applying the mode filter E_s. On the other hand, the linear semigroup in (4.22) could not balance the missing weight without losing powers of L^{-n}.

Lemma 4.9. *Let $r \geq 3$. For $p \in (0, 1/2)$ there exists a positive constant C such that $H_n^s(\alpha_n, L^{-(1-p)n}w_n)$ can be split according to the order of x-derivatives in the form*

$$L^{(3-p)n}H_n^s(\alpha_n, L^{-(1-p)n}w_n) = \sum_{j=0}^{3}\begin{pmatrix} 0 \\ h_{n,j}^s(\alpha_n, L^{-(1-p)n}w_n)\end{pmatrix},$$

where

$$\begin{aligned} &\|h^s_{n,j}(\alpha_n, L^{-(1-p)n}w_n)\|_{B_{L^n}(2,r-j,2)} \\ &\leq C\Big(L^{(2-p)n}\|\alpha_n\|^2_{B_{L^n}(2,r,2)} + L^n\|\alpha_n\|_{B_{L^n}(2,r,2)}\|w_n\|_{\mathcal{B}_{L^n}(2,r,2)} \\ &\quad + L^{pn}\|w_n\|^2_{\mathcal{B}_{L^n}(2,r,2)}\Big) \end{aligned} \quad (4.29)$$

for all $(\alpha_n, w_n) \in X_{L^n}(2, r, 2)$ with $(\alpha_n, w_n)_{X_{L^n}(2,r,2)} \leq 1$.

Proof. The proof works along similar lines as for the critical part. Again, it is sufficient to estimate the quadratic terms. The appropriate estimates for higher order terms follow a fortiori since each convolution yields an additional factor L^{-n}. Moreover, every w_n gives a factor $L^{-(1-p)n}$. We only have to pay attention to the different regularities in space. Since in (2.18) the highest derivative $\partial_x^3 h$ occurs only linearly (i.e., the IBL is quasilinear), the second component of H_n^s maps to $B_{L^n}(2, r-j, 2)$, $j \in \{0, 3\}$ due to Lemma 4.6. Inequality (4.29) then follows by counting the respective powers of L^{-n}. $\square$

Remark 4.10. At first view, estimate (4.29) for the stable part seems worse than those for the critical part in Lemma 4.9, since the powers of L in the coefficients do not converge to zero for $n \to \infty$. However, applying the linear semigroup in (4.20) yields an additional factor L^{-2n}, which in §4.3 allows to prove that the stable component decays polynomially for $t \to \infty$. Furthermore, the nonlinearity $h^s_{n,3}$ only lies in $B_{L^n}(2, r-3, 2)$. But since in the second component the phase space is $B_{L^n}(2, r-1, 2)$, this can be smoothed out by the linear semigroup.

4.3. **Splitting, iteration, and conclusion.** The result of the formal calculation in §2.6 was

$$\partial_t\tilde{\alpha}(t,\ell) = -c_2\ell^2\tilde{\alpha}(t,\ell) + \mathrm{i}d\ell\chi(\ell)\tilde{\alpha}^{*2}(t,\ell) + \text{h.o.t.} \tag{4.30}$$

Since $\tilde{\alpha}$ is independent of x, (4.30) is reminiscent of the Fourier transform $\partial_t\hat{v} = -c_2\ell^2\hat{v} + \mathrm{i}d\ell\hat{v}^{*2} + \text{h.o.t.}$ of the Burgers equation. According to §2.5, the higher order perturbations are asymptotically irrelevant, and the renormalized solution $t^{1/2}v(t,t^{1/2}x)$ converges for $t\to\infty$ towards $f_z(x)$, see (2.54). In Fourier space, this corresponds to $\hat{v}(t,t^{-1/2}\ell)\to\hat{f}_z(\ell)$. If we consider the initial condition $v(1,x) = v_0(x)$, the parameter z is given by $\ln(z+1) = \frac{d}{c_2}\int_{\mathbb{R}} v_0(x)\,\mathrm{d}x = 2\pi\frac{d}{c_2}\hat{v}_0(0)$.

Transferring this result to (4.30), we expect the rescaled critical component $\tilde{\alpha}(t,t^{-1/2}\ell)$ to converge towards $\hat{f}_{z_0}(\ell)$ for $t\to\infty$, where

$$\ln(z_0+1) = 2\pi\frac{d}{c_2}\tilde{\alpha}(1,0). \tag{4.31}$$

Thus, for fixed times $t = L^{2n}$, $n\in\mathbb{N}$, the renormalized solution $\alpha_n(1,\ell) = \tilde{\alpha}(L^{2n},L^{-n}\ell)$ is expected to converge towards $\hat{f}_{z_0}(\ell)$ for $n\to\infty$.

Splitting. The formal considerations above give reason to split α_n into

$$\alpha_n(t,\ell) = \alpha_n^{(z)}(t,\ell) + L^{-(1-p)n}\gamma_n(t,\ell)$$

with the Fourier transformed profile

$$\alpha_n^{(z)}(t,\ell) := \chi(L^{-n}\ell)\hat{v}_{z_0}(t,\ell) = \chi(L^{-n}\ell)\hat{f}_{z_0}(t^{1/2}\ell).$$

Then, according to (4.5), the correction term γ_n satisfies

$$\begin{aligned}\partial_t\gamma_n = L^{2n}\mu_1(L^{-n}\cdot)\gamma_n + L^{(3-p)n}\Big(&B_n^c(\alpha_n) - B_n^c(\alpha_n^{(z)})\\ &+ H_n^c(\alpha_n, L^{-n(1-p)}w_n)\Big) + L^{(1-p)n}\mathrm{Res}_n,\end{aligned} \tag{4.32}$$

where $\mathrm{Res}_n := -\partial_t\alpha_n^{(z)} + L^{2n}\mu_1(L^{-n}\cdot)\alpha_n^{(z)} + L^{2n}B_n^c(\alpha_n^{(z)})$.

Lemma 4.11. *Let $|z_0|<1$. Then there exists a $C>0$ such that*

$$\sup_{t\in[L^{-2},1]}\|\mathrm{Res}_n\|_{B_{L^n}(2,r,2)} \le CL^{-n}|z_0|.$$

Proof. By construction we have $L^{2n}B_n^c(\alpha_n^{(z)})(\ell) = \mathrm{i}d\ell\chi(L^{-n}\ell)(\alpha_n^{(z)}*\alpha_n^{(z)})(\ell)$, while the renormalization of the largest eigenvalue reads

$$L^{2n}\mu_1(L^{-n}\ell) = -c_2\ell^2 + \mathcal{O}(L^{-n}\ell^3).$$

As $\hat{v}_{z_0}$ is an exact solution of $\partial_t\hat{v}_{z_0}(t,\ell) = -c_2\ell^2\hat{v}_{z_0}(t,\ell) + \mathrm{i}d\ell(\hat{v}_{z_0}*\hat{v}_{z_0})(t,\ell)$, we obtain

$$\begin{aligned}\mathrm{Res}_n(\ell) &= \mathrm{i}d\ell\chi(L^{-n}\ell)\left((\alpha_n^{(z)}*\alpha_n^{(z)})(\ell) - (\hat{v}_{z_0}*\hat{v}_{z_0})(\ell)\right)\\ &= \mathrm{i}d\ell\chi(L^{-n}\ell)\int_{-L^n/2}^{L^n/2}\left(\chi\left(\frac{\ell-m}{L^n}\right)\chi\left(\frac{m}{L^n}\right)-1\right)\hat{v}_{z_0}(\ell-m)\hat{v}_{z_0}(m)\,\mathrm{d}m.\end{aligned}$$

This can be estimated in $B_{L^n}(2,r,2)$ by $CL^{-n}|z_0|$ since the first factor in the integral is zero for both $\ell-m$ and m small, and since $\hat{v}_{z_0}$ is a smooth and exponentially decaying function. $\square$

Next we study the evolution of γ_n at the fixed wave number $\ell = 0$. Due to the definition of the critical mode filter $\tilde{E}_c^*$ we obtain $\tilde{\alpha}(t,0) = \langle \tilde{w}(t,0,\cdot), \psi^1(0,\cdot)\rangle$ where $\tilde{w} = e^{-i\ell c_1 t}(\tilde{h}, \tilde{q})^\top$, see (2.36) and (2.42). Since according to (2.41) we have $\psi^1(0,x) = (c_0, 0)^\top$, we obtain

$$\begin{aligned}\tilde{\alpha}(t,0) &= c_0 \int_0^\gamma \tilde{h}(t,0,x)\,\mathrm{d}x = c_0 \int_0^\gamma \sum_{j\in\mathbb{Z}} e^{ijk_0x}\mathcal{F}h(t,k_0 j)\,\mathrm{d}x \\ &= c_0\gamma\mathcal{F}h(t,0) = \frac{1}{2\pi}c_0\gamma \int_{\mathbb{R}} h(t,x)\,\mathrm{d}x.\end{aligned}$$

The perturbation's mass $\int_{\mathbb{R}} h\,\mathrm{d}x$ is conserved in the IBL, cf. §2.2. Thus, we have $\tilde{\alpha}(t,0) = \tilde{\alpha}(1,0)$ for all $t \geq 1$, which yields

$$\begin{aligned}L^{-(1-p)n}\gamma_n(t,0) &= \alpha_n(t,0) - \alpha_n^{(z)}(t,0) = \tilde{\alpha}(L^{2n}t,0) - \hat{f}_{z_0}(0) \\ &= \tilde{\alpha}(1,0) - \frac{c_2}{2\pi d}\ln(z_0+1) = 0\end{aligned}$$

for all $t \in [L^{-2}, 1]$, cf. (4.31). The following lemma shows a contraction property of the rescaled linear semigroup when acting on the remainder γ_n with $\gamma_n(0) = 0$, and explains why we require some regularity in ℓ in the spaces $B(m,r,b)$.

Lemma 4.12. *Let $g \in H^2(2)$ with $g(0) = 0$. Then*

$$\|e^{(1-L^{-2})L^{2n}\mu_1(L^{-n}\cdot)}\mathcal{R}_{1/L}g\|_{H^2(2)} \leq CL^{-1}\|g\|_{H^2(2)}.$$

Proof. We state here only the estimates for the $L^2(2)$-norm, as the additional factor L^{-1} coming from $\frac{\mathrm{d}}{\mathrm{d}\ell}(\mathcal{R}_{1/L}g)(\ell) = L^{-1}g'(L^{-1}\ell)$ leads to easier estimates in case of derivatives. Since $g(0) = 0$, we have for a $\tilde{\ell} \in [0, L^{-1}\ell]$

$$|g(L^{-1}\ell)| = L^{-1}\ell g'(\tilde{\ell}) \leq L^{-1}\ell\|g\|_{C^1} \leq CL^{-1}\ell\|g\|_{H^2(2)}$$

due to standard Sobolev embedding. Thus, we obtain

$$\begin{aligned}&\|e^{(1-L^{-2})L^{2n}\mu_1(L^{-2n}\cdot)}\mathcal{R}_{1/L}g\|_{L^2(2)}^2 \\ &= \int_{\mathbb{R}} e^{2(1-L^{-2})L^{2n}\mu_1(L^{-n}\ell)}(g(L^{-1}\ell))^2(1+\ell)^2\,\mathrm{d}\ell \\ &\leq CL^{-2}\|g\|_{H^2(2)}^2 \int_{\mathbb{R}} e^{2(1-L^{-2})L^{2n}\mu_1(L^{-n}\ell)}\ell^2(1+\ell)^2\,\mathrm{d}\ell,\end{aligned}$$

where the integral can be estimated independently of n since $L^{2n}\mu_1(L^{-n}\ell) = -c_2\ell^2 + \mathcal{O}(L^{-n}\ell^3)$. □

Let

$$\begin{gathered}g_{n,c}(\ell) := \gamma_n(1,\ell), \quad \rho_{n,c} := \|g_{n,c}\|_{B_{L^n}(2,r,2)}, \\ g_{n,s}(\ell,x) := w_n(1,\ell,x), \quad \rho_{n,s} := \|g_{n,s}\|_{\mathcal{B}_{L^n}(2,r,2)}, \\ \rho_n := \|\alpha_n(1)\|_{B_{L^n}(2,r,2)} + \|w_n(1)\|_{\mathcal{B}_{L^n}(2,r,2)}.\end{gathered}$$

Hence, $\rho_n \leq L^{-(1-p)n}\rho_{n,c} + \|\alpha_n^{(z)}(1)\|_{H^2(2)} + \rho_{n,s}$, and to prove Theorem 4.2 we will show that both $\rho_{n,c}$ and $\rho_{n,s}$ are bounded for $n \to \infty$.

Proof of Theorem 4.2. Taking L_0 and C_1 from Theorem 4.1, we assume the initial condition (α_0, w_0) to be small enough to fulfill

$$\rho_0 \leq L^{-m_0-1}, \quad L^{-m_0} \leq C_1 L^{-5/2}, \tag{4.33}$$

where $L > L_0$ and $m_0 \in \mathbb{N}$ are specified later. In particular, this yields

$$|z_0| \leq C\|\alpha_0^{(z)}(1,\cdot)\|_{H^2(2)} \leq CL^{-m_0-1}. \tag{4.34}$$

By (4.32) we have

$$\begin{aligned}\gamma_n(t,\ell) &= \mathrm{e}^{(t-L^{-2})L^{2n}\mu_1(L^{-n}\ell)}\gamma_n(L^{-2},\ell)\\ &\quad + L^{(3-p)n}\int_{L^{-2}}^t \mathrm{e}^{(t-s)L^{2n}\mu_1(L^{-n}\ell)}\Big(B_n^c(\alpha_n(s))(\ell) - B_n^c(\alpha_n^{(z)}(s))(\ell)\Big)\,\mathrm{d}s\\ &\quad + L^{(3-p)n}\int_{L^{-2}}^t \mathrm{e}^{(t-s)L^{2n}\mu_1(L^{-n}\ell)}H_n^c(\alpha_n(s), L^{-(1-p)n}w_n(s))(\ell)\,\mathrm{d}s\\ &\quad + L^{(1-p)n}\int_{L^{-2}}^t \mathrm{e}^{(t-s)L^{2n}\mu_1(L^{-n}\ell)}\mathrm{Res}_n(s,\ell)\,\mathrm{d}s,\end{aligned} \tag{4.35}$$

while w_n is obtained from (4.20). In order to apply an iteration scheme, we now assume

$$\rho_{n-1} \leq L^{-m_0}, \quad \rho_{n-1,c} \leq L^{-m_0}, \tag{4.36}$$

which is obviously true for $n = 1$. Since $\rho_{n-1} \leq C_1 L^{-5/2}$, Theorem 4.1 implies

$$\begin{aligned}\|(\alpha_n, w_n)\|_{\mathcal{X}_{L^n}^{r+1}((1/L^2,1),2,2)} &\leq CL^{5/2}\rho_{n-1},\\ \|(\alpha_n, w_n)\|_{\mathcal{X}_{L^n}^{r+2}((1/2,1),2,2)} &\leq CL^{5/2}\rho_{n-1}\end{aligned}$$

for a $C > 0$. Due to Corollary 3.12, these estimates yield

$$\sup_{t\in[L^{-2},1]}\|\alpha_n\|_{B_{L^n}(2,r,2)} \leq CL^{5/2}\rho_{n-1}, \tag{4.37}$$

$$\sup_{t\in[L^{-2},1]}\|w_n\|_{\mathcal{B}_{L^n}(2,r,2)} \leq CL^{5/2}\rho_{n-1}, \tag{4.38}$$

$$\sup_{t\in[1/2,1]}\|w_n\|_{\mathcal{B}_{L^n}(2,r+1,2)} \leq CL^{5/2}\rho_{n-1}. \tag{4.39}$$

First, we show an a-priori estimate for $\sup_{t\in[L^{-2},1]}\|\gamma_n(t,\cdot)\|_{B_{L^n}(2,r,2)}$ by estimating (4.35). We start with the first term on the right-hand side of (4.35). The initial condition (4.8) yields

$$\begin{aligned}\gamma_n(L^{-2},\ell) &= L^{(1-p)n}\Big(\alpha_{n-1}(1,L^{-1}\ell) - \chi(L^{-n}\ell)\hat{f}_{z_0}(L^{-1}\ell)\Big)\\ &= L^{1-p}\gamma_{n-1}(1,L^{-1}\ell) + L^{(1-p)n}\Big(\chi(L^{-(n-1)}\ell) - \chi(L^{-n}\ell)\Big)\hat{f}_{z_0}(L^{-1}\ell).\end{aligned}$$

Since $\chi(L^{-(n-1)}\ell) - \chi(L^{-n}\ell) = 0$ for $|\ell| \leq L^{n-1}r_\chi$ and since $\hat{f}_{z_0}$ decays exponentially, we obtain

$$\begin{aligned}\|\gamma_n(L^{-2},\cdot)\|_{B_{L^n}(2,r,2)} &\leq CL^{1-p}L^{5/2}\|\gamma_{n-1}(1,\cdot)\|_{B_{L^{n-1}}(2,r,2)} + CL^{-1}|z_0|\\ &\leq CL^{7/2-p}\rho_{n-1,c} + CL^{-1}|z_0|,\end{aligned} \tag{4.40}$$

where the factor $L^{5/2}$ comes from the different scalings of γ_{n-1} and γ_n, see (4.3). Next we estimate the first integral in (4.35). Due to the definitions of $\tilde{B}_c$ in (2.48) and of the convolution $*_{L^n}$ in (4.23) we have $L^{2n}B_n^c(\alpha_n)(\ell) = \mathrm{i}d\ell\chi(\ell)(\alpha_n *_{L^n} \alpha_n)(\ell)$, and thus

$$\begin{aligned} &L^{(3-p)n}\Big(B_n^c(\alpha_n) - B_n^c(\alpha_n^{(z)})\Big)(\ell) \\ &= L^{(1-p)n}\mathrm{i}d\ell\chi(\ell)\Big(L^{-2(1-p)n}\gamma_n *_{L^n} \gamma_n + L^{-(1-p)n}\gamma_n *_{L^n} \alpha_n^{(z)}\Big)(\ell). \end{aligned}$$

Therefore,

$$\begin{aligned} &L^{(3-p)n}\sup_{t\in[L^{-2},1]}\|B_n^c(\alpha_n) - B_n^c(\alpha_n^{(z)})\|_{B_{L^n}(2,r,1)} \\ &\le CL^{-(1-p)n}\sup_{t\in[L^{-2},1]}\|\gamma_n(t,\cdot)\|^2_{B_{L^n}(2,r,2)} + C|z_0|\sup_{t\in[L^{-2},1]}\|\gamma_n(t,\cdot)\|_{B_{L^n}(2,r,2)}. \end{aligned}$$

The missing weight in ℓ can be balanced by the linear semigroup in Lemma 4.3, which gives

$$\begin{aligned} &L^{(3-p)n}\sup_{t\in[L^{-2},1]}\Big\|\int_{L^{-2}}^t \mathrm{e}^{(t-s)L^{2n}\mu_1(L^{-n}\cdot)}\Big(B_n^c(\alpha_n(s)) - B_n^c(\alpha_n^{(z)}(s))\Big)\,\mathrm{d}s\Big\|_{B_{L^n}(2,r,2)} \\ &\le L^{(3-p)n}\sup_{t\in[L^{-2},1]}\|B_n^c(\alpha_n) - B_n^c(\alpha_n^{(z)})\|_{B_{L^n}(2,r,1)}\sup_{t\in[L^{-2},1]}\int_{L^{-2}}^t (t-s)^{-1/2}\,\mathrm{d}s \\ &\le CL^{-(1-p)n}\sup_{t\in[L^{-2},1]}\|\gamma_n(t,\cdot)\|^2_{B_{L^n}(2,r,2)} + C|z_0|\sup_{t\in[L^{-2},1]}\|\gamma_n(t,\cdot)\|_{B_{L^n}(2,r,2)}. \end{aligned} \tag{4.41}$$

Similarly, we can estimate the second integral in (4.35) by applying Lemma 4.8 and the estimates for α_n, w_n in (4.37), (4.39), which gives

$$L^{(3-p)n}\sup_{t\in[L^{-2},1]}\|H_n^c(\alpha_n(t), L^{-(1-p)n}w_n(t))\|_{B_{L^n}(2,r,p)} \le C(L^{5/2}\rho_{n-1})^2.$$

Using the properties of the linear semigroup in Lemma 4.3 yields

$$\begin{aligned} &L^{(3-p)n}\sup_{t\in[L^{-2},1]}\Big\|\int_{L^{-2}}^t \mathrm{e}^{(t-s)L^{2n}\mu_1(L^{-n}\cdot)}H_n^c(\alpha_n(s), L^{-(1-p)n}w_n(s))\,\mathrm{d}s\Big\|_{B_{L^n}(2,r,2)} \\ &\le CL^5\rho_{n-1}^2. \end{aligned} \tag{4.42}$$

Finally, by Lemma 4.11, we obtain

$$L^{(1-p)n}\sup_{t\in[L^{-2},1]}\Big\|\int_{L^{-2}}^t \mathrm{e}^{(t-s)L^{2n}\mu_1(L^{-n}\ell)}\mathrm{Res}_n(s,\ell)\,\mathrm{d}s\Big\|_{B_{L^n}(2,r,2)} \le CL^{-pn}|z_0|. \tag{4.43}$$

Combining (4.35) and (4.40)-(4.43), we achieve

$$\begin{aligned} &\sup_{t\in[L^{-2},1]}\|\gamma_n(t,\cdot)\|_{B_{L^n}(2,r,2)} \\ &\le CL^{7/2-p}\rho_{n-1,c} + CL^{-1}|z_0| + CL^{-(1-p)n}\sup_{t\in[L^{-2},1]}\|\gamma_n(t,\cdot)\|^2_{B_{L^n}(2,r,2)} \\ &\quad + C|z_0|\sup_{t\in[L^{-2},1]}\|\gamma_n(t,\cdot)\|_{B_{L^n}(2,r,2)} + CL^5\rho_{n-1}^2 + CL^{-pn}|z_0|. \end{aligned} \tag{4.44}$$

By choosing m_0 large enough we obtain $C|z_0| \le CL^{-m_0-1} \le 1/3$, and thus

$$L^{-(1-p)n}\sup_{t\in[L^{-2},1]}\|\gamma_n(t,\cdot)\|_{B_{L^n}(2,r,2)}$$

$$\begin{aligned}&\le \sup_{t\in[L^{-2},1]} \|\alpha_n(t,\cdot)-\alpha_n^{(z)}(t,\cdot)\|_{B_{L^n}(2,r,2)}\\&\le C(L^{5/2}\rho_{n-1}+|z_0|)\le CL^{5/2}L^{-m_0}+CL^{-m_0-1}\le 1/(3C).\end{aligned}$$

Finally, (4.44) yields the a-priori estimate

$$\begin{aligned}&1/3 \sup_{t\in[L^{-2},1]} \|\gamma_n(t,\cdot)\|_{B_{L^n}(2,r,2)}\\&\le CL^{7/2-p}\rho_{n-1,c}+CL^{-1}|z_0|+CL^5\rho_{n-1}^2+CL^{-pn}|z_0|.\end{aligned} \tag{4.45}$$

Iteration. To conclude $\rho_n\le L^{-m_0}$ and $\rho_{n,c}\le L^{-m_0}$ from assumption (4.36), the first term on the right-hand side of (4.45) is not yet small enough. Thus, we have to use (4.35) once more for the fixed time $t=1$. In this case, as $\gamma_{n-1}(t,0)=0$, we can apply Lemma 4.12. In contrast to (4.40), we achieve

$$\begin{aligned}&\|e^{(1-L^{-2})L^{2n}\mu_1(L^{-n}\cdot)}\gamma_n(L^{-2},\cdot)\|_{B_{L^n}(2,r,2)}\\&\le CL^{-p}\|\gamma_{n-1}(1,\cdot)\|_{B_{L^{n-1}}(2,r,2)}+CL^{(1-p)n-1}\|\\&\quad\times\Big(\chi(L^{-(n-2)}\cdot)-\chi(L^{-(n-1)}\cdot)\Big)\hat f_{z_0}\|_{H^2(2)}\\&\le CL^{-p}\rho_{n-1,c}+CL^{-1}\|\hat f_{z_0}\|_{H^2(2)}\le CL^{-p}\rho_{n-1,c}+CL^{-1}|z_0|.\end{aligned}$$

Similar to (4.44), we obtain

$$\begin{aligned}\rho_{n,c}&\le CL^{-p}\rho_{n-1,c}+CL^{-1}|z_0|+CL^{-(1-p)n}\sup_{t\in[L^{-2},1]}\|\gamma_n(t,\cdot)\|^2_{B_{L^n}(2,r,2)}\\&\quad+C|z_0|\sup_{t\in[L^{-2},1]}\|\gamma_n(t,\cdot)\|_{B_{L^n}(2,r,2)}+CL^5\rho_{n-1}^2+CL^{-pn}|z_0|.\end{aligned} \tag{4.46}$$

From (4.45) we obtain

$$\begin{aligned}&\sup_{t\in[L^{-2},1]}\|\gamma_n(t,\cdot)\|_{B_{L^n}(2,r,2)}\\&\le C\Big(L^{7/2-p}L^{-m_0}+L^{-1}L^{-m_0-1}+L^{5-2m_0}+L^{-pn}L^{-m_0-1}\Big)\le CL^{7/2-p-m_0},\end{aligned}$$

and plugging this estimate into (4.46) yields

$$\begin{aligned}\rho_{n,c}\le C\Big(&L^{-p}L^{-m_0}+L^{-1}L^{-m_0-1}+L^{-(1-p)n}L^{7-2p-2m_0}+L^{-m_0-1}L^{7/2-p-m_0}\\&+L^5L^{-2m_0}+L^{-pn}L^{-m_0-1}\Big).\end{aligned}$$

Choosing now $L>L_0$ such that $C\le L^p/18$, we obtain

$$\rho_{n,c}\le\frac{1}{18}\Big(L^{-m_0}+L^{-m_0-1}+L^{7-2m_0}+L^{3/2-2m_0}+L^{5-2m_0}+L^{-m_0-1}\Big).$$

Choosing finally $m_0>7$ such that $L^{7-m_0}\le 1$ leads to

$$\rho_{n,c}\le\frac13 L^{-m_0}. \tag{4.47}$$

Next, we estimate the stable part $\rho_{n,s}$. From (4.20) we obtain

$$\begin{aligned}g_{n,s}(\ell,x)&=L^{1-p}e^{(1-L^{-2})L^{2n}\tilde A_s(L^{-n}\ell)}g_{n-1,s}(L^{-1}\ell,x)\\&\quad+L^{(3-p)n}\int_{L^{-2}}^1 e^{(1-s)L^{2n}\tilde A_s(L^{-n}\ell)}H_n^s(\alpha_n(s),L^{-(1-p)n}w_n(s))(\ell,x)\,ds.\end{aligned} \tag{4.48}$$

Applying Lemma 4.3 we can estimate the first term on the right-hand side of (4.48) as

$$\begin{aligned}
&L^{1-p}\|\mathrm{e}^{(1-L^{-2})L^{2n}\tilde{A}_s(L^{-n}\cdot)}\mathcal{R}_{L^{-1}}g_{n-1,s}\|_{\mathcal{B}_{L^n}(2,r,2)}\\
&\le L^{1-p}\mathrm{e}^{-\sigma_1(1-L^{-2})L^{2n}}\|g_{n-1,s}\|_{\mathcal{B}_{L^{n-1}}(2,r,2)}\\
&\le CL^{-1}\rho_{n-1,s}.
\end{aligned}$$

To estimate the integral in (4.48) we use Lemma 4.9, which gives

$$\begin{aligned}
&L^{(3-p)n}\Big\|\int_{L^{-2}}^{1}\mathrm{e}^{(1-s)L^{2n}\tilde{A}_s(L^{-n}\cdot)}H_n^s(\alpha_n(s),L^{-(1-p)n}w_n(s))\,\mathrm{d}s\Big\|_{\mathcal{B}_{L^n}(2,r,2)}\\
&\le\sum_{j=0}^{3}\Big\|\int_{L^{-2}}^{1}\mathrm{e}^{(1-s)L^{2n}\tilde{A}_s(L^{-n}\cdot)}\begin{pmatrix}0\\ h_{n,j}^s(\alpha_n,L^{-(1-p)n}w_n)\end{pmatrix}\mathrm{d}s\Big\|_{\mathcal{B}_{L^n}(2,r,2)}\\
&=:\sum_{j=0}^{3}\|I_j\|_{\mathcal{B}_{L^n}(2,r,2)}.
\end{aligned}$$

For the integrals I_0 and I_1, the smoothing properties of the linear semigroup are not required since $\mathcal{B}_L(m,r,b):=B_L(m,r,b)\times B_L(m,r-1,b)$; i.e., the regularity needed for the second component is only $r-1$. Applying Lemma 4.3 and Lemma 4.4 we achieve

$$\begin{aligned}
\|I_j\|_{\mathcal{B}_{L^n}(2,r,2)}&\le C\int_{L^{-2}}^{1}\mathrm{e}^{-\sigma_1(1-s)L^{2n}}\|h_{n,j}^s\|_{B_{L^n}(2,r-j,2)}\,\mathrm{d}s\\
&\le CL^{(2-p)n}(L^{5/2}\rho_{n-1})^2\int_{L^{-2}}^{1}\mathrm{e}^{-\sigma_1(1-s)L^{2n}}\,\mathrm{d}s\le CL^{-pn}L^5\rho_{n-1}^2
\end{aligned}$$

for $j\in\{0,1\}$. For the estimate of I_2, the linear semigroup has to smooth out one x-derivative, which yields

$$\begin{aligned}
\|I_2\|_{\mathcal{B}_{L^n}(2,r,2)}&\le C\int_{L^{-2}}^{1}\mathrm{e}^{-\sigma_1(1-s)L^{2n}}(1+L^{-n}(1-s)^{-1/2})\\
&\quad\times\|h_{n,2}^s(\alpha_n(s),L^{-(1-p)n}w_n(s))\|_{B_{L^n}(2,r-2,2)}\,\mathrm{d}s\\
&\le CL^{(2-p)n}(L^{5/2}\rho_{n-1})^2\int_{L^{-2}}^{1}\mathrm{e}^{-\sigma_1(1-s)L^{2n}}(1+L^{-n}(1-s)^{-1/2})\,\mathrm{d}s\\
&\le CL^{-pn}L^5\rho_{n-1}^2.
\end{aligned}$$

Treating I_3 the same way would lead to factor $(1-s)^{-1}$ and therefore to a non-integrable singularity. Thus, we split I_3 into

$$I_3=\int_{L^{-2}}^{1/2}\dots\,\mathrm{d}s+\int_{1/2}^{1}\dots\,\mathrm{d}s.$$

While on the first time interval $[L^{-2},1/2]$ the singularity does not occur, we can use the higher regularity of w_n on the second time interval as stated in (4.39). Note again that α_n is independent of x, thus the value of r plays no role in the spaces for α_n. We obtain

$$\begin{aligned}
\|I_3\|_{\mathcal{B}_{L^n}(2,r,2)}&\le C\int_{L^{-2}}^{1/2}\mathrm{e}^{-\sigma_1(1-s)L^{2n}}(1+L^{-n}(1-s)^{-1})\\
&\quad\times\|h_{n,3}^s(\alpha_n(s),L^{-(1-p)n}w_n(s))\|_{B_{L^n}(2,r-3,2)}\,\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&+ C\int_{1/2}^{1} \mathrm{e}^{-\sigma_1(1-s)L^{2n}}(1+L^{-n}(1-s)^{-1/2})\\
&\times \|h^s_{n,3}(\alpha_n(s), L^{-(1-p)n}w_n(s))\|_{B_{L^n}(2,r-2,2)}\,\mathrm{d}s\\
&\le CL^{-pn}L^5\rho_{n-1}^2.
\end{aligned}
$$

Collecting the estimates for I_j gives

$$
\begin{aligned}
\rho_{n,s} &\le CL^{-1}\rho_{n-1} + CL^{-pn}L^5\rho_{n-1}^2\\
&\le CL^{-1}L^{-m_0} + CL^{-p}L^5L^{-2m_0} \le \frac{1}{3}L^{-m_0}.
\end{aligned}
\tag{4.49}
$$

Combining (4.34), (4.47), and (4.49), we finally obtain

$$
\rho_n = \|\alpha_n(1)\|_{B_{L^n}(2,r,2)} + \|w_n(1)\|_{\mathcal{B}_{L^n}(2,r,2)} \le L^{-(1-p)n}\rho_{n,c} + C|z_0| + \rho_{n,s} \le L^{-m_0}.
$$

Conclusion. So far we have shown that if $\rho_0 \le L^{-m_0-1}$, then $\rho_{n,c}$, $\rho_{n,s}$, and ρ_n stay smaller than L^{-m_0} for all $n \in \mathbb{N}$. In order to prove (4.17), we estimate

$$
(\ell, x) \mapsto \tilde{w}(t, t^{-1/2}\ell, x) - \chi(t^{-1/2}\ell)\hat{f}_{z_0}(\ell)\phi^1(0,x)
$$

at the discrete times $t = L^{2n}$, $n \in \mathbb{N}$; i.e.,

$$
\begin{aligned}
&\tilde{w}(L^{2n}, L^{-n}\ell, x) - \chi(L^{-n}\ell)\hat{f}_{z_0}(\ell)\phi^1(0,x)\\
&= \tilde{\alpha}(L^{2n}, L^{-n}\ell)\phi^1(L^{-n}\ell, x) + \tilde{w}_s(L^{2n}, L^{-n}\ell, x) - \chi(L^{-n}\ell)\hat{f}_{z_0}(\ell)\phi^1(0,x)\\
&= \alpha_n(1,\ell)\phi^1(L^{-n}\ell, x) + L^{-(1-p)n}w_n(1,\ell,x) - \chi(L^{-n}\ell)\hat{f}_{z_0}(\ell)\phi^1(0,x)\\
&= \Big(\alpha_n(1,\ell) - \chi(L^{-n}\ell)\hat{f}_{z_0}(\ell)\Big)\phi^1(0,x) + \alpha_n(1,\ell)\left(\phi^1(L^{-n}\ell,x) - \phi^1(0,x)\right)\\
&\quad + L^{-(1-p)n}w_n(1,\ell,x)\\
&= L^{-(1-p)n}\gamma_n(1,\ell)\phi^1(0,x) + \alpha_n(1,\ell)\left(\phi^1(L^{-n}\ell,x) - \phi^1(0,x)\right)\\
&\quad + L^{-(1-p)n}w_n(1,\ell,x).
\end{aligned}
\tag{4.50}
$$

Taking the $\mathcal{B}_{\sqrt{t}}(2,r,2)$-norm at $t = L^{2n}$, the first term on the right-hand side of (4.50) yields

$$
\begin{aligned}
&\|(\ell,x) \mapsto \gamma_n(1,\ell)\phi^1(0,x)\|_{\mathcal{B}_{L^n}(2,r,2)}\\
&= \Big(\sum_{j=0}^{2}\int_{-1/2k_0L^n}^{1/2k_0L^n}(1+\ell^2)^2|\partial_\ell^j\gamma_n(1,\ell)|^2\|\phi^1(0,\cdot)\|^2_{H^r(I_\gamma)}\,\mathrm{d}\ell\Big)^{1/2}\\
&\le C\|\gamma_n(1,\cdot)\|_{B_{L^n}(2,r,2)} \le CL^{-m_0}.
\end{aligned}
$$

The second term in (4.50) is estimated as follows. We have

$$
\begin{aligned}
&\|(\ell,x) \mapsto \alpha_n(1,\ell)\left(\phi^1(L^{-n}\ell,x) - \phi^1(0,x)\right)\|_{\mathcal{B}_{L^n}(2,r,1)}\\
&= \Big(\sum_{j=0}^{2}\int_{-1/2k_0L^n}^{1/2k_0L^n}(1+\ell^2)\Big\|\frac{\mathrm{d}^j}{\mathrm{d}\ell^j}[\alpha_n(1,\ell)\left(\phi^1(L^{-n}\ell,\cdot) - \phi^1(0,\cdot)\right)]\Big\|^2_{H^r(I_\gamma)}\,\mathrm{d}\ell\Big)^{1/2}.
\end{aligned}
\tag{4.51}
$$

We now have to distinguish between terms in which the eigenvector ϕ^1 is differentiated with respect to ℓ and those in which only α_n is differentiated. For the first group, we obtain estimates of the type

$$\begin{aligned}&\int_{-1/2k_0L^n}^{1/2k_0L^n}(1+\ell^2)|\partial_\ell^{j_1}\alpha_n(1,\ell)|^2\Big\|\frac{\mathrm{d}^{j_2}}{\mathrm{d}\ell^{j_2}}\phi^1(L^{-n}\ell,\cdot)\Big\|_{H^r(I_\gamma)}^2\,\mathrm{d}\ell\\&\le L^{-2n}\int_{-1/2k_0L^n}^{1/2k_0L^n}(1+\ell^2)|\partial_\ell^{j_1}\alpha_n(1,\ell)|^2\Big\|\frac{\mathrm{d}^{j_2-1}}{\mathrm{d}\ell^{j_2-1}}(\partial_\ell\phi^1(L^{-n}\ell,\cdot))\Big\|_{H^r(I_\gamma)}^2\,\mathrm{d}\ell\\&\le CL^{-2n}\|\alpha_n(1,\cdot)\|_{B_{L^n}(2,r,2)}^2,\end{aligned}$$

where j_2 is at least one. For the terms without a derivative of ϕ^1 we can use Taylor expansion, which leads to

$$\phi^1(L^{-n}\ell,x)-\phi^1(0,x)=L^{-n}\ell\partial_\ell\phi^1(\tilde{\ell}(\ell),x),$$

where $|\tilde{\ell}(\ell)|\le L^{-n}|\ell|$. Thus, in (4.51) there also occur terms of the type

$$\begin{aligned}&\int_{-1/2k_0L^n}^{1/2k_0L^n}(1+\ell^2)\|\partial_\ell^{j}\alpha_n(1,\ell)L^{-n}\ell\partial_\ell\phi^1(\tilde{\ell}(\ell),\cdot)\|_{H^r(I_\gamma)}^2\,\mathrm{d}\ell\\&\le L^{-2n}\int_{-1/2k_0L^n}^{1/2k_0L^n}(1+\ell^2)^2|\partial_\ell^{j}\alpha_n(1,\ell)|^2\|\partial_\ell\phi^1(\tilde{\ell}(\ell),\cdot)\|_{H^r(I_\gamma)}^2\,\mathrm{d}\ell\\&\le CL^{-2n}\|\alpha_n(1,\cdot)\|_{B_{L^n}(2,r,2)}^2.\end{aligned}$$

Note that the additional factor ℓ in this estimate is the reason why we have to lower the weight in ℓ from $\mathcal{B}_{\sqrt{t}}(2,r,2)$ to $\mathcal{B}_{\sqrt{t}}(2,r,1)$ in Theorem 4.2. Altogether, we obtain

$$\begin{aligned}\|(\ell,x)\mapsto\alpha_n(1,\ell)\left(\phi^1(L^{-n}\ell,x)-\phi^1(0,x)\right)\|_{\mathcal{B}_{L^n}(2,r,1)}&\le CL^{-n}\|\alpha_n(1,\cdot)\|_{B_{L^n}(2,r,2)}\\&\le CL^{-m_0}L^{-n}.\end{aligned}$$

The $\mathcal{B}_{L^n}(2,r,2)$-norm of third term on the right-hand side of (4.50) can be easily estimated by $L^{-(1-p)n}\rho_{n,s}$. Combining all these estimates, we obtain

$$\|(\ell,x)\mapsto\tilde{w}(L^{2n},L^{-n}\ell,x)-\chi(L^{-n}\ell)\hat{f}_{z_0}(\ell)\phi^1(0,x)\|_{\mathcal{B}_{L^n}(2,r,1)}\le CL^{-(1-p)n}.$$

This is (4.17) for $t=L^{2n}$, and the local existence Theorem 4.1 yields the result for all $t\in[L^{2n},L^{2(n+1)}]$.

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Tobias Häcker
Institut für Analysis, Dynamik und Modellierung, Universität Stuttgart, Pfaffenwaldring 57, D-70569 Stuttgart, Germany
E-mail address: `tobias.haecker@mathematik.uni-stuttgart.de`

Guido Schneider
Institut für Analysis, Dynamik und Modellierung, Universität Stuttgart, Pfaffenwaldring 57, D-70569 Stuttgart, Germany
E-mail address: `guido.schneider@mathematik.uni-stuttgart.de`

Hannes Uecker
Institut für Mathematik, Carl von Ossietzky Universität Oldenburg, D-26111 Oldenburg, Germany
E-mail address: `hannes.uecker@uni-oldenburg.de`