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VARIATIONAL APPROACH FOR WEAK QUASIPERIODIC SOLUTIONS OF QUASIPERIODICALLY EXCITED LAGRANGIAN SYSTEMS ON RIEMANNIAN MANIFOLDS

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Abstract. We apply a variational method to prove the existence of weak Besicovitch quasiperiodic solutions for natural Lagrangian system on Riemannian manifold with time-quasiperiodic force function. In contrast to previous papers, our approach does not require non-positiveness condition for sectional Riemannian curvature. As an application of obtained results, we find conditions for the existence of weak quasiperiodic solutions in spherical pendulum system under quasiperiodic forcing.

1. Introduction

Let $\mathcal{M}$ be a smooth complete connected m-dimensional Riemannian manifold equipped with an inner product $\langle\cdot,\cdot\rangle$ on fibers $T_x\mathcal{M}$ of tangent bundle $T\mathcal{M}$ as well as with Levi-Civita connection ∇. A natural system on $\mathcal{M}$ is a Lagrangian system with Lagrangian density of the form $L\big|_{T_x\mathcal{M}} = \frac{1}{2}\langle\dot{x},\dot{x}\rangle - \Pi(t,x)$ where the terms $\frac{1}{2}\langle\dot{x},\dot{x}\rangle$ and $\Pi(t,x)$ stand for kinetic and potential energy respectively. In this paper, we consider the special case of potential energy represented as $\Pi := -W(\omega t,x)$ where $W(\omega t,x)$ is ω-quasiperiodic force function generated by a function $W(\cdot,\cdot) \in \mathrm{C}^{0,2}(\mathbb{T}^k \times \mathcal{M},\mathbb{R})$ ($W(\cdot,\cdot)$ is continuous together with $W''_{xx}(\cdot,\cdot)$); here $\mathbb{T}^k = \mathbb{R}^k/2\pi\mathbb{Z}^k$ is k-dimensional torus and $\omega = (\omega_1,\dots,\omega_k) \in \mathbb{R}^k$ is a frequencies vector with rationally independent components. The problem is to detect in such a system ω-quasiperiodic oscillations.

In local coordinates $(x_1,\dots,x_m)$, $i = 1,\dots,m$, the system is governed by the equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\sum_{j=1}^{m} g_{ij}(x)\dot{x}_j\Big) = \frac{\partial W(\omega t,x)}{\partial x_i}, \quad i = 1,\dots,m,$$

where g_{ij} is a metric tensor. When $\mathcal{M}$ is a Euclidean space $\mathbb{E}^m$, and hence, $g_{ij}(x) = \delta_{ij}$ (the Kronecker symbol), the above mentioned problem has been studied even in more general case of almost periodic second order systems. Non-local existence results for such systems are usually obtained using topological principles and

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methods of nonlinear analysis under certain monotonicity, convexity and coercivity conditions (see, e.g., [12, 30, 9, 13, 14, 11]).

In periodic case, variational methods of finding and constructing periodic solutions have been developed in details (see, e.g., [28, 15, 27, 24, 22]). Blot in his series of papers [5, 6, 7, 8] applied variational method to establish the existence of weak almost periodic solutions for systems in $\mathbb{E}^m$. Later, this method was used in [3, 20, 32, 2, 25, 1, 19] to prove the existence of weak and classical almost periodic solutions for various systems of variational type. In [33, 34], weak and classical quasiperiodic solutions were found for natural mechanical systems in convex compact subsets of Riemannian manifolds with non-positive sectional curvature. The goal of the present paper is to extend these results to natural systems on arbitrary Riemannian manifolds.

This paper is organized as follows. In Sections 2–4 we improve results announced in [26] on variational approach for searching weak quasiperiodic solutions of quasiperiodically excited Lagrangian systems on Riemannian manifold. In particular, in Sect. 2 we define the weak quasiperiodic solution as Besicovitch quasiperiodic function generated by extremal point of functional J (see (2.2)). Here we also discuss about the difficulties that occur in application of variational approach when we reject the requirements of non-positiveness of Riemannian curvature. In Section 3 we show how one can ensure convexity properties of functional J by means of geodesics of conformally equivalent metric associated with inner product $\mathrm{e}^{V(x)}\langle\cdot,\cdot\rangle\big|_{T_x\mathcal{M}}$ where $V(\cdot):\mathcal{M}\to\mathbb{R}$ is appropriately chosen function. The conditions we impose on this auxiliary function are less restrictive that of [26]. At the same time, with respect to the force function $W(\cdot,\cdot)$, the function $V(\cdot)$ plays a role which is, to some extent, analogous to that of guiding function in [18, 21]. In Section 4 we give the proof of the main existence theorem based on variational approach. In Section 5 we describe a searching procedure for weak quasiperiodic solution; the latter one is associated with a minimum point of the mean $\bar{W}(x)$ of force function $W(\omega t,x)$, while the oscillating part $\tilde{W}(\omega t,x):=W(\omega t,x)-\bar{W}(x)$ plays a role of perturbation. Finally, in Section 6 we show how the developed approach works when studying quasiperiodic forcing of natural systems on hypersurfaces of Euclidean space. In particular, we consider quasiperiodic forcing of physical pendulum and derive simple sufficient condition for the existence of weak quasiperiodic solutions to corresponding Lagrangian system.

2. Variational approach

One can interpret a natural system on $\mathcal{M}$ as a natural system in Euclidean space $\mathbb{E}^n$ (of appropriate dimension n) with holonomic constraint. Namely, by famous Nash theorem, for some natural number n, there exists a smooth isometric embedding $\iota:\mathcal{M}\to\mathbb{E}^n$. Denote by $\hat{W}(\cdot,\cdot)\in\mathrm{C}^{0,2}(\mathbb{T}^k\times\mathbb{E}^n,\mathbb{R})$ an extension of the function $W(\cdot,\cdot)$. Let the set $\iota(\mathcal{M})$ play the role of holonomic constraint for natural system in $\mathbb{E}^n$ with kinetic energy $K=\frac{1}{2}\langle\dot{y},\dot{y}\rangle_{\mathbb{E}^n}$ and potential energy $-\hat{W}(t\omega,y)$. Then the Lagrangian density of the above natural system on $\mathcal{M}$ is represented in the form $\frac{1}{2}\langle\iota_*\dot{x},\iota_*\dot{x}\rangle_{\mathbb{E}^n}+\hat{W}(t\omega,\iota(x))$ (here ι_* stands for the tangent map generated by ι).

In what follows we shall use identical notations for $\mathcal{M}$ and $\iota(\mathcal{M})$, for vectors $\xi\in T\mathcal{M}$ and $\iota_*\xi\in\mathbb{E}^n$, for inner product $\langle\cdot,\cdot\rangle_{\mathbb{E}^n}$ of $\mathbb{E}^n$ and the induced inner

product
$$\langle\cdot,\cdot\rangle = \iota^*\langle\cdot,\cdot\rangle_{\mathbb{E}^n} := \langle\iota_*\cdot,\iota_*\cdot\rangle_{\mathbb{E}^n}$$
as well as for the function $W(\cdot,\cdot)$ on $\mathbb{T}^k\times\mathcal{M}$ and its extension $\hat{W}(\cdot,\cdot)$ on $\mathbb{T}^k\times\mathbb{E}^n$.

Denote by $\mathrm{H}(\mathbb{T}^k,\mathbb{E}^n) := L^2(\mathbb{T}^k,\mathbb{E}^n)$ the space of $\mathbb{E}^n$-valued functions on k-torus which are integrable with the square of Euclidean norm $\|\cdot\| := \sqrt{\langle\cdot,\cdot\rangle}$. Define on $\mathrm{H}(\mathbb{T}^k,\mathbb{E}^n)$ the standard scalar product $\langle\cdot,\cdot\rangle_0 = (2\pi)^{-k}\int_{\mathbb{T}^k}\langle\cdot,\cdot\rangle \mathrm{d}\varphi$ and the corresponding semi-norm $\|\cdot\|_0 := \sqrt{\langle\cdot,\cdot\rangle_0}$. By $\mathrm{H}^1_\omega(\mathbb{T}^k,\mathbb{E}^n)$ denote the space of functions $f(\cdot)\in\mathrm{H}(\mathbb{T}^k,\mathbb{E}^n)$ each of which has weak (Sobolev) derivative $D_\omega f(\cdot)\in\mathrm{H}(\mathbb{T}^k,\mathbb{E}^n)$ in the direction of vector ω. Recall that $D_\omega f(\cdot)$ is characterized by the following property
$$\int_{\mathbb{T}^k}\langle D_\omega f(\varphi), g(\varphi)\rangle\,\mathrm{d}\varphi = -\int_{\mathbb{T}^k}\langle f(\varphi), D_\omega g(\varphi)\rangle\,\mathrm{d}\varphi \quad \forall g(\cdot)\in\mathrm{C}^1(\mathbb{T}^k,\mathbb{E}^n),$$
where $D_\omega g(\varphi) := \sum_{j=1}^k \frac{\partial g(\varphi)}{\partial\varphi_j}\omega_j$. Recall also that a function $u(\cdot)\in\mathrm{H}(\mathbb{T}^k,\mathbb{E}^n)$ with Fourier series $\sum_{\mathbf{n}\in\mathbb{Z}^k} u_{\mathbf{n}}\mathrm{e}^{\mathrm{i}\mathbf{n}\cdot\varphi}$ has a weak derivative if and only if the series $\sum_{\mathbf{n}\in\mathbb{Z}^k}|\mathbf{n}\cdot\omega|^2\|u_{\mathbf{n}}\|^2$ converges and the Fourier series of $D_\omega u(\cdot)$ is $\sum_{\mathbf{n}\in\mathbb{Z}^k}\mathrm{i}(\mathbf{n}\cdot\omega)u_{\mathbf{n}}\mathrm{e}^{\mathrm{i}\mathbf{n}\cdot\varphi}$ (see, e.g., [10, 29]).

The space $\mathrm{H}^1_\omega(\mathbb{T}^k,\mathbb{E}^n)$ is equipped with the semi-norm $\|\cdot\|_1$ generated by the scalar product $\langle D_\omega\cdot, D_\omega\cdot\rangle_0 + \langle\cdot,\cdot\rangle_0$. After identification of functions coinciding a.e., both spaces $\mathrm{H}(\mathbb{T}^k,\mathbb{E}^n)$ and $\mathrm{H}^1_\omega(\mathbb{T}^k,\mathbb{E}^n)$ becomes Hilbert spaces with norms $\|\cdot\|_0$ and $\|\cdot\|_1$ respectively.

To any function $u(\cdot)\in\mathrm{H}(\mathbb{T}^k,\mathbb{E}^n)$ with Fourier series $\sum_{\mathbf{n}\in\mathbb{Z}^k}u_{\mathbf{n}}\mathrm{e}^{\mathrm{i}\mathbf{n}\cdot\varphi}$, one can put into correspondence a Besicovitch quasiperiodic function $x(t) := u(t\omega)$ defined by its Fourier series $\sum_{\mathbf{n}\in\mathbb{Z}^k}u_{\mathbf{n}}\mathrm{e}^{\mathrm{i}(\mathbf{n}\cdot\omega)t}$ (see, e.g., [10, 29]). If $u(\cdot)\in\mathrm{H}^1_\omega(\mathbb{T}^k,\mathbb{E}^n)$ then $\dot{x}(t)$ denotes a Besicovitch quasiperiodic function $D_\omega u(t\omega)$.

We define weak solution of Lagrangian system on $\mathcal{M}$ with density $L = \frac{1}{2}\langle\dot{x},\dot{x}\rangle + W(t\omega, x)$ in a slightly different way then in [32]. First, for any bounded subset $\mathcal{A}\subseteq\mathcal{M}$, put
$$\mathcal{S}_{\mathcal{A}} := \mathrm{C}^\infty(\mathbb{T}^k,\mathcal{A}).$$
Observe that if $u_j(\cdot)\in\mathcal{S}_{\mathcal{A}}$ is a sequence bounded in $\mathrm{H}^1_\omega(\mathbb{T}^k,\mathbb{E}^n)$ and convergent to a function $u(\cdot)$ by norm of the space $\mathrm{H}(\mathbb{T}^k,\mathbb{E}^n)$ (recall that we consider the set $\mathcal{A}\subseteq\mathcal{M}$ also as a subset of $\mathbb{E}^n$), then for any $\mathbf{n}\in\mathbb{Z}^k$ the sequence of Fourier series coefficients $u_{j,\mathbf{n}}$ converges to $u_{\mathbf{n}}$ and for some $K>0$ we have
$$\begin{aligned}\sum_{|\mathbf{n}|\le N}|\mathbf{n}\cdot\omega|^2\|u_{\mathbf{n}}\|^2 &= \lim_{j\to\infty}\sum_{|\mathbf{n}|\le N}|\mathbf{n}\cdot\omega|^2\|u_{j,\mathbf{n}}\|^2\\ &\le \liminf_{j\to\infty}\sum_{\mathbf{n}\in\mathbb{Z}^k}|\mathbf{n}\cdot\omega|^2\|u_{j,\mathbf{n}}\|^2 \le K \quad \forall N\in\mathbb{N},\end{aligned}$$
where $|\mathbf{n}| := \max_i|n_i|$. Hence, $u(\cdot)\in\mathrm{H}^1_\omega(\mathbb{T}^k,\mathbb{E}^n)$ and
$$\|D_\omega u\|_0 \le \liminf_{j\to\infty}\|D_\omega u_j\|_0.$$
Moreover, $u_j(\cdot)$ converges to $u(\cdot)$ weakly in $\mathrm{H}^1_\omega(\mathbb{T}^k,\mathbb{E}^n)$. In fact, there exists $K_1>0$ such that $\|u_j\|_1\le K_1$ and for any $g(\cdot)\in\mathrm{H}^1_\omega(\mathbb{T}^k,\mathbb{E}^n)$ and $\varepsilon>0$ there exists a trigonometric polynomial $p(\cdot)$ such that $\|g-p\|_1<\varepsilon$. Then, since $u_{j,\mathbf{n}}\to u_{\mathbf{n}}$, we have
$$\lim_{j\to\infty}|\langle u_j-u,g\rangle_1| \le \lim_{j\to\infty}|\langle u_j-u,p\rangle_1| + (K+\|u\|_1)\varepsilon = (K+\|u\|_1)\varepsilon.$$

Hence, $\langle u_j - u, g\rangle_1 \to 0$. Besides, it is well known that if in addition $\|D_\omega u_j\|_0 \to \|D_\omega u\|_0$, then $u_j(\cdot)$ converges to $u(\cdot)$ strongly in $\mathrm{H}^1_\omega(\mathbb{T}^k, \mathbb{E}^n)$.

Next, for any bounded subset $\mathcal{A} \subseteq \mathcal{M}$ define a functional space $\mathcal{H}_\mathcal{A}$ in a following way: $u(\cdot) \in \mathcal{H}_\mathcal{A}$ if and only if there exists a sequence $u_j(\cdot) \in \mathcal{S}_\mathcal{A}$ bounded in $\mathrm{H}^1_\omega(\mathbb{T}^k, \mathbb{E}^n)$ and convergent to $u(\cdot)$ by norm of the space $\mathrm{H}(\mathbb{T}^k, \mathbb{E}^n)$ (recall that we consider the set $\mathcal{A} \subseteq \mathcal{M}$ both as a subset of $\mathbb{E}^n$). As it was noted above $\mathcal{H}_\mathcal{A} \subset \mathrm{H}^1_\omega(\mathbb{T}^k, \mathbb{E}^n)$. We shall say that $h(\cdot) \in \mathrm{H}^1_\omega(\mathbb{T}^k, \mathbb{E}^n)$ is a vector field along the map $u(\cdot) \in \mathcal{H}_\mathcal{A}$ defined in the above sens by a sequence $u_j(\cdot)$ if there exists a sequence $h_j(\cdot) \in \mathrm{C}^\infty(\mathbb{T}^k, T\mathcal{M})$ such that $h_j(\varphi) \in T_{u_j(\varphi)}\mathcal{M}$, the sequences $\max_{\varphi\in\mathbb{T}^k} \|h_j(\varphi)\|$, $\|h_j\|_1$ are bounded, and $\lim_{j\to\infty} \|h - h_j\|_1 = 0$.

Definition 2.1. A Besicovitch quasiperiodic function $t \to x(t) := u(t\omega)$ generated by a function $u(\cdot) \in \mathcal{H}_\mathcal{A}$ is called a weak quasiperiodic solution of the natural system on $\mathcal{M}$ if $u(\cdot)$ satisfies the equality

$$\langle D_\omega u(\varphi), D_\omega h(\varphi)\rangle_0 + \langle W'_x(\varphi, u(\varphi)), h(\varphi)\rangle_0 = 0 \tag{2.1}$$

for any vector field $h(\cdot)$ along $u(\cdot)$.

This definition is natural since the equality (2.1) holds true for any classical quasiperiodic solution $u(t\omega)$ and continuous vector field $h(\varphi)$ along $u(\cdot)$ with continuous derivative $D_\omega h(\cdot)$. It should be also noted the following fact. Let ∇_ξ stands for the covariant differentiation of Levi-Civita connection in the direction of vector $\xi \in T\mathcal{M}$, and let ∇f stands for gradient vector field of a scalar function $f(\cdot) : \mathcal{M} \to \mathbb{R}$, i.e $\langle \nabla f(x), \xi\rangle = \mathrm{d}f(x)(\xi)$ for any $\xi \in T_x\mathcal{M}$. Then for any smooth $u(\cdot) : \mathbb{T}^k \to \mathcal{M}$ one can consider $v(t) := \frac{\mathrm{d}}{\mathrm{d}t} u(t\omega) = D_\omega u(t\omega)$ as a tangent vector field along the curve $x = u(t\omega)$ and $h(t\omega)$ — as a vector fields along this curve. Hence there holds the equality

$$\langle D_\omega u(t\omega), D_\omega h(t\omega)\rangle = \langle D_\omega u(t\omega), \nabla_{v(t)} h(t\omega)\rangle \quad \forall t \in \mathbb{R}$$

which yields

$$\langle D_\omega u(\varphi), D_\omega h(\varphi)\rangle = \langle D_\omega u(\varphi), \nabla_{D_\omega u(\varphi)} h(\varphi)\rangle \quad \forall \varphi \in \mathbb{T}^k$$

From this it follows that for a classical solution the equality (2.1) in terms of inner geometry can be rewritten in the form

$$\langle D_\omega u(\varphi), \nabla_{D_\omega u(\varphi)} h(\varphi)\rangle_0 + \langle \nabla W(\varphi, u(\varphi)), h(\varphi)\rangle_0 = 0$$

where $\nabla W(\varphi, x)$ denotes the gradient of function $W(\varphi, \cdot) : \mathcal{M} \to \mathbb{R}$ when $\varphi \in \mathbb{T}^k$ is fixed.

The application of variational approach for searching a weak quasiperiodic solution consists in finding a function $u_*(\cdot) \in \mathcal{H}_\mathcal{A}$ which takes values in appropriately chosen bounded subset $\mathcal{A} \subset \mathcal{M}$ and which is a strong limit in $\mathrm{H}(\mathbb{T}^k, \mathbb{E}^n)$ of minimizing sequence for the functional (the averaged Lagrangian)

$$J[u] = \int_{\mathbb{T}^k} [\frac{1}{2}\|D_\omega u(\varphi)\|^2 + W(\varphi, u(\varphi))] d\varphi \tag{2.2}$$

restricted to $\mathcal{S}_\mathcal{A}$. It is naturally to expect that the first variation of J at $u_*(\cdot)$ vanishes, i.e.

$$J'[u_*](h) := \langle D_\omega u_*(\varphi), D_\omega h(\varphi)\rangle_0 + \langle W'_x(\varphi, u_*(\varphi)), h(\varphi)\rangle_0 = 0 \tag{2.3}$$

for any vector field $h(\cdot)$ along $u_*(\cdot)$. In such a case $u_*(t\omega)$ is a weak quasiperiodic solution.

To guarantee the convergence of a minimizing sequence $u_j(\cdot) \in \mathcal{S}_{\mathcal{A}}$ for $J\big|_{\mathcal{S}_{\mathcal{A}}}$ by norm $\|\cdot\|_0$ it is naturally to impose some convexity conditions both on the set $\mathcal{A}$ and on the functional J. Usually, such conditions are formulated by means of geodesics. But in the case where $(\mathcal{M}, \langle\cdot,\cdot\rangle)$ is not a Riemannian manifold of non-positive sectional curvature, we are not able to determine whether the functional of averaged kinetic energy, namely $J_1[u] := \frac{1}{2}\int_{\mathbb{T}^k} \|D_\omega u(\varphi)\|^2 d\varphi$, is convex using geodesics of Levi-Civita connection ∇. To clarify this fact consider a pair of functions $u_i(\cdot) \in \mathcal{S}_{\mathcal{A}}$, $i = 0, 1$. Under certain conditions imposed on $\mathcal{A}$, for any fixed $\varphi \in \mathbb{T}^k$, one can define a smooth homotopy $[0,1] \times \mathbb{R} \ni (s,t) \to \gamma(s,t) \in \mathcal{A}$ between two functions $t \to u_i(\varphi + t\omega)$, $i = 0, 1$, in such a way that $\gamma(i,t) = u_i(\varphi + t\omega)$ for all $t \in \mathbb{R}$, $i = 0, 1$, and for any fixed t the mapping $\gamma(\cdot, t) : [0,1] \to \mathcal{A}$ is a minimal geodesic connecting $u_0(\varphi + t\omega)$ with $u_1(\varphi + t\omega)$. Obviously that $\frac{\partial}{\partial t}\big|_{t=0}\gamma(i,t) = D_\omega u_i(\varphi)$. The problem is whether the function $g(s) := \|\frac{\partial}{\partial t}\big|_{t=0}\gamma(s,t)\|^2$ is convex. Put $\eta(s,t) := \frac{\partial}{\partial t}\gamma(s,t)$ and $\xi(s,t) := \frac{\partial}{\partial s}\gamma(s,t)$. Then

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}\|\eta\|^2 = 2\frac{\mathrm{d}}{\mathrm{d}s}\langle\nabla_\xi \eta, \eta\rangle = 2[\langle\nabla_\xi^2 \eta, \eta\rangle + \|\nabla_\xi \eta\|^2].$$

In view of geodesic equation $\nabla_\xi \xi = 0$ and equalities

$$\nabla_\eta \xi = \nabla_\xi \eta, \quad \nabla_\eta \nabla_\xi \xi - \nabla_\xi \nabla_\eta \xi = R(\eta, \xi)\xi \tag{2.4}$$

where R is the Riemann curvature tensor of $(\mathcal{M}, \langle\cdot,\cdot\rangle)$, we have $\nabla_\xi^2 \eta = -R(\eta,\xi)\xi$. This implies

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}s^2}\|\eta\|^2 &= 2[\|\nabla_\xi \eta\|^2 - \langle R(\eta,\xi)\xi, \eta\rangle] \\ &= 2[\|\nabla_\xi \eta\|^2 - K(\sigma_x(\xi,\eta))(\|\eta\|^2\|\xi\|^2 - \langle\eta,\xi\rangle^2)]\end{aligned}$$

where $\sigma_x(\xi,\eta)$ is a plane defined by vectors $\xi, \eta \in T_x\mathcal{M}$ and $K(\sigma_x(\xi,\eta))$ is a sectional curvature in direction $\sigma_x(\xi,\eta)$ [17]. In general case, it may happen that $\nabla_\xi \eta = 0$ for some s. Thus, one can guarantee the convexity of g(s) if $(\mathcal{M}, \langle\cdot,\cdot\rangle)$ is a Riemannian manifold of non-positive sectional curvature. It is this case that was considered in [33, 34].

To overcome the above difficulty we introduce a conformally equivalent inner product of the form $\langle\cdot,\cdot\rangle_V\big|_{T_x\mathcal{M}} := \mathrm{e}^{V(x)}\langle\cdot,\cdot\rangle\big|_{T_x\mathcal{M}}$ with appropriately chosen smooth function $V(\cdot) : \mathcal{M} \to \mathbb{R}$. With this approach we succeed in establishing a required convexity properties of averaged Lagrangian under certain convexity conditions imposed on functions $V(\cdot)$ and $W(\varphi, \cdot)$.

3. Convexity of averaged Lagrangian

It is easily seen that if $V(\cdot) \in \mathrm{C}^\infty(\mathcal{M}, \mathbb{R})$ is a bounded function on $\mathcal{M}$ then the Riemannian manifold $(\mathcal{M}, \langle\cdot,\cdot\rangle_V)$ equipped with corresponding Levi-Civita connection is complete. In fact, by definition, the standard distance between any two points $x, y \in (\mathcal{M}, \langle\cdot,\cdot\rangle)$ is defined as

$$\rho(x,y) := \inf\{l(c) : c \in \mathcal{C}_{x,y}\},$$

where $\mathcal{C}_{x,y}$ is the set of all piecewise differentiable paths $c : [0,1] \to \mathcal{M}$ connecting x with y, and $l(c)$ is the length of c on $(\mathcal{M}, \langle\cdot,\cdot\rangle)$. If we denote by $l_V(c)$ the length of path c on $(\mathcal{M}, \langle\cdot,\cdot\rangle_V)$, then

$$\inf_{x\in\mathcal{M}} \sqrt{\mathrm{e}^{V(x)}}\, l(c) \le l_V(c) \le \sup_{x\in\mathcal{M}} \sqrt{\mathrm{e}^{V(x)}}\, l(c).$$

Hence, the metric $\rho(\cdot,\cdot)$ and the metric $\rho_V(\cdot,\cdot)$ of $(\mathcal{M},\langle\cdot,\cdot\rangle_V)$ are equivalent. Now it remains only to apply the Hopf-Rinow theorem (see, e.g., [17, sect. 5.3]).

To distinguish geodesics of metrics ρ and ρ_V we shall call them ρ-geodesic and ρ_V-geodesic respectively.

For $x \in \mathcal{M}$, let $\exp_x(\cdot) : T_x\mathcal{M}\to \mathcal{M}$ denotes the exponential mapping for Riemannian manifold $(\mathcal{M},\langle\cdot,\cdot\rangle)$ with Levi-Civita connection ∇ and let $\exp_x^V(\cdot) : T_x\mathcal{M} \to \mathcal{M}$ be the analogous mapping for Riemannian manifold $(\mathcal{M},\langle\cdot,\cdot\rangle_V)$ with corresponding Levi-Civita connection ∇^V. Note that since both manifolds are complete the domains of both exponential mappings coincide with entire $T_x\mathcal{M}$.

Recall that for the function $V(\cdot)$, the Hesse form $H_V(x)$ at point x (see, e.g., [17]) is defined by the equality

$$\langle H_V(x)\xi,\eta\rangle := \langle\nabla_\xi\nabla V(x),\eta\rangle \quad \forall\xi,\eta\in T_x\mathcal{M}.$$

Let us introduce yet another quadratic form

$$\langle G_V(x)\xi,\xi\rangle := \langle H_V(x)\xi,\xi\rangle - \frac{1}{2}\langle\nabla V(x),\xi\rangle^2 \quad \forall\xi\in T_x\mathcal{M},$$

and the notation

$$\begin{gathered}\lambda_V(x) := \min_{\xi\in T_x\mathcal{M}\setminus\{0\}}\langle H_V(x)\xi,\xi\rangle/\|\xi\|^2,\\ \mu_V(x) := \min_{\xi\in T_x\mathcal{M}\setminus\{0\}}\langle G_V(x)\xi,\xi\rangle/\|\xi\|^2,\\ \mathcal{D} := \{x\in\mathcal{M} : \lambda_V(x)+\frac{1}{2}\|\nabla V(x)\|^2>0\}.\end{gathered}$$

Now we state the following hypotheses concerning convexity properties of functions $V(\cdot)$ and $W(\cdot,\cdot)$:

(H1) there exist a noncritical value $v\in V(\mathcal{D})$ and a bounded connected component Ω of open sublevel set $V^{-1}((-\infty,v))$ with the following properties:
  (a) $\bar\Omega := \Omega\cup\partial\Omega\subseteq\mathcal{D}$ and for any $x,y\in\Omega$ the set $\bar{\mathcal{D}}$ contains at least one ρ_V-geodesic segment with endpoints x,y;
  (b) the second fundamental form of $\partial\Omega$ is positive at each point $x\in\partial\Omega$ (i.e. for any $x\in\partial\Omega$ the restriction of $H_V(x)$ to $T_x\partial\Omega$ is positive definite);
  (c) the function $V(\cdot)$ satisfies the inequality

$$\mu_V(x)\ge 2K^*(x) \quad \forall x\in\Omega \tag{3.1}$$

  where

$$K^*(x) := \max_{\sigma_x(\xi,\eta)}\frac{\langle R(\eta,\xi)\xi,\eta\rangle}{\|\eta\|^2\|\xi\|^2-\langle\eta,\xi\rangle^2}$$

  is the maximum sectional curvature at point x;

(H2) the function $W(\cdot,\cdot)$ satisfies the following inequalities

$$\begin{gathered}\lambda_W(\varphi,x)+\frac{1}{2}\langle\nabla W(\varphi,x),\nabla V(x)\rangle>0 \quad \forall(\varphi,x)\in\mathbb{T}^k\times\bar\Omega \quad (\bar\Omega:=\Omega\cup\partial\Omega),\\ \langle\nabla W(\varphi,x),\nabla V(x)\rangle>0 \quad \forall(\varphi,x)\in\mathbb{T}^k\times\partial\Omega\end{gathered}$$

  where $\lambda_W(\varphi,x)$ is minimal eigenvalue of Hesse form $H_W(\varphi,x)$ for the function $W(\varphi,\cdot):\mathcal{M}\to\mathbb{R}$.

Remark 3.1. Let us give some arguments in order to justify the above hypotheses. Recall that a set of a Riemannian manifold is called convex if together with any two points x_1, x_2 this set contains a (unique) minimal geodesic segment connecting x_1 with x_2(see, e.g., [4, sect. 11.8] or [17, sect. 5.2]). It is well known that for any point x_0 an open ball of sufficiently small radius centered at point x_0 is convex. A function $f : \mathcal{D}_f \to \mathbb{R}$ with convex domain $\mathcal{D}_f \subset \mathcal{M}$ is convex if and only if its superposition with any naturally parametrized geodesic containing in $\mathcal{D}_f$ is convex. Now suppose that the function $V(\cdot)$ reaches its local minimum at a non-degenerate stationary point $x_* \in \mathcal{M}$. This implies $\nabla V(x_*) = 0$ and $\lambda_V(x_*) > 0$. For sufficiently small $b > 0$, there exists $d > 0$ such that the ball

$$B_V(x_*; d) := \{x \in \mathcal{M} : \rho_V(x, x_*) < d\}$$

is a convex subset of $(\mathcal{M}, \langle\cdot,\cdot\rangle_V)$ and there holds inequalities

$$\lambda_V(x) + \frac{1}{2}\|\nabla V(x)\|^2 > 0, \quad \mu_V(x) > b \quad \forall x \in B_V(x_*; d).$$

Moreover, for arbitrary $b > 0$ one can choose $a > 0$ and $d > 0$ in such a way that the same inequality holds true if we replace $V(\cdot)$ with $aV(\cdot)$.The inequality (3.1) is required to provide convexity of averaged kinetic energy functional $\frac{1}{2}\int_{\mathbb{T}^k} \|D_\omega u(\varphi)\|^2 d\varphi$. The first inequality of Hypothesis (H2) is required to provide the convexity of force function, and the second one implies local growth of $W(\varphi, \cdot)$ in direction of external normal to $\partial\Omega$.

It may happen that the direct verification of condition (a) of Hypothesis (H1) involving ρ_V-geodesic is rather difficult. In such a case the following statement which make use of ρ-geodesics might be useful.

Proposition 3.2. *Set $\mathcal{G} := \{x \in \mathcal{M} : \lambda_V(x) > 0\}$. Let there exists a non-critical value $v \in V(\mathcal{G})$ such that the set $V^{-1}((-\infty, v])$ is a bounded and connected subset of $\mathcal{G}$. Suppose also that for any pair of points $x_0, x_1 \in V^{-1}(v)$ the closure of $\mathcal{G}$ contains at least one minimal ρ-geodesic segment connecting x_0, x_1. Then the conditions (a) and (b) of Hypothesis (H1) hold for $\Omega := V^{-1}((-\infty, v))$. Moreover, any minimal ρ_V-geodesic segment connecting points $x, y \in \Omega$ belongs to $\bar{\Omega}$.*

Proof. It is easily seen that $\mathcal{G} \subseteq \mathcal{D}$. Let $r = \rho(x_0, x_1)$ and $\gamma(\cdot) : [0, r] \to \bar{\mathcal{G}}$ be a minimal naturally parametrized ρ-geodesic segment with endpoints $x_0, x_1 \in V^{-1}(v)$. Then $\nabla_{\dot\gamma(s)}\dot\gamma(s) \equiv 0$ and

$$\begin{aligned}\frac{d^2}{ds^2} e^{V\circ\gamma(s)} &= [e^{V(x)}(\langle\nabla V(x), \dot x\rangle^2 + \langle H_V(x)\dot x, \dot x\rangle)]_{x=\gamma(s)} \\ &\geq e^{V\circ\gamma(s)}\lambda_V \circ \gamma(s) \geq 0.\end{aligned}$$

Hence, the function $\exp\circ V \circ \gamma(\cdot)$ is convex. This imply that

$$e^{V\circ\gamma(\theta r)} \leq (1-\theta)e^{V(x_0)} + \theta e^{V(x_1)} = e^v \quad \forall\theta \in [0, 1],$$

and thus $\gamma(s) \in \bar\Omega$ for all $s \in [0, r]$. If now $c(\cdot) : [0, 1] \to \mathcal{M}$ is arbitrary piecewise differentiable path with endpoints x_0, x_1 such that $c(t) \in \mathcal{M} \setminus \bar\Omega$ for all $t \in (0, 1)$, then $V \circ c(t) > v$ for all $t \in (0, 1)$ and

$$\begin{aligned}l_V(\gamma) &= \int_0^r \sqrt{e^{V\circ\gamma(s)}}\|\dot\gamma(s)\| ds \leq e^{v/2} l(\gamma) \leq e^{v/2} l(c) \\ &< \int_0^1 \sqrt{e^{V\circ c(t)}}\|\dot c(t)\| dt = l_V(c).\end{aligned}$$

Now consider a minimal ρ_V-geodesic segment γ_V connecting points $x, y \in \Omega$. Let us show that $\gamma_V \in \bar{\Omega}$. If we suppose that $\gamma_V \not\subset \bar{\Omega}$, then γ_V must contain at least one segment $\tilde{\gamma}_V$ with endpoints $\tilde{x}_0, \tilde{x}_1 \in V^{-1}(v)$ and with the property that

$$\tilde{\gamma}_V \setminus (\{\tilde{x}_0\} \cup \{\tilde{x}_1\}) \subset \mathcal{M} \setminus \bar{\Omega}.$$

Replace $\tilde{\gamma}_V$ by a minimal ρ-geodesic segment $\tilde{\gamma}$ connecting the points $\tilde{x}_0, \tilde{x}_1$. As has been already shown above, $l_V(\tilde{\gamma}) < l_V(\tilde{\gamma}_V)$, and this yields

$$\begin{aligned}\rho_V(x, y) &:= l_V(\gamma_V) = l_V(\gamma_V \setminus \tilde{\gamma}_V) + l_V(\tilde{\gamma}_V)\\ &> l_V(\gamma_V \setminus \tilde{\gamma}_V) + l_V(\tilde{\gamma}) = l_V([\gamma_V \setminus \tilde{\gamma}_V] \cup \tilde{\gamma}).\end{aligned}$$

Thus, on Riemannian manifold $(\mathcal{M}, \langle\cdot,\cdot\rangle_V)$ the length of piecewise differentiable path $[\gamma_V \setminus \tilde{\gamma}_V] \cup \tilde{\gamma}$ connecting points x, y is less than $\rho_V(x, y)$. We arrive at contradiction with definition of metric $\rho_V(\cdot,\cdot)$. Hence, $\gamma_V \in \bar{\Omega}$.

Finally, the inequality $\lambda_V\big|_{\partial\Omega} > 0$ ensures the fulfillment of condition (b). $\square$

The next technical statement on the convexity property of the functional J plays a key role in existence proof of weak quasiperiodic solution.

Theorem 3.3. *Let the Hypotheses (H1)–(H2) hold. Then there exist positive constants C, C_1 and c such that for any $u_0(\cdot), u_1(\cdot) \in \mathrm{C}^\infty(\mathbb{T}^k, \Omega)$ one can choose a vector field $h(\cdot) \in \mathrm{C}^\infty(\mathbb{T}^k, T\mathcal{M})$ along $u_0(\cdot)$ (this implies that $h(\varphi) \in T_{u_0(\varphi)}\mathcal{M}$ for all $\varphi \in \mathbb{T}^k$) in such a way that the following inequalities hold*

$$\begin{gathered}c\rho(u_0(\varphi), u_1(\varphi)) \le \|h(\varphi)\| \le C \quad \forall \varphi \in \mathbb{T}^k,\\ \|D_\omega h(\varphi)\| \le C_1[\|D_\omega u_0(\varphi)\| + \|D_\omega u_1(\varphi)\|] \quad \forall \varphi \in \mathbb{T}^k,\\ J[u_1] - J[u_0] - J'[u_0](h) \ge \frac{\kappa c^2}{2}\int_{\mathbb{T}^k} \rho^2(u_0, u_1)\mathrm{d}\varphi\end{gathered}$$

where $\kappa := \min\{\lambda_W(\varphi, x) + \frac{1}{2}\langle\nabla W(\varphi, x), \nabla V(x)\rangle : (\varphi, x) \in \mathbb{T}^k \times \bar{\Omega}\}$.

The proof of this theorem needs several auxiliary statements and will be given at the end of present Section.

Proposition 3.4. *The Euler-Lagrange equation for ρ_V-geodesic on Riemannian manifold $(\mathcal{M}, \langle\cdot,\cdot\rangle)$ has the form*

$$\nabla_{\dot{x}}\dot{x} = -\langle\nabla V(x), \dot{x}\rangle\dot{x} + \frac{\|\dot{x}\|^2}{2}\nabla V(x), \tag{3.2}$$

Proof. A ρ_V-geodesic segment with endpoints $x_0, x_1 \in \mathcal{M}$ is an extremal of functional $\Phi[x(\cdot)] = \int_0^1 \mathrm{e}^{V\circ x(t)}\|\dot{x}(t)\|^2\mathrm{d}t$ defined on the space $\mathcal{C}^2_{x_0x_1}$ of twice continuously differentiable curves $x = x(t)$, $t \in [0, 1]$, such that $x(0) = x_0$, $x(1) = x_1$. We are going to derive the Euler-Lagrange equation using the connection ∇. Consider a variation of $x(\cdot)$ defined by a smooth mapping $y(\cdot,\cdot) : [0, 1] \times (-\varepsilon, \varepsilon) \to \mathcal{M}$ such that $y(\cdot, \lambda) \in \mathcal{C}^\infty_{x_0x_1}$ for any fixed $\lambda \in (-\varepsilon, \varepsilon)$ and $y(t, 0) \equiv x(t)$. Put

$$\dot{y}(t, \lambda) := \frac{\partial}{\partial t}y(t, \lambda), \quad y'(t, \lambda) := \frac{\partial}{\partial\lambda}y(t, \lambda).$$

Obviously, $\dot{y}(t, 0) = \dot{x}(t)$, $y(i, \lambda) \equiv x_i$ and $y'(i, \lambda) = 0$, $i = 0, 1$. Since $\nabla_{y'}\dot{y} = \nabla_{\dot{y}}y'$, we have

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\Big|_{\lambda=0}\int_0^1 \mathrm{e}^{V\circ y}\|\dot{y}\|^2\mathrm{d}t = \int_0^1 [\mathrm{e}^{V\circ y}\langle\nabla V \circ y, y'\rangle\|\dot{y}\|^2 + 2\mathrm{e}^{V\circ y}\langle\nabla_{y'}\dot{y}, \dot{y}\rangle]_{\lambda=0}\mathrm{d}t$$

$$= \int_0^1 [\mathrm{e}^{V\circ y}\langle \nabla V \circ y, y'\rangle \|\dot{y}\|^2 + 2\mathrm{e}^{V\circ y}\langle \nabla_{\dot{y}} y', \dot{y}\rangle]_{\lambda=0}\mathrm{d}t.$$

Taking into account that

$$\frac{\partial}{\partial t}\mathrm{e}^{V\circ y}\langle y', \dot{y}\rangle = \mathrm{e}^{V\circ y}\langle \nabla V \circ y, \dot{y}\rangle\langle y', \dot{y}\rangle + \mathrm{e}^{V\circ y}\langle \nabla_{\dot{y}} y', \dot{y}\rangle + \mathrm{e}^{V\circ y}\langle y', \nabla_{\dot{y}}\dot{y}\rangle$$

and $\mathrm{e}^{V\circ y}\langle y', \dot{y}\rangle\big|_{t=0,1} = 0$, we obtain

$$\int_0^1 \mathrm{e}^{V\circ y}\langle \nabla_{\dot{y}} y', \dot{y}\rangle \mathrm{d}t = -\int_0^1 \mathrm{e}^{V\circ y}[\langle \nabla V \circ y, \dot{y}\rangle\langle y', \dot{y}\rangle + \langle y', \nabla_{\dot{y}}\dot{y}\rangle]\mathrm{d}t.$$

From this it follows that the first variation on functional Φ is

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}\lambda}\Big|_{\lambda=0}\Phi[y(\cdot,\lambda)] = \Phi'[x(\cdot)](y'(\cdot,0)) \\ &= \int_0^1 [\mathrm{e}^{V}(\langle \nabla V, y'\rangle\|\dot{x}\|^2 - 2\langle \nabla V, \dot{x}\rangle\langle \dot{x}, y'\rangle - 2\langle \nabla_{\dot{x}}\dot{x}, y'\rangle)]\big|_{x=x(t),\lambda=0}\mathrm{d}t, \end{aligned}$$

and the Euler-Lagrange equation is exactly (3.2). □

Proposition 3.5. *Let Hypothesis* (H1) *hold. If a ρ_V-geodesic segment connecting points $x, y \in \Omega$ belongs to $\bar{\mathcal{D}}$, then this segment belongs to Ω.*

Proof. Let $x(\cdot) \in \mathcal{C}^2_{x_0x_1}$ satisfies (3.2) and let $x(t) \in \bar{\mathcal{D}}$ for all $t \in [0,1]$. Then

$$\begin{aligned} \frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathrm{e}^{V}\Big|_{x=x(t)} &= [\mathrm{e}^{V}(\langle \nabla_{\dot{x}}\nabla V, \dot{x}\rangle + \langle \nabla V, -\langle \nabla V, \dot{x}\rangle\dot{x} + \|\dot{x}\|^2\nabla V/2\rangle + \langle \nabla V, \dot{x}\rangle^2)]\big|_{x=x(t)} \\ &= [\mathrm{e}^{V}(\langle \nabla_{\dot{x}}\nabla V, \dot{x}\rangle + \|\dot{x}\|^2\|\nabla V\|^2/2)]\big|_{x=x(t)} \\ &\geq [\mathrm{e}^{V}\|\dot{x}\|^2(\lambda_V + \|\nabla V\|^2/2)]\big|_{x=x(t)} \geq 0. \end{aligned}$$

Hence, $\mathrm{e}^{V\circ x(\cdot)}$ is convex and this implies $V \circ x(t) < v$ for all $t \in [0,1]$; i.e., $x(t) \in \Omega$ for all $t \in [0,1]$. □

Corollary 3.6. *For any solution $x(\cdot) : [0,1] \to \mathcal{D}$ of equation* (3.2)*, the function $\mathrm{e}^{V\circ x(\cdot)}$ has positive second derivative.*

Proposition 3.7. *Under Hypothesis* (H1) *the set Ω contains a unique non-degenerate minimum point of function $V(\cdot)$, and there are no other stationary points of this function in Ω. The domain Ω is simply connected.*

Proof. Under Hypothesis (H1) $\nabla V(x) \neq 0$ on $\partial\Omega$. Hence, $V(\cdot)$, as well as $\mathrm{e}^{V(\cdot)}$, has at least one minimum point $x_* \in \Omega$. The inequality $\lambda_V(x_*) > 0$ yields that this point is non-degenerate. Any other stationary point $y_* \in \Omega$, if it exists, is also a non-degenerate minimum point. But this is impossible, since, as it follows from Corollary (3.6), the function $\mathrm{e}^{V(\cdot)}$ is strictly convex along ρ_V-geodesic segment connecting x_* with y_* and containing in $\Omega \subseteq \mathcal{D}$. Hence, x_* is unique, and the domain Ω can be shrunk to x_* by means of the flow of vector field $-\nabla V/\|\nabla V\|^2$. □

Proposition 3.8. *Under Hypotheses (H1), any two points $x, y \in \Omega$ are the end-points of unique ρ_V-geodesic segment containing in Ω.*

Proof. It is known (see. [17, sect. 3.6]) that the sectional curvature in direction $\sigma_x(\xi_1, \xi_2)$ on Riemannian manifold $(\mathcal{M}, \mathrm{e}^{V}\langle\cdot,\cdot\rangle)$ is represented in the form

$$K_V(\sigma_x(\xi_1, \xi_2))$$

$$= \mathrm{e}^{-V}[K(\sigma_x(\xi_1,\xi_2)) - \frac{1}{2}\sum_{i=1}^{2}[\langle H_V(x)\xi_i,\xi_i\rangle - \frac{1}{2}\langle\nabla V(x),\xi_i\rangle^2] - \frac{1}{4}\|\nabla V(x)\|^2]$$

where ξ_1,ξ_2 is an orthonormal basis of the plane $\sigma_x(\xi_1,\xi_2)$, and the inequality (3.1) yields that this curvature is non-positive for any $x\in\bar\Omega$. Under Hypothesis (H1), taking into account Proposition 3.5, there exists a ρ_V-geodesic segment $\gamma_V\subset\Omega$ connecting $x,y\in\Omega$. By the Morse-Schoenberg theorem [17, sect. 6.2] any ρ_V-geodesic segment which belongs to $\bar\Omega$, in particular γ_V, does not contain conjugate points. Hence, the image of the set

$$\mathcal{Z}_x := \{\xi\in T_x\mathcal{M} : \exp_x^V(s\xi)\in\Omega\ \forall s\in[0,1]\}$$

under the mapping $\exp_x^V(\cdot)$ coincides with Ω, and this mapping is a local diffeomorphism at any point of the closure $\bar{\mathcal{Z}}_x$. Let us show, that the set $\bar{\mathcal{Z}}_x$ is bounded. It follows from the proof of Proposition 3.5 that for arbitrary $\xi\in\mathcal{Z}_x\setminus\{0\}$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\exp[V\circ\exp_x^V(t\xi/\|\xi\|_V)] \ge \min_{x\in\bar\Omega}\mathrm{e}^{V(x)}(\lambda_V(x)+\|\nabla V(x)\|^2/2) =: \sigma > 0$$

while $t\xi/\|\xi\|_V\in\bar{\mathcal{Z}}_x$, and thus for such t we have

$$\exp[V\circ\exp_x^V(t\xi/\|\xi\|_V)] \ge \frac{\sigma t^2}{2} - t\max_{x\in\bar\Omega}\|\nabla V(x)\|_V + \min_{x\in\bar\Omega}\mathrm{e}^{V(x)}.$$

This yields that there exist $T>0$ and sufficiently small $\varepsilon>0$ with the property that for any $\xi\in\mathcal{Z}_x$ one can point out $t_\varepsilon(\xi)\in(0,T]$ such that $V\circ\exp_x^V(t_\varepsilon(\xi)\xi/\|\xi\|_V) = v+\varepsilon$. Hence, $\|\xi\|_V<T$.

Now it is not hard to see that for any $y\in\Omega$ the set $\mathcal{Z}_x\cap[\exp_x^V]^{-1}(y)$ is finite. In fact, otherwise there would exist a sequence $\xi_k\in\mathcal{Z}_x$ converging to $\xi_*\in\bar{\mathcal{Z}}_x$ such that $\xi_i\ne\xi_k$, $i\ne k$, and $\exp_x^V(\xi_k)=\exp_x^V(\xi_*)=y$. But this is impossible since $\exp_x^V(\cdot)$ is local diffeomorphism at ξ_*.

From the above reasoning it is clear that any point $y\in\Omega$ has a neighborhood U such that $[\exp_x^V]^{-1}(U)$ is a finite disjoint union of open sets of $\mathcal{Z}_x$ each of which is mapped diffeomorphically onto U by $\exp_x^V(\cdot)$, i.e. U is evenly covered by the map $\exp_x^V(\cdot)$. Hence, $\exp_x^V(\cdot):\mathcal{Z}_x\to\Omega$ is a finite-fold covering mapping. This mapping is bijection since Ω is simply connected (Proposition 3.7) and $\mathcal{Z}_x$ is path-connected (see., e.g., [31, sect. 3.23]). Thus, for any point $y\in\Omega$, there exists a unique $\zeta\in\mathcal{Z}_x$ such that $\exp_x^V(\zeta)=y$, and then Ω contains a unique ρ_V-geodesic segment, namely $\cup_{s\in[0,1]}\{\exp_x^V(s\zeta)\}$, with endpoints x,y. □

As a corollary of above proposition and the implicit function theorem we obtain the following statement.

Proposition 3.9. *Under Hypotheses* (H1) *there exist a smooth mapping* $\zeta(\cdot,\cdot):\Omega\times\Omega\to T\mathcal{M}$ *and a constant* $T>0$ *such that* $\zeta(x,y)\in T_x\mathcal{M}$ *and*

$$\begin{gathered}\exp_x^V(\zeta(x,y))=y,\quad \rho_V(x,y)\le \mathrm{e}^{V(x)/2}\|\zeta(x,y)\|\le T,\\ \exp_x^V(t\zeta(x,y))\in\Omega\quad\forall t\in[0,1].\end{gathered}\tag{3.3}$$

If we define the mapping

$$\gamma_V(\cdot,\cdot,\cdot):[0,1]\times\Omega\times\Omega\to\Omega,\quad \gamma_V(t,x,y):=\exp_x^V(t\zeta(x,y)),$$

then for any $x,y\in\mathcal{D}$ the mapping $\gamma_V(\cdot,x,y):[0,1]\to\mathcal{D}$ satisfies the equation (3.2) together with boundary conditions $\gamma_V(0,x,y)=x$, $\gamma_V(1,x,y)=y$. The following scalar differential equation

$$\frac{\mathrm{d}\tau}{\mathrm{d}s} = \exp(V \circ \gamma_V(\tau, x, y)) \int_0^1 \exp(-V \circ \gamma_V(t, x, y))\mathrm{d}t.$$

has a unique strictly monotonically increasing solution

$$\tau(\cdot, x, y) : [0, 1] \to [0, 1], \quad \tau(0, x, y) = 0, \quad \tau(1, x, y) = 1. \tag{3.4}$$

By means of reparametrization $t = \tau(s, x, y)$ we define a smooth mapping

$$\chi(\cdot, \cdot, \cdot) : [0, 1] \times \Omega \times \Omega \to \Omega, \quad \chi(s, x, y) := \gamma_V(\tau(s, x, y), x, y)$$

which plays an important role in subsequent reasoning. In [33] $\chi(\cdot, \cdot, \cdot)$ is called the connecting mapping.

Proposition 3.10. *For any $x, y \in \Omega$ the mapping $\chi(\cdot, x, y) : [0, 1] \to \Omega$ is a solution of boundary value problem*

$$\nabla_{x'} x' = \frac{\|x'\|^2}{2} \nabla V(x), \quad \chi(0, x, y) = x, \quad \chi(1, x, y) = y \tag{3.5}$$

where $x' = \frac{\mathrm{d}x}{\mathrm{d}s}$.

Proof. The boundary conditions follow from definition of γ_V and (3.4). Let us show that (3.5) is obtained from (3.2) after the change of independent variable $t = \tau(s)$. In fact, let $\chi(s) = x \circ \tau(s)$. Then (3.2) takes the form

$$\frac{1}{\tau'} \nabla_{\chi'} (\frac{1}{\tau'} \chi') = -\frac{1}{(\tau')^2} \langle \nabla V \circ \chi, \chi' \rangle \chi' + \frac{\|\chi'\|^2}{2(\tau')^2} \nabla V \circ \chi,$$

or

$$-\frac{\tau''}{\tau'} \chi' + \nabla_{\chi'} \chi' = -[\frac{\mathrm{d}}{\mathrm{d}s} V \circ \chi] \chi' + \frac{\|\chi'\|^2}{2} \nabla V \circ \chi.$$

From this it follows (3.5) since $\tau''/\tau' = (V \circ \chi)'$. □

Proposition 3.11. *Under Hypothesis* (H1) *the following inequality is valid*

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2} \|D_\omega \chi(s, u_0(\varphi), u_1(\varphi))\|^2 \geq 0 \quad \forall s \in [0, 1],\ \forall \varphi \in \mathbb{T}^k,\ \forall u_i(\cdot) \in \mathcal{S}_\Omega,\ i = 0, 1.$$

Proof. For any fixed $\varphi \in \mathbb{T}^k$ put

$$\eta(s, t) := \frac{\partial}{\partial t} \chi(s, u_0(\varphi + t\omega), u_1(\varphi + t\omega)) \equiv D_\omega \chi(s, u_0(\varphi + t\omega), u_1(\varphi + t\omega)),$$

$$\xi(s, t) := \frac{\partial}{\partial s} \chi(s, u_0(\varphi + t\omega), u_1(\varphi + t\omega)).$$

Then in view of (2.4) and (3.5), we have

$$\begin{aligned} \nabla_\xi^2 \eta &= \nabla_\eta \nabla_\xi \xi - R(\eta, \xi)\xi \\ &= \langle \nabla_\eta \xi, \xi \rangle \nabla V \circ \chi + \frac{\|\xi\|^2}{2} \nabla_\eta \nabla V \circ \chi - R(\eta, \xi)\xi \end{aligned}$$

and hence

$$\begin{aligned} &\frac{\mathrm{d}^2}{\mathrm{d}s^2} \|\eta\|^2 \\ &= 2[\langle \nabla_\xi^2 \eta, \eta \rangle + \|\nabla_\xi \eta\|^2] \\ &= 2\|\nabla_\xi \eta\|^2 + 2\langle \nabla_\xi \eta, \xi \rangle \langle \nabla V \circ \chi, \eta \rangle + \|\xi\|^2 \langle \nabla_\eta \nabla V \circ \chi, \eta \rangle - 2\langle R(\eta, \xi)\xi, \eta \rangle \\ &\geq 2\|\nabla_\xi \eta\|^2 - 2\|\nabla_\xi \eta\| \|\xi\| |\langle \nabla V \circ \chi, \eta \rangle| + \|\xi\|^2 \langle \nabla_\eta \nabla V \circ \chi, \eta \rangle - 2K^* \circ \chi \|\xi\|^2 \|\eta\|^2. \end{aligned}$$

Once Hypothesis (H1) holds, we obtain

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}\|\eta\|^2 \geq 2\|\xi\|^2\|\eta\|^2[r^2 - |\langle \nabla V \circ \chi, \mathbf{e}\rangle| r + \frac{1}{2}\langle \nabla_{\mathbf{e}} \nabla V \circ \chi, \mathbf{e}\rangle - K^* \circ \chi] \geq 0$$

where $r := \frac{\|\nabla_\xi \eta\|}{\|\xi\|\|\eta\|}$. □

Now we are in position to prove Theorem 3.3. Let $u_i(\cdot) \in \mathcal{S}_\Omega$, $i = 0, 1$. By means of connecting mapping we obtain the following representation

$$J[\chi(s, u_0, u_1)] = J[u_0] + sJ'[u_0](\chi'_s(0, u_0, u_1)) + \frac{s^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}s^2}\Big|_{s=\theta} J[\chi(s, u_0, u_1)] \quad (3.6)$$

with some $\theta \in (0, 1)$. To estimate from below the term with second derivative we make use of Proposition 3.11 which together with Hypothesis (H3) implies

$$\begin{aligned}
&\frac{\mathrm{d}^2}{\mathrm{d}s^2}[\frac{1}{2}\|D_\omega \chi(s, u_0(\varphi), u_1(\varphi))\|^2 + W(\varphi, \chi(s, u_0, u_1))] \\
&\geq \frac{\mathrm{d}}{\mathrm{d}s}\langle \nabla W(\varphi, \chi), \chi'_s\rangle \\
&= \langle \nabla_{\chi'_s} \nabla W(\varphi, \chi), \chi'_s\rangle + \langle \nabla W(\varphi, \chi), \nabla_{\chi'_s}\chi'_s\rangle \\
&= \langle \nabla_{\chi'_s} \nabla W(\varphi, \chi), \chi'_s\rangle + \frac{\|\chi'_s\|^2}{2}\langle \nabla W(\varphi, \chi), \nabla V(\chi)\rangle \geq \kappa\|\chi'_s\|^2.
\end{aligned}$$

By the definition of χ we have

$$\begin{aligned}
&\chi'_s(s, u_0, u_1) \\
&= \tau'(s)\dot{\gamma}_V(\tau(s), u_0, u_1) \\
&= \exp(V \circ \gamma_V(\tau(s), u_0, u_1)) \int_0^1 \exp(-V \circ \gamma_V(t, u_0, u_1))\mathrm{d}t \dot{\gamma}_V(\tau(s), u_0, u_1).
\end{aligned}$$

Since $\gamma_V(t, x, y)$ is ρ_V−geodesic, then $\exp(V \circ \gamma_V)\|\dot{\gamma}_V\|^2$ does not depend on t and

$$\mathrm{e}^{V(x)/2}\|\dot{\gamma}_V(0, x, y)\| = \mathrm{e}^{V(x)/2}\|\zeta(x, y)\|.$$

Hence

$$\begin{aligned}
&\|\chi'_s(s, u_0, u_1)\|^2 \\
&= [\int_0^1 \exp(-V \circ \gamma_V(t, u_0, u_1))\mathrm{d}t]^2 \exp(V \circ \gamma_V(\tau(s), u_0, u_1))\mathrm{e}^{V(u_0)}\|\zeta(u_0, u_1)\|^2,
\end{aligned}$$

and (3.3) implies that there exist positive constants C, c dependent only on $V(\cdot)$ and Ω such that

$$c\rho(u_0, u_1) \leq \|\chi'_s(s, u_0, u_1)\| \leq C. \quad (3.7)$$

Define $h(\varphi) := \chi'_s(0, u_0(\varphi), u_2(\varphi))$. Then (3.6) with $s = 1$ yields

$$J[u_1] - J[u_0] - J'[u_0](\chi'_s(0, u_0, u_1)) \geq \frac{\kappa c^2}{2}\int_{\mathbb{T}^k} \rho^2(u_0, u_1)\mathrm{d}\varphi.$$

Finally, since the set Ω is bounded and the mapping χ is smooth, there exists positive constant C_1 such that

$$\|D_\omega h(\varphi)\| \leq C_1[\|D_\omega u_0(\varphi)\| + \|D_\omega u_1(\varphi)\|] \quad \forall \varphi \in \mathbb{T}^k.$$

The proof is complete.

4. Main existence theorem

Now we proceed to the main result of this paper.

Theorem 4.1. *Let hypotheses* (H1) *and* (H2) *hold. Then the natural system on Riemannian manifold* $(\mathcal{M}, \langle\cdot,\cdot\rangle)$ *with Lagrangian density* $L = \frac{1}{2}\langle\dot{x},\dot{x}\rangle + W(t\omega, x)$ *has a weak quasiperiodic solution.*

Proof. The proof consists of three steps.

Step 1. Construction of a projection mapping and its smooth approximation. Put $\Omega+\delta = (\cup_{x\epsilon\Omega} B(x;\delta))$ where $B(x;\delta)$ stands for an open ball of radius δ centered at $x \in \mathcal{M}$ on Riemannian manifold $(\mathcal{M}, \langle\cdot,\cdot\rangle)$. Since by Hypothesis (H1) v is a noncritical value, then $\partial\Omega = V^{-1}(v)$ is a regular hypersurface with unit normal field $\boldsymbol{\nu} := \frac{\nabla V}{\|\nabla V\|}$. As is well known (see, e.g., [4, sect. 8.1]), for sufficiently small $\delta > 0$, one can correctly define the projection mapping $P_\Omega : \Omega+\delta \to \bar{\Omega}$ such that $P_\Omega x \in \bar{\Omega}$ is the nearest point to $x \in \Omega+\delta$. If $x = X(q)$, $q \in \mathcal{Q} \subset \mathbb{R}^{m-1}$, is a smooth local parametric representation of $\partial\Omega$ in a neighborhood of a point $x_0 \in \partial\Omega$, then for sufficiently small $\delta_0 > 0$ the mapping

$$\mathcal{Q} \times (-\delta_0, \delta_0) \ni (q,z) \to \exp_{X(q)}(z\boldsymbol{\nu} \circ X(q))$$

introduces local coordinates with the following properties: local equation of $\partial\Omega$ is $z = 0$; each naturally parametrized ρ-geodesic

$$\gamma(s) = \exp_{X(q)}(s\boldsymbol{\nu} \circ X(q))$$

is orthogonal to each hypersurface $z = \text{const}$; the Riemannian metric takes the form $\sum_{i,j=1}^{m-1} b_{ij}(q,z)\mathrm{d}q_i\mathrm{d}q_j + \mathrm{d}z^2$, where $B(q,z) = \{b_{ij}(q,z)\}_{i,j=1}^{m-1}$ is positive definite symmetric matrix; the function $V(\cdot)$ is represented in the form $V(q,z) = v + a(q)z + b(q,z)z^2$; the mapping P_Ω has the form

$$P_\Omega(q,z) := \begin{cases} (q,0) & \text{if } z \in (0,\delta_0),\\ (q,z) & \text{if } z \in (-\delta_0, 0]. \end{cases}$$

The projection mapping is continuous on $\Omega+\delta$ and continuously differentiable on $(\Omega+\delta)\backslash\partial\Omega$. Moreover, it turns out that for sufficiently small $\delta > 0$ the derivative $P_{\Omega*}$ is contractive on $(\Omega+\delta)\backslash\partial\Omega$; i.e.

$$\|P_{\Omega*}\xi\| \le \|\xi\| \quad \forall \xi \in T_x\mathcal{M},\ x \in (\Omega+\delta)\backslash\partial\Omega. \tag{4.1}$$

It is sufficiently to prove this inequality for any $x \in (\Omega+\delta)\backslash\partial\Omega$. Let $q = q(s)$, $z = z(s)$ be natural equations of ρ-geodesic which starts at a point $x_0 = (q_0, 0) \in \partial\Omega$ in direction of vector $\eta =(\dot{q}_0, 0) \in T_{x_0}\partial\Omega$. The hypothesis (H1) implies that

$$\langle\nabla_\eta\nabla V(x_0), \eta\rangle = \frac{\mathrm{d}^2}{\mathrm{d}s^2}\Big|_{s=0} V(q(s), z(s)) > 0 \quad \Leftrightarrow \quad a(q_0)\ddot{z}(0) > 0.$$

Since $a(q_0) > 0$ ($\boldsymbol{\nu}$ is external normal to $\partial\Omega$) and z-component of geodesic equations is

$$\ddot{z} = \frac{1}{2}\frac{\partial}{\partial z}\sum_{i,j=1}^{m-1} b_{ij}(q,z)\dot{q}_i^2\dot{q}_j^2,$$

then the matrix $B'_z(q_0,0)$ is positive definite. From this it follows that $B(q,z_1) > B(q,z_2)$ for all q from a neighborhood of q_0 and all $z_1, z_2 \in (-\delta,\delta)$, $z_1 > z_2$ if

$\delta \in (0, \delta_0)$ is sufficiently small. Let $\xi = (\dot q, \dot z)$ be a tangent vector at point (q, z) where $z \in (0, \delta)$. Then

$$\begin{aligned}\|\xi\|^2 &= \sum_{i,j=1}^{m-1} b_{ij}(q,z)\dot q_i \dot q_j + \dot z^2 \ge \sum_{i,j=1}^{m-1} b_{ij}(q,z)\dot q_i \dot q_j \\ &\ge \sum_{i,j=1}^{m-1} b_{ij}(q,0)\dot q_i \dot q_j = \|(\dot q, 0)\|^2 = \|P_{\Omega *}\xi\|^2.\end{aligned}$$

Let us introduce a smooth approximation of projection mapping in the following way. For $\varepsilon \in (0, \delta)$, define

$$\varpi_\varepsilon(z) := \begin{cases} \exp(1/z - 1/(z+\varepsilon)), & z \in (-\varepsilon, 0), \\ 0, & z \in \mathbb{R} \setminus (-\varepsilon, 0), \end{cases}$$

$$Z_\varepsilon(z) := \int_{-\varepsilon}^{z} \frac{\int_s^0 \varpi_\varepsilon(t)\mathrm{d}t}{\int_{-\varepsilon}^0 \varpi_\varepsilon(t)\mathrm{d}t}\mathrm{d}s - \varepsilon, \quad z \in (-\delta_0, \delta_0).$$

Obviously the function $Z_\varepsilon(\cdot)$ is smooth, its derivative, $Z'_\varepsilon(z)$, equals 1 for $z \in (-\delta_0, -\varepsilon]$, monotonically decreases from 1 to 0 on $[-\varepsilon, 0]$, and equals 0 for $z \ge 0$. From this it follows that $Z_\varepsilon(z)$ equals z for $z \in (-\delta_0, -\varepsilon]$ monotonically increases from $-\varepsilon$ to $Z_\varepsilon(0) \in (-\varepsilon, 0)$ on $[-\varepsilon, 0]$, and equals $Z_\varepsilon(0)$ for $z \in [0, \delta_0)$. Now locally define

$$P_{\varepsilon,\Omega}(q, z) := \begin{cases} (q, Z_\varepsilon(0)) & \text{if } z \in (0, \delta_0), \\ (q, Z_\varepsilon(z)) & \text{if } z \in (-\delta_0, 0] \end{cases}$$

and for each point $x \in \Omega$ such that $B(x; \delta) \subset \Omega$ put $P_{\varepsilon,\Omega}(x) = x$. Since $Z_\varepsilon(0) < 0$, then

$$P_{\varepsilon,\Omega}(\Omega + \delta) \subset \Omega$$

and since $|Z'_\varepsilon(z)| \le 1$, then for any $z \in (-\delta, \delta)$, and for any tangent vector $\xi = (\dot q, \dot z)$ at point (q, z) we have

$$\begin{aligned}\|\xi\|^2 &= \sum_{i,j=1}^{m-1} b_{ij}(q,z)\dot q_i \dot q_j + \dot z^2 \ge \sum_{i,j=1}^{m-1} b_{ij}(q, Z_\varepsilon(z))\dot q_i \dot q_j + (Z'_\varepsilon(z)\dot z)^2 \\ &= \|(\dot q, Z'_\varepsilon(z)\dot z)\|^2 = \|P_{\varepsilon,\Omega *}\xi\|.\end{aligned}$$

From this it follows that

$$\|P_{\varepsilon,\Omega *}\xi\| \le \|\xi\| \quad \forall x \in \Omega + \delta,\ \forall \xi \in T_x\mathcal{M}. \tag{4.2}$$

Also, Hypothesis (H2) implies

$$W(\varphi, P_{\varepsilon,\Omega}x) \le W(\varphi, x) \quad \forall \varphi \in \mathbb{T}^m,\ \forall x \in \Omega + \delta \tag{4.3}$$

for sufficiently small δ and $\varepsilon \in (0, \delta)$.

Step 2. Minimization of functional J on $\mathcal{S}_{\Omega+\delta}$. Obviously that the functional J restricted to $S_{\Omega+\delta}$ is bounded from below. Let us show that

$$J_* := \inf J[\mathcal{S}_{\Omega+\delta}] = \inf J[\mathcal{S}_\Omega]. \tag{4.4}$$

In fact, if $v_j(\cdot) \in \mathcal{S}_{\Omega+\delta}$ is such a sequence that $J[v_j]$ monotonically decreases to J_*, then (4.2) and (4.3) implies

$$J_* \le J[P_{\varepsilon/j,\Omega}v_j] \le J[v_j].$$

Hence, the sequence $u_j(\cdot) := P_{\varepsilon/j,\Omega}v_j(\cdot)$ is minimizing both for $J\big|_{\mathcal{S}_\Omega}$ and for $J\big|_{\mathcal{S}_{\Omega+\delta}}$.

Step 3. Convergence of minimizing sequence to a weak solution. Let $u_j(\cdot) \in \mathcal{S}_\Omega$ be a minimizing sequence for $J\big|_{\mathcal{S}_\Omega}$. Without loss of generality, we may consider that

$$\|D_\omega u_j\|_0^2 \le M := \frac{2}{(2\pi)^k}\sup_{x\in\Omega}\int_{\mathbb{T}^k} W(\varphi,x)\mathrm{d}\varphi - \frac{2}{(2\pi)^k}\int_{\mathbb{T}^k}\inf_{x\in\Omega} W(\varphi,x)\mathrm{d}\varphi. \tag{4.5}$$

Let $h_j(\cdot) \in \mathrm{C}^\infty(\mathbb{T}^k, T\mathcal{M})$ be a sequence of smooth mappings such that $h_j(\varphi) \in T_{u_j(\varphi)}\mathcal{M}$ for any $\varphi \in \mathbb{T}^k$ and besides there exist positive constants $K,\ K_1$ such that

$$\|h_j\|_1 \le K_1, \quad \|h_j(\varphi)\| \le K \quad \forall \varphi \in \mathbb{T}^k, \quad \forall j = 1, 2, \dots \tag{4.6}$$

Let us show that

$$\lim_{j\to\infty} J'[u_j](h_j) = 0. \tag{4.7}$$

On one hand, $J[u_j]$ decreases to $J_* := \inf J[\mathcal{S}_\Omega]$. On the other hand, for sufficiently small $s_0 \le 1$ and for any $j \in \mathbb{N}$ there exists a number $\theta_j \in [-s_0, s_0]$ such that

$$J[\exp_{u_j}(sh_j)] = J[u_j] + sJ'[u_j](h_j) + \frac{s^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}s^2}\Big|_{s=\theta_j} J[\exp_{u_j}(sh_j)]$$

for aa $s \in [-s_0, s_0]$ and all $j \in \mathbb{N}$; and, there exists a constant $K_2 > 0$ such that

$$\Big|\frac{\mathrm{d}^2}{\mathrm{d}s^2} J[\exp_{u_j}(sh_j)]\Big| \le K_2 \quad \forall s \in [-s_0, s_0], \quad \forall j \in \mathbb{N}.$$

If now we suppose that $\limsup_{j\to\infty} |J'[u_j](h_j)| > 0$ then one can choose j and $s_j \in [-s_0, s_0]$ in such a way that

$$\exp_{u_j}(s_j h_j) \in \mathcal{S}_{\Omega+\delta}, \quad J[\exp_{u_j}(s_j h_j)] < J_*.$$

Thus, in view of (4.4), we arrive at a contradiction with definition of J_*.

Now by Theorem 3.3 for any pair $u_{i+j}(\cdot),\ u_j(\cdot)$ there exists a vector field $h_{ij}(\cdot)$ along $u_j(\cdot)$ such that

$$J[u_{i+j}] - J[u_j] - J'[u_j](h_{ij}) \ge \frac{\kappa c^2}{2}\int_{\mathbb{T}^k}\rho^2(u_j, u_{i+j})\mathrm{d}\varphi \ge \frac{(2\pi)^k\kappa c^2}{2}\|u_{i+j} - u_j\|_0^2.$$

Since (4.7) implies $J'[u_j](h_{ij}) \to 0$ as $j \to \infty$, then the sequence $u_j(\cdot)$ is fundamental in $\mathrm{H}(\mathbb{T}^k, \mathbb{E}^n)$ and in view of (4.5) converges to a function $u_*(\cdot)$ strongly in $\mathrm{H}(\mathbb{T}^k, \mathbb{E}^n)$ and weakly in $\mathrm{H}^1_\omega(\mathbb{T}^k, \mathbb{E}^n)$. Without loss of generality we may consider that $u_*(\cdot)$ is defined by a minimizing sequence which converges a.e.

Now it remains only to prove that $u_*(\cdot)$ is a weak solution; i.e., that there holds (2.3). Let $h(\cdot)$ be a vector field along $u_*(\cdot)$. By definition, there exists a sequence of smooth mappings $h_j(\varphi) \in T_{u_j(\varphi)}\mathcal{M}$ which satisfies (4.6) and (4.7). Then, in view of (4.5), we obtain

$$\begin{aligned} &\lim_{j\to\infty} |\langle D_\omega u_*, D_\omega h\rangle_0 - \langle D_\omega u_j, D_\omega h_j\rangle_0| \\ &\le \lim_{j\to\infty} |\langle D_\omega(u_* - u_j), D_\omega h\rangle_0| + \sqrt{M}\lim_{j\to\infty}\|D_\omega(h - h_j)\|_0 = 0. \end{aligned}$$

As was noted above, we may consider that $u_j(\cdot)$ is a minimizing sequence that converges to $u_*(\cdot)$ a.e. on $\mathbb{T}^k$. Then $W(\varphi, u_j(\varphi))$ converges to $W(\varphi, u_*(\varphi))$ for almost all $\varphi \in \mathbb{T}^k$. By definition of $\mathcal{S}_\Omega$, each function $u_j(\cdot)$ takes values in bounded domain Ω. For this reason

$$|W(\varphi, u_j(\varphi))| \leq \max_{(\varphi,u)\in\mathbb{T}^k\times\bar{\Omega}} |W(\varphi, u)|.$$

By the dominated convergence Lebesgue theorem, the function $\varphi \to W(\varphi, u_*(\varphi))$ is integrable on $\mathbb{T}^k$ and

$$\lim_{j\to\infty} \int_{\mathbb{T}^k} [W(\varphi, u_j(\varphi)) - W(\varphi, u_*(\varphi))] \mathrm{d}\varphi = 0.$$

Hence, $J'[u_*](h) = \lim_{j\to\infty} J'[u_j](h_j) = 0$. □

5. Searching for weak quasiperiodic solutions by means of averaged force function

There naturally arise a question of how to choose a conformally equivalent metric. One of the ways is to seek the function $V(\cdot)$ in the form $V(\cdot) = Y \circ \bar{W}(\cdot)$ where

$$\bar{W}(x) := \frac{1}{(2\pi)^k} \int_{\mathbb{T}^k} W(\varphi, x) \mathrm{d}\varphi$$

is the averaged force function and $Y(\cdot) \in \mathrm{C}^\infty(\bar{W}(\mathcal{M}), \mathbb{R})$ is an unknown function. Here we suppose that $\bar{W}(\cdot)$ is smooth, otherwise we replace this function by a smooth approximation.

A searching procedure for quasiperiodic solutions can be as follows.

Step 1. Searching for non-degenerate points of local minimum for $\bar{W}(\cdot)$. Let x_* be one of such points. Without loss of generality, we assume that $\bar{W}(x_*) = 0$. A minimal eigenvalue of quadratic form $\langle H_{\bar{W}}(x_*)\xi, \xi\rangle$ on $T_{x_*}\mathcal{M}$ is $\lambda_{\bar{W}}(x_*) > 0$. Let

$$\mathcal{E} := \{x \in \mathcal{M} : \lambda_{\bar{W}}(x) > 0,\ \nabla\bar{W} \neq 0\ \forall x \neq x_*\},$$

and for the sake of definiteness consider the case where $K^*(x) > 0$ for all $x \in \mathcal{E}$. Denote by Θ_w a connected component of sublevel set $\bar{W}^{-1}((-\infty, w))$ such that $x_* \in \Theta_w$, and put

$$w_1 := \sup\{w > 0 : \bar{\Theta}_w \subset \mathcal{E}\}. \tag{5.1}$$

Step 2. Constructing a smooth function $Y(\cdot)$ in order to satisfy (3.1) with $V(\cdot) = Y \circ \bar{W}(\cdot)$.

Lemma 5.1. *Put* $p(w) := \max_{x\in\bar{\Theta}_w} \frac{\|\nabla\bar{W}(x)\|^2}{\lambda_{\bar{W}}(x)}$, $q(w) := \max_{x\in\bar{\Theta}_w} \frac{2K^*(x)}{\lambda_{\bar{W}}(x)}$. *Let* $z(\cdot) : [0, w_2) \to \mathbb{R}_+$, *where* $w_2 \in (0, w_1]$, *be a smooth function satisfying the inequalities*

$$z' \geq \frac{z^2}{2} + \frac{q(w) - z}{p(w)}, \quad \forall w \in (0, w_2), \quad z \geq q(w) \quad \forall w \in [0, w_2), \tag{5.2}$$

and let $Y(w) = \int_0^w z(s)\mathrm{d}s$. *Then* $\mu_{Y\circ\bar{W}}(x) \geq 2K^*(x)$ *for all* $x \in \Theta_w$ *and all* $w \in (0, w_2)$.

Proof. If $V(\cdot) = Y \circ W(\cdot)$ then $\nabla V(x) = Y' \circ \bar{W}(x)\nabla\bar{W}(x)$ and

$$\langle H_V(x)\xi, \xi\rangle = Y' \circ \bar{W}(x)\langle H_{\bar{W}}(x)\xi, \xi\rangle + Y'' \circ \bar{W}(x)\langle\nabla\bar{W}(x), \xi\rangle^2,$$

$$\langle G_V(x)\xi, \xi\rangle = Y' \circ \bar{W}(x)\langle H_{\bar{W}}(x)\xi, \xi\rangle + [Y'' - \frac{1}{2}(Y')^2] \circ \bar{W}(x)\langle\nabla\bar{W}(x), \xi\rangle^2,$$

for $\xi \in T_x\mathcal{M}$. Any $\xi \in T_x\mathcal{M}$, $\|\xi\| = 1$, can be represented in the form $\xi = \cos\alpha\eta + \sin\alpha\varsigma$ where $\varsigma := \frac{\nabla\bar{W}(x)}{\|\nabla\bar{W}(x)\|}$ and $\eta \in T_x\mathcal{M}$ is such that $\|\eta\| = 1$, $\langle\eta,\varsigma\rangle = 0$. Then it is not hard to show that

$$\begin{aligned}\mu_{Y\circ\bar{W}}(x) &\geq \lambda_{\bar{W}}(x)Y'\circ\bar{W}(x) + [Y'' - \frac{1}{2}(Y')^2]\circ\bar{W}(x)\|\nabla\bar{W}(x)\|^2\sin^2\alpha\\ &\geq \lambda_{\bar{W}}(x)Y'\circ\bar{W}(x)\cos^2\alpha\\ &\quad + ([Y'' - \frac{1}{2}(Y')^2]\circ\bar{W}(x)\|\nabla\bar{W}(x)\|^2 + \lambda_{\bar{W}}(x)Y'\circ\bar{W}(x))\sin^2\alpha\\ &\geq 2K^*(x)\end{aligned}$$

for all $x \in \Theta_w$, $w \in (0, w_2)$. $\square$

Remark 5.2. Observe that $p(w) \to +0$ when $w \to +0$, the functions $p(\cdot)$ and $q(\cdot)$ are non-decreasing, and the roots $z_\pm(w)$ of equation $\frac{z^2}{2} + \frac{q(w)-u}{p(w)} = 0$ have asymptotic representations

$$\begin{aligned}z_+(w) &:= \frac{1}{p(w)}[1 + \sqrt{1 - 2p(w)q(w)}] = \frac{2}{p(w)} - q(w) + O(p(w)),\\ z_-(w) &:= \frac{1}{p(w)}[1 - \sqrt{1 - 2p(w)q(w)}] = q(w) + O(p(w)),\end{aligned}$$

as $w \to +0$. Set $w_3 := \sup\{w \in (0, w_1) : 2p(w)q(w) < 1\}$. One can show that $z_+(\cdot)$ is non-increasing, $z_-(\cdot)$ is non-decreasing, $z_+(w) > z_-(w) > q(w)$ for all $w \in (0, w_3)$, and integral curves of the the equation

$$z' = \frac{z^2}{2} + \frac{q(w) - z}{p(w)} \tag{5.3}$$

have negative slope in the domain bounded by the graphs of $z_-(\cdot)$, $z_+(\cdot)$, and the ordinate axis. If $w_3 = w_1$, then we can put $w_2 = w_1$ and $z(w) \equiv \frac{1}{p(w_2)}$. Let now $w_3 < w_1$. For any $w_0 \in (0, w_3]$, let $z(\cdot; w_0) : (0, w_4) \to \mathbb{R}$ be a non-continuable solution of (5.3) satisfying the initial condition $z(w_0; w_0) = z_-(w_0) > q(w_0)$. It is naturally to choose w_0 in such a way that

$$w_5 := \sup\{w \in (w_0, w_4) :\ z(s; w_0) \geq q(s)\ \forall s \in (w_0, w)\}$$

be maximally large. One can put $w_2 := \min\{w_1, w_5\}$ and take for the solution of deferential inequality (5.2) the following function: $z(\cdot)\big|_{[0,w_0]}$ is a non-decreasing smooth function satisfying the conditions

$$\begin{gathered}z_-(w) < z(w) < z_+(w) \quad \forall w \in [0, w_0),\\ z^{(i)}(w_0) = \frac{\partial^i}{\partial w^i}\Big|_{w=w_0} z(w; w_0) \quad \forall i = 0, 1, 2, \dots,\end{gathered}$$

and $z(w)\big|_{(w_0,w_2)} := z(w; w_0)$.

Finally, let $w^* \in (0, w_2)$ be a number arbitrarily close to w_2. Define $V(\cdot)\big|_{\Theta_{w^*}} := Y \circ \bar{W}(\cdot)$ and smoothly extend this function on the entire $\mathcal{M}$ in such a way that $V(x) \geq Y(w^*)$ for all $x \in \mathcal{M} \setminus \Theta_{w^*}$.

Step 3. Choosing w_* to ensure ρ_V-convexity of Θ_{w_*}. Observe that

$$\lambda_{Y\circ\bar{W}}(x) \geq \mu_{Y\circ\bar{W}}(x) \geq 2K^*(x) > 0$$

if $x \in \Theta_{w^*}$. By propositions 3.2 and 3.8, in order that Θ_{w_*} be ρ_V-convex it is sufficient that $w_* \in (0, w^*)$ be such a number that for any $x_0, x_1 \in \partial\Theta_{w_*}$ the set

$\bar\Theta_{w^*}$ contains at least one minimal ρ-geodesic segment with endpoints x_0, x_1. Such a choice of w_* is always possible (see Remark 3.1).

So we managed to construct a function $V(\cdot)$ for which Hypothesis (H1) holds true with $v := Y(w_*)$ and $\mathcal{D} = \Omega = \Theta_{w_*}$.

Finally, to prove the existence of weak quasiperiodic solution by means of Theorem 4.1, it remains to verify Hypothesis (H2). Let $\tilde W(\varphi, x) := W(\varphi, x) - \bar W(x)$. In order that Hypothesis (H2) holds it is sufficient that the force function satisfies the following two conditions:

$$\min_{(\varphi,x)\in\mathbb{T}^k\times\partial\Theta_w}[\lambda_W(\varphi,x)+\frac{z(w)}{2}(\|\nabla\bar W(x)\|^2+\langle\nabla\bar W,\nabla\tilde W(\varphi,x)\rangle)]>0 \quad \forall w\in[0,w_*), \tag{5.4}$$

$$\|\nabla\bar W(x)\| > \|\nabla\tilde W(\varphi,x)\| \quad \forall(\varphi,x)\in\mathbb{T}^k\times\partial\Theta_{w_*}, \tag{5.5}$$

where $z(\cdot)$ is the function defined in Lemma 5.1 and $\lambda_W(\varphi,\cdot)$ is the minimal eigenvalue of Hesse form of function $W(\varphi,\cdot):\mathcal{M}\to\mathbb{R}$ for each $\varphi\in\mathbb{T}^k$. Thus we arrive at the following result.

Theorem 5.3. *Let $z(\cdot)$ and $V(\cdot)$ be the functions constructed above and let the force function $W(\cdot,\cdot)$ satisfy the inequalities* (5.4), (5.5). *Then the natural system on Riemannian manifold $(\mathcal{M},\langle\cdot,\cdot\rangle)$ with Lagrangian density $L=\frac{1}{2}\langle\dot x,\dot x\rangle+W(t\omega,x)$ has a weak quasiperiodic solution.*

6. Quasiperiodic forcing of natural system on hypersurface

In coordinate Euclidean space $\mathbb{E}^n$, consider a natural conservative system which undergo quasiperiodic forcing. The Lagrangian density of such a system has the form

$$L := \frac{\|\dot y\|^2}{2} + U(y) + \langle f(t\omega), y\rangle$$

where $U(\cdot)\in C^\infty(\mathbb{E}^n,\mathbb{R})$, $f(\cdot)\in C(\mathbb{T}^k,\mathbb{E}^n)$, $\int_{\mathbb{T}^k} f(\varphi)d\varphi=0$. Suppose that the system is constrained to a regular connected compact hypersurface represented as a level set $\mathcal{M} := F^{-1}(0)$ of a smooth function $F(\cdot):\mathbb{E}^n\to\mathbb{R}$ such that $\operatorname{grad} F(y)\neq 0$ for any $y\in\mathcal{M}$. Let us show how one can verify the inequalities of Hypotheses (H1)–(H2) in the case where $\bar W(\cdot)=U(\cdot)\big|_{\mathcal{M}}$ and $\tilde W(\varphi,\cdot)=\langle f(\varphi),\cdot\rangle\big|_{\mathcal{M}}$.

In the rest of this article, we shall use the notation $\operatorname{grad} U(y)$ and $\operatorname{Hess} U(y)$, respectively, for gradient and Hessian matrix of function $U(y)$ in $\mathbb{E}^n$, while $\nabla U(x)$ and $H_U(x)$ will denote the same for the restriction of $U(\cdot)$ to $\mathcal{M}$ (let us agree to denote current point of the hypersurface $\mathcal{M}$ by x).

Determine the normal vector field and the second fundamental form of hypersurface $\mathcal{M}$:

$$\nu_x := \frac{\operatorname{grad} F(x)}{\|\operatorname{grad} F(x)\|},$$

$$II_x(\xi,\eta) := \langle d\nu_x(\xi),\eta\rangle = \frac{\langle \operatorname{Hess} F(x)\xi,\eta\rangle}{\|\operatorname{grad} F(x)\|} \quad \xi,\eta\in T_x\mathcal{M}.$$

Taking into account that the metric tensor and the Levi-Civita connection for $\mathcal{M}$ are induced by scalar product of $\mathbb{E}^n$, we have

$$\nabla U(x) = \operatorname{grad} U(x) - \langle \operatorname{grad} U(x),\nu_x\rangle\nu_x,$$

$$\langle H_U(x)\xi,\xi\rangle := \langle\nabla_\xi\nabla U(x),\xi\rangle = \langle \operatorname{Hess} U(x)\xi,\xi\rangle - \langle \operatorname{grad} U(x),\nu_x\rangle II_x(\xi,\xi),$$

$$\lambda_U(x) = \min\{\langle H_U(x)\xi, \xi\rangle :\ \xi \in \mathbb{E}^n,\ \langle \nu_x, \xi\rangle = 0,\ \|\xi\| = 1\}.$$

Put

$$\kappa_F(x) := \max\{|II_x(\xi,\xi)| : \xi \in T_x\mathcal{M},\ \|\xi\| = 1\}.$$

At each point $x \in \mathcal{M}$, we split the vector $f(\varphi)$ into its tangential and normal components with respect to $T_x\mathcal{M}$, namely

$$f(\varphi) = f^{\top}(\varphi, x) + f^{\perp}(\varphi, x), \quad f^{\top}(\varphi, x) := f(\varphi) - \langle f(\varphi), \nu_x\rangle\nu_x,$$
$$f^{\perp}(\varphi, x) := \langle f(\varphi), \nu_x\rangle\nu_x,$$

Since in our case $\tilde{W}(\varphi, y) = \langle f(\varphi), y\rangle$ and

$$\nabla\tilde{W}(\varphi, x) = f^{\top}(\varphi, x), \quad \langle H_{\tilde{W}}(\varphi, x)\xi, \xi\rangle = -\langle f(\varphi), \nu_x\rangle II_x(\xi, \xi) \quad \xi \in T_x\mathcal{M},$$

we obtain

$$\|\nabla\tilde{W}(\varphi, x)\| = \|f^{\top}(\varphi, x)\|, \quad \|H_{\tilde{W}}(\varphi, x)\| \le \kappa_F(x)|\langle f(\varphi), \nu_x\rangle|. \tag{6.1}$$

Finally, let us determine $K^*(x)$. It is known (see, e.g., [17, sect. 3.7]) that

$$K^*(x) = \max\Big\{ \begin{vmatrix} II_x(\xi_1,\xi_1) & II_x(\xi_1,\xi_2) \\ II_x(\xi_1,\xi_2) & II_x(\xi_2,\xi_2) \end{vmatrix} : \\ \xi_i \in \mathbb{E}^n,\ \langle \xi_i, \nu_x\rangle = 0,\ \langle \xi_i, \xi_j\rangle = \delta_{ij},\ i,j = 1,2\Big\}$$

From this it follows that if $\kappa_i(x)$, $i = 1, \dots, n-1$, are $n-1$ principal curvatures of hypersurface $\mathcal{M}$ at point x, then

$$K^*(x) = \max_{1 \le i < j \le n-1} \kappa_i(x)\kappa_j(x).$$

Thus, we have determined all the functions involved in searching procedure for weak quasiperiodic solutions. After having found the function $z(\cdot)$ from Lemma 5.1 and the number w_*, it remains only to verify the inequalities (5.4) and (5.5). In order for these inequalities hold it is sufficient that

$$\begin{gathered} 2\lambda_U(x) + z(w)(\|\nabla U(x)\|^2 + \langle \nabla U(x), f^{\top}(\varphi, x)\rangle) \ge 2\kappa_F(x)|\langle f(\varphi), \nu_x\rangle| \\ \forall(\varphi, x) \in \mathbb{T}^k \times \partial\Theta_w,\ \forall w \in [0, w_*], \\ \|\nabla U(x)\| > \sqrt{\|f(\varphi)\|^2 - \langle f(\varphi), \nu_x\rangle^2} \quad \forall x \in \mathbb{T}^k \times \partial\Theta_{w_*}. \end{gathered} \tag{6.2}$$

As an application of the technique developed, we consider the motion of spherical pendulum under quasiperiodic forcing. The spherical pendulum is a natural system on 2-D sphere with functions $F(\cdot)$ and $U(\cdot)$ of the form

$$F(x) = x_1^2 + x_2^2 + x_3^2 - 1, \quad U(y) = g \cdot (1 - y_3), \quad g = \text{const} > 0.$$

Restrict our study to the case where the forcing function is $f(t\omega) = a(t\omega)b$ where

$$b = (b_1, b_2, b_3) \in \mathbb{E}^3 \quad b_3 > 0, \quad \max_{\varphi \in \mathbb{T}^k} |a(\varphi)| = 1, \quad \int_{\mathbb{T}^k} a(\varphi)\mathrm{d}\varphi = 0. \tag{6.3}$$

Theorem 6.1. *Let the following inequality hold:*

$$g \ge 2.42 \max\{\sqrt{b_1^2 + b_2^2}, b_3\} + b_3. \tag{6.4}$$

Then the spherical pendulum system with forcing function $t \to f(t\omega) := a(t\omega)b$ *satisfying the conditions* (6.3) *has a weak quasiperiodic solution containing in the upper hemisphere.*

Proof. In the case under consideration we have

$$
\begin{gathered}
\operatorname{grad} U = (0,0,-g), \quad \nu_x = (x_1, x_2, x_3), \\
\nabla U(x) = \operatorname{grad} U(x) - \langle \operatorname{grad} U(x), \nu_x \rangle \nu_x = g \cdot (x_1 x_3, x_2 x_3, x_3^2 - 1), \\
\|\nabla U(x)\| = g\sqrt{1 - x_3^2}, \quad II_x(\xi,\xi) = \langle \xi, \xi \rangle, \quad \langle H_U(x)\xi, \xi \rangle = g x_3 \langle \xi, \xi \rangle, \\
\lambda_U(x) = g x_3, \quad \kappa_F(x) = K^*(x) = 1, \\
\begin{aligned}
\langle \nabla U(x), f^\top(\varphi, x) \rangle &= \langle \operatorname{grad} U(x), f(\varphi) \rangle - \langle f(\varphi), \nu_x \rangle \langle \operatorname{grad} U(x), \nu_x \rangle \\
&= g a(\varphi)[x_3 \langle b, x \rangle - b_3],
\end{aligned} \\
\|f^\top(\varphi, x)\|^2 = a^2(\varphi)[\|b\|^2 - \langle b, x \rangle^2].
\end{gathered}
$$

For the stationary point $x_* = (0,0,1)$ of $U(x)$ we have $\lambda_U(x_*) = g > 0$. The set $\mathcal{E}$ in our case is the upper hemisphere, and $w_1 = g$ (see (5.1)). Further,

$$
\begin{gathered}
p(w) = g \max_{1 - w/g \le x_3 \le 1} \big(\frac{1 - x_3^2}{x_3} \big) = \frac{2gw - w^2}{g - w}, \\
q(w) = \frac{2}{g} \max_{1 - w/g \le x_3 \le 1} \frac{1}{x_3} = \frac{2}{g - w}.
\end{gathered}
$$

Hence, the inequalities of Lemma 5.1 take the form

$$
z' \ge \frac{z^2}{2} + \frac{2 - (g - w)z}{2gw - w^2}, \quad z \ge \frac{2}{g - w}.
$$

The natural solution of these inequalities is $z(w) = \frac{2}{g-w}$. Since ρ-geodesics on the sphere are great circles, then the spherical cap

$$
\Theta_w = U^{-1}((-\infty, w)) = \{x \in \mathbb{R}^3 : \sum_{i=1}^3 x_i^2 = 1, \ x_3 > 1 - w/g\}
$$

is ρ-convex for any $w \in (0, g)$. For this reason one can put $w_* = w^*$ where $w^* \in (0, g)$ is a number arbitrarily close to g.

Further, the parametric representation of boundary $\partial \Theta_w$ is

$$
x_1 = \sqrt{1 - x_3^2} \cos \alpha, \quad x_2 = \sqrt{1 - x_3^2} \sin \alpha, \quad x_3 = 1 - w/g.
$$

On this curve, we have

$$
|\langle b, x \rangle| \le \sqrt{(b_1^2 + b_2^2)(1 - x_3^2)} + b_3 x_3,
$$

and it is easily seen that the inequalities (6.2) hold once

$$
\begin{gathered}
g \ge 2x_3 \big(\sqrt{(b_1^2 + b_2^2)(1 - x_3^2)} + b_3 x_3 \big) + b_3 \quad \forall x_3 \in [1 - w_*/g, 1], \\
g\sqrt{1 - (1 - w_*/g)^2} > \sqrt{b_1^2 + b_2^2 + b_3^2}.
\end{gathered}
$$

To complete the proof it remains only to observe that

$$
\max_{x_3 \in [0,1]} x_3 \big(\sqrt{(b_1^2 + b_2^2)(1 - x_3^2)} + b_3 x_3 \big) \le 1.21 \max\{ \sqrt{b_1^2 + b_2^2}, b_3 \},
$$

and to choose w_* sufficiently close to g. □

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References

[1] M. Ayachi; *Variational methods and almost periodic solutions of second order functional differential equations with infinite delay*, Commun. Math. Anal., **9** (2010), no. 1, 15–31.

[2] M. Ayachi, J. Blot; *Variational methods for almost periodic solutions of a class of neutral delay equations*, Abstr. Appl. Anal., **2008** (2008), Article ID 153285, 13 pp.

[3] M. S. Berger, Luping Zhang; *A new method for large quasiperiodic nonlinear oscillations with fixed frequencies for nondissipative second order conservative systems of second type*, Commun. Appl. Nonlinear Anal., **3** (1996), no. 1, 25–49.

[4] R. L. Bishop, R.J. Crittenden; *Geometry of manifolds*, Academic Press, New York - London (1964).

[5] J. Blot; *Calculus of variations in mean and convex Lagrangians,* J. Math. Anal. Appl., **134** (1988), no. 2, 312–321.

[6] J. Blot; *Calculus of variations in mean and convex Lagrangians*. II, Bull. Aust. Math. Soc., **40** (1989), 457–463.

[7] J. Blot; *Calculus of variations in mean and convex Lagrangians* III, Israel J. Math., **67** (1989), no. 3, 337–344.

[8] J. Blot; *Almost periodically forced pendulum*, Funkcial. Ekvac., **36** (1993), 235–250.

[9] J. Blot, P. Cieutat, J. Mawhin; *Almost-periodic oscillations of monotone second-order systems*, Adv. Differential Equations, **2** (1997), 693–714.

[10] J. Blot, D. Pennequin; *Spaces of quasi-periodic functions and oscillations in differential equations*, Acta. Appl. Math, **65** (2001), 83–113.

[11] D. Cheban, C. Mammana; *Invariant manifolds, almost periodic and almost automorphic solutions of second-order monotone equations*, Int. J. Evol. Equ., **1** (2005), no. 4, 319–343.

[12] V. M. Cheresiz; *Stable and conditionally stable almost-periodic solutions of V-monotone systems*, Sib. Math. J., **15** (1974), no. 1, 116–125.

[13] P. Cieutat; *Almost periodic solutions of second-order systems with monotone fields on a compact subset*, Nonlin. Anal., **53** (2003), no. 6, 751–763.

[14] P. Cieutat; *Bounded and almost periodic solutions of convex Lagrangian systems*, J. Differential Equations, **190** (2003), 108–130.

[15] David C. Clark; *Periodic solutions of variational systems of ordinary differential equations*, J. Differential Equations, **28**, no. 3, 1978, 354–368.

[16] B. A. Dubrovin, A. T. Fomenko, S. P. Novikov; *Modern Geometry. Methods and Applications*, Springer-Verlag, GTM 93, Part 1, (1984).

[17] D. Gromol, W. Klingenberg, W. Meyer; *Riemannsche geometrie im grossen*, Springer-Verlag, Heidelberg-New York (1968).

[18] M. A. Krasnosel'skii, P. P. Zabreiko; *Geometrical Methods of Nonlinear Analysis*, Berlin-Heidelberg-New York-Tokio: Springer-Verlag, A Series of Comprehensive Studies in Mathematics, **263** (1984), 409 pp.

[19] J. Kuang; *Variational approach to quasi-periodic solution of nonautonomous second-order Hamiltonian systems*, Abstr. Appl. Anal., **2012** (2012), Article ID 271616, 14 pp.

[20] J. Mawhin; *Bounded and almost periodic solutions of nonlinear differential equations: variational vs nonvariational approach*, in: Calculus of Variations and Differential Equations (Haifa, 1998), Chapman & Hall/CRC Research Notes in Mathematics, **410**, Boca Raton, Raton, FL, 2000, 167–184.

[21] J. Mawhin, James R. Ward, Jr.; *Guiding-like functions for periodic or bounded solutions of ordinary differential equations*, Discrete Contin. Dyn. Syst., **8** (2002), 39–54.

[22] I. Meghea, V. Stanciu; *Existence of the solutions of forced pendulum equation by variational methods*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys., **71** (2009), no. 4, 115–124.

[23] J. Nash; *The imbedding problem for Riemannian manifolds*, Ann. of Math. **63**, no. 1 (1956), 2063.

[24] A. Nowakowski and A. Rogowski; *Periodic solutions of Lagrange equations*, Topol. Methods Nonlinear Anal., **22** (2003), 167–180.

[25] R. Ortega; *The pendulum equation: from periodic to almost periodic forcings*, Differential Integral Equations, **22** (2009), no. 9–10, 801–814.

[26] I. Parasyuk, A. Rustamova; *Existence theorem for weak quasiperiodic solutions of Lagrangian systems on Riemannian manifolds,*arXiv:1112.1550v1 [math-ph] (2011), 14 p.

[27] A. I. Perov, T. I. Smagina, V. L. Khatskevich, *Variational approach to the periodic solutions problem*, Sib. Math. J., **25** (1984), no. 1, 88–100.
[28] P. Rabinowitz; *Periodic solutions of Hamiltonian systems*, Comm. Pure. Appl. Math, **31** (1978), 157–184.
[29] A. M. Samoilenko; *Elements of the mathematical theory of multi-frequency oscillations*, Kluwer Academic Publishers, Dordrecht (1991).
[30] Yu. V. Trubnikov, A. I. Perov; *Differential equations with monotone nonlinearities*, Nauka i Tekhnika, Minsk (1986) (in Russian).
[31] F. W. Warner; *Foundations of differentiable manifolds and Lie groups*, Berlin-Heidelberg-New York-Tokio: Springer-Verlag (1968).
[32] S. F. Zakharin, I. O. Parasyuk; *Generalized and classical almost periodic solutions of Lagrangian systems*, Funkcial. Ekvac., **42** (1999), 325–338.
[33] S. F. Zakharin, I. O. Parasyuk; *Generalized quasiperiodic solutions of Lagrangian systems on Riemannian manifolds of nonpositive curvature*, Bull. Kyiv. Univ. (Visnyk Kyiv. Univ.), (1999), Iss. 3, 15–20. (in Ukrainian)
[34] S. F. Zakharin, I. O. Parasyuk; *On smoothness of generalized quasiperiodic solutions of Lagrangian systems on Riemannian manifolds of nonpositive curvature*, Nonlinear Oscil. (Nelinijni Kolyvannya), **2** (1999), no. 2, 180-193. (in Ukrainian)

7. Addendum posted on December 28, 2012

In this addendum, we show that if a function $u_*(\cdot)$ is determined by a minimizing sequence $u_j(\cdot) \in \mathcal{S}_\Omega$ of the functional J, then this sequence converges strongly to $u_*(\cdot)$ not only in $\mathrm{H}(\mathbb{T}^k, \mathbb{E}^n)$, but also in $\mathrm{H}^1_\omega(\mathbb{T}^k, \mathbb{E}^n)$. Our motivation for doing this is as follows. Recall that a function $u(\cdot) \in \mathcal{H}_\mathcal{A}$ is defined as a strong limit in $\mathrm{H}(\mathbb{T}^k, \mathbb{E}^n)$ of a sequence $u_j(\cdot) \in \mathcal{S}_\mathcal{A}$ bounded in $\mathrm{H}^1_\omega(\mathbb{T}^k, \mathbb{E}^n)$, but at the same time the definition of vector field $h(\cdot)$ along $u(\cdot)$ (page 4) requires that a sequence of vector fields $h_j(\cdot) \in \mathrm{C}^\infty(\mathbb{T}^k, T\mathcal{M})$ along $u_j(\cdot)$ converges to $h(\cdot)$ strongly in $\mathrm{H}^1_\omega(\mathbb{T}^k, \mathbb{E}^n)$. This brings up a question whether such a definition of vector field along $u(\cdot) \in \mathcal{H}_\mathcal{A}$ is correct enough. Obviously, the answer is positive for a subset of $\mathcal{H}_\mathcal{A}$ which is strong closure of $\mathcal{S}_\mathcal{A}$ by norm of the space $\mathrm{H}^1_\omega(\mathbb{T}^k, \mathbb{E}^n)$.

Suppose that $W(\cdot,\cdot) \in \mathrm{C}^{0,3}(\mathbb{T}^k \times \mathcal{M}, \mathbb{R})$ and that there holds the following hypothesis which slightly strengthen Hypothesis (H1):

(H1') the conditions of Hypothesis (H1) are valid together with strict inequality

$$\mu_* := \min_{x\in\bar{\Omega}}[\mu_V(x) - 2K^*(x)] > 0.$$

Now for any $u_i(\cdot), u_j(\cdot) \in \mathcal{S}_\Omega$, define

$$\eta(s,t) := \frac{\partial}{\partial t}\chi(s, u_i(\varphi+\omega t), u_j(\varphi+\omega t)) \equiv D_\omega\chi(s, u_i(\varphi+\omega t), u_j(\varphi+\omega t)),$$

$$\xi(s,t) := \frac{\partial}{\partial s}\chi(s, u_i(\varphi+\omega t), u_j(\varphi+\omega t)).$$

Then in view of Proposition 3.10, we have

$$\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}s^2}\|\eta\|^2 &= 2[\langle \nabla_\xi^2\eta, \eta\rangle + \|\nabla_\xi\eta\|^2] \\
&= 2\|\nabla_\xi\eta\|^2 + 2\langle \nabla_\xi\eta, \xi\rangle\langle \nabla V\circ\chi, \eta\rangle \\
&\quad + \|\xi\|^2\langle \nabla_\eta\nabla V\circ\chi, \eta\rangle - 2\langle R(\eta,\xi)\xi, \eta\rangle \\
&\geq 2\|\nabla_\xi\eta\|^2 - 2\|\nabla_\xi\eta\|\|\xi\||\langle \nabla V\circ\chi, \eta\rangle| \\
&\quad + \|\xi\|^2\langle \nabla_\eta\nabla V\circ\chi, \eta\rangle - 2K^*\circ\chi\|\xi\|^2\|\eta\|^2 \\
&= 2\|\nabla_\xi\eta\|^2 - 2|\langle \nabla V\circ\chi, \mathbf{e}\rangle|\|\nabla_\xi\eta\|\|\xi\|\|\eta\|
\end{aligned}$$

$$+ [\langle \nabla_{\mathbf{e}} \nabla V \circ \chi, \mathbf{e} \rangle - 2K^* \circ \chi] \|\xi\|^2 \|\eta\|^2$$

where $\mathbf{e} := \eta/\|\eta\|$. It is easily seen that once the Hypothesis (H1') holds, then the quadratic form

$$z_1^2 - |\langle \nabla V \circ \chi, \mathbf{e} \rangle| z_1 z_2 + [\frac{1}{2} \langle \nabla_{\mathbf{e}} \nabla V \circ \chi, \mathbf{e} \rangle - K^* \circ \chi] z_2^2$$

of variables z_1, z_2 is positive definite for any unit vector field $\mathbf{e}$. Hence there exists a positive number $\nu > 0$ such that

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2} \|\eta\|^2 \ge 2\nu [\|\nabla_\xi \eta\|^2 + \|\xi\|^2 \|\eta\|^2],$$

and taking into account Hypothesis (H2) we get

$$\begin{aligned}
&\frac{\mathrm{d}^2}{\mathrm{d}s^2} [\frac{1}{2} \|D_\omega \chi(s, u_i, u_j)\|^2 + W(\varphi, \chi(s, u_i(\varphi), u_j(\varphi)))] \\
&\ge \nu [\|\nabla_{\chi_s'} D_\omega \chi\|^2 + \|\chi_s'\|^2 \|D_\omega \chi\|^2] \\
&\quad + \langle \nabla_{\chi_s'} \nabla W(\varphi, \chi), \chi_s' \rangle + \langle \nabla W(\varphi, \chi), \nabla_{\chi_s'} \chi_s' \rangle \\
&= \nu [\|\nabla_{\chi_s'} D_\omega \chi\|^2 + \|\chi_s'\|^2 \|D_\omega \chi\|^2] \\
&\quad + \langle \nabla_{\chi_s'} \nabla W(\varphi, \chi), \chi_s' \rangle + \frac{\|\chi_s'\|^2}{2} \langle \nabla W(\varphi, \chi), \nabla V(\chi) \rangle \\
&\ge \nu [\|\nabla_{\chi_s'} D_\omega \chi\|^2 + \|\chi_s'\|^2 \|D_\omega \chi\|^2] + \kappa \|\chi_s'\|^2.
\end{aligned} \tag{7.1}$$

From this it follows that the function $f_{ij}(s) = J[\chi(s, u_i, u_j)]$ is convex (downward):

$$f_{ij}(s) \le s f_{ij}(1) + (1-s) f_{ij}(0) \implies J[\chi(s, u_i, u_j)] \le s J[u_j] + (1-s) J[u_i].$$

Now let $u_i(\cdot) \in \mathcal{S}_\Omega$, $i = 1, 2, \dots$ be a minimizing sequence for $J[u]$, and the sequence $J[u_i]$ monotonically decreases to J_*. Then, taking into account the convexity of $f_{ij}(s)$, the sequence $J[\chi(s, u_i, u_{i+l})]$ is minimising for any $l \in \mathbb{N}$. Together with the property (4.7) on page 15, this assures that for any $\varepsilon > 0$ and $s_1, s_2 \in [0, 1]$ there exists $i(\varepsilon, s_1, s_2) \in \mathbb{N}$ such that

$$\begin{gathered}
J_* \le J[\chi(s_k, u_i, u_j)] \le J_* + \frac{(s_1 - s_2)^2}{8} \varepsilon, \\
|J'[\chi(s_k, u_i, u_j)](\chi_s'(s_k, u_i, u_j))| < \frac{|s_2 - s_1| \varepsilon}{4}
\end{gathered}$$

for all $j \ge i \ge i(\varepsilon, s_1, s_2)$, $k = 1, 2$. But

$$\begin{aligned}
J[\chi(s_2, u_i, u_j)] &= J[\chi(s_1, u_i, u_j)] + (s_2 - s_1) J'[\chi(s_1, u_i, u_j)](\chi_s'(s_1, u_i, u_j)) \\
&\quad + \frac{(s_2 - s_1)^2}{2} \frac{\mathrm{d}^2}{\mathrm{d}s^2} \Big|_{s = \theta_{ij}} J[\chi(s, u_i, u_j)]
\end{aligned}$$

with some θ_{ij} belonging to the interval with endpoints s_1 and s_2. Then

$$0 \le \frac{\mathrm{d}^2}{\mathrm{d}s^2} \Big|_{s = \theta_{ij}} J[\chi(s, u_i, u_j)] \le \varepsilon.$$

Since $u_i(\cdot) \in \mathcal{S}_\Omega$ and $\|D_\omega u_i\|_0^2 \le M$ (see (4.7), page 15), then there exists a constant $K_3 > 0$ such that

$$|\frac{\mathrm{d}^3}{\mathrm{d}s^3} J[\chi(s, u_i, u_j)]| \le K_3 \quad \forall i, j \in \mathbb{N},\ \forall s \in [0, 1].$$

Hence for some σ_{ij} belonging to the interval with endpoints s_1 and θ_{ij}, we have

$$\begin{aligned}&\frac{\mathrm{d}^2}{\mathrm{d}s^2}\Big|_{s=\theta_{ij}} J[\chi(s,u_i,u_j)]\\ &= \frac{\mathrm{d}^2}{\mathrm{d}s^2}\Big|_{s=s_1} J[\chi(s,u_i,u_j)] + (\theta_{ij}-s_1)\frac{\mathrm{d}^3}{\mathrm{d}s^3}\Big|_{s=\sigma_{ij}} J[\chi(s,u_i,u_j)].\end{aligned}$$

If we now assume that $\varepsilon < K_3/2$ and put

$$s_2(s_1,\varepsilon) := \begin{cases} s_1+\varepsilon/K_3 & \text{if } s_1\in[0,1/2],\\ s_1-\varepsilon/K_3 & \text{if } s_1\in(1/2,1],\end{cases}$$
$$i_*(\varepsilon,s_1) := i(\varepsilon,s_1,s_2(s_1,\varepsilon)),$$

then

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}\Big|_{s=s_1} J[\chi(s,u_i,u_j)] \le 2\varepsilon \quad \forall j\ge i\ge i_*(\varepsilon,s_1).$$

In view of (7.1) and $\|\chi_s'(s,u_i,u_j)\| \ge c\|u_i-u_j\|$ (see (3.7), page 12), by letting $s_1=0$ and $s_1=1$, we get

$$\|u_i-u_j\|_0^2 + \|\|u_i-u_j\|\|D_\omega u_l\|\|_0^2 \le C_1\varepsilon, \tag{7.2}$$

$$\|\nabla_{\chi_s'} D_\omega\chi(s,u_i,u_j)\|_0^2\big|_{s=0} = \|\nabla_{D_\omega u_i}\chi_s'(0,u_i,u_j)\|_0^2 \le C_2\varepsilon, \tag{7.3}$$

for all $j\ge i\ge\max\{i_*(\varepsilon,0), i_*(\varepsilon,1)\}$, $l=i,j$, with some positive constants C_1,C_2 independent of i,j,ε.

Now let us estimate $\|\nabla_{D_\omega u_i}\chi_s'(0,u_i,u_j)\|$ from below. First, observe that under Hypothesis (H1$'$) the mapping $\zeta(\cdot,\cdot)$ (see Propositions 3.8, 3.9) is correctly defined and smooth not only in $\Omega\times\Omega$ but also in a neighborhood of this domain. If we fix a point $x_0\in\Omega$, then by means of mapping $\exp^V_{x_0}(\cdot)$ one can introduce local coordinates in a neighborhood of Ω. Namely, for any y belonging to such a neighborhood of Ω we put into one-to-one correspondence the vector $\xi(y) := \zeta(x_0,y)\in T_{x_0}\mathcal{M}$ such that $\exp^V_{x_0}(\xi(y)) = y$.

Next, observe that from the equalities

$$\exp^V_x(\zeta(x,y)) = y, \quad \zeta(x,x)=0, \quad (\exp^V_x)_*(0) = \mathrm{Id}$$

it follows that

$$(\exp^V_x)_*(\zeta(x,y))\zeta_y'(x,y) = \mathrm{Id} \implies \zeta_y'(x,y)\big|_{x=y} = \mathrm{Id},$$
$$\zeta_x'(x,y)\big|_{x=y} + \zeta_y'(x,y)\big|_{x=y} = 0 \implies \zeta_x'(x,y)\big|_{x=y} = -\mathrm{Id}.$$

Since $\tau_s'(0,x,x)=1$ and

$$\chi_s'(0,x,y) = \tau'(0,x,y)\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\exp^V_x(t\zeta(x,y)) = \tau'(0,x,y)\zeta(x,y),$$

then

$$\chi_{sx}''(0,x,y)\big|_{y=x} = \zeta_x'(x,y)\big|_{x=y} = -\mathrm{Id},$$
$$\chi_{sy}''(0,x,y)\big|_{y=x} = \zeta_y'(x,y)\big|_{x=y} = \mathrm{Id}.$$

Now it is easily seen that there exists a constant $C_3>0$ such that

$$\max\{\|\mathrm{Id}+\chi_{sx}''(s,x,y)\|, \|\mathrm{Id}-\chi_{sy}''(s,x,y)\|\} \le C_3\|x-y\| \tag{7.4}$$

for any $(x,y)\in\bar\Omega\times\bar\Omega$.

In local coordinates introduced above, points of Ω and tangent vectors at such points are represented as m-dimensional vectors of coordinate space $\mathbb{E}^m$, and there exists a bilinear mapping

$$\Gamma_x(\cdot,\cdot) : \mathbb{E}^m \times \mathbb{E}^m \to \mathbb{E}^m$$

smoothly depending on x such that

$$\nabla_{D_\omega u_i}\chi'_s(0,u_i,u_j) = D_\omega\chi'_s(0,u_i,u_j) + \Gamma_{u_i}(D_\omega u_i, \chi'_s(0,u_i,u_j))$$

(this bilinear mapping is expressed via the Christoffel symbols). It is not hard to show that there exists a constant $C_4 > 0$ such that

$$\|\Gamma_{u_i}(D_\omega u_i, \chi'_s(0,u_i,u_j))\| \le C_4\|D_\omega u_i\|\|u_i - u_j\| \quad \forall i,j \in \mathbb{N}.$$

Now the equality

$$\begin{aligned} D_\omega\chi'_s(0,u_i,u_j) &= D_\omega u_j - D_\omega u_i + [\mathrm{Id} + \chi''_{sx}(s,u_i,u_j)]D_\omega u_i \\ &\quad + [\chi''_{sx}(s,u_i,u_j) - \mathrm{Id}]D_\omega u_j \end{aligned}$$

together with (7.4) yields

$$\|D_\omega\chi'_s(0,u_i,u_j)\| \ge \|D_\omega u_j - D_\omega u_i\| - C_3\|u_i - u_j\|[\|D_\omega u_i\| + \|D_\omega u_j\|]$$

and finally,

$$\begin{aligned} \|\nabla_{D_\omega u_i}\chi'_s(0,u_i,u_j)\| &\ge \|D_\omega u_j - D_\omega u_i\| \\ &\quad - \|u_i - u_j\|[(C_3 + C_4)\|D_\omega u_i\| + \|D_\omega u_j\|]. \end{aligned}$$

From this it follows that there exists a constant $C_5 > 0$ such that

$$\begin{aligned} \|\nabla_{D_\omega u_i}\chi'_s(0,u_i,u_j)\|^2 &\ge \|D_\omega u_j - D_\omega u_i\|^2 \\ &\quad - C_5[\|D_\omega u_j - D_\omega u_i\|\|u_i - u_j\|(\|D_\omega u_i\| + \|D_\omega u_i\|) \\ &\quad + \|u_i - u_j\|^2(\|D_\omega u_i\|^2 + \|D_\omega u_i\|^2)], \end{aligned}$$

and after applying the Schwartz inequality we obtain

$$\begin{aligned} &\|\nabla_{D_\omega u_i}\chi'_s(0,u_i,u_j)\|_0^2 \\ &\ge \|D_\omega u_j - D_\omega u_i\|_0^2 \\ &\quad - C_5\Big[\|D_\omega u_j - D_\omega u_i\|_0(\|\|u_i - u_j\|\|D_\omega u_i\|\|_0 + \|\|u_i - u_j\|\|D_\omega u_j\|\|_0) \\ &\quad + \|\|u_i - u_j\|\|D_\omega u_i\|\|_0^2 + \|\|u_i - u_j\|\|D_\omega u_j\|\|_0^2\Big]. \end{aligned}$$

Taking into account the inequalities (7.2), (7.3) we see that there exist $\varepsilon_0 > 0$ and $C_6 > 0$ such that

$$\|D_\omega u_j - D_\omega u_i\|_0 \le C_6\sqrt{\varepsilon}$$

for all $\varepsilon \in (0,\varepsilon_0)$ and all $j \ge i \ge \max\{i_*(\varepsilon,0), i_*(\varepsilon,1)\}$. Hence under Hypothesis (H1'), the minimizing sequence $u_i(\cdot)$ is fundamental not only in $\mathrm{H}(\mathbb{T}^k,\mathbb{E}^n)$ but also in $\mathrm{H}^1_\omega(\mathbb{T}^k,\mathbb{E}^n)$.

We also want to correct some misprints.
In the last formula on page 3, the constant K should be replaced with K_1.
In formula (2.3), page 4, the left hand side of equality must be $\frac{1}{(2\pi)^k}J'[u_*]h$.

On line 8, page 12, Hypothesis (H3) should be replaced with Hypothesis (H2).
In Theorem 4.1, the same arguments we used to show that

$$\lim_{j\to\infty}\int_{\mathbb{T}^k}[W(\varphi,u_j(\varphi))-W(\varphi,u_*(\varphi))]\mathrm{d}\varphi=0$$

should be used to prove that

$$\lim_{j\to\infty}\int_{\mathbb{T}^k}\|W'_x(\varphi,u_j(\varphi))-W'_x(\varphi,u_*(\varphi))\|\mathrm{d}\varphi=0.$$

End of addendum.

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