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REDUCIBILITY OF SYSTEMS AND EXISTENCE OF SOLUTIONS FOR ALMOST PERIODIC DIFFERENTIAL EQUATIONS

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Abstract. We establish the reducibility of linear systems of almost periodic differential equations into upper triangular systems of a. p. differential equations. This is done while the number of independent a. p. solutions is conserved. We prove existence and uniqueness of a. p. solutions of a nonlinear system with an a. p. linear part. Also we prove the continuous dependence of a. p. solutions of a nonlinear system with respect to an a. p. control term.

1. Introduction

First we consider the almost periodic, in the Bohr sense, system of linear ordinary differential equations

$$x'(t) = A(t)x(t) \tag{1.1}$$

where A is an almost periodic (a.p.) real $n \times n$ matrix. In Theorem 3.5 below, we establish that when all the solutions of (1.1) are a.p., there exist an a.p. transformation between the solutions of (1.1) and the solutions of

$$y'(t) = B(t)y(t) \tag{1.2}$$

where $B(t)$ is an a.p. real $n \times n$ matrix that is upper triangular for all $t \in \mathbb{R}$.

When there are k linearly independent a. p. solutions of (1.1), we can build a continuous matrix $B(t)$ such that (1.2) also possesses k linearly independent a.p. solutions, see Theorem 4.1 below.

In Section 5 we consider the nonlinear equation

$$u'(t) = B(t)u(t) + f(t, u(t)) \tag{1.3}$$

where B is an a.p. matrix such that the homogeneous equation of (1.3) does not possess any nonzero a.p. solution and f is uniformly a.p. (Theorem 5.4). In a previous work [2], we considered the case where $u'(t) = B(t)u(t)$ does not possess any nonzero a.p. solution and where this homogeneous system can be transformed into a linear system with a constant matrix in the quasi- periodic case under diophantine conditions.

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In Section 6, by using results of Section 5, we build a parametrized fixed point approach to obtain an existence result and a continuous dependence results on a. p. solutions of the equation

$$x'(t) = A(t)x(t) + f(t, x(t), u(t)), \tag{1.4}$$

where u is a control term (see Theorem 6.1).

2. Preliminaries and notation

The usual inner product of $\mathbb{R}$ is denoted by $(\cdot|\cdot)$ and $\|\cdot\|$ will be the associated norm. $\mathbb{R}^{\mathbb{N}}$ denotes the set of real sequences, and $\mathcal{S}(\mathbb{N},\mathbb{N})$ denotes the space of the (strictly) increasing functions from $\mathbb{N}$ into $\mathbb{N}$.

When $(E, \|\cdot\|)$ is a Banach space, $C^0(\mathbb{R}, E)$ denotes the space of the continuous functions from $\mathbb{R}$ into E and $BC^0(\mathbb{R}, E)$ denotes the space of the $u \in C^0(\mathbb{R}, E)$ witch are bounded on $\mathbb{R}$. Endowed with the norm $\|u\|_\infty = \sup_{t\in\mathbb{R}} \|u(t)\|$, $BC^0(\mathbb{R}, E)$ is a Banach space.

When $k \in \mathbb{N}_* = \mathbb{N}\backslash\{0\}$, $C^k(\mathbb{R}, E)$ is the space of the k-times differentiable functions from $\mathbb{R}$ into E.

Following a result by Bochner [8, Definition 1.1, p.1], we define an a.p. function $u : \mathbb{R} \to E$ saying that $u \in BC^0(\mathbb{R}, E)$ and for all $(r_m)_m \in \mathbb{R}^{\mathbb{N}}$ there exists $\sigma \in \mathcal{S}(\mathbb{N},\mathbb{N})$ such that the sequence of the translated functions $(f(. + r_{\sigma(m)}))_m$ is uniformly convergent on $\mathbb{N}$. We denote by $AP^0(E)$ the space of the a.p. functions from $\mathbb{R}$ into E; it is a Banach subspace of $(BC^0(\mathbb{R}, E), \|.\|)$. When $k \in \mathbb{N}_*$, $AP^k(E)$ is the space of functions $u \in C^k(\mathbb{R}, E) \cap AP^0(E)$ such that $u^{(j)} = \frac{d^j u}{dt^j} \in AP^0(E)$ for all $j \in \{1, \dots, k\}$.

Endowed with the norm $\|u\|_{C^k} = \|u\|_\infty + \sum_{j=1}^k \|u^{(j)}\|_\infty$, the space $AP^k(E)$ is a Banach space. When $u \in AP^0(E)$, its mean value

$$\mathcal{M}\{u\} = \mathcal{M}\{u(t)\}_t = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} u(t)\, dt$$

exists in E.

For all real number λ, there exists $a(u, \lambda) = \mathcal{M}\{e^{-i\lambda t}u(t)\}_t$ in E; these vectors are the Fourier-Bohr coefficients of u. We set $\Lambda(u) = \{\lambda \in \mathbb{R} : a(u, \lambda) \neq 0\}$ which is at most countable, and we denote by $\mathrm{Mod}(u)$ the $\mathbb{Z}$-submodule of $\mathbb{R}$ which is spanned by $\Lambda(u)$. For all these notions on the a.p. functions, we refer to [6, 8, 18].

When M is a $\mathbb{Z}$-submodule in $\mathbb{R}$, $AP^k(\mathbb{R}^n, M) = \{u \in AP^k(\mathbb{R}^n) : \mathrm{Mod}(u) \subset M\}$. We denote by $\mathbb{M}(n, \mathbb{R})$ the space of the $n \times n$ real matrices. The transpose of $M \in \mathbb{M}(n, \mathbb{R})$ is denoted by M^*.

The following result is a corollary of a powerfull theorem, due to Bochner, proven in [8, Theorem 1.17, p.12].

Theorem 2.1. *Let $f \in AP^0(\mathbb{R}^n)$ and $(r_m)_m \in \mathbb{R}^{\mathbb{N}}$. Then there exists $\sigma \in \mathcal{S}(\mathbb{N},\mathbb{N})$ such that $\lim_{m\to\infty} f(t + r_{\sigma(m)}) = g(t)$ uniformly on $\mathbb{R}$ and $\lim_{m\to\infty} g(t - r_{\sigma(m)}) = f(t)$ uniformly on $\mathbb{R}$.*

Theorem 2.2 ([8, Theorem 5.7, p. 85])**.** *Let $A \in AP^0(\mathbb{M}(n, \mathbb{R}))$ and $x \in AP^1(\mathbb{R})$ be the solution of $x'(t) = A(t)x(t)$. Then we have $\inf_{t\in\mathbb{R}} \|x(t)\| > 0$ or $x = 0$.*

Theorem 2.3 ([8, Theorem 4.5 , p. 61])**.** *Let $f \in AP^0(\mathbb{R}^n)$ and $g \in AP^0(\mathbb{R}^k)$. If for all $(\tau_m)_m \in \mathbb{R}^{\mathbb{N}}$ which is convergent in $[-\infty, \infty]$, $((f(. + \tau_m))_m$ uniformly* *convergent on* $\mathbb{R}$) $\Longrightarrow$ $((g(.+\tau_m))_m$ *uniformly convergent on* $\mathbb{R}$), *then* $\mathrm{Mod}(g) \subset \mathrm{Mod}(f)$.

A consequence of the above theorem, we have the following result.

Corollary 2.4. *Let* $f \in AP^0(\mathbb{R}^n)$ *and if* ϕ *is a continuous mapping from* $\overline{f(\mathbb{R})}$ *into* $\mathbb{R}^k$, *then* $\mathrm{Mod}(\phi \circ f) \subset \mathrm{Mod}(f)$.

3. First result in reducibility

In this section we establish that (1.1) is reducible to a upper triangular system (1.2) under the following assumptions.

(A1) $A \in AP^0(\mathbb{M}(n,\mathbb{R}), M)$

(A2) All the solutions of (1.1) are into $AP^1(\mathbb{R}^n, M)$,

where M is a fixed $\mathbb{Z}$-submodule of $\mathbb{R}$.

Lemma 3.1. *Let* $u \in AP^1(\mathbb{R}^n; M)$ *such that* $\inf_{t\in\mathbb{R}} \|u(t)\| > 0$. *Then* $t \mapsto \frac{1}{\|u(t)\|} \in AP^1(\mathbb{R}; M)$.

Proof. We know that $\|\cdot\|$ is of class C^1 on $\mathbb{R}^n \setminus \{0\}$. Denoting $N(z) = \|z\|$ and $N_1(z) = \frac{1}{\|z\|}$, we have that for all $z, h \in \mathbb{R}^n$, $DN(z)h = \frac{1}{\|z\|}(z\|h)$, $DN_1(z)h = \frac{-1}{\|z\|^3}(z\|h)$. Using the Chain Rule we establish that $\frac{d}{dt}(\frac{1}{\|u(t)\|}) = \frac{-1}{\|u(t)\|^3}(u(t)\|u'(t))$. Since $u, u' \in AP^0(\mathbb{R}^n; M)$, since $\inf_{t\in\mathbb{R}}(\frac{1}{\|u(t)\|^3}) > 0$ and using [8, Theorem 1.9, p. 5] we have that $\frac{d}{dt}(\frac{1}{\|u\|}) \in AP^0(\mathbb{R}; M)$ and so $\frac{1}{\|u\|} \in AP^1(\mathbb{R}; M)$. □

Lemma 3.2. *Assume* (A1)–(A2), *and let* $x_1, \dots, x_n \in AP^1(\mathbb{R}^n, M)$ *be linearly independent solutions of* (1.1). *Then there exist* $w_1, \dots, w_n \in AP^1(\mathbb{R}^n, M)$ *which satisfy the following conditions.*

(i) *for* $j, k \in \{1, \dots, n\}$ *such that* $j \neq k$, *for all* $t \in \mathbb{R}$, $(w_j(t)\|w_k(t)) = 0$.

(ii) *for* $k = 1, \dots, n$, $\forall t \in \mathbb{R}$, $\mathrm{span}\{w_j(t) : 1 \le j \le k\} = \mathrm{span}\{x_j(t) : 1 \le j \le k\}$.

(iii) $x_1 = w_1$ *and* $\forall k \in \{2, \dots, n\}$, $\forall t \in \mathbb{R}$, $w_k(t) = x_k(t) - \sum_{j=1}^{k-1} \lambda_{j,k}(t) x_j(t)$ *where* $\lambda_{j,k} \in AP^1(\mathbb{R}; M)$.

(iv) $x_1 = w_1$ *and* $\forall k \in \{2, \dots, n\}$, $\forall t \in \mathbb{R}$, $x_k(t) = w_k(t) - \sum_{j=1}^{k-1} \mu_{j,k}(t) w_j(t)$ *where* $\mu_{j,k} \in AP^1(\mathbb{R}; M)$.

(v) *for* $k = 1, \dots, n$, $\inf_{t\in\mathbb{R}} \|w_k(t)\| > 0$.

Proof. We proceed by induction on $k \in \{1, \dots, n\}$.

First step: $k = 1$. We set $w_1 = x_1$. Since $x_1, \dots, x_n$ are linearly independent, we have $x_1(t) \neq 0$, and since x_1 is a solution of (1.1) we have $x_1(t) \neq 0$ for all $t \in \mathbb{R}$. Condition (i) has no content for one function, conditions (ii), (iii) and (iv) are obvious and (v) is a consequence of Theorem 2.2.

Second step: Induction assumption on $k \in \{1, \dots, n-1\}$. We assume that there exist $w_1, \dots, w_k \in AP^1(\mathbb{R}^n; M)$ such that the following assertions hold.

$(i)_k$ $\forall i \neq j \in \{1, \dots, k\}$, $\forall t \in \mathbb{R}$, $(w_i(t)\|w_j(t)) = 0$.

$(ii)_k$ $x_1 = w_1$ and $\forall j \in \{2, \cdots, n\}$, $\forall t \in \mathbb{R}$, $\mathrm{span}\{w_i(t) : 1 \le i \le j\} = \mathrm{span}\{x_i(t) : 1 \le i \le j\}$.

$(iii)_k$ $x_1 = w_1$ and $\forall j \in \{2, \cdots, n\}$, $\forall t \in \mathbb{R}$, $w_j(t) = x_j(t) - \sum_{i=1}^{j-1} \lambda_{i,j}(t) x_i(t)$ where $\lambda_{i,j} \in AP^1(\mathbb{R}; M)$.

$(iv)_k$ $\forall j = 1, \dots, k$, $\forall t \in \mathbb{R}$, $x_j(t) = w_j(t) - \sum_{i=1}^{j-1} \mu_{i,j}(t) w_i(t)$ where $\mu_{i,j} \in AP^1(\mathbb{R}; M)$.

$(v)_k$ $\forall j = 1, \dots, k$, $\inf_{t \in \mathbb{R}} \|w_j(t)\| > 0$.

Third step: we prove the existence of $w_{k+1} \in AP^1(\mathbb{R}^n; M)$ such that $w_1, \dots, w_{k+1}$ satisfy $(i)_{k+1}$, $(ii)_{k+1}$, $(iii)_{k+1}$, $(iv)_{k+1}$, $(v)_{k+1}$. We consider $P_{k,t}$ the orthogonal projection on $\operatorname{span}\{x_i(t) : 1 \le i \le k\} = \operatorname{span}\{w_i(t) : 1 \le i \le k\}$ (after $(ii)_k$). Using $(i)_k$, it is well known [10, p. 136-138] that

$$P_{k,t}(x_{k+1}(t)) = \sum_{j=1}^{k} \frac{(x_{k+1}(t) \| w_j(t))}{\|w_j(t)\|^2} w_j(t). \tag{3.1}$$

We define

$$w_{k+1}(t) = x_{k+1}(t) - P_{k,t}(x_{k+1}(t)). \tag{3.2}$$

By using the characterization of orthogonal projection [10, p. 136-138] and $(i)_k$) we obtain $(i)_{k+1}$.

Using $(v)_k$ we can assure that $t \mapsto \|w_j(t)\|^{-2} \in AP^1(\mathbb{R}; M)$. Since $(.\|.)$ is bilinear continuous and since x_{k+1}, $w_j \in AP^1(\mathbb{R}^n; M)$, using Corollary 2.4, we obtain that $t \mapsto (x_{k+1}(t) \| w_j(t)) \in AP^1(\mathbb{R}; M)$. Since $AP^1(\mathbb{R}; M)$ is an algebra and since $(r, \xi) \mapsto r\xi$ is bilinear continuous from $\mathbb{R} \times \mathbb{R}^n$ into $\mathbb{R}^n$, using (3.1) we obtain

$$w_{k+1} \in AP^1(\mathbb{R}^n; M). \tag{3.3}$$

Using (3.1) and the previous arguments we see that $(iv)_{k+1}$ holds.

The upper index q denoting the q-th coordinate of a vector of $\mathbb{R}^n$, the relation in $(iv)_{k+1}$ is equivalent to following system, for $q = 1, \dots, n$ and $j = 1, \dots, k+1$

$$x_j^q(t) = w_j^q(t) - \sum_{i=1}^{j-1} \mu_{i,j}(t) w_i^q(t). \tag{3.4}$$

Setting $T(t) = (\tau_{i,j}(t))_{1 \le i,j \le k+1}$ with $\tau_{i,j}(t) = 0$ when $j > i$, $\tau_{i,i}(t) = 1$ and $\tau_{i,j}(t) = -\mu_{i,j}(t)$ when $j < i$, (3.4) is equivalent to the following system, for $q = 1, \dots, n$,

$$\begin{pmatrix} x_1^q(t) \\ \vdots \\ x_{k+1}^q(t) \end{pmatrix} = T(t) \begin{pmatrix} w_1^q(t) \\ \vdots \\ w_{k+1}^q(t) \end{pmatrix}. \tag{3.5}$$

We see that $\det T(t) = \prod_{i=1}^{k+q} \tau_{ii}(t) = 1$ since $T(t)$ is triangular lower, and so $T(t)$ is invertible, and the inverse of $T(t)$ is $T(t)^{-1} = cof\ T(t)^*$ the matrix of the cofactors of $T(t)$. Denoting $T(t)^{-1} = (\sigma_{i,j}(t))_{1 \le i,j \le k+1}$, we have

$$\sigma_{i,j} = (-1)^{i+j} cof_{i,j} T(t) = (-1)^{i+j} \det T(t)_{\widehat{i},\widehat{j}},$$

where $T(t)_{\widehat{i},\widehat{j}}$ is the $k \times k$ matrix obtained by deleting the i-th row and the j-th column, [9, Définition 4.15, p. 117].

Since the $\tau_{i,j} \in AP^1(\mathbb{R}; M)$ and since a determinant is multilinear continuous, by using Corollary 2.4 we obtain that $\sigma_{i,j} \in AP^1(\mathbb{R}; M)$ for all i, j.

Since $T(t)$ is lower triangular, $T(t)^{-1}$ is also lower triangular, and from (3.5) we obtain, for all $q = 1, \dots, n$,

$$\begin{pmatrix} w_1^q(t) \\ \vdots \\ w_{k+1}^q(t) \end{pmatrix} = T(t)^{-1} \begin{pmatrix} x_1^q(t) \\ \vdots \\ x_{k+1}^q(t) \end{pmatrix}, \tag{3.6}$$

that implies $(iii)_{k+1}$.

Using $(iii)_{k+1}$ and $(iv)_{k+1}$ we see that $(ii)_{k+1}$ holds.

It remains to prove that $(v)_{k+1}$ holds. For this, using $(v)_k$, it suffices to prove that $\inf_{t\in\mathbb{R}}\|w_{k+1}(t)\|>0$. We proceed by contradiction, assume that $\inf_{t\in\mathbb{R}}\|w_{k+1}(t)\|=0$. Consequently, there exists $(r_m)_m\in\mathbb{R}^{\mathbb{N}}$ such that $\lim_{m\to\infty}w_{k+1}(r_m)=0$. Using Theorem 2.1, there exists $\sigma\in\mathcal{S}(\mathbb{N},\mathbb{N})$ such that , for all $j=1,\dots,k+1$, and $i=1,\dots,j$, we have

$$\begin{gathered}
\lim_{m\to\infty}x_j(t+r_{\sigma(m)})=y_j(t),\quad \lim_{m\to\infty}y_j(t-r_{\sigma(m)})=x_j(t),\\
\lim_{m\to\infty}x_j'(t+r_{\sigma(m)})=y_j'(t),\quad \lim_{m\to\infty}y_j'(t-r_{\sigma(m)})=x_j'(t),\\
\lim_{m\to\infty}A(t+r_{\sigma(m)})=L(t),\quad \lim_{m\to\infty}L(t-r_{\sigma(m)})=A(t),\\
\lim_{m\to\infty}\lambda_{i,j}(t+r_{\sigma(m)})=\mu_{i,j}(t),\quad \lim_{m\to\infty}\mu_{i,j}(t-r_{\sigma(m)})=\lambda_{i,j}(t),
\end{gathered}$$

where all these convergences are uniform on $\mathbb{R}$.

Therefore, for all $j=1,\dots,k+1$,

$$y_j'(t)=L(t)y_j(t). \tag{3.7}$$

Note that

$$\begin{aligned}
0&=\lim_{m\to\infty}w_{k+1}(r_{\sigma(m)})=\lim_{m\to\infty}[x_{k+1}(r_{\sigma(m)})-\sum_{j=1}^{k}\lambda_{j,k}(r_{\sigma(m)})x_j(r_{\sigma(m)})]\\
&=y_{k+1}(0)-\sum_{j=1}^{k}\nu_{j,k}(0)y_j(0);\ \text{and so } y_{k+1}(0)=\sum_{j=1}^{k}\mu_{j,k}(0)y_j(0).
\end{aligned}$$

Since the y_j are solutions of (3.7) we have, for all $t\in\mathbb{R}$, $y_{k+1}(t)=\sum_{j=1}^{k}\mu_{j,k}(0)y_j(t)$. Consequently,

$$x_{k+1}(t)=\lim_{m\to\infty}y_{k+1}(t-r_{\sigma(m)})=\sum_{j=1}^{k}\mu_{j,k}(0)\lim_{m\to\infty}y_j(t-r_{\sigma(m)})=\sum_{j=1}^{k}\mu_{j,k}(0)x_j(t)$$

for all $t\in\mathbb{R}$, that is impossible since $x_1,\dots,x_{k+1}$ are linearly independent. And so the proof is achieved. $\square$

Lemma 3.3. *Assume* (A1)–(A2)*, and let $t\mapsto X(t)$ be a fundamental matrix of* (1.1)*. Then there exist $R\in AP^1(\mathbb{M}(n,\mathbb{R});M)$ and $Q\in AP^1(\mathbb{M}(n,\mathbb{R});M)$ such that $Q(t)$ is orthogonal, $R(t)$ is upper triangular and $Q(t)=X(t)R(t)$ for all $t\in\mathbb{R}$.*

Proof. We denote by $x_1(t),\dots,x_n(t)$ the columns of $X(t)$. Note that $x_1,\dots,x_n$ satisfy the assumptions of Lemma 3.2. Let $w_1,\dots,w_n$ be provided by Lemma 3.2. We set $v_k(t)=\|w_k(t)\|^{-1}w_k(t)$ for all $k\in\{1,\dots,n\}$ and for $t\in\mathbb{R}$. Using (v) of Lemma 3.2 and Corollary 2.4, we obtain that $\|w_k(.)\|^{-1}\in AP^1(\mathbb{R};M)$, and $v_k\in AP^1(\mathbb{R}^n;M)$. Since $w_1(t),\dots,w_n(t)$ are orthogonal we obtain

$$\forall j,k=1,\dots,n,\ \forall t\in\mathbb{R},\quad (v_j(t)\|v_k(t))=\delta_j^k\quad \text{(Kronecker symbol)}. \tag{3.8}$$

We define $Q(t)$ as the matrix whom the columns are $v_1(t),\dots,v_n(t)$. From (3.8) we deduce that $Q(t)^*Q(t)=I$; i.e., $Q(t)$ is orthogonal. Since $v_k\in AP^1(\mathbb{R}^n;M)$, $Q\in AP^1(\mathbb{M}(n,\mathbb{R});M)$.

From (iii) in Lemma 3.2, we deduce that, for all $k=1,\dots,n$ and for all $t\in\mathbb{R}$, we have $v_k(t)=\|w_k(t)\|^{-1}x_k(t)-\sum_{j=1}^{n}\|w_k(t)\|^{-1}\lambda_{j,k}(t)x_j(t)$ with $\lambda_{j,k}(t)=0$ when $j>k$.

The upper index denoting the coordinate of the vectors, we obtain, for all $k = 1, \dots, n$ and for all $i = 1, \dots n$,

$$\begin{aligned} v_k^i(t) &= \|w_k(t)\|^{-1} x_k^i(t) - \sum_{j=1}^{n} \|w_k(t)\|^{-1} \lambda_{j,k}(t) x_j^i(t) \\ &= (x_1^i(t) \dots x_k^i(t) \dots x_n^i(t)) \begin{pmatrix} \|w_k(t)\|^{-1}\lambda_{1,k}(t) \\ \dots \\ \|w_k(t)\|^{-1}\lambda_{k-1,k}(t) \\ \|w_k(t)\|^{-1} \\ 0 \\ \dots \\ 0 \end{pmatrix} \end{aligned} \tag{3.9}$$

and so, setting

$$r_{j,k}(t) = \begin{cases} \|w_k(t)\|^{-1}\lambda_{1,k}(t) & \text{when } j \le k-1 \\ \|w_k(t)\|^{-1} & \text{when } j = k \\ 0 & \text{when } j > k \end{cases}$$

the matrix $R(t) = (r_{j,k}(t))_{1 \le j,k \le n}$ is upper triangular, and (3.9) means that $Q(t) = X(t)R(t)$. Using Lemma 3.2, we obtain that $R \in AP^1(\mathbb{M}(n,\mathbb{R}); M)$ since its entries belong to $AP^1(\mathbb{R}; M)$. $\square$

Lemma 3.4. *Assume* (A1)–(A2) *and lett* $t \mapsto X(t)$ *be a fundamental matrix of* (1.1). *Let* Q *and* R *be provided by Lemma 3.2. We set*

$$B(t) = -Q^{-1}(t)Q'(t) + Q^{-1}(t)A(t)Q(t)$$

for all $t \in \mathbb{R}$. *Then* $B \in AP^0(\mathbb{M}(n,\mathbb{R}); M)$ *and* $B(t)$ *is upper triangular for all* $t \in \mathbb{R}$.

Proof. For all $t \in \mathbb{R}$,

$$Q'(t) = X'(t)R(t) + X(t)R'(t) = A(t)X(t)R(t) + X(t)R'(t) = A(t)Q(t) + X(t)R'(t)$$

which implies

$$\begin{aligned} -Q^{-1}(t)Q'(t) &= -Q^{-1}(t)A(t)Q(t) - Q^{-1}(t)X(t)R'(t) \\ &= -Q^{-1}(t)A(t)Q(t) - R^{-1}(t)R'(t) \end{aligned}$$

which in turn implies

$$B(t) = -R^{-1}(t)R'(t). \tag{3.10}$$

Since $R(t)$ is upper triangular, $R^{-1}(t)$ and $R'(t)$ are also upper triangular, and since a product of upper triangular matrices is upper triangular, we obtain from (3.10) that $B(t)$ is upper triangular.

Since $Q(t)$ is orthogonal, we have $B(t) = -Q^*(t)Q'(t) + Q^*(t)A(t)Q(t)$. Since Q, Q^*, $A \in AP^0(\mathbb{M}(n,\mathbb{R}); M)$, we have obtain that $B \in AP^0(\mathbb{M}(n,\mathbb{R}); M)$. $\square$

Theorem 3.5. *Under* (A1) *and* (A2), *there exist* $Q \in AP^1(\mathbb{M}(n,\mathbb{R}); M)$ *and* $B = (b_{jk})_{1 \le j,k \le n} \in AP^0(\mathbb{M}(n,\mathbb{R}); M)$ *such that the following conditions hold:*

(i) $Q(t)$ *is orthogonal for all* $t \in \mathbb{R}$.
(ii) $B(t)$ *is upper triangular for all* $t \in \mathbb{R}$.

(iii) *If x is a solution of* (1.1) *then y defined by $y(t) = Q^{-1}(t)x(t)$ is a solution of* (1.2) *and conversely if y is a solution of* (1.2) *then x defined by $x(t) = Q(t)y(t)$ is a solution of* (1.1).

(iv) *For all $k = 1, \dots, n$, $t \mapsto \int_0^t b_{kk}(s)\,ds \in AP^1(\mathbb{R}; M)$.*

Proof. Let X be a fundamental matrix of (1.1), let Q and R be provided by Lemma 3.3, and let B be provided by Lemma 3.4. After Lemma 3.3, (i) holds and after Lemma 3.4 we know that (ii) holds.

To prove (iii), if $x'(t) = A(t)x(t)$ and $y(t) = Q^{-1}(t)x(t)$, then

$$\begin{aligned}
&y'(t)\\
&= (Q^{-1})'(t)x(t) + Q^{-1}(t)x'(t)\\
&= -Q^{-1}(t)Q'(t)Q^{-1}(t)x(t) + Q^{-1}(t)A(t)x(t)\\
&= -Q^{-1}(t)X'(t)R(t)Q^{-1}(t)x(t) - Q^{-1}(t)X(t)R'(t)Q^{-1}(t)x(t) + Q^{-1}(t)A(t)x(t)\\
&= -Q^{-1}(t)A(t)X(t)R(t)Q^{-1}(t)x(t) - R^{-1}(t)R'(t)y(t) + Q^{-1}(t)A(t)x(t)\\
&= Q^{-1}(t)A(t)x(t) + Q^{-1}(t)A(t)x(t) - R^{-1}(t)R'(t)y(t)\\
&= B(t)y(t),
\end{aligned}$$

using (3.10). Conversely, if $y'(t) = B(t)y(t)$ and $x(t) = Q(t)y(t)$, then

$$\begin{aligned}
x'(t) &= Q'(t)y(t) + Q(t)y'(t)\\
&= Q'(t)Q^{-1}(t)x(t) + Q(t)B(t)y(t)\\
&= Q'(t)Q^{-1}(t)x(t) + Q(t)[-Q^{-1}(t)Q'(t) + Q^{-1}(t)A(t)Q(t)]Q^{-1}(t)x(t)\\
&= Q'(t)Q^{-1}(t)x(t) - Q'(t)Q^{-1}(t)x(t) + A(t)x(t)\\
&= A(t)x(t).
\end{aligned}$$

And so (iii) is proven.

Last, to prove (iv) note that (1.2) is equivalent to

$$y_k'(t) = \sum_{j=k}^{n} b_{kj}(t)y_j(t), \quad 1 \le k \le n.$$

Now we proceed following a decreasing induction.

First step: $k = n$. Since all the solutions of the scalar equation $y_n'(t) = b_{nn}(t)y_n(t)$ are a.p., by using [5] we necessarily have that $t \mapsto \int_0^t b_{nn}(s)\,ds$ is bounded and consequently it is a.p.

Second step: the induction assumption $k \in \{2, \dots, n\}$ is $t \mapsto \int_0^t b_{jj}(s)\,ds$ is a.p. for all $j \in \{k, \dots, n\}$.

Third step: the case $k-1$. We consider the subsystem

$$y_j'(t) = \sum_{i=j}^{n} b_{ji}(t)y_i(t), \quad i = k-1, \dots, n. \tag{3.11}$$

Since the b_{ji} are a.p., and since all the solutions of (3.11) are a.p., using [5] we know that $t \mapsto \int_0^t \sum_{i=k-1}^n b_{ii}(s)\,ds$ is a.p., and by using the induction assumption we know that $t \mapsto \int_0^t \sum_{i=k}^n b_{ii}(s)\,ds$ is a.p. as a sum of a.p. functions. Consequently $t \mapsto \int_0^t b_{k-1,k-1}(s)\,ds$ is a.p. as a difference of two a.p. functions. $\square$

Remark 3.6. In the setting of the previous theorem, $Y(t) = Q^{-1}(t)X(t)$ is a fundamental matrix for (1.2). Since $Q(t) = X(t)R(t)$ we have $Y(t) = R^{-1}(t)$ which is upper triangular since the inverse of a regular upper triangular matrix is also upper triangular.

Remark 3.7. Such a construction of $B(t)$ from $A(t)$ is made in the continuous case in [13, Theorem 1.4, p.4], in the periodic case in [13] (in the proof of the Theorem 1.7, p.12-13) and in the quasi-periodic case in [13] (in the proof of the Lemma 4.3, p.134-135) under diophantine conditions. Theorem 3.5 contains the quasi-periodic case since we can choose M as a $\mathbb{Z}$-submodule of $\mathbb{R}$ having a finite basis, and we have not need any diophantine condition.

Remark 3.8. In the Floquet-Lin theory for quasi-periodic systems developed by Lin [11, 12, 13], the Floquet characteristic exponents (FL-CER) of (1.1), denoted by $\beta_1, \dots, \beta_n$, satisfy $\beta_k = \mathcal{M}\{b_{kk}\}_t$ [13, p. 137]. A consequence of (iv) in Theorem 3.5 is $\beta_k = 0$ for all $k = 1, \dots, n$. If there exists a real $n \times n$ constant upper triangular matrix Ω provided by the Lin theory [13, Theorem 4.1, p. 139] then (1.1) is reducible to $z'(t) = \Omega\, z(t)$ and the eigenvalues of Ω are $\beta_1, \dots, \beta_n$. And so, under ($\mathbf{A1} - \mathbf{A2}$), we can easily verify that $\Omega = 0$ since all the solutions of $z'(t) = \Omega z(t)$ are a.p.

4. Second result of reducibility

To study (1.1) we consider the condition

(A3) Equation (1.1) possesses k linearly independent almost periodic solutions in $AP^1(\mathbb{R}^n; M)$, where M is a$\mathbb{Z}$-submodule of $\mathbb{R}$ and $k \in \{1, \dots, n\}$.

Theorem 4.1. *Under assumptions* (A1), (A3)*, there exist* $Q \in C^1(\mathbb{R}, \mathbb{M}(n, \mathbb{R}))$, $B \in C^0(\mathbb{R}, \mathbb{M}(n, \mathbb{R}))$ *such that the following conditions hold.*

(i) $Q(t)$ *is orthogonal for all* $t \in \mathbb{R}$.
(ii) $B(t)$ *is upper triangular for all* $t \in \mathbb{R}$.
(iii) *If* x *is a solution of* (1.1) *then* y *defined by* $y(t) = Q^{-1}(t)x(t)$ *is a solution of* (1.2) *and conversely if* y *is a solution of* (1.2) *then* x *defined by* $x(t) = Q(t)y(t)$ *is a solution of* (1.1).
(iv) *If* $Q(t) = \mathrm{col}(v_1(t), \dots, v_n(t))$ *then* $v_1(t), \dots, v_k(t) \in AP^1(\mathbb{R}, \mathbb{R}^n; M)$.
(v) *Equation* (1.2) *possesses* k *linearly independent a.p. solutions.*

Proof. We denote by $x_1, \dots, x_k \in AP^1(\mathbb{R}^n; M)$ k linearly independent solutions of (1.1). We choose $x_{k+1}, \dots, x_n \in C^1(\mathbb{R}, \mathbb{R}^n)$ solutions of (1.1) such that $x_1, \dots, x_n$ are linearly independent.

We set $X(t) = \mathrm{col}(x_1(t), \dots, x_n(t))$, and so it is a fundamental matrix of (1.1). We set $w_1(t) = x_1(t)$ and, for all $k \in \{2, \dots, n\}$,

$$w_k(t) = x_k(t) - \sum_{j=1}^{k-1} \frac{(x_k(t) \| w_j(t))}{\|w_j(t)\|^2} w_j(t).$$

We set $v_k(t) = \frac{1}{\|w_k(t)\|} w_k(t)$ for all $k \in \{1, \dots, n\}$.

We have $v_1, \dots, v_n \in C^1(\mathbb{R}, \mathbb{R}^n)$ since $x_1, \dots, x_n \in C^1(\mathbb{R}, \mathbb{R}^n)$. We define $Q(t) = \mathrm{col}(v_1(t), \dots, v_n(t))$. We verify that $Q(t) = X(t)R(t)$ where $R \in C^1(\mathbb{R}, \mathbb{M}(n, \mathbb{R}))$ and $R(t)$ is upper triangular for all $t \in \mathbb{R}$. Then we set $B(t) = -Q^{-1}(t)Q'(t) + Q^{-1}(t)A(t)Q(t)$. $B(t)$ is upper triangular for all $t \in \mathbb{R}$ and $B \in C^0(\mathbb{R}, \mathbb{M}(n, \mathbb{R}))$.

This construction is proven in [13, Theorem 1.4, p.4], and the assertions (i), (ii), (iii) result of this theorem.

For all $j \in \{1, \dots, k\}$ we set $y_j(t) = Q^{-1}(t)x_j(t) = Q^*(t)x_j(t)$ for all $t \in \mathbb{R}$. Then $y_1, \dots, y_k$ are solutions of (1.2). Following the definition of v_j for $j \in \{1, \dots, k\}$ and reasoning as in the proof of Lemma 3.2, we verify that $v_1, \dots, v_k \in AP^1(\mathbb{R}^n; M)$ that proves (iv).

For all $p \in \{2, \dots, n\}$ and for all $t \in \mathbb{R}$ we know that $v_p(t)$ is orthogonal to $\{x_q(t) : 1 \le q \le p-1\}$, and so we have

$$\forall p \in \{2, \dots, n\}, \forall q \in \{1, \dots, p-1\}, \forall t \in \mathbb{R}, \quad (v_p(t) \| x_q(t)) = 0. \tag{4.1}$$

When $j \in \{1, \dots, k\}$, since $y_j(t) = Q^*(t)x_j(t)$ we have, for all $i \in \{1, \dots, n\}$, $y_j^i(t) = (v_i(t) \| x_j(t))$, and so, using (4.1), we have $y_j^i(t) = 0$ when $i > j$ and therefore $y_j^i(t) = 0$ when $i > k$.

When $i \le k$ we have $y_j^i \in AP^1(\mathbb{R}; M)$ since $v_i, v_j \in AP^1(\mathbb{R}^n; M)$. And so all the coordinates of y_j belong to $AP^1(\mathbb{R}; M)$ that implies that $y_j \in AP^1(\mathbb{R}^n; M)$. And so $y_1, \dots, y_k$ are solutions of (1.2) which belong to $AP^1(\mathbb{R}^n; M)$. Moreover they are linearly independent since $\sum_{j=1}^k \xi_j y_j = 0$ implies $0 = \sum_{j=1}^k \xi_j Q^{-1}(.)x_j = Q^{-1}(.)\big(\sum_{j=1}^k \xi_j x_j\big)$ implies $\sum_{j=1}^k \xi_j x_j = 0$ that implies $\xi_1 = \dots = \xi_k = 0$ since $x_1, \dots, x_k$ are linearly independent. And so (v) is proven. $\square$

5. Existence result

In this section we study the existence of a.p. solutions of (1.3). First we establish results on linear systems.

Lemma 5.1. *Let $a \in AP^0(\mathbb{R}; M)$ such that $\mathcal{M}\{a\} \neq 0$. Then the following two assertions hold.*

- (i) *The scalar equation $x'(t) = a(t)x(t)$ does not possess any almost periodic solution.*
- (ii) *For all $b \in AP^0(\mathbb{R}; M)$ there exists a unique $x \in AP^0(\mathbb{R}; M)$ which is a solution of $x'(t) = a(t)x(t) + b(t)$. Moreover there exists a constant $\alpha \in (0, \infty)$ such that*

$$\|x_b\|_\infty \le \alpha \|b\|_\infty. \tag{5.1}$$

Proof. We consider the following two systems

$$x'(t) = a(t)x(t) \tag{5.2}$$

$$x'(t) = a(t)x(t) + b(t), \tag{5.3}$$

and distinguish the cases: $\mathcal{M}\{a\} > 0$ and $\mathcal{M}\{a\} < 0$.

5.1. **Case $\mathcal{M}\{a\} > 0$.** (i) By the existence of mean value we have for all $\epsilon \in (0, \mathcal{M}\{a\})$, there exists $t_\epsilon > 0$, such that ll $t \ge t_\epsilon$, $\mathcal{M}\{a\} - \epsilon \le \frac{1}{t}\int_0^t a(r)dr \le \mathcal{M}\{a\} + \epsilon$. This implies that $\forall t \ge t_\epsilon$, $\int_0^t a(r)dr \ge t(\mathcal{M}\{a\} - \epsilon)$. Hence $\exp\big(\int_0^t a(r)dr\big) \ge \exp\big(t(\mathcal{M}\{a\} - \epsilon)\big) \to \infty$ when $t \to \infty$, and consequently

$$\lim_{t\to\infty} \int_0^t a(r)dr = \infty. \tag{5.4}$$

A consequence of (5.4) is that all the solutions of (5.2), which is in the form $x(t) = \exp\big(\int_0^t a(r)dr\big)x(0)$ are not bounded and consequently are not a.p.

Since the difference of two a.p. solutions of (5.3) are necessarily an a.p. solution of (5.2), (5.3) cannot possess more than one a.p. solution.

(ii) Now we prove the assertion that

$$\int_0^\infty \exp\Big(-\int_0^s a(r)dr\Big)ds \quad \text{exists in } \mathbb{R}_+. \tag{5.5}$$

Since $\lim_{s\to\infty} \frac{1}{s}\int_0^s a(r)dr = \mathcal{M}\{a\} > 0$, for all $\epsilon \in (0, \mathcal{M}\{a\})$, there exists $s_\epsilon > 0$, such that for $s \geq s_\epsilon$, $\mathcal{M}\{a\} - \epsilon \leq \frac{1}{s}\int_0^s a(r)dr \leq \mathcal{M}\{a\} + \epsilon$ implies for all $s \geq s_\epsilon$, $s\big(\mathcal{M}\{a\} - \epsilon\big) \leq \int_0^s a(r)dr$ which implies for all $s \geq s_\epsilon$, $-\int_0^s a(r)dr \leq -s\big(\mathcal{M}\{a\} - \epsilon\big)$ which implies for all $s \geq s_\epsilon$, $\exp\big(-\int_0^s a(r)dr\big) \leq \exp\big(-s(\mathcal{M}\{a\} - \epsilon)\big)$ which implies

$$\begin{aligned}
&\int_{s_\epsilon}^\infty \exp\Big(-\int_0^s a(r)dr\Big)ds \leq \int_{s_\epsilon}^\infty \exp(-s(\mathcal{M}\{a\} - \epsilon))ds \\
&= \frac{1}{\mathcal{M}\{a\} - \epsilon}\exp\big(-s_\epsilon(\mathcal{M}\{a\} - \epsilon)\big) = \xi_\epsilon.
\end{aligned}$$

Since $s \mapsto \exp\big(-\int_0^s a(r)dr\big)$ is continuous on the compact interval $[0, s_\epsilon]$, it follows that $\int_0^{s_\epsilon} \exp\big(-\int_0^{s_\epsilon} a(r)dr\big)\,ds \leq \infty$, and so

$$\begin{aligned}
&\int_0^\infty \exp\Big(-\int_0^s a(r)dr\Big)ds \\
&= \int_0^{s_\epsilon} \exp\Big(-\int_0^s a(r)dr)ds + \int_{s_\epsilon}^\infty \exp\Big(-\int_0^s a(r)dr\Big)ds \\
&\leq \int_0^{s_\epsilon} \exp\Big(-\int_0^s a(r)dr\Big)ds + \xi_\epsilon < \infty.
\end{aligned}$$

And so (5.5) is proven.

Since $s \mapsto \exp\big(-\int_0^s a(r)dr\big)b(s)$ is continuous on $\mathbb{R}_+$, it is Borel-mesurable, and using the Lebesgue integral for nonnegative functions on $\mathbb{R}_+$, we have

$$\int_{\mathbb{R}_+} \|\exp\Big(-\int_0^s a(r)dr\Big)b(s)\|ds \leq \|b\|_\infty \int_{\mathbb{R}_+} \exp\Big(-\int_0^s a(r)dr\Big)ds$$

by using (5.5). Thus $s \mapsto \|\exp\big(-\int_0^s a(r)dr\big)\,b(s)\|$ is Lebesgue integrable on $\mathbb{R}_+$; therefore $s \mapsto \exp\big(-\int_0^s a(r)dr\big)\,b(s)$ is Lebesgue integrable on $\mathbb{R}_+$, and we have

$$\int_0^\infty \exp\Big(-\int_0^s a(r)dr\Big)b(s)ds \quad \text{exists in } \mathbb{R}. \tag{5.6}$$

Now for $t \in \mathbb{R}$, we set

$$\begin{aligned}
\hat{x}(t) = \exp\Big(\int_0^t a(r)dr\Big)\Big[&-\int_0^\infty \exp\Big(-\int_0^s a(r)dr\Big)b(s)ds \\
&+ \int_0^t \exp\Big(-\int_0^s a(r)dr\Big)b(s)ds].
\end{aligned} \tag{5.7}$$

Using a calculation formula, called variation of constants, we obtain that

$$\hat{x} \text{ is a solution on } \mathbb{R} \text{ of } (NH). \tag{5.8}$$

In the following step, we want to prove that $\hat{x}$ is bounded on $\mathbb{R}_+$. Using the Chasles relation we deduce from (5.7) the equality

$$\hat{x}(t) = \exp\Big(-\int_0^t a(r)dr\Big)\Big[-\int_t^\infty \exp\Big(-\int_0^s a(r)dr\Big)b(s)ds\Big]$$

$$= -\int_t^{\infty} \exp\Big(\int_0^t a(r)dr - \int_0^s a(r)dr\Big) b(s)ds, \quad \forall t \geq 0.$$

Therefore,

$$\hat{x}(t) = -\int_t^{\infty} \exp\Big(-\int_t^s a(r)dr\Big) b(s)ds, \quad \forall t \geq 0. \tag{5.9}$$

Introducing the change of variables $\sigma : \mathbb{R}_+ \to [t, \infty]$, $\sigma(\rho) = \rho + t$, from (5.9) and using the change of variable formula, we have

$$\begin{aligned}
\hat{x}(t) &= -\int_{\sigma(0)}^{\sigma(\infty)} \exp\Big(-\int_t^s a(r)dr\Big) b(s)ds \\
&= -\int_0^{\infty} \exp\Big(-\int_t^{\sigma(\rho)} a(r)dr\Big) b(\sigma(\rho))\sigma'(\rho)d\rho \\
&= -\int_0^{\infty} \exp\Big(-\int_t^{t+\rho} a(r)dr\Big) b(t+\rho)d\rho.
\end{aligned}$$

Using the mean value theorem for integrals,

$$\|\hat{x}(t)\| \leq \Big(\int_0^{\infty} \exp\Big(-\int_t^{t+\rho} a(r)dr\Big)d\rho\Big)\|b\|_{\infty}, \quad \forall t \geq 0. \tag{5.10}$$

Using a result by Bohr [4, p.44] we have for all $\epsilon \in (0, \mathcal{M}\{a\})$, there exists $\rho_\epsilon > 0$, $\forall \rho \geq \rho_\epsilon$, $\forall t \in \mathbb{R}$,

$$\mathcal{M}\{a\} - \epsilon \leq \frac{1}{\rho}\int_t^{t+\rho} a(r)dr \leq \mathcal{M}\{a\} + \epsilon.$$

$\implies \forall \rho \geq \rho_\epsilon$, $\forall t \in \mathbb{R}$, $\rho\left(\mathcal{M}\{a\} - \epsilon\right) \leq \int_t^{t+\rho} a(r)dr \leq \rho\left(\mathcal{M}\{a\} + \epsilon\right)$.
$\implies \forall \rho \geq \rho_\epsilon$, $\forall t \in \mathbb{R}$, $-\rho\left(\mathcal{M}\{a\} - \epsilon\right) \geq -\int_t^{t+\rho} a(r)dr$.
$\implies \forall \rho \geq \rho_\epsilon$, $\forall t \in \mathbb{R}$, $\exp\Big(-\int_t^{t+\rho} a(r)dr\Big) \leq \exp\left(-\rho\left(\mathcal{M}\{a\} - \epsilon\right)\right)$, which implies

$$\begin{aligned}
\int_{\rho_\epsilon}^{\infty} \exp\Big(-\int_t^{t+\rho} a(r)dr\Big)d\rho &\leq \int_{\rho_\epsilon}^{\infty} \exp(-\rho(\mathcal{M}\{a\} - \epsilon))d\rho \\
&= \frac{1}{\mathcal{M}\{a\} - \epsilon}\exp(-\rho_\epsilon(\mathcal{M}\{a\} - \epsilon)) = \xi_\epsilon, \quad \forall t \in \mathbb{R},
\end{aligned}$$

Moreover, when $\rho \in [0, \rho_\epsilon]$,

$$-\int_t^{t+\rho} a(r)dr \leq \|\int_t^{t+\rho} a(r)dr\| \leq \rho \sup_{s \in [0,\rho_\epsilon]} \|a(s)\| \leq \rho\|a\|_{\infty}$$

implies

$$\exp\Big(-\int_t^{t+\rho} a(r)dr\Big) \leq \exp(\rho\|a\|_{\infty}) \quad \forall t \in \mathbb{R}$$

which implies

$$\int_0^{\rho_\epsilon} \exp\Big(-\int_t^{t+\rho} a(r)dr\Big)d\rho \leq \int_0^{\rho_\epsilon} \exp(\rho\|a\|_{\infty})d\rho = \frac{1}{\|a\|_{\infty}}(\exp(\rho_\epsilon\|a\|_{\infty}) - 1) = \xi_\epsilon^1.$$

Now using the Chasles relation we obtain,

$$\int_0^{\infty} \exp\Big(-\int_t^{t+\rho} a(r)dr\Big)d\rho \leq \xi_\epsilon + \xi_\epsilon^1 = \xi_\epsilon^2 < \infty, \quad \text{for all } t \in \mathbb{R}. \tag{5.11}$$

Note that ξ_ϵ and ξ_ϵ^1 do not depend of t. A consequence of (5.10) and (5.11) is the assertion that

$$\hat{x} \text{ is bounded on } \mathbb{R}_+. \tag{5.12}$$

In the following step we want to prove that $\hat{x}$ is bounded on $\mathbb{R}_- = (-\infty, 0]$. We introduce

$$\begin{aligned} x_1(t) &= \exp\Big(\int_0^t a(r)dr\Big)\Big(-\int_0^\infty \exp\Big(-\int_0^s a(r)dr\Big)b(s)ds\Big) \\ x_2(t) &= \exp\Big(\int_0^t a(r)dr\Big)\Big(\int_0^t \exp\Big(-\int_0^s a(r)dr\Big)b(s)ds\Big). \end{aligned}$$

Therefore,

$$\hat{x}(t) = x_1(t) + x_2(t) \quad \text{for all } t \in \mathbb{R}_-. \tag{5.13}$$

We know that $\mathcal{M}\{a\} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^T a(r)dr = \lim_{T\to\infty} \frac{1}{T}\int_0^T a(r)dr$ [4, p. 44], and since $\frac{1}{2T}\int_{-T}^T a(r)dr = \frac{1}{2}(\frac{1}{T}\int_0^T a(r)dr + \frac{1}{T}\int_{-T}^0 a(r)dr)$, taking $t = -T$, we obtain

$$\lim_{t\to-\infty} \frac{1}{-t}\int_t^0 a(r)dr = \mathcal{M}\{a\}. \tag{5.14}$$

Note that, for $t \le 0$, $\int_0^t a(r)dr = -\int_t^0 a(r)dr = t\frac{1}{-t}\int_t^0 a(r)dr$ which implies $\exp\big(\int_0^t a(r)dr\big) = \exp\big(t.\frac{1}{-t}\int_t^0 a(r)dr\big) \to \exp(-\infty.\mathcal{M}\{a\}) = 0$ as $t \to -\infty$, and so we have proven that

$$\lim_{t\to-\infty} \exp\Big(\int_0^t a(r)dr\Big) = 0. \tag{5.15}$$

Then using (5.6) and (5.15), we obtain $\lim_{t\to-\infty} x_1(t) = 0$, and, since x_1 is continuous on $\mathbb{R}_-$, we obtain

$$x_1 \text{ is bounded on } \mathbb{R}_-. \tag{5.16}$$

For all $t \le 0$,

$$\begin{aligned} x_2(t) &= \int_0^t \exp\Big(\int_0^t a(r)dr\Big)\exp\Big(-\int_0^s a(r)dr\Big)b(s)ds \\ &= -\int_t^0 \exp\Big(-\int_t^s a(r)dr\Big)b(s)ds. \end{aligned}$$

Hence $\|x_2(t)\| \le \int_t^0 \exp\big(-\int_t^s a(r)dr\big)\,ds\|b\|_\infty$. Introducing $\gamma : [0, -t] \to [t, 0]$, $\gamma(\rho) = \rho + t$ and using the change of variable formula, we obtain

$$\begin{aligned} \int_t^0 \exp\Big(-\int_t^s a(r)dr\Big)ds &= \int_{\gamma(0)}^{\gamma(-t)} \exp\Big(-\int_t^s a(r)dr\Big)ds \\ &= \int_0^{-t} \exp\Big(-\int_t^{\gamma(\rho)} a(r)dr\Big)\gamma'(\rho)d\rho \\ &= \int_0^{-t} \exp\Big(-\int_t^{t+\rho} a(r)dr\Big)d\rho \\ &\le \int_0^{\infty} \exp\Big(-\int_t^{t+\rho} a(r)dr\Big)d\rho \le \xi_\epsilon^2, \end{aligned}$$

for all $t \le 0$ after (5.11) where ξ_ϵ^2 is independent of t. Consequently, $\forall t \le 0$, $\|x_2(t)\| \le \xi_\epsilon^2\|b\|_\infty$, that proves that

$$x_2 \text{ is bounded on } \mathbb{R}_-. \tag{5.17}$$

From (5.13), (5.16) and (5.17), we have that $\hat{x}$ is bounded on $\mathbb{R}_-$, and with (5.12)

$$\hat{x} \text{ is bounded on } \mathbb{R}. \tag{5.18}$$

Using [8, Theorem 6.3, p.100], $\hat{x}$ is an a.p. solution of (5.3), and it is the unique solution. So the proof of the lemma is complete in the case $\mathcal{M}\{a\} > 0$.

5.2. **Case $\mathcal{M}\{a\} < 0$.** To treat this case, we consider the additional equation

$$y'(t) = -a(-t)y(t) - b(-t) \tag{5.19}$$

and we note that $y(t) = x(-t)$ is a solution of (5.19) when and only when x is a solution of (5.3). Also note that $\mathcal{M}_t\{-a(-t)\} = -\mathcal{M}_t\{a(-t)\} = -\mathcal{M}\{a\}$. When $\mathcal{M}\{a\} < 0$, then $\mathcal{M}_t\{-a(-t)\} > 0$ and using the previous reasoning, (5.19) possesses a unique a.p. solution y. Consequently $x(t) = y(-t)$ is the unique a.p. solution of (5.3). This completes the proof of the lemma. $\square$

Now we consider the linear ordinary differential equation

$$x'(t) = A(t)x(t) + b(t) \tag{5.20}$$

where $A = (A_{ij})_{1\le i,j\le n} \in AP^0(\mathbb{M}(n,\mathbb{R}))$ and $b \in AP^0(\mathbb{R}^n)$ such that

(A4) A is upper triangular s.t. $\mathcal{M}\{A_{ii}\} \neq 0$ for $i = 1,\dots,n$.

Lemma 5.2. *Let $A \in AP^0(\mathbb{M}(n,\mathbb{R}))$ which satisfies* (A4) *and $b \in AP^0(\mathbb{R}^n)$. Then* (5.20) *possesses a unique solution in $AP^0(\mathbb{R}^n)$. Moreover there exists $\alpha \in (0,\infty)$ such that*

$$\|x\|_\infty \le \alpha\|b\|_\infty.$$

Proof. Equation (5.20) can be written as

$$\begin{gathered} x_1'(t) = A_{11}(t)x_1(t) + A_{12}(t)x_2(t) + \dots + A_{1n}(t)x_n(t) + b_1(t) \\ x_2'(t) = A_{22}(t)x_2(t) + \dots + A_{2n}(t)x_n(t) + b_2(t) \\ \dots \\ x_n'(t) = A_{nn}(t)x_n(t) + b_n(t) \end{gathered} \tag{5.21}$$

where $x = (x_1,\dots,x_n)$ and $b = (b_1,\dots,b_n)$.

Since $\mathcal{M}\{A_{nn}\} \neq 0$ and by using Lemma 5.1, we deduce that the last scalar equation in (5.21),

$$x_n'(t) = A_{nn}(t)x_n(t) + b_n(t), \tag{5.22}$$

has a unique solution $\hat{x}_n \in AP^0(\mathbb{R})$ such that

$$\|x_n\|_\infty \le \alpha_n\|b_n\|_\infty \tag{5.23}$$

where α_n is a positive constant. The $(n-1)$-th equation of system (5.21) is

$$x_{n-1}'(t) = A_{n-1,n-1}(t)x_{n-1}(t) + d_{n-1}(t), \tag{5.24}$$

where $d_{n-1}(t) = A_{n-1,n}(t)x_n(t) + b_{n-1}(t)$ for all $t \in \mathbb{R}$. It is clear that $d_{n-1} \in AP^0(\mathbb{R})$ as a sum and product of a.p. functions $A_{n-1,n}$, x_n and b_{n-1}. Using always Lemma 5.1 and the fact that $\mathcal{M}\{A_{n-1,n-1}\} \neq 0$, we conclude that equation (5.24) has a unique solution $\hat{x}_{n-1} \in AP^0(\mathbb{R})$ and there exists $\alpha_{n-1} \in (0,\infty)$ such that

$$\|\hat{x}_{n-1}\|_\infty \le \alpha_{n-1}\|d_{n-1}\|_\infty. \tag{5.25}$$

And so using the same reasoning as above, we can prove by induction that for $k = 1, \dots, n$ the k-th equation of (5.21) has a unique solution $\hat{x}_k \in AP^0(\mathbb{R})$ and there exists $\alpha_k \in (0, \infty)$ such that

$$\|\hat{x}_k\|_\infty \le \alpha_k \|d_k\|_\infty, \tag{5.26}$$

where $d_k(t) = A_{k,k+1}(t)x_{k+1}(t) + \dots + A_{k,n}(t)x_n(t) + b_k(t)$, for all $t \in \mathbb{R}$. Therefore (5.21) has a unique solution $\hat{x} \in AP^0(\mathbb{R}^n)$.

Now we shall prove that there exists $\alpha \in (0, \infty)$ such that

$$\|\hat{x}\|_\infty \le \alpha \|b\|_\infty, \tag{5.27}$$

Since on $BC^0(\mathbb{R}, \mathbb{R}^n)$ the norm $\|.\|_\infty$ is equivalent to the norm $\|\cdot\|_0$, where $\|x\|_0 = \sum_{j=1}^n \|x_j\|_\infty$ and $\|x\|_\infty = \sup_{t\in\mathbb{R}} \|x(t)\|_2 = \sup_{t\in\mathbb{R}} \sqrt{\sum_{j=1}^n \|x_j(t)\|^2}$, with $x = (x_1, \dots, x_n)$, it is enough to prove (5.27) for $\|\cdot\|_0$. For this we proceed by induction on the order $k \in \{1, \dots, n\}$ of A.

First step: $k = 1$. So (5.21) is the scalar equation (5.22) and by (5.23), (5.27) is obtained.

Second step: $k = n - 1$. We assume that there exists $\gamma_{n-1} \in (0, \infty)$ such that

$$\|\hat{x}_n\|_\infty + \dots + \|\hat{x}_2\|_\infty \le \gamma_{n-1}(\|b_n\|_\infty + \dots + \|b_2\|_\infty). \tag{5.28}$$

Third step: $k = n$. By (5.28) we obtain $\|\hat{x}_n\|_\infty + \dots + \|\hat{x}_1\|_\infty \le \gamma_{n-1}(\|b_n\|_\infty + \dots + \|b_2\|_\infty) + \|\hat{x}_1\|_\infty$, and by (5.26) we know that $\|\hat{x}_1\|_\infty \le \alpha_1 \|d_1\|_\infty$, where $d_1(t) = A_{12}(t)x_2(t) + \dots + A_{1n}(t)x_n(t) + b_1(t)$ for all $t \in \mathbb{R}$ that implies

$$\begin{aligned}
\|\hat{x}_1\|_\infty &\le \alpha_1(\|A_{12}\|_\infty \|\hat{x}_2\|_\infty + \dots + \|A_{1n}\|_\infty \|\hat{x}_n\|_\infty + \|b_1\|_\infty) \\
&\le \alpha_1 \left(\max(\|A_{12}\|_\infty, \dots, \|A_{1n}\|_\infty)(\|\hat{x}_2\|_\infty + \dots + \|\hat{x}_n\|_\infty) + \|b_1\|_\infty\right).
\end{aligned}$$

After using induction assumption and noting $c_n = \max(\|A_{12}\|_\infty, \dots, \|A_{1n}\|_\infty)$, we obtain

$$\begin{aligned}
\|\hat{x}_1\|_\infty &\le \alpha_1(c_n\gamma_{n-1}(\|b_2\|_\infty + \dots + \|b_n\|_\infty) + \|b_1\|_\infty) \\
&\le \alpha_1 k_n(\|b_2\|_\infty + \dots + \|b_n\|_\infty + \|b_1\|_\infty) \\
&\le M_n(\|b_1\|_\infty + \dots + \|b_n\|_\infty)
\end{aligned}$$

where $k_n = \max(c_n\gamma_{n-1}, 1)$ and $M_n = \alpha_1 k_n$. Hence we conclude that

$$\|\hat{x}_1\|_\infty + \dots + \|\hat{x}_n\|_\infty \le \gamma_{n-1}(\|b_2\|_\infty + \dots + \|b_n\|_\infty) + M_n(\|b_1\|_\infty + \dots + \|b_n\|_\infty)$$

with $\gamma_n = \max(\gamma_{n-1}, M_n) \in (0, \infty)$; i.e., $\|\hat{x}\|_0 \le \gamma_n \|b\|_0$. And so (5.27) is proven. $\square$

Definition 5.3. We so-call the Bohr-Neugebauer constant is the least constant α which satisfies the last assertion of Lemma 5.2.

Now let $A \in AP^0(\mathbb{M}(n, \mathbb{R}))$ and define the two matrices $T = (T_{ij})_{1\le i,j\le n}$ and $R = (R_{ij})_{1\le i,j\le n}$ as follows, for $t \in \mathbb{R}$,

$$T_{ij}(t) = \begin{cases} A_{ij}(t) & \text{if } j \ge i \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad R(t) = A(t) - T(t). \tag{5.29}$$

Note that $T(t)$ is upper triangular.

Theorem 5.4. *Let $A \in AP^0(\mathbb{M}(n,\mathbb{R}))$ such that $\mathcal{M}\{A_{ii}\} \neq 0$ for $i = 1,\dots,n$ and let $f \in APU(\mathbb{R}\times\mathbb{R}^n)$. We also assume that*

$$\|R\|_\infty < \frac{1}{\|T\|_\infty \alpha + 1 + \alpha}, \tag{5.30}$$

and that for all $t \in \mathbb{R}$ and all $x, y \in \mathbb{R}^n$, there exists $k \in (0, (\|T\|_\infty\alpha + 1 + \alpha)^{-1} - \|R\|_\infty)$ such that

$$\|f(t,x) - f(t,y)\| \le k\|x - y\|, \tag{5.31}$$

where T and R are defined as in (5.29). Then the equation

$$x'(t) = A(t)x(t) + f(t, x(t)) \tag{5.32}$$

possess a unique solution in $AP^1(\mathbb{R}^n)$.

Proof. First we remark that (5.32) can be written as

$$x'(t) = T(t)x(t) + g(t, x(t)) \tag{5.33}$$

where $g(t,x(t)) = R(t)x(t) + f(t,x(t))$ for all $t \in \mathbb{R}$.

Consider the linear operator $L : AP^1(\mathbb{R}^n) \to AP^0(\mathbb{R}^n)$ defined by $Lx = [t \mapsto x'(t) - T(t)x(t)]$. Since $\mathcal{M}\{A_{ii}\} \neq 0$ for $i = 1,\dots,n$, the operator T satisfy the assumption in Lemma 5.2 and we deduce that for $b \in AP^0(\mathbb{R}^n)$ there exists a unique solution of the differential equation

$$x'(t) = T(t)x(t) + b(t). \tag{5.34}$$

Then L is invertible, we denote by $x[b]$ the unique solution of (5.34), and so we have $L^{-1}(b) = x[b]$. By Lemma 5.2, there exists $\alpha \in (0,\infty)$ such that $\|x[b]\|_\infty \le \alpha\|b\|_\infty$ and using (5.34), we obtain

$$\|x'[b]\|_\infty \le \|T\|_\infty\|x[b]\|_\infty + \|b\|_\infty \le \|T\|_\infty\alpha\|b\|_\infty + \|b\|_\infty = (\|T\|_\infty\alpha + 1)\|b\|_\infty.$$

This implies

$$\|L^{-1}(b)\|_{C^1} \le (\|T\|_\infty\alpha + 1 + \alpha)\|b\|_\infty.$$

Consequently,

$$\|L^{-1}\|_{\mathcal{L}} \le (\|T\|_\infty\alpha + 1 + \alpha). \tag{5.35}$$

Now we consider the superposition operator $N_g : AP^0(\mathbb{R}^n) \to AP^0(\mathbb{R}^n)$, $N_g(x) = [t \mapsto g(t, x(t))]$. N_g is well defined, [3].

Using assumption (5.31), we have

$$\|N_g(x) - N_g(y)\|_\infty \le (\|R\|_\infty + k)\|x - y\|_\infty, \tag{5.36}$$

for all $x, y \in AP^0(\mathbb{R}^n)$. Now, from (5.35) and (5.36) it is easy to verify that for $x, y \in AP^0(\mathbb{R}^n)$,

$$\|L^{-1}\circ N_g(x) - L^{-1}\circ N_g(y)\|_\infty \le k_1\|x - y\|_\infty,$$

where $k_1 = (\|T\|_\infty\alpha + 1 + \alpha)(\|R\|_\infty + k)$. From (5.30) and (5.31), it is clear to see that $k_1 \in (0,1)$ and hence the operator $L^{-1}\circ N_g : AP^0(\mathbb{R}^n) \to AP^0(\mathbb{R}^n)$ is a contraction. Then by using the Picard-Banach Fixed Point Theorem, we obtain that there exists a unique $x \in AP^0(\mathbb{R}^n)$ such that

$$L^{-1}\circ N_g(x) = x.$$

This is equivalent to saying that x is a solution of (5.33) in $AP^1(\mathbb{R}^n)$ and so it is a unique solution of (5.32) in $AP^1(\mathbb{R}^n)$. $\square$

6. A continuous dependence result

Theorem 6.1. *Let $A \in AP^0(\mathbb{M}(n,\mathbb{R}))$ such that $\mathcal{M}\{A_{ii}\} \neq 0$ for $i = 1,\dots,n$ and $f \in APU(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^p)$ which satisfy the following condition: For all $t \in \mathbb{R}$ and for $(x,y,u) \in \mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^p$, there exists $c \in (0, (\|T\|_\infty\alpha + 1 + \alpha)^{-1})$ such that*

$$\|f(t,x,u) - f(t,y,u)\| \leq c\|x-y\|, \tag{6.1}$$

where α is the Bohr-Neugebauer constant, and T is defined as above. Then, for all $u \in AP^0(\mathbb{R}^p)$, there exists a unique solution $\tilde{x}[u]$ of

$$x'(t) = A(t)x(t) + f(t,x(t),u(t)) \tag{6.2}$$

which is in $AP^1(\mathbb{R}^n)$. Moreover the mapping $u \mapsto \tilde{x}[u]$ is continuous from $AP^0(\mathbb{R}^p)$ into $AP^1(\mathbb{R}^n)$.

Proof. Let L the operator be defined in the proof of the above theorem, and N_f the superposition operator defined by $N_f : AP^0(\mathbb{R}^n)\times AP^0(\mathbb{R}^n) \to AP^0(\mathbb{R}^n)$, $N_f(x,u) = [t \mapsto f(t,x(t),u(t))]$. N_f is well defined and continuous (see [3]). This implies that the mapping $u \mapsto \Phi(x,u)$ is continuous on $AP^0(\mathbb{R}^p)$ for all $x \in AP^0(\mathbb{R}^n)$, where $\Phi : AP^0(\mathbb{R}^n)\times AP^0(\mathbb{R}^p) \to AP^0(\mathbb{R}^n)$, $\Phi(x,u) = L^{-1}\circ N_f(x,u)$.

Now by (6.1) it follows that

$$\|N_f(x,u) - N_f(y,u)\|_\infty \leq c\|x-y\|_\infty \tag{6.3}$$

for all $x,y \in AP^0(\mathbb{R}^n)$ and for all $u \in AP^0(\mathbb{R}^p)$. With (5.35), this implies that

$$\|\Phi(x,u) - \Phi(y,u)\|_\infty \leq c(\|T\|_\infty\alpha + 1 + \alpha)\|x-y\|_\infty$$

for all $x,y \in AP^0(\mathbb{R}^n)$ and for all $u \in AP^0(\mathbb{R}^p)$. Therefore we can apply the Theorem of parametrized fixed point in [15, p. 103] to conclude that equation (6.2) possess a unique solution $\tilde{x}[u] \in AP^1(\mathbb{R}^n)$ and the mapping $u \mapsto \tilde{x}[u]$ is continuous from $AP^0(\mathbb{R}^p)$ into $AP^0(\mathbb{R}^n)$. □

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