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SOLITARY WAVES FOR THE COUPLED NONLINEAR KLEIN-GORDON AND BORN-INFELD TYPE EQUATIONS

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Abstract. In this article we study the existence of solutions for a nonlinear Klein-Gordon-Maxwell equation coupled with a Born-Infeld equation.

1. Introduction

It is well known that the gauge potential $(\phi, \mathbf{A})$ can be coupled to a complex order parameter ψ through the minimal coupling rule; that is the formal substitution

$$\begin{gathered}\frac{\partial}{\partial t} \mapsto \frac{\partial}{\partial t} + ie\phi, \\ \nabla \mapsto \nabla - ie\mathbf{A},\end{gathered}$$

where e is the electric charge, $\mathbf{A} : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$ is a magnetic vector potential and $\phi : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ is an electric potential. Therefore, in a flat Minkowskian space-time with metric $(g_{\mu\nu}) = \mathrm{diag}[1, -1, -1, -1]$, we can define the Klein-Gordon-Maxwell Lagrangian density

$$\mathcal{L}_{KGM} = \frac{1}{2}\big[|\frac{\partial \psi}{\partial t} + ie\phi\psi|^2 - |\nabla\psi - ie\mathbf{A}|^2 - m^2|\psi|^2\big] + \frac{1}{q}|\psi|^q,$$

where $m \geq 0$ represents the mass of the charged field. The total action of the system is thus given by

$$\mathcal{S} = \iint (\mathcal{L}_{KGM} + \mathcal{L}_{\mathrm{emf}})\, dx\, dt, \tag{1.1}$$

where $\mathcal{L}_{\mathrm{emf}}$ is the Lagrangian density of the electro-magnetic field. In the Born-Infeld theory (see [8]), with a suitable choice of constants, $\mathcal{L}_{\mathrm{emf}}$ can be written as

$$\mathcal{L}_{\mathrm{emf}} = \mathcal{L}_{BI} := \frac{b^2}{4\pi}\Big(1 - \sqrt{1 - \frac{1}{b^2}(|\mathbf{E}|^2 - |\mathbf{B}|^2)}\Big),$$

where b is the so-called Born-Infeld parameter, $b \gg 1$. By the Maxwell equations,

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}$$

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is the electric field, and

$$\mathbf{B} = \nabla \times \mathbf{A}$$

is the magnetic induction field. If, as in [4], we consider the electrostatic solitary wave:

$$\psi(x,t) = u(x)e^{-i\omega t}, \quad \mathbf{A} = 0, \quad \phi = \phi(x),$$

where $u : \mathbb{R}^3 \to \mathbb{R}$ and $\omega \in \mathbb{R}$, then the total action in (1.1) takes the form

$$\begin{aligned} F_{\rm BI}(u,\phi) = \frac{1}{2}\int_{\mathbb{R}^3} |Du|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^3} \left(m^2 - (e\phi-\omega)^2\right)u^2\,dx \\ - \frac{1}{q}\int_{\mathbb{R}^3} |u|^q\,dx - \frac{b^2}{4\pi}\int_{\mathbb{R}^3}\Big(1 - \sqrt{1 - \frac{1}{b^2}|\nabla\phi|^2}\Big)\,dx. \end{aligned} \tag{1.2}$$

The critical point (u,ϕ) of $F_{\rm BI}$ satisfies the Euler-Lagrange equations associted to (1.2). By standard calculations, we obtain:

$$\begin{gathered} -\Delta u + [m^2 - (\phi-\omega)^2]u = |u|^{q-2}u, \quad \text{in } \mathbb{R}^3, \\ \nabla \cdot \frac{\nabla\phi}{\sqrt{1 - \frac{1}{b^2}|\nabla\phi|^2}} = 4\pi(\phi-\omega)u^2, \quad \text{in } \mathbb{R}^3, \end{gathered} \tag{1.3}$$

where we have taken $e = 1$. We can see that the sign ω is not relevant for the existence of solutions for problem (1.3). In fact, if (u,ϕ) is a solution of (1.3) with ω, then $(u,-\phi)$ is also a solution corresponding to $-\omega$. So, without loss of generality, we can assume $\omega > 0$.

As we know, a large number of works have been devoted to the problem like (1.3). In the following we review some assumptions and the corresponding results.

In [2, 3, 4, 5, 6, 7, 9, 10, 15], the authors consider the first-order expansion of the second formula of (1.3) for $b \to +\infty$. Therefore (1.3) becomes

$$\begin{gathered} -\Delta u + [m^2 - (\phi-\omega)^2]u = |u|^{q-2}u, \quad \text{in } \mathbb{R}^3, \\ \Delta\phi = 4\pi(\phi-\omega)u^2, \quad \text{in } \mathbb{R}^3. \end{gathered} \tag{1.4}$$

About the problem (1.4), the pioneering work is given by Benci and Fortunato [4]. They showed that (1.4) has infinitely many solutions when $q \in (4,6)$ and $0 < \omega < m$. In [10] d'Aprile and Mugnai proved the existence of nontrivial solutions of (1.4) whenever $q \in (2,4]$ and

$$\frac{q-2}{2}m^2 > \omega^2.$$

d'Aprile and Mugnai [9] also showed that (1.4) has no nontrivial solutions when $q \geq 6$ and $0 < \omega \leq m$ or $q \leq 2$. Recently, in [2], under the following conditions:

$$\begin{gathered} (q-2)(4-q)m^2 > \omega^2, \quad p \in (2,3), \\ m > \omega > 0, \quad p \in [3,6), \end{gathered}$$

Azzollini, Pisani and Pomponio showed that (1.4) admits a nontrivial solution. It is easy to see that $(p-2)(4-p) > (p-2)/2$ for $p \in (2,3]$.

In [11, 12, 14], the authors consider the second-order expansion of the second formula of (1.3) for $b \to +\infty$. Therefore (1.3) becomes

$$\begin{gathered} -\Delta u + [m^2 - (\phi-\omega)^2]u = |u|^{q-2}u, \quad \text{in } \mathbb{R}^3, \\ \Delta\phi + \beta_2\Delta_4\phi = 4\pi(\phi-\omega)u^2, \quad \text{in } \mathbb{R}^3, \end{gathered} \tag{1.5}$$

where $\beta_2 = 1/(2b^2) \to 0$ and $\Delta_4\phi = D(|D\phi|^2 D\phi)$. In [12], Fortunato, Orsina and Pisani showed the existence of electrostatic solutions with finite energy, while in [11] d'Avenia and Pisani proved that (1.5) has infinitely many solutions, provided that $4 < q < 6$ and $0 < \omega < m$. In [14] Mugnai established the same results under the following assumptions: $4 \le q < 6$ and $0 < \omega < m$ or $2 < q < 4$ and

$$\frac{q-2}{2} m^2 > \omega^2.$$

Recently, Yu [18] studied the original Born-Infeld equations, i.e. (1.3). He proved the existence of the least-action solitary waves in both bounded smooth domain case and $\mathbb{R}^3$ case whenever $q \in (2, 6)$ and

$$\frac{q-2}{q} m^2 > \omega^2.$$

In the present paper we consider the nonlinear Klein-Gordon equations coupled with the N-th order expansion of the second formula of (1.3) for $b \to +\infty$:

$$\begin{gathered}
-\Delta u + [m^2 - (\phi - \omega)^2]u = |u|^{q-2}u, \quad \text{in } \mathbb{R}^3, \\
\sum_{k=1}^{N} (\beta_k \Delta_{2k}\phi) = 4\pi(\phi - \omega)u^2, \quad \text{in } \mathbb{R}^3,
\end{gathered} \tag{1.6}$$

where $\beta_1 = 1$, $\beta_k = \frac{1\cdot 3\cdot 5 \dots (2k-3)}{2^{k-1}(k-1)!} \frac{1}{b^{2(k-1)}}$ and $\Delta_{2k}\phi = D(|D\phi|^{2k-2}D\phi)$, for $k = 2, 3, \dots, N$.

It is well-known that $H^1(\mathbb{R}^3)$ is the usual Sobolev space endowed with the norm

$$\|u\|_{H^1(\mathbb{R}^3)} = \Big(\int_{\mathbb{R}^3} [|Du|^2 + u^2]\, dx \Big)^{1/2}$$

(see [1], [17, Theorem 1.8]). $D^N(\mathbb{R}^3)$ denotes the completion of $C_0^\infty(\mathbb{R}^3, \mathbb{R})$ with respect to the norm

$$\|\phi\|_{D^N(\mathbb{R}^3)} = \Big(\int_{\mathbb{R}^3} |D\phi|^2\, dx \Big)^{1/2} + \Big(\int_{\mathbb{R}^3} |D\phi|^{2N}\, dx \Big)^{1/(2N)}.$$

By a solution (u, ϕ) of (1.6), we understand $(u, \phi) \in H^1(\mathbb{R}^3) \times D^N(\mathbb{R}^3)$ satisfying (1.6) in the weak sense. Obviously, $(u, \phi) = (0, 0)$ is a trivial solution of (1.6). We define a functional $F_N : H^1(\mathbb{R}^3) \times D^N(\mathbb{R}^3) \to \mathbb{R}$ by

$$F_N(u, \phi) = \int_{\mathbb{R}^3} \big[\frac{1}{2}|Du|^2 - \frac{1}{4\pi} \sum_{k=1}^{N} \big(\frac{1}{2k}\beta_k |D\phi|^{2k}\big) + \frac{1}{2}(m^2 - (\phi - \omega)^2)u^2 - \frac{1}{q}|u|^q\big]\, dx\,.$$

It is easy to see that $F_N \in C^1(H^1(\mathbb{R}^3) \times D^N(\mathbb{R}^3), \mathbb{R})$. Therefore solutions of (1.6) correspond to critical points of the functional F_N. Next we give our main result.

Theorem 1.1. *Problem* (1.6) *has at least a nontrivial solution* $(u, \phi) \in H^1(\mathbb{R}^3) \times D^N(\mathbb{R}^3)$*, provided one of the following conditions is satisfied*

(i) $q \in (3, 6)$ *and* $m > \omega > 0$.
(ii) $q \in (2, 3]$ *and* $(q-2)(4-q)m^2 > \omega^2 > 0$.

Set $|u|_q := \{\int_{\mathbb{R}^3} |u|^q dx\}^{1/q}$ for $1 < q < \infty$. We say that $\{u_n\} \subset H^1(\mathbb{R}^3)$ is a Palais-Smale sequence for $\Phi \in C^1\big(H^1(\mathbb{R}^3), \mathbb{R}\big)$ at level $c \in \mathbb{R}$ (the $(PS)_c$-sequence for short), if and only if $\{u_n\}$ satisfies $\Phi(u_n) \to c$ and $\Phi'(u_n) \to 0$ as $n \to \infty$.

To find the critical points of the functional $F_N(u, \phi)$ we will overcome two difficulties. The first difficulty is that $F_N(u, \phi)$ is strongly indefinite (unbounded both from below and from above on infinite dimensional subspaces). To avoid this difficulty, we use the reduction method just like in [12, 11, 14]. The reduction method consists in reducing the study of $F_N(u,\phi)$ to the study of a functional $J(u)$ in the only variable u. The second difficulty is that the embedding of $H^1(\mathbb{R}^3)$ into $L^q(\mathbb{R}^3)$ is not compact, where $2<q<2^*(=6)$. So $J(u)$ does not in general satisfy the Palais-Smale condition. We will study $J(u)$ in $H^1_r(\mathbb{R}^3)$, where

$$H^1_r(\mathbb{R}^3)=\{u\in H^1(\mathbb{R}^3):u(x)=u(|x|)\}.$$

By the Principle of symmetric criticality (see [16] or [17, Theorem 1.28]), a critical point $u\in H^1_r(\mathbb{R}^3)$ for $J(u)$ is also a critical point in $H^1(\mathbb{R}^3)$. We construct a bounded $(PS)_c$-sequence following the methods of Jeanjean [13]. Then there exists a subsequence of $\{u_n\}$ which converges strongly in $H^1_r(\mathbb{R}^3)$.

This paper is organized as follows: in Section 2, we make some preliminaries; in Section 3, we obtain that the solutions of (1.6) must verify some suitable Pohožaev identity; in Section 4, we give the proof of Theorem 1.1.

2. Preliminaries

In the following we give some lemmas, whose similar proofs can be founded in [9, 11, 14].

Lemma 2.1. *For every $u\in H^1(\mathbb{R}^3)$ there is a unique $\phi=\Phi(u)\in D^N(\mathbb{R}^3)$ which solves*

$$\sum_{k=1}^{N}(\beta_k\Delta_{2k}\phi)=4\pi(\phi-\omega)u^2. \tag{2.1}$$

Lemma 2.2. *For any $u\in H^1(\mathbb{R}^3)$, on the set $\{x\in\mathbb{R}^3:u(x)\neq 0\}$,*

$$0\le\Phi(u)\le\omega.$$

Proof. Set $\Phi^-=\min\{\Phi,0\}$. Multiplying (2.1) by Φ^-, we have

$$-\frac{1}{4\pi}\sum_{k=1}^{N}\Big(\beta_k\int_{\mathbb{R}^3}|D\Phi^-|^{2k}\,dx\Big)=\int_{\mathbb{R}^3}(\Phi^-)^2u^2\,dx-\omega\int_{\mathbb{R}^3}\Phi^-u^2\,dx\ge 0.$$

So we obtain $D\Phi^-\equiv 0$. Hence, $\Phi\ge 0$.

When we multiply (2.1) by $(\Phi(u)-\omega)^+=\max\{\Phi(u)-\omega,0\}$, we obtain

$$\int_{\Phi(u)\ge\omega}(\Phi(u)-\omega)^2u^2\,dx=-\frac{1}{4\pi}\sum_{k=1}^{N}\Big(\beta_k\int_{\Phi(u)\ge\omega}|D\Phi(u)|^{2k}\,dx\Big)\ge 0,$$

so that $(\Phi(u)-\omega)^+=0$ for $u\neq 0$. Hence $\Phi(u)\le\omega$. $\square$

Lemma 2.3. *The pair $(u,\phi)\in H^1(\mathbb{R}^3)\times D^N(\mathbb{R}^3)$ is a solution of (1.6) if and only if u is a critical point of*

$$\begin{aligned}J_N(u):=F_N(u,\Phi(u))=\int_{\mathbb{R}^3}\Big[&\frac{1}{2}|Du|^2-\frac{1}{4\pi}\sum_{k=1}^{N}\big(\frac{1}{2k}\beta_k|D\Phi(u)|^{2k}\big)\\&+\frac{1}{2}(m^2-(\Phi(u)-\omega)^2)u^2-\frac{1}{q}\int_{\mathbb{R}^3}|u|^q\Big]\,dx\end{aligned}$$

and $\phi=\Phi(u)$.

The functional of (1.6) is

$$F_N(u,\phi)=\int_{\mathbb{R}^3}\Big[\frac{1}{2}|Du|^2-\frac{1}{4\pi}\sum_{k=1}^{N}\big(\frac{1}{2k}\beta_k|D\phi|^{2k}\big)+\frac{1}{2}(m^2-(\phi-\omega)^2)u^2-\frac{1}{q}\int_{\mathbb{R}^3}|u|^q\Big]\,dx.$$

From Lemma 2.1, for fixed $u\in H^1(\mathbb{R}^3)$, we have

$$-\frac{1}{4\pi}\sum_{k=1}^{N}\Big(\beta_k\int_{\mathbb{R}^3}|D\Phi(u)|^{2k}\,dx\Big)=\int_{\mathbb{R}^3}\Phi^2(u)u^2\,dx-\omega\int_{\mathbb{R}^3}\Phi(u)u^2\,dx,$$

where $\Phi(u)$ appears in Lemma 2.1. Then

$$\begin{aligned}J_N(u)&=F_N(u,\Phi(u))\\&=\frac{1}{2}\int_{\mathbb{R}^3}|Du|^2\,dx+\frac{1}{2}(m^2-\omega^2)\int_{\mathbb{R}^3}u^2\,dx+\frac{\omega}{2}\int_{\mathbb{R}^3}\Phi(u)u^2\,dx\\&\quad+\frac{1}{4\pi}\sum_{k=2}^{N}\Big(\frac{k-1}{2k}\beta_k\int_{\mathbb{R}^3}|D\Phi(u)|^{2k}\,dx\Big)-\frac{1}{q}\int_{\mathbb{R}^3}|u|^q\,dx.\end{aligned}$$

By the definition of $J_N(u)$, we have

$$\begin{aligned}\langle J_N'(u),u\rangle&=\int_{\mathbb{R}^3}|Du|^2\,dx+(m^2-\omega^2)\int_{\mathbb{R}^3}u^2\,dx-\int_{\mathbb{R}^3}\Phi^2(u)u^2\,dx\\&\quad+2\omega\int_{\mathbb{R}^3}\Phi(u)u^2\,dx-\int_{\mathbb{R}^3}|u|^q\,dx.\end{aligned}$$

From Lemmas 2.1 and 2.3, to obtain a solution of (1.6), we need only to find a critical point of J_N in $H^1(\mathbb{R}^3)$. Note that the functional J_N depends only on u. Set

$$H^1_r(\mathbb{R}^3)=\{u\in H^1(\mathbb{R}^3):u(x)=u(|x|)\}.$$

By standard arguments (Principle of symmetric criticality) one sees that a critical point $u\in H^1_r(\mathbb{R}^3)$ for the functional J_N in $H^1_r(\mathbb{R}^3)$ is also a critical point for J_N in $H^1(\mathbb{R}^3)$.

3. The Pohožaev identity

In this section we obtain that the solutions of (1.6) must verify some suitable Pohožaev identity, as was proved in [9], which provides necessary conditions to prove the existence of nontrivial solutions.

Lemma 3.1. *Let $u\in H^2_{loc}(\mathbb{R}^n)$, $\phi\in H^{2k}_{loc}(\mathbb{R}^n)$ and $a,b\geq 0$. Then, for any ball $B_R=\{x\in\mathbb{R}^n:|x|\leq R>0\}$, the following equalities hold:*

$$\begin{aligned}&\int_{B_R}-\Delta u\langle x,Du\rangle\,dx\\&=\frac{2-n}{2}\int_{B_R}|Du|^2\,dx-\frac{1}{R}\int_{\partial B_R}\langle x,Du\rangle^2\,d\sigma+\frac{R}{2}\int_{\partial B_R}|Du|^2\,d\sigma;\end{aligned}\tag{3.1}$$

$$
\begin{aligned}
&\int_{B_R} (a+b\phi)\phi u\langle x, Du\rangle\,dx \\
&= -\int_{B_R} \big(\frac{a}{2}+b\phi\big)u^2\langle x, D\phi\rangle\,dx \\
&\quad -\frac{n}{2}\int_{B_R}(a+b\phi)\phi u^2\,dx + \frac{R}{2}\int_{\partial B_R}(a+b\phi)\phi u^2\,d\sigma;
\end{aligned}
\tag{3.2}
$$

$$
\int_{B_R} g(u)\langle x, Du\rangle\,dx = -n\int_{B_R} G(u)\,dx + R\int_{\partial B_R} G(u)d\sigma; \tag{3.3}
$$

$$
\begin{aligned}
\int_{B_R} \Delta_{2k}\phi\,\langle x, D\phi\rangle\,dx &= \int_{B_R} D(|D\phi|^{2k-2}D\phi)\langle x, D\phi\rangle\,dx \\
&= \frac{n-2k}{2k}\int_{B_R}|D\phi|^{2k}dx - \frac{R}{2k}\int_{\partial B_R}|D\phi|^{2k}d\sigma \\
&\quad + \frac{1}{R}\int_{\partial B_R}|D\phi|^{2k-2}\langle x, D\phi\rangle^2\,d\sigma,
\end{aligned}
\tag{3.4}
$$

where $\Delta_{2k}\phi = D(|D\phi|^{2k-2}|D\phi|)$ *and* $g:\mathbb{R}\to\mathbb{R}$ *is a continuous function such that* $g(0)=0$ *and* $G(s)=\int_0^s g(t)\,dt$.

Proof. The proofs of (3.1), (3.2) and (3.3) can be found in [9, Lemma 3.1]. In the following we show (3.4). For fix $i_1,\dots,i_{k-1}$, $j,l=1,2,\dots,n$, we see from the integration by parts formula that

$$
\begin{aligned}
&\int_{B_R}\phi^2_{x_{i_1}}\dots\phi^2_{x_{i_{k-1}}}\phi_{x_jx_j}x_l\phi_{x_l}dx \\
&= -\int_{B_R}(\phi^2_{x_{i_1}}\dots\phi^2_{x_{i_{k-1}}}x_l\phi_{x_l})_{x_j}\phi_{x_j}dx + \int_{\partial B_R}\phi^2_{x_{i_1}}\dots\phi^2_{x_{i_{k-1}}}x_l\phi_{x_l}\phi_{x_j}\frac{x_j}{|x|}d\sigma \\
&= -\int_{B_R}(\phi^2_{x_{i_1}}\dots\phi^2_{x_{i_{k-1}}})_{x_j}x_l\phi_{x_l}\phi_{x_j}dx - \int_{B_R}\phi^2_{x_{i_1}}\dots\phi^2_{x_{i_{k-1}}}\phi_{x_l}\phi_{x_j}\delta_{lj}dx \\
&\quad -\int_{B_R}\phi^2_{x_{i_1}}\dots\phi^2_{x_{i_{k-1}}}x_l\phi_{x_lx_j}\phi_{x_j}dx + \int_{\partial B_R}\phi^2_{x_{i_1}}\dots\phi^2_{x_{i_{k-1}}}x_l\phi_{x_l}\phi_{x_j}\frac{x_j}{|x|}d\sigma,
\end{aligned}
$$

where $d\sigma$ indicates the $(n-1)$-dimensional area element in ∂B_R and δ_{lj} are the Kroneker symbols. Summing up for $i_1,\dots,i_{k-1}$, $j,l=1,2,\dots,n$, we have

$$
\begin{aligned}
&\int_{B_R}|D\phi|^{2k-2}\Delta\phi\langle x, D\phi\rangle\,dx \\
&= -\int_{B_R}\langle D|D\phi|^{2k-2}, D\phi\rangle\langle x, D\phi\rangle\,dx - \int_{B_R}|D\phi|^{2k}\,dx \\
&\quad -\int_{B_R}|D\phi|^{2k-2}\langle x, D^2\phi D\phi\rangle\,dx + \frac{1}{R}\int_{\partial B_R}|D\phi|^{2k-2}\langle x, D\phi\rangle^2\,d\sigma.
\end{aligned}
\tag{3.5}
$$

Similarly, for fix $i_1,\dots,i_{k-1}$, $j,l=1,2,\dots,n$, we see from the integration by parts formula that

$$
\begin{aligned}
&2\int_{B_R}\phi^2_{x_{i_1}}\dots\phi^2_{x_{i_{k-1}}}x_l\phi_{x_lx_j}\phi_{x_j}dx = \int_{B_R}\phi^2_{x_{i_1}}\dots\phi^2_{x_{i_{k-1}}}x_l(\phi^2_{x_j})_{x_l}dx \\
&= -\int_{B_R}(\phi^2_{x_{i_1}}\dots\phi^2_{x_{i_{k-1}}}x_l)_{x_l}\phi^2_{x_j}dx + \int_{\partial B_R}\phi^2_{x_{i_1}}\dots\phi^2_{x_{i_{k-1}}}\phi^2_{x_j}\frac{x_l^2}{|x|}d\sigma
\end{aligned}
$$

$$
\begin{aligned}
&= -\int_{B_R} (\phi^2_{x_{i_1}} \dots \phi^2_{x_{i_{k-1}}})_{x_l} x_l \phi^2_{x_j}\, dx - \int_{B_R} (\phi^2_{x_{i_1}} \dots \phi^2_{x_{i_{k-1}}})_{x_l} \phi^2_{x_j}\, dx \\
&\quad + \int_{\partial B_R} \phi^2_{x_{i_1}} \dots \phi^2_{x_{i_{k-1}}} \phi^2_{x_j} \frac{x_l^2}{|x|} d\sigma.
\end{aligned}
$$

Summing up for $i_1, \dots, i_{k-1}$, $j, l = 1, 2, \dots, n$, we have

$$
\begin{aligned}
&2\int_{B_R} |D\phi|^{2k-2} \langle x, D^2\phi D\phi\rangle dx \\
&= -\int_{B_R} \langle x, D(|D\phi|^{2k-2})\rangle |D\phi|^2 dx - n\int_{B_R} |D\phi|^{2k} dx + R\int_{\partial B_R} |D\phi|^{2k} d\sigma \\
&= -2(k-1)\int_{B_R} |D\phi|^{2k-2} \langle x, D^2\phi D\phi\rangle dx - n\int_{B_R} |D\phi|^{2k} dx + R\int_{\partial B_R} |D\phi|^{2k} d\sigma.
\end{aligned}
$$

Then

$$
\int_{B_R} |D\phi|^{2k-2} \langle x, D^2\phi D\phi\rangle dx = -\frac{n}{2k}\int_{B_R} |D\phi|^{2k} dx + \frac{R}{2k}\int_{\partial B_R} |D\phi|^{2k} d\sigma. \tag{3.6}
$$

Using (3.5) and (3.6), we obtain

$$
\begin{aligned}
&\int_{B_R} \Delta_{2k}\phi\, \langle x, D\phi\rangle\, dx \\
&= \int_{B_R} D(|D\phi|^{2k-2} D\phi) \langle x, D\phi\rangle\, dx \\
&= \int_{B_R} |D\phi|^{2k-2} \Delta\phi\, \langle x, D\phi\rangle\, dx + \int_{B_R} \langle D|D\phi|^{2k-2}, D\phi\rangle \langle x, D\phi\rangle\, dx \\
&= -\int_{B_R} |D\phi|^{2k}\, dx - \int_{B_R} |D\phi|^{2k-2} \langle x, D^2\phi D\phi\rangle\, dx + \frac{1}{R}\int_{\partial B_R} |D\phi|^{2k-2} \langle x, D\phi\rangle^2\, d\sigma \\
&= \frac{n-2k}{2k}\int_{B_R} |D\phi|^{2k} dx - \frac{R}{2k}\int_{\partial B_R} |D\phi|^{2k} d\sigma + \frac{1}{R}\int_{\partial B_R} |D\phi|^{2k-2} \langle x, D\phi\rangle^2\, d\sigma.
\end{aligned}
$$

□

Set $\Omega = m^2 - w^2$. From the above Lemma we have the following result.

Lemma 3.2. *If (u, ϕ) is a solution of the system (1.6), then (u, ϕ) satisfies the Pohožaev type identity:*

$$
\begin{aligned}
&\int_{\mathbb{R}^3} |Du|^2\, dx + 3\int_{\mathbb{R}^3} u^2\, dx + \frac{1}{4\pi}\sum_{k=2}^{N} \Big(\beta_k \frac{3(k-1)}{k}\int_{\mathbb{R}^3} |D\phi|^{2k} dx\Big) \\
&\quad - 2\int_{\mathbb{R}^3} \phi^2 u^2\, dx + 5\int_{\mathbb{R}^3} \omega\phi u^2\, dx - \frac{6}{q}\int_{\mathbb{R}^3} |u|^q\, dx = 0.
\end{aligned} \tag{3.7}
$$

Proof. Multiplying the first formula of (1.6) by $\langle x, Du\rangle$, integrating on B_R and using the above Lemma, we conclude that

$$\begin{aligned}
&-\frac{1}{2}\int_{B_R}|Du|^2\,dx-\frac{3}{2}\Omega\int_{B_R}u^2\,dx\\
&+\int_{B_R}(\phi-\omega)u^2\langle x,D\phi\rangle\,dx+\frac{3}{2}\int_{B_R}(\phi-2\omega)\phi u^2\,dx+\frac{3}{q}\int_{B_R}|u|^q\,dx\\
&=\frac{1}{R}\int_{\partial B_R}\langle x,Du\rangle^2 d\sigma-\frac{R}{2}\int_{\partial B_R}|Du|^2 d\sigma\\
&\quad-\frac{\Omega R}{2}\int_{\partial B_R}u^2 d\sigma+\frac{R}{2}\int_{B_R}(\phi-2\omega)\phi u^2\,d\sigma+\frac{R}{q}\int_{\partial B_R}|u|^q\,dx.
\end{aligned}\tag{3.8}$$

Multiplying the second formula of (1.6) by $\langle x, D\phi\rangle$, integrating on B_R and using the above Lemma, we obtain

$$\begin{aligned}
&4\pi\int_{B_R}(\phi-\omega)u^2\langle x,D\phi\rangle\,dx\\
&=\int_{B_R}\sum_{k=1}^{N}(\beta_k\Delta_{2k}\phi)\langle x,D\phi\rangle\,dx\\
&=\sum_{k=1}^{N}\beta_k\int_{B_R}\Delta_{2k}\phi\langle x,D\phi\rangle\,dx\\
&=\sum_{k=1}^{N}\beta_k\Big(\frac{3-2k}{2k}\int_{B_R}|D\phi|^{2k}dx-\frac{R}{2k}\int_{\partial B_R}|D\phi|^{2k}d\sigma\\
&\quad+\frac{1}{R}\int_{\partial B_R}|D\phi|^{2k-2}\langle x,D\phi\rangle^2\,d\sigma\Big).
\end{aligned}\tag{3.9}$$

By (3.8), (3.9) and the proof of [9, Theorem 1.1, pp. 316-317], we deduce the equality

$$\begin{aligned}
&-\frac{1}{2}\int_{\mathbb{R}^3}|Du|^2\,dx-\frac{3}{2}\Omega\int_{\mathbb{R}^3}u^2\,dx+\frac{1}{4\pi}\sum_{k=1}^{N}\Big(\beta_k\frac{3-2k}{2k}\int_{\mathbb{R}^3}|D\phi|^{2k}dx\Big)\\
&+\frac{3}{2}\int_{\mathbb{R}^3}(\phi-2\omega)\phi u^2\,dx+\frac{3}{q}\int_{\mathbb{R}^3}|u|^q\,dx=0.
\end{aligned}$$

Then, noting (1.6), we have

$$\begin{aligned}
&\int_{\mathbb{R}^3}|Du|^2\,dx+3\Omega\int_{\mathbb{R}^3}u^2\,dx+\frac{1}{2\pi}\sum_{k=2}^{N}\Big(\beta_k\frac{3(k-1)}{2k}\int_{\mathbb{R}^3}|D\phi|^{2k}dx\Big)\\
&-2\int_{\mathbb{R}^3}\phi^2u^2\,dx+5\omega\int_{\mathbb{R}^3}\phi u^2\,dx-\frac{6}{q}\int_{\mathbb{R}^3}|u|^q\,dx=0.
\end{aligned}$$

□

4. Proof of the main theorem

First, we give a abstract result which is due to Jeanjean [13].

Proposition 4.1. *Let $(X, \|\cdot\|)$ be a Banach space and let $I \subset \mathbb{R}^+$ be an interval. Consider the family of C^1 functionals on X*

$$\Psi_\lambda(u) = A(u) - \lambda B(u), \quad \forall \lambda \in I,$$

with $B(u)$ nonnegative and either $A(u) \to +\infty$ or $B(u) \to +\infty$, as $\|u\| \to \infty$ and such that $\Psi_\lambda(0) = 0$. For any $\lambda \in I$ we set

$$\Gamma_\lambda = \{\gamma \in C([0,1], X) : \gamma(0) = 0, \Psi_\lambda(\gamma(1)) \leq 0\}.$$

If for every $\lambda \in I$ the set Γ_λ is nonempty and

$$c_\lambda := \inf_{\gamma \in \Gamma_\lambda} \max_{t \in [0,1]} \Psi_\lambda(\gamma(t)) > 0,$$

then for almost every $\lambda \in I$ there is a sequence $\{(u_\lambda)_n\} \subset X$ such that

(i) *$\{(u_\lambda)_n\}$ is bounded in X;*
(ii) *$\Psi_\lambda((u_\lambda)_n) \to c_\lambda$;*
(iii) *$\Psi'_\lambda((u_\lambda)_n) \to 0$ in the dual X^* of X.*

Proof Theorem 1.1. Denote

$$M(\phi) := \frac{1}{4\pi} \sum_{k=2}^{N} \Big(\beta_k \frac{k-1}{k} \int_{\mathbb{R}^3} |D\phi|^{2k}\Big) dx.$$

Then, noting the definition of $\Phi(u)$ we can write (3.7) and $J(u)$ by:

$$\begin{aligned}
&\int_{\mathbb{R}^3} |Du|^2\,dx + 3\Omega \int_{\mathbb{R}^3} u^2\,dx + 3M(\Phi(u)) - 2\int_{\mathbb{R}^3} \Phi^2(u)u^2\,dx \\
&+ 5\omega \int_{\mathbb{R}^3} \Phi(u)u^2\,dx - \frac{6}{q} \int_{\mathbb{R}^3} |u|^q\,dx = 0
\end{aligned}$$

and

$$\begin{aligned}
J_N(u) = {}& \frac{1}{2} \int_{\mathbb{R}^3} |Du|^2\,dx + \frac{1}{2}\Omega \int_{\mathbb{R}^3} u^2\,dx + \frac{\omega}{2} \int_{\mathbb{R}^3} \Phi(u)u^2\,dx \\
&+ \frac{1}{2} M(\Phi(u)) - \frac{1}{q} \int_{\mathbb{R}^3} |u|^q\,dx,
\end{aligned}$$

respectively.

For $\lambda \in [\frac{1}{2}, 1]$, we define the family of functionals $J_{N,\lambda} : H^1_r(\mathbb{R}^3) \to \mathbb{R}$ by

$$\begin{aligned}
J_{N,\lambda}(u) = {}& \frac{1}{2} \int_{\mathbb{R}^3} |Du|^2\,dx + \frac{1}{2}\Omega \int_{\mathbb{R}^3} u^2\,dx + \frac{\omega}{2} \int_{\mathbb{R}^3} \Phi(u)u^2\,dx \\
&+ \frac{1}{2} M(\Phi(u)) - \frac{\lambda}{q} \int_{\mathbb{R}^3} |u|^q\,dx
\end{aligned}$$

Using a slightly modified version of [2, Lemmas 2.3 and 2.4], it can be proved that: for every $\lambda \in [\frac{1}{2}, 1]$, there exist $\alpha_\lambda, \rho_\lambda > 0$ and $\nu_\lambda \in H^1_r(\mathbb{R}^3)$ such that

(i) $\inf_{\|u\| = \rho_\lambda} J_{N,\lambda}(u) > \alpha_\lambda$.
(ii) $\|\nu_\lambda\| > \rho_\lambda$ and $J_{N,\lambda}(\nu_\lambda) < 0$.

Thus $J_{N,\lambda}$ has the mountain pass geometry. So we can define the Mountain Pass level c_λ by

$$c_\lambda := \inf_{\gamma \in \Gamma_\lambda} \max_{0 \leq t \leq 1} J_{N,\lambda}(\gamma(t)),$$

where

$$\Gamma_\lambda = \{\gamma \in C([0,1], H^1_r(\mathbb{R}^3)) : \gamma(0) = 0, \gamma(1) = \nu_\lambda\}.$$

Set $X = H_r^1(\mathbb{R}^3), I = [\frac{1}{2}, 1], \Psi_\lambda = J_{N,\lambda}$,

$$A(u) = \frac{1}{2}\int_{\mathbb{R}^3} |Du|^2\,dx + \frac{1}{2}\Omega\int_{\mathbb{R}^3} u^2\,dx + \frac{\omega}{2}\int_{\mathbb{R}^3} \Phi(u)u^2\,dx + \frac{1}{2}M(\Phi(u))$$

and

$$B(u) = \frac{1}{q}\int_{\mathbb{R}^3} |u|^q\,dx.$$

It is easy to see that $B(u) \geq 0$ for all $u \in H_r^1(\mathbb{R}^3)$ and $A(u) \to +\infty$ as $\|u\| \to \infty$. Thus, by Proposition 4.1, for almost every $\lambda \in I$ there is a sequence $\{(u_\lambda)_n\} \subset X$ such that

(i) $\{(u_\lambda)_n\}$ is bounded in $H_r^1(\mathbb{R}^3)$;
(ii) $J_{N,\lambda}((u_\lambda)_n) \to c_\lambda$;
(iii) $J'_{N,\lambda}((u_\lambda)_n) \to 0$ in the dual $(H_r^1(\mathbb{R}^3))^*$ of $H_r^1(\mathbb{R}^3)$.

There exists $u_\lambda \in H_r^1(\mathbb{R}^3)$ such that

$$J'_\lambda(u_\lambda) = 0, \quad J_\lambda(u_\lambda) = c_\lambda,$$

for almost every $\lambda \in I$. Now we can choose a suitable $\lambda_n \to 1$ and u_{λ_n} such that

$$J'_{\lambda_n}(u_{\lambda_n}) = 0, \quad J_{\lambda_n}(u_{\lambda_n}) = c_{\lambda_n} \to c_1,$$

For simplicity we denoted u_{λ_n} by u_n. Since $J'_{\lambda_n}(u_n) = 0$, u_n satisfies the Pohožaev equality

$$\begin{aligned}&\int_{\mathbb{R}^3} |Du_n|^2\,dx + 3\Omega\int_{\mathbb{R}^3} u_n^2\,dx + 3M(\Phi(u_n)) - 2\int_{\mathbb{R}^3} \Phi^2(u_n)u_n^2\,dx \\ &+ 5\omega\int_{\mathbb{R}^3} \Phi(u_n)u_n^2\,dx - \frac{6\lambda_n}{q}\int_{\mathbb{R}^3} |u_n|^q\,dx = 0.\end{aligned} \tag{4.1}$$

By $J'_{\lambda_n}(u_n) = 0$ and $J_{\lambda_n}(u_n) = c_{\lambda_n} \to c_1$, we have

$$\begin{aligned}&\int_{\mathbb{R}^3} |\nabla u_n|^2\,dx + \Omega\int_{\mathbb{R}^3} u_n^2\,dx + 2\omega\int_{\mathbb{R}^3} \Phi(u_n)u_n^2\,dx \\ &- \int_{\mathbb{R}^3} \Phi^2(u_n)u_n^2\,dx - \lambda_n\int_{\mathbb{R}^3} |u_n|^q\,dx = 0\end{aligned} \tag{4.2}$$

and, for n large enough,

$$\begin{aligned}&\frac{1}{2}\int_{\mathbb{R}^3} |\nabla u_n|^2\,dx + \frac{1}{2}\Omega\int_{\mathbb{R}^3} u_n^2\,dx + \frac{1}{2}M(\Phi(u_n)) \\ &+ \frac{\omega}{2}\int_{\mathbb{R}^3} \Phi(u_n)u_n^2\,dx - \frac{\lambda_n}{q}\int_{\mathbb{R}^3} |u_n|^q\,dx \leq c_1 + 1.\end{aligned}$$

Set α and β two real number (which we will estimate later). Then from $\alpha \times (4.1) + \beta \times (4.2)$, we obtain

$$\begin{aligned}&\frac{\lambda_n}{q}\int_{\mathbb{R}^3} |u_n|^q\,dx \\ &= \frac{1}{6\alpha + q\beta}\Big\{(\alpha+\beta)\int_{\mathbb{R}^3} |\nabla u_n|^2\,dx + (3\alpha+\beta)\Omega\int_{\mathbb{R}^3} u_n^2\,dx + 3\alpha M(\Phi(u_n)) \\ &\quad + (5\alpha+2\beta)\int_{\mathbb{R}^3} \omega\Phi_{u_n}u_n^2\,dx - (2\alpha+\beta)\int_{\mathbb{R}^3} \Phi_{u_n}^2 u_n^2\,dx\Big\}.\end{aligned}$$

Thus

$$c_1 + 1 \geq J_{\lambda_n}(u_n)$$

$$
\begin{aligned}
&= \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u_n|^2\,dx + \frac{1}{2}\Omega\int_{\mathbb{R}^3} u_n^2\,dx \\
&\quad + \frac{1}{2}M(\Phi(u_n)) + \frac{1}{2}\int_{\mathbb{R}^3} \omega\phi_{u_n}u_n^2\,dx - \frac{\lambda_n}{q}\int_{\mathbb{R}^3} |u_n|^q\,dx \\
&= \Big(\frac{1}{2} - \frac{\alpha+\beta}{6\alpha+q\beta}\Big)\int_{\mathbb{R}^3} |\nabla u_n|^2\,dx + \Big(\frac{1}{2} - \frac{3\alpha+\beta}{6\alpha+q\beta}\Big)\Omega\int_{\mathbb{R}^3} u_n^2\,dx \\
&\quad + \Big(\frac{1}{2} - \frac{5\alpha+2\beta}{6\alpha+q\beta}\Big)\int_{\mathbb{R}^3} \omega\phi_{u_n}u_n^2\,dx + \Big(\frac{1}{2} - \frac{3\alpha}{6\alpha+q\beta}\Big)M(\Phi(u_n)) \\
&\quad + \frac{2\alpha+\beta}{6\alpha+q\beta}\int_{\mathbb{R}^3} \Phi^2(u_n)u_n^2\,dx \\
&= \Big(\frac{1}{2} - \frac{\tau+1}{6\tau+q}\Big)\int_{\mathbb{R}^3} |\nabla u_n|^2\,dx \\
&\quad + \Big(\frac{1}{2} - \frac{3\tau}{6\tau+q}\Big)M(\Phi(u_n)) + \frac{2\tau+1}{6\tau+q}\int_{\mathbb{R}^3} \Phi^2(u_n)u_n^2\,dx \\
&\quad + \Big(\frac{1}{2} - \frac{3\tau+1}{6\tau+q}\Big)\Omega\int_{\mathbb{R}^3} u_n^2\,dx + \Big(\frac{1}{2} - \frac{5\tau+2}{6\tau+q}\Big)\int_{\mathbb{R}^3} \omega\Phi(u_n)u_n^2\,dx,
\end{aligned}
$$

where $\tau = \frac{\alpha}{\beta}$. Under one of the following conditions:

(i) $q \in (4,6)$, $\tau \in ((2-q)/4, -1/2)$ and $m > \omega > 0$;
(ii) $q \in (3,4]$, $\tau \in ((2-q)/4, (q-4)/4)$ and $m > \omega > 0$;
(iii) $q \in (2,3]$, $\tau \in ((2-q)/4, +\infty)$ and $m\sqrt{(q-2)(4-q)} > \omega > 0$,

we conclude that

$$
\frac{1}{2} - \frac{\tau+1}{6\tau+q} > 0, \quad \frac{1}{2} - \frac{3\tau}{6\tau+q} > 0
$$

and

$$
\frac{2\tau+1}{6\tau+q}t^2 + \Big(\frac{1}{2} - \frac{5\tau+2}{6\tau+q}\Big)\omega t + \Big(\frac{1}{2} - \frac{3\tau+1}{6\tau+q}\Big)\Omega \geq 0, \quad \text{for } t \in [0,\omega].
$$

So we obtain that $\int_{\mathbb{R}^3} |\nabla u_n|^2\,dx$ is bounded for all n. Then, as in [2, Proof of Teorem 1.1, pp. 9] we have $\{u_n\}$ is bounded in $H_r^1(\mathbb{R}^3)$. Thus $\{u_n\}$ is a bounded $(PS)_{c_1}$-sequence for J_N. So J_N has a nontrivial critical point u_N.

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