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# EXISTENCE OF SOLUTIONS FOR A TWO-POINT BOUNDARY-VALUE PROBLEM OF A FOURTH-ORDER STURM-LIOUVILLE TYPE

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Abstract. In this work, we establish the existence of two intervals for a parameter $\lambda$ for which a two-point boundary-value problem of fourth-order Sturm-Liouville type admits three weak solutions whose norms are uniformly bounded with respect to $\lambda$. Employing two three critical point theorems, existence of at least three weak solutions is ensured. This approach is based on variational methods and critical point theory.

## 1. Introduction

Consider the fourth-order Sturm-Liouville type problem

$$\begin{gathered}(p(x)u''(x))'' - (q(x)u'(x))' + r(x)u(x) = \lambda f(x,u) \quad x \in (0,1),\\ u(0) = u(1) = u''(0) = u''(1) = 0\end{gathered} \tag{1.1}$$

where $p, q, r \in L^\infty([0,1])$ with $p^- := \operatorname{ess\,inf}_{x\in[0,1]} p(x) > 0$, $\lambda$ is a positive parameter and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is an $L^1$-Carathéodory function.

Due to the importance of fourth-order two-point boundary value problems in describing a large class of elastic deflection, many researchers have studied the existence and multiplicity of solutions for such a problem, we refer the reader to [1, 4, 9, 10, 14, 20, 26, 27] and references therein. In this article we are interested in establishing the existence of two intervals for the positive real parameter $\lambda$ for which (1.1) admits three weak solutions whose norms are uniformly bounded with respect to $\lambda$ belonging to one of the two intervals. A basic tool in our arguments is a three critical points theorem due to Bonanno [5], which extends previous results established by Ricceri [24, 25]. The starting point for such properties is the pioneering *three critical points* theorem due to Pucci and Serrin [22, 23] (see also [18]), which asserts that a function $f \in C^1(E;\mathbb{R})$ has at least three critical points, provided that $f$ satisfies the Palais-Smale condition and has two local minima, where $E$ is a Banach space. Critical point methods of the type appealed to in this paper originate in the fundamental extension by Leggett and Williams [19] of the multiple fixed-point technique pioneered by Krasnosell'ski and Stecenko [17].

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For basic notation and definitions, and for a through on the subject, we refer the reader to [7, 8, 11, 12, 15, 16, 18, 25, 24, 28].

The rest of this article is organized as follows. Section 2 contains our main tools, that are, some three critical points results and some basic definitions. Whereas, Section 3 is devoted to the existence of three solutions for the problem (1.1). To be precise, our main results (Theorems 3.1-3.3), Corollaries 3.4-3.6 and Corollaries 3.7-3.9 which present verifiable criteria for applying our main results, and the proof of the corollaries are there presented. Again in Section 3, the applicability of our results is illustrated by an example. Section 4 consists of the proofs of our main results.

We note that some of the ideas used here were motivated by corresponding ideas in [10].

## 2. Preliminaries and basic notation

First we here recall for the reader's convenience our main tools to prove the results; in the first one and the second one the coercivity of the functional $\Phi - \lambda\Psi$ is required, in the third one a suitable sign hypothesis is assumed. The first result has been obtained in [5], the second one in [3] and the third one in [2]. We recall the second and the third as given in [6].

**Theorem 2.1** ([5, Theorem 3.1]). *Let $X$ be a separable and reflexive real Banach space, $\Phi : X \to \mathbb{R}$ a nonnegative continuously Gâteaux differentiable and sequentially weakly lower semi-continuous functional whose Gâteaux derivative admits a continuous inverse on $X^*$, $\Psi : X \to \mathbb{R}$ a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Assume that there exists $x_0 \in X$ such that $\Phi(x_0) = \Psi(x_0) = 0$ and that*

$$\lim_{\|x\|\to+\infty}(\Phi(x) - \lambda\Psi(x)) = +\infty \quad \textit{for all } \lambda \in [0, +\infty[.$$

*Further, assume that there are $r > 0$, $x_1 \in X$ such that $r < \Phi(x_1)$ and*

$$\sup_{x\in\overline{\Phi^{-1}(]-\infty,r[)}^w} \Psi(x) < \frac{r}{r+\Phi(x_1)}\Psi(x_1);$$

*here $\overline{\Phi^{-1}(]-\infty,r[)}^w$ denotes the closure of $\Phi^{-1}(]-\infty,r[)$ in the weak topology (in particular note $\Psi(x_1) \geq 0$ since $x_0 \in \overline{\Phi^{-1}(]-\infty,r[)}^w$ (note $\Psi(x_0) = 0$) so $\sup_{x\in\overline{\Phi^{-1}(]-\infty,r[)}^w} \Psi(x) \geq 0$). Then, for each*

$$\lambda \in \Lambda_1 = \Big]\frac{\Phi(x_1)}{\Psi(x_1) - \sup_{x\in\overline{\Phi^{-1}(]-\infty,r[)}^w}\Psi(x)}, \frac{r}{\sup_{x\in\overline{\Phi^{-1}(]-\infty,r[)}^w}\Psi(x)}\Big[,$$

*the equation*

$$\Phi'(u) - \lambda\Psi'(u) = 0 \tag{2.1}$$

*has at least three solutions in $X$ and, moreover, for each $h > 1$, there exist an open interval*

$$\Lambda_2 \subseteq [0, \frac{hr}{r\frac{\Psi(x_1)}{\Phi(x_1)} - \sup_{x\in\overline{\Phi^{-1}(-\infty,r[)}^w}\Psi(x)}]$$

*and a positive real number $\sigma$ such that, for each $\lambda \in \Lambda_2$, the equation (2.1) has at least three solutions in $X$ whose norms are less than $\sigma$.*

**Theorem 2.2** ([6, Theorem 3.2]). *Let $X$ be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a coercive and continuously Gâteaux differentiable functional whose derivative admits a continuous inverse on $X^*$, $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose derivative is compact, such that*

$$\inf_X \Phi = \Phi(0) = \Psi(0) = 0.$$

*Assume that there is a positive constant $r$ and $\overline{v} \in X$, with $2r < \Phi(\overline{v})$, such that*

(a1) $\dfrac{\sup_{u\in\Phi^{-1}(]-\infty,r[)}\Psi(u)}{r} < \frac{2}{3}\frac{\Psi(\overline{v})}{\Phi(\overline{v})}$;

(a2) *for all* $\lambda \in \Big]\dfrac{2\Phi(\overline{v})}{3\Psi(\overline{v})}, \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r[)}\Psi(u)}\Big[$*, the functional $\Phi - \lambda\Psi$ is coercive.*

*Then, for each $\lambda \in \Big]\frac{3}{2}\frac{\Phi(\overline{v})}{\Psi(\overline{v})}, \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r[)}\Psi(u)}\Big[$ the functional $\Phi - \lambda\Psi$ has at least three distinct critical points.*

**Theorem 2.3** ([6, Theorem 3.3]). *Let $X$ be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a convex, coercive and continuously Gâteaux differentiable functional whose derivative admits a continuous inverse on $X^*$, $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose derivative is compact, such that*

(1) $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$;
(2) *for each $\lambda > 0$ and for every $u_1$, $u_2$ which are local minimum for the functional $\Phi - \lambda\Psi$ and such that $\Psi(u_1) \geq 0$ and $\Psi(u_2) \geq 0$, one has*

$$\inf_{s\in[0,1]} \Psi(su_1 + (1-s)u_2) \geq 0.$$

*Assume that there are two positive constants $r_1, r_2$ and $\overline{v} \in X$, with $2r_1 < \Phi(\overline{v}) < \frac{r_2}{2}$, such that*

(b1) $\frac{\sup_{u\in\Phi^{-1}(]-\infty,r_1[)}\Psi(u)}{r_1} < \frac{2\Psi(\overline{v})}{3\Phi(\overline{v})}$;
(b2) $\frac{\sup_{u\in\Phi^{-1}(]-\infty,r_2[)}\Psi(u)}{r_2} < \frac{1}{3}\frac{\Psi(\overline{v})}{\Phi(\overline{v})}$.

*Then, for each $\lambda \in \Big]\frac{3}{2}\frac{\Phi(\overline{v})}{\Psi(\overline{v})},\ \min\{\frac{r_1}{\sup_{u\in\Phi^{-1}(]-\infty,r_1[)}\Psi(u)},\ \frac{\frac{r_2}{2}}{\sup_{u\in\Phi^{-1}(]-\infty,r_2[)}\Psi(u)}\}\Big[$, the functional $\Phi - \lambda\Psi$ has at least three distinct critical points which lie in $\Phi^{-1}(]-\infty, r_2[)$.*

Let us introduce some notation which will be used later. Assume that

$$\min\big\{\frac{q^-}{\pi^2}, \frac{r^-}{\pi^4}, \frac{q^-}{\pi^2} + \frac{r^-}{\pi^4}\big\} > -p^-, \tag{2.2}$$

where $p^- := \operatorname{ess\,inf}_{x\in[0,1]} p(x) > 0$, $q^- := \operatorname{ess\,inf}_{x\in[0,1]} q(x)$, $r^- := \operatorname{ess\,inf}_{x\in[0,1]} r(x)$. Moreover, set

$$\sigma := \min\big\{\frac{q^-}{\pi^2}, \frac{r^-}{\pi^4}, \frac{q^-}{\pi^2} + \frac{r^-}{\pi^4}, 0\big\},$$
$$\delta := \sqrt{p^- + \sigma}.$$

Let $X := H^2([0,1]) \cap H_0^1([0,1])$ be the Sobolev space endowed with the usual norm. We recall the following Poincaré type inequalities (see, for instance, [21, Lemma 2.3]):

$$\|u'\|^2_{L^2([0,1])} \leq \frac{1}{\pi^2}\|u''\|^2_{L^2([0,1])}, \tag{2.3}$$

$$\|u\|^2_{L^2([0,1])} \le \frac{1}{\pi^4}\|u''\|^2_{L^2([0,1])} \tag{2.4}$$

for all $u \in X$. Therefore, taking into account (2.2)-(2.4), the norm

$$\|u\| = (\int_0^1 (p(x)|u''(x)|^2 + q(x)|u'(x)|^2 + r(x)|u(x)|^2)dx)^{1/2}$$

is equivalent to the usual norm, and, in particular,

$$\|u''\|_{L^2([0,1])} \le \frac{1}{\delta}\|u\|. \tag{2.5}$$

**Proposition 2.4.** *Let $u \in X$. Then*

$$\|u\|_\infty \le \frac{1}{2\pi\delta}\|u\|.$$

*Proof.* Taking (2.3) and (2.5) into account, the conclusion follows from the well-known inequality $\|u\|_\infty \le \frac{1}{2}\|u'\|_{L^2([0,1])}$. □

## 3. Main results

We say that $u$ is a weak solution of (1.1) if $u \in X$ and

$$\int_0^1 (p(x)u''(x)v''(x)+q(x)u'(x)v'(x)+r(x)u(x)v(x))dx-\lambda\int_0^1 f(x,u(x))v(x)dx = 0$$

for every $v \in X$. If $f$ is continuous in $[0,1]\times\mathbb{R}$, then the weak and the classical solutions of the problem (1.1) coincide. Put

$$F(x,t) = \int_0^t f(x,\xi)d\xi \tag{3.1}$$

for $(x,t) \in [0,1]\times\mathbb{R}$. We state our main results as follows:

**Theorem 3.1.** *Assume that there exists a function $w \in X$ and a positive constants $r$ such that*

- (A1) $\|w\|^2 > 2r$;
- (A2) *for $F(x,t)$ given in* (3.1),
$$\int_0^1 \sup_{t\in[-\frac{1}{\delta\pi}\sqrt{\frac{r}{2}},\frac{1}{\delta\pi}\sqrt{\frac{r}{2}}]} F(x,t)dx < r\frac{\int_0^1 F(x,w(x))dx}{r+\|w\|^2/2};$$
- (A3) $\frac{2}{\delta^2\pi^4}\limsup_{|t|\to+\infty}\frac{F(x,t)}{t^2} < \Theta_1$ *uniformly with respect to $x \in [0,1]$ where*
$$\Theta_1 := \max\Big\{\frac{\int_0^1 \sup_{t\in[-\frac{1}{\delta\pi}\sqrt{\frac{r}{2}},\frac{1}{\delta\pi}\sqrt{\frac{r}{2}}]} F(x,t)dx}{r}, \frac{\frac{2r}{\|w\|^2}\int_0^1 F(x,w(x))dx - \int_0^1 \sup_{t\in[-\frac{1}{\delta\pi}\sqrt{\frac{r}{2}},\frac{1}{\delta\pi}\sqrt{\frac{r}{2}}]} F(x,t)dx}{hr}\Big\}$$
*with $h > 1$.*

*Then, for each*

$$\lambda \in \Lambda_1 = \Big]\frac{\frac{1}{2}\|w\|^2}{\int_0^1 F(x,w(x))dx - \int_0^1 \sup_{t\in[-\frac{1}{\delta\pi}\sqrt{\frac{r}{2}},\frac{1}{\delta\pi}\sqrt{\frac{r}{2}}]} F(x,t)dx}, \frac{r}{\int_0^1 \sup_{t\in[-\frac{1}{\delta\pi}\sqrt{\frac{r}{2}},\frac{1}{\delta\pi}\sqrt{\frac{r}{2}}]} F(x,t)dx}\Big[,$$

*problem* (1.1) *admits at least three weak solutions in* $X$ *and, moreover, for each* $h > 1$, *there exist an open interval*

$$\Lambda_2 \subseteq \Big[0, \frac{hr}{2r\frac{\int_0^1 F(x,w(x))dx}{\|w\|^2} - \int_0^1 \sup_{t\in[-\frac{1}{\delta\pi}\sqrt{\frac{r}{2}},\frac{1}{\delta\pi}\sqrt{\frac{r}{2}}]} F(x,t)dx}\Big]$$

*and a positive real number* $\sigma$ *such that, for each* $\lambda \in \Lambda_2$, *the problem* (1.1) *admits at least three weak solutions in* $X$ *whose norms are less than* $\sigma$.

**Theorem 3.2.** *Assume that there exist a function* $w \in X$ *and a positive constant* $r$ *such that*

(B1) $\|w\|^2 > 4r$;

(B2) $\frac{\int_0^1 \sup_{t\in[-\frac{1}{\delta\pi}\sqrt{\frac{r}{2}},\frac{1}{\delta\pi}\sqrt{\frac{r}{2}}]} F(x,t)dx}{r} < \frac{4}{3}\frac{\int_0^1 F(x,w(x))dx}{\|w\|^2}$;

(B3) $\frac{2}{\delta^2\pi^4}\limsup_{|t|\to+\infty}\frac{F(x,t)}{t^2} < \frac{\int_0^1 \sup_{t\in[-\frac{1}{\delta\pi}\sqrt{\frac{r}{2}},\frac{1}{\delta\pi}\sqrt{\frac{r}{2}}]} F(x,t)dx}{r}$.

*Then, for each*

$$\lambda \in \Big]\frac{3}{4}\frac{\|w\|^2}{\int_0^1 F(x,w(x))dx}, \ \frac{r}{\int_0^1 \sup_{t\in[-\frac{1}{\delta\pi}\sqrt{\frac{r}{2}},\frac{1}{\delta\pi}\sqrt{\frac{r}{2}}]} F(x,t)dx}\Big[,$$

*problem* (1.1) *admits at least three weak solutions.*

**Theorem 3.3.** *Suppose that* $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ *satisfies the condition* $f(x,t) \geq 0$ *for all* $x \in [0,1]$ *and* $t \geq 0$. *Assume that there exist a function* $w \in X$ *and two positive constants* $r_1$ *and* $r_2$ *with* $4r_1 < \|w\|^2 < r_2$ *such that*

(C1) $\frac{\int_0^1 \sup_{t\in[-\frac{1}{\delta\pi}\sqrt{\frac{r_1}{2}},\frac{1}{\delta\pi}\sqrt{\frac{r_1}{2}}]} F(x,t)dx}{r_1} < \frac{4}{3}\frac{\int_0^1 F(x,w(x))dx}{\|w\|^2}$;

(C2) $\frac{\int_0^1 \sup_{t\in[-\frac{1}{\delta\pi}\sqrt{\frac{r_2}{2}},\frac{1}{\delta\pi}\sqrt{\frac{r_2}{2}}]} F(x,t)dx}{r_2} < \frac{2}{3}\frac{\int_0^1 F(x,w(x))dx}{\|w\|^2}$.

*Then, for each*

$$\lambda \in \Big]\frac{3}{4}\frac{\|w\|^2}{\int_0^1 F(x,w(x))dx}, \ \Theta_2\Big[,$$

*where*

$$\Theta_2 := \min\Big\{\frac{r_1}{\int_0^1 \sup_{t\in[-\frac{1}{\delta\pi}\sqrt{\frac{r_1}{2}},\frac{1}{\delta\pi}\sqrt{\frac{r_1}{2}}]} F(x,t)dx}, \frac{\frac{r_2}{2}}{\int_0^1 \sup_{t\in[-\frac{1}{\delta\pi}\sqrt{\frac{r_2}{2}},\frac{1}{\delta\pi}\sqrt{\frac{r_2}{2}}]} F(x,t)dx}\Big\},$$

*problem* (1.1) *admits at least three non-negative weak solutions* $v^1, v^2, v^3$ *such that*

$$|v^j(x)| < \frac{1}{\delta\pi}\sqrt{\frac{r_2}{2}} \quad \textit{for each } x \in [0,1], \ j = 1,2,3.$$

Put

$$k := \Big(\|p\|_\infty + \frac{1}{\pi^2}\|q\|_\infty + \frac{1}{\pi^4}\|r\|_\infty\Big)^{1/2}.$$

It is easy to see that $k > 0$ and $\delta < k$. Let us give a particular consequence of Theorems 3.1-3.3 for a fixed test function $w$.

**Corollary 3.4.** *Assume that there exist two positive constants* $c$ *and* $d$ *with* $c < \frac{32}{3\sqrt{3}\pi}d$ *such that*

(A4) $F(x,t) \geq 0$ *for a.e.* $x \in [0,3/8] \cup [5/8,1]$ *and all* $t \in [0,d]$*;*

(A5) $\int_0^1 \sup_{t\in[-c,c]} F(x,t)dx < (\delta\pi c)^2 \frac{\int_{3/8}^{5/8} F(x,d)dx}{(\delta\pi c)^2 + \frac{1024}{27}k^2d^2}$*;*

(A6) $\frac{2}{\delta^2\pi^4} \limsup_{|t|\to+\infty} \frac{F(x,t)}{t^2} < \Theta_3$ *uniformly with respect to* $x \in [0,1]$ *where* $\Theta_3$

$$:= \max\Big\{ \frac{\int_0^1 \sup_{t\in[-c,c]} F(x,t)dx}{2(\delta\pi c)^2},\ \frac{(\delta\pi c)^2 \frac{\int_{3/8}^{5/8} F(x,d)dx}{\frac{256}{27}k^2d^2} - \int_0^1 \sup_{t\in[-c,c]} F(x,t)dx}{2h(\delta\pi c)^2} \Big\}$$

*with* $h > 1$.

*Then, for each*

$$\lambda \in \Lambda_1' = \Big] \frac{\frac{2048}{27}k^2d^2}{\int_{3/8}^{5/8} F(x,d)dx - \int_0^1 \sup_{t\in[-c,c]} F(x,t)dx}, \frac{2(\delta\pi c)^2}{\int_0^1 \sup_{t\in[-c,c]} F(x,t)dx} \Big[,$$

*problem* (1.1) *admits at least three weak solutions in* $X$ *and, moreover, for each* $h > 1$*, there exist an open interval*

$$\Lambda_2' \subseteq \Big[0, \frac{2(\delta\pi c)^2 h}{(\delta\pi c)^2 \frac{\int_{3/8}^{5/8} F(x,d)dx}{\frac{1024}{27}k^2d^2} - \int_0^1 \sup_{t\in[-c,c]} F(x,t)dx} \Big]$$

*and a positive real number* $\sigma$ *such that, for each* $\lambda \in \Lambda_2'$*, problem* (1.1) *admits at least three weak solutions in* $X$ *whose norms are less than* $\sigma$*.*

*Proof.* We claim that all the assumptions of Theorem 3.1 are fulfilled with $w$ given by

$$w(x) = \begin{cases} -\frac{64d}{9}(x^2 - \frac{3}{4}x) & x \in [0, \frac{3}{8}[, \\ d & x \in [\frac{3}{8}, \frac{5}{8}], \\ -\frac{64d}{9}(x^2 - \frac{5}{4}x + \frac{1}{4}) & x \in ]\frac{5}{8}, 1] \end{cases} \tag{3.2}$$

and $r = 2(\delta\pi c)^2$. It is easy to verify that $w \in X$, and in particular,

$$\frac{4096}{27}\delta^2d^2 \leq \|w\|^2 \leq \frac{4096}{27}k^2d^2.$$

Hence, taking into account that $c < \frac{32}{3\sqrt{3}\pi}d$,

$$\|w\|^2 > 2r.$$

Since, $0 \leq w(x) \leq d$ for each $x \in [0,1]$, the condition (A4) ensures that

$$\int_0^{3/8} F(x,w(x))dx + \int_{5/8}^1 F(x,w(x))dx \geq 0,$$

so from (A5),

$$\begin{aligned} \int_0^1 \sup_{t\in[-c,c]} F(x,t)dx &< (\delta\pi c)^2 \frac{\int_{3/8}^{5/8} F(x,d)dx}{(\delta\pi c)^2 + \frac{1024}{27}k^2d^2} \\ &= 2(\delta\pi c)^2 \frac{\int_{3/8}^{5/8} F(x,d)dx}{2(\delta\pi c)^2 + \frac{2048}{27}k^2d^2} \\ &\leq 2(\delta\pi c)^2 \frac{\int_0^1 F(x,w(x))dx}{2(\delta\pi c)^2 + \frac{2048}{27}k^2d^2} \end{aligned}$$

$$\leq r\frac{\int_0^1 F(x,w(x))dx}{r+\frac{1}{2}\|w\|^2},$$

so (A2) holds (note $c^2 = r/(\delta\pi)^2$). Next notice that

$$\begin{aligned}
&\frac{\frac{1}{2}\|w\|^2}{\int_0^1 F(x,w(x))dx - \int_0^1 \sup_{t\in[-\frac{1}{\delta\pi}\sqrt{\frac{r}{2}},\frac{1}{\delta\pi}\sqrt{\frac{r}{2}}]} F(x,t)dx}\\
&\leq \frac{\frac{2048}{27}k^2d^2}{\int_{3/8}^{5/8} F(x,d)dx - \int_0^1 \sup_{t\in[-c,c]} F(x,t)dx}
\end{aligned}$$

and

$$\frac{r}{\int_0^1 \sup_{t\in[-\frac{1}{\delta\pi}\sqrt{\frac{r}{2}},\frac{1}{\delta\pi}\sqrt{\frac{r}{2}}]} F(x,t)dx} = \frac{2(\delta\pi c)^2}{\int_0^1 \sup_{t\in[-c,c]} F(x,t)dx}.$$

In addition note

$$\begin{aligned}
&\frac{\frac{2048}{27}k^2d^2}{\int_{3/8}^{5/8} F(x,d)dx - \int_0^1 \sup_{t\in[-c,c]} F(x,t)dx}\\
&< \frac{\frac{2048}{27}k^2d^2}{\big(\frac{2(\delta\pi c)^2+\frac{2048}{27}k^2d^2}{2(\delta\pi c)^2} - 1\big)\int_a^b \sup_{t\in[-c,c]} F(x,t)dx}\\
&= \frac{2(\delta\pi c)^2}{\int_0^1 \sup_{t\in[-c,c]} F(x,t)dx}.
\end{aligned}$$

Finally note that

$$\begin{aligned}
&\frac{hr}{2r\frac{\int_0^1 F(x,w(x))dx}{\|w\|^2} - \int_0^1 \sup_{t\in[-\frac{1}{\delta\pi}\sqrt{\frac{r}{2}},\frac{1}{\delta\pi}\sqrt{\frac{r}{2}}]} F(x,t)dx}\\
&\leq \frac{2(\delta\pi c)^2 h}{2(\delta\pi c)^2\frac{\int_{3/8}^{5/8} F(x,d)dx}{\frac{4096}{27}k^2d^2} - \int_0^1 \sup_{t\in[-c,c]} F(x,t)dx},
\end{aligned}$$

and taking into account that $\Lambda_1' \subseteq \Lambda_1$ and $\Lambda_2 \subseteq \Lambda_2'$ we have the desired conclusion directly from Theorem 3.1. □

**Corollary 3.5.** *Assume that there exist two positive constants $c$ and $d$ with $c < \frac{16\sqrt{2}}{3\sqrt{3}\pi}d$ such that the assumption* (A4) *in Corollary 3.4 holds. Furthermore, suppose that*

(B4) $\frac{\int_0^1 \sup_{t\in[-c,c]} F(x,t)dx}{2(\delta\pi c)^2} < \frac{4}{3}\frac{\int_{3/8}^{5/8} F(x,d)dx}{\frac{4096}{27}k^2d^2}$;

(B5) $\frac{2}{\delta^2\pi^4}\limsup_{|t|\to+\infty}\frac{F(x,t)}{t^2} < \frac{\int_0^1 \sup_{t\in[-c,c]} F(x,t)dx}{2(\delta\pi c)^2}$.

*Then, for each*

$$\lambda \in \Big]\frac{3}{4}\frac{\frac{4096}{27}k^2d^2}{\int_{3/8}^{5/8} F(x,d)dx},\ \frac{2(\delta\pi c)^2}{\int_0^1 \sup_{t\in[-c,c]} F(x,t)dx}\Big[,$$

*problem* (1.1) *admits at least three weak solutions.*

*Proof.* All the assumptions of Theorem 3.2 are fulfilled by choosing $w$ as given in (3.2) and $r = 2(\delta\pi c)^2$. Indeed, bearing in mind that

$$\frac{4096}{27}\delta^2 d^2 \leq \|w\|^2 \leq \frac{4096}{27}k^2 d^2$$

and recalling

$$\int_0^{3/8} F(x, w(x))dx + \int_{5/8}^1 F(x, w(x))dx \geq 0,$$

we clearly observe that our hypotheses guarantee that all assumptions of Theorem 3.2 are satisfied. Hence, by applying Theorem 3.2 we have the conclusion. □

**Corollary 3.6.** *Suppose that $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ satisfies the condition $f(x, t) \geq 0$ for all $x \in [0, 1]$ and $t \geq 0$. Assume that there exist three positive constants $c_1$, $c_2$ and $d$ with $c_1 < \frac{32}{3\sqrt{3}\pi} d$ and $\frac{64}{3\sqrt{3}\pi}\frac{kd}{\delta} < c_2$ such that*

(C3) $\frac{\int_0^1 \sup_{t\in[-c_1,c_1]} F(x,t)dx}{2(\delta\pi c_1)^2} < \frac{4}{3}\frac{\int_{3/8}^{5/8} F(x,d)dx}{\frac{4096}{27}k^2 d^2}$;

(C4) $\frac{\int_0^1 \sup_{t\in[-c_2,c_2]} F(x,t)dx}{2(\delta\pi c_2)^2} < \frac{2}{3}\frac{\int_{3/8}^{5/8} F(x,d)dx}{\frac{4096}{27}k^2 d^2}$.

*Then, for each*

$$\lambda \in \Big]\frac{3}{4}\frac{\frac{4096}{27}k^2 d^2}{\int_{3/8}^{5/8} F(x, d)dx},\ \Theta_4\Big[,$$

*where* $\Theta_4 := \min\Big\{\frac{2(\delta\pi c_1)^2}{\int_0^1 \sup_{t\in[-c_1,c_1]} F(x,t)dx},\ \frac{(\delta\pi c_2)^2}{\int_0^1 \sup_{t\in[-c_2,c_2]} F(x,t)dx}\Big\}$, *problem* (1.1) *admits at least three non-negative weak solutions $v^1$, $v^2$, $v^3$ such that*

$$|v^j(x)| < c_2 \quad \text{for each } x \in [0, 1],\ j = 1, 2, 3.$$

*Proof.* Following the same way as in the proof of Corollary 3.5, we achieve the stated assertion by applying Theorem 3.3 with $w$ as given in (3.2), $r_1 = 2(\delta\pi c_1)^2$ and $r_2 = 2(\delta\pi c_2)^2$. □

The following results give the existence of at least three classical solutions in $X$ to problem (1.1) in the autonomous case. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, and put $F(t) = \int_0^t f(\xi)d\xi$ for all $t \in \mathbb{R}$. We have the following results as direct consequences of Corollaries 3.4-3.6, respectively.

**Corollary 3.7.** *Assume that there exist two positive constants $c$ and $d$ with $c < \frac{32}{3\sqrt{3}\pi} d$ such that*

(A7) $f(t) \geq 0$ *for all* $t \in [0, d]$;

(A8) $\max_{t\in[-c,c]} F(t) < (\delta\pi c)^2 \frac{\frac{1}{4}F(d)}{(\delta\pi c)^2 + \frac{1024}{27}k^2 d^2}$;

(A9) $\frac{2}{\delta^2\pi^4} \limsup_{|t|\to+\infty} \frac{F(t)}{t^2} < \Theta_5$ *where*

$$\Theta_5 := \max\Big\{\frac{\max_{t\in[-c,c]} F(t)}{2(\delta\pi c)^2},\ \frac{(\delta\pi c)^2 \frac{\frac{1}{4}F(d)}{\frac{256}{27}k^2 d^2} - \max_{t\in[-c,c]} F(t)}{2h(\delta\pi c)^2}\Big\}$$

*with $h > 1$.*

*Then, for each*

$$\lambda \in \Lambda_1' = \Big]\frac{\frac{2048}{27}k^2 d^2}{\frac{1}{4}F(d) - \max_{t\in[-c,c]} F(t)}, \frac{2(\delta\pi c)^2}{\max_{t\in[-c,c]} F(t)}\Big[,$$

*the problem*

$$\begin{gathered}(p(x)u''(x))'' - (q(x)u'(x))' + r(x)u(x) = \lambda f(u) \quad x \in (0,1), \\ u(0) = u(1) = u''(0) = u''(1) = 0\end{gathered} \tag{3.3}$$

*admits at least three classical solutions in $X$ and, moreover, for each $h > 1$, there exist an open interval*

$$\Lambda_2'' \subseteq \Big[0, \frac{2(\delta\pi c)^2 h}{(\delta\pi c)^2 \frac{F(d)}{\frac{256}{27}k^2d^2} - \max_{t\in[-c,c]} F(t)}\Big]$$

*and a positive real number $\sigma$ such that, for each $\lambda \in \Lambda_2''$, problem* (3.3) *admits at least three classical solutions in $X$ whose norms are less than $\sigma$.*

**Corollary 3.8.** *Assume that there exist two positive constants $c$ and $d$ with $c < \frac{16\sqrt{2}}{3\sqrt{3}\pi}d$ such that the assumption* (A7) *in Corollary 3.7 holds. Furthermore, suppose that*

(B6) $\frac{\max t\in[-c,c]F(t)}{2(\delta\pi c)^2} < \frac{1}{3}\frac{F(d)}{\frac{4096}{27}k^2d^2}$;

(B7) $\frac{2}{\delta^2\pi^4}\limsup_{|t|\to+\infty}\frac{F(t)}{t^2} < \frac{\max_{t\in[-c,c]}F(t)}{2(\delta\pi c)^2}$.

*Then, for each*

$$\lambda \in \Big]3\frac{\frac{4096}{27}k^2d^2}{F(d)},\ \frac{2(\delta\pi c)^2}{\max_{t\in[-c,c]} F(t)}\Big[$$

*problem* (3.3) *admits at least three weak solutions.*

**Corollary 3.9.** *Suppose that $f : \mathbb{R} \to \mathbb{R}$ satisfies the condition $f(t) \geq 0$ for all $t \geq 0$. Assume that there exist three positive constants $c_1$, $c_2$ and $d$ with $c_1 < \frac{32}{3\sqrt{3}\pi}d$ and $\frac{64}{3\sqrt{3}\pi}\frac{kd}{\delta} < c_2$ such that*

(C5) $\frac{\max_{t\in[-c_1,c_1]}F(t)}{2(\delta\pi c_1)^2} < \frac{1}{3}\frac{F(d)}{\frac{4096}{27}k^2d^2}$;

(C6) $\frac{\max_{t\in[-c_2,c_2]}F(t)}{2(\delta\pi c_2)^2} < \frac{1}{6}\frac{F(d)}{\frac{4096}{27}k^2d^2}$.

*Then, for each*

$$\lambda \in \Big]3\frac{\frac{4096}{27}k^2d^2}{F(d)},\ \Theta_6\Big[,$$

*where $\Theta_6 := \min\big\{\frac{2(\delta\pi c_1)^2}{\max_{t\in[-c_1,c_1]}F(t)}, \frac{(\delta\pi c_2)^2}{\max_{t\in[-c_2,c_2]}F(t)}\big\}$, problem* (3.3) *admits at least three non-negative weak solutions $v^1$, $v^2$, $v^3$ such that*

$$|v^j(x)| < c_2 \quad \textit{for each } x \in [0,1],\ j = 1,2,3.$$

We conclude this section by presenting an example to illustrate our results applying by Corollary 3.7.

**Example 3.10.** Put $p(x) = 3e^x$, $q(x) = x - \pi^2$ and $r(x) = x^2 - \pi^4$ for every $x \in [0,1]$, and $f(t) = e^{-t}t^9(10-t)$ for each $t \in \mathbb{R}$. It is easy to verify that with $c = 1$ and $d = 5$, taking into account that $\delta = 1$ and $k = \sqrt{3e+2}$, since $\limsup_{|t|\to+\infty}\frac{F(t)}{t^2} = 0$, all assumptions of Corollary 3.7 are satisfied. Then, for each

$$\lambda \in \Lambda_1''' = \Big]\frac{\frac{51200}{27}(3e+2)}{\frac{1}{4}(e^{-5}5^{10}) - e},\ \frac{2\pi^2}{e}\Big[$$

problem (3.3) admits at least three classical solutions in $X$ and, moreover, for each $h > 1$, there exist an open interval

$$\lambda \in \Lambda_2''' \subseteq [0, \frac{2\pi^2 h}{\pi^2 \frac{e^{-5}5^{10}}{\frac{25600}{27}(3e+2)} - e}]$$

and a positive real number $\sigma$ such that, for each $\lambda \in \Lambda_2'''$, problem (3.3) admits at least three classical solutions in $X$ whose norms are less than $\sigma$.

**Remark 3.11.** The weak solutions of problem (1.1) where $f$ is a continuous function, by using standard methods, are classical solutions. Namely, the classical and the weak solutions of problem (1.1) coincide.

## 4. Proofs the main results

*Proof of Theorem 3.1.* With the purpose of applying Theorem 2.1, arguing as in [16], we begin by setting

$$\Phi(u) = \frac{1}{2}\|u\|^2, \quad \Psi(u) = \int_a^b F(x, u(x))dx$$

for $u \in X$. It is well known that $\Psi$ is a Gâteaux differentiable functional whose Gâteaux derivative at the point $u \in X$ is the functional $\Psi'(u) \in X^*$, given by

$$\Psi'(u)(v) = \int_a^b f(x, u(x))v(x)dx$$

for every $v \in X$, and that $\Psi' : X \to X^*$ is a continuous and compact operator. Moreover, $\Phi$ is a continuously Gâteaux differentiable and sequentially weakly lower semi continuous functional whose Gâteaux derivative at the point $u \in X$ is the functional $\Phi'(u) \in X^*$, given by

$$\Phi'(u)(v) = \int_0^1 (p(x)u''(x)v''(x) + q(x)u'(x)v'(x) + r(x)u(x)v(x))dx$$

for $v \in X$. Moreover, $\Phi'$ admits a continuous inverse on $X^*$. Furthermore from (A3) there exist two constants $\gamma, \tau \in \mathbb{R}$ with $\gamma < \Theta_1$ such that

$$\frac{2}{\delta^2\pi^4} F(x,t) \le \gamma t^2 + \tau \text{ for all } x \in (0,1) \text{ and all } t \in \mathbb{R}.$$

Fix $u \in X$. Then

$$F(x, u(x)) \le \frac{\delta^2\pi^4}{2}(\gamma|u(x)|^2 + \tau) \quad \text{for all } x \in (0,1). \tag{4.1}$$

Now, to prove the coercivity of the functional $\Phi - \lambda\Psi$, first we assume that $\gamma > 0$. So, for any fixed $\lambda \in ]0, \frac{1}{\Theta_1}]$, since $\|u\|_{L_2([0,1])} \le \frac{1}{\delta\pi^2}\|u\|$ (see [5, pg 1168]), using (4.1), we have

$$\begin{aligned}
\Phi(u) - \lambda\Psi(u) &= \frac{\|u\|^2}{2} - \lambda \int_0^1 F(x, u(x))dx \\
&\ge \frac{1}{2}\|u\|^2 - \frac{\delta^2\pi^4}{2\Theta_1}(\gamma \int_0^1 |u(x)|^2 dx + \tau) \\
&\ge \frac{1}{2}(1 - \frac{\gamma}{\Theta_1})\|u\|^2 - \frac{\delta^2\pi^4}{2\Theta_1}\tau,
\end{aligned}$$

and so

$$\lim_{||u||\to+\infty}(\Phi(u)-\lambda\Psi(u))=+\infty.$$

On the other hand, if $\gamma\le 0$, Clearly, we obtaain $\lim_{\|u\|\to+\infty}(\Phi(u)-\lambda\Psi(u))=+\infty$. Both cases lead to the coercivity of functional $\Phi-\lambda\Psi$. Also according to (A1) we achieve $\Phi(w)>r$. Note $\|u\|_\infty\le\frac{1}{2\pi\delta}\|u\|$ for each $u\in X$, from the definition of $\Phi$ we observe

$$\begin{aligned}
\Phi^{-1}(]-\infty,r[)&=\{u\in X;\Phi(u)<r\}\\
&\subseteq\{u\in X;\|u\|<\sqrt{2r}\}\\
&\subseteq\{u\in X;|u(x)|\le\frac{1}{2\pi\delta}\sqrt{2r},\quad\text{for all }x\in[0,1]\}\\
&=\{u\in X:|u(x)|<\frac{1}{\delta\pi}\sqrt{\frac{r}{2}}\quad\text{for all }x\in[0,1]\},
\end{aligned}$$

so, we have

$$\sup_{u\in\overline{\Phi^{-1}(]-\infty,r[)}^{w}}\Psi(u)\le\int_0^1\sup_{t\in[-\frac{1}{\delta\pi}\sqrt{\frac{r}{2}},\frac{1}{\delta\pi}\sqrt{\frac{r}{2}}]}F(x,t)dx.$$

Therefore, from (A2) we have

$$\begin{aligned}
\sup_{u\in\overline{\Phi^{-1}(]-\infty,r[)}^{w}}\Psi(u)&\le\int_0^1\sup_{t\in[-\frac{1}{\delta\pi}\sqrt{\frac{r}{2}},\frac{1}{\delta\pi}\sqrt{\frac{r}{2}}]}F(x,t)dx\\
&<\frac{r}{r+\frac12\|w\|^2}\int_0^1F(x,w(x))dx\\
&=\frac{r}{r+\Phi(w)}\Psi(w).
\end{aligned}$$

Now, we can apply Theorem 2.1. Note for each $x\in[0,1]$,

$$\begin{aligned}
&\frac{\Phi(w)}{\Psi(w)-\sup_{u\in\overline{\Phi^{-1}(]-\infty,r[)}^{w}}\Psi(u)}\\
&\le\frac{\frac12\|w\|^2}{\int_a^bF(x,w(x))dx-\int_a^b\sup_{t\in[-\frac{1}{\delta\pi}\sqrt{\frac{r}{2}},\frac{1}{\delta\pi}\sqrt{\frac{r}{2}}]}F(x,t)dx}
\end{aligned}$$

and

$$\frac{r}{\sup_{u\in\overline{\Phi^{-1}(]-\infty,r[)}^{w}}\Psi(u)}\ge\frac{r}{\int_0^1\sup_{t\in[-\frac{1}{\delta\pi}\sqrt{\frac{r}{2}},\frac{1}{\delta\pi}\sqrt{\frac{r}{2}}]}F(x,t)dx}.$$

Note also that (A2) implies

$$\begin{aligned}
&\frac{\frac12\|w\|^2}{\int_0^1F(x,w(x))dx-\int_0^1\sup_{t\in[-\frac{1}{\delta\pi}\sqrt{\frac{r}{2}},\frac{1}{\delta\pi}\sqrt{\frac{r}{2}}]}F(x,t)dx}\\
&<\frac{\frac12\|w\|^2}{(\frac{r+\frac12\|w\|^2}{r}-1)\int_0^1\sup_{t\in[-\frac{1}{\delta\pi}\sqrt{\frac{r}{2}},\frac{1}{\delta\pi}\sqrt{\frac{r}{2}}]}F(x,t)dx}\\
&=\frac{r}{\int_0^1\sup_{t\in[-\frac{1}{\delta\pi}\sqrt{\frac{r}{2}},\frac{1}{\delta\pi}\sqrt{\frac{r}{2}}]}F(x,t)dx}.
\end{aligned}$$

Also

$$\begin{aligned}
&\frac{hr}{r\frac{\Psi(w)}{\Phi(w)} - \sup_{u\in\overline{\Phi^{-1}(-\infty,r[)}^w}\Psi(u)}\\
&\le \frac{hr}{2r\frac{\int_0^1 F(x,w(x))dx}{\|w\|^2} - \int_0^1 \sup_{t\in[-\frac{1}{\delta\pi}\sqrt{\frac{r}{2}},\frac{1}{\delta\pi}\sqrt{\frac{r}{2}}]} F(x,t)dx} = \rho.
\end{aligned}$$

Note from (A2) that

$$\begin{aligned}
&2r\frac{\int_0^1 F(x,w(x))dx}{\|w\|^2} - \int_0^1 \sup_{t\in[-\frac{1}{\delta\pi}\sqrt{\frac{r}{2}},\frac{1}{\delta\pi}\sqrt{\frac{r}{2}}]} F(x,t)dx\\
&> \Big(\frac{2r}{\|w\|^2} - \frac{r}{r+\frac{1}{2}\|w\|^2}\Big)\int_0^1 F(x,w(x))dx\\
&\ge \Big(\frac{2r}{\|w\|^2} - \frac{2r}{\|w\|^2}\Big)\int_0^1 F(x,w(x))dx = 0,
\end{aligned}$$

since $\int_0^1 F(x,w(x))dx \ge 0$ (note $F(x,0)=0$ so $\int_0^1 \sup_{t\in[-\frac{1}{\delta\pi}\sqrt{\frac{r}{2}},\frac{1}{\delta\pi}\sqrt{\frac{r}{2}}]} F(x,t)dx \ge 0$ and now apply (A2)). Now with $x_0=0, x_1=w$ from Theorem 2.1 (note $\Psi(0)=0$ ) it follows that, for each $\lambda\in\Lambda_1$, problem (1.1) admits at least three weak solutions and there exist an open interval $\Lambda_2\subseteq[0,\rho]$ and a real positive number $\sigma$ such that, for each $\lambda\in\Lambda_2$, problem (1.1) admits at least three weak solutions whose norms in $X$ are less than $\sigma$. $\square$

*Proof of Theorem 3.2.* To apply Theorem 2.2 to our problem, we take the functionals $\Phi,\Psi : X\to\mathbb{R}$ as given in the proof of Theorem 3.1. Let us prove that the functionals $\Phi$ and $\Psi$ satisfy the conditions required in Theorem 2.2. The regularity assumptions on $\Phi$ and $\Psi$, as requested in Theorem 2.2 hold. According to (B1) we deduce $\Phi(w) > 2r$. Note that $\|u\|_\infty \le \frac{1}{2\pi\delta}\|u\|$ for each $u\in X$, we observe

$$\Phi^{-1}(]-\infty,r[) = \{u\in X : |u(x)| < \frac{1}{\delta\pi}\sqrt{\frac{r}{2}} \text{ for all } x\in[0,1]\},$$

and it follows that

$$\begin{aligned}
\sup_{(u)\in\Phi^{-1}(]-\infty,r[)}\Psi(u) &= \sup_{(u_1,\dots,u_n)\in\Phi^{-1}(]-\infty,r[)}\int_0^1 F(x,u)dx\\
&\le \int_0^1 \sup_{t\in[-\frac{1}{\delta\pi}\sqrt{\frac{r}{2}},\frac{1}{\delta\pi}\sqrt{\frac{r}{2}}]} F(x,t)dx.
\end{aligned}$$

Therefore, due to assumption (B2), we have

$$\begin{aligned}
\frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)}\Psi(u)}{r} &= \frac{\sup_{(u_1,\dots,u_n)\in\Phi^{-1}(]-\infty,r[)}\int_0^1 F(x,u(x))dx}{r}\\
&\le \frac{\int_0^1 \sup_{t\in[-\frac{1}{\delta\pi}\sqrt{\frac{r}{2}},\frac{1}{\delta\pi}\sqrt{\frac{r}{2}}]} F(x,t)dx}{r}\\
&< \frac{4}{3}\frac{\int_a^b F(x,w(x))dx}{\|w\|^2} = \frac{2\Psi(w)}{3\Phi(w)}.
\end{aligned}$$

Furthermore, from (B3) there exist two constants $\eta, \vartheta \in R$ with
$$\eta < \frac{\int_0^1 \sup_{t\in[-\frac{1}{\delta\pi}\sqrt{\frac{r}{2}},\frac{1}{\delta\pi}\sqrt{\frac{r}{2}}]} F(x,t)dx}{r}$$
such that
$$\frac{2}{\delta^2\pi^4} F(x,t) \le \eta t^2 + \vartheta \text{ for all } x \in [0,1] \text{ and for all } t \in \mathbb{R}.$$
Fix $u \in X$. Then
$$F(x,u(x)) \le \frac{\delta^2\pi^4}{2}(\eta|u(x)|^2 + \vartheta) \quad \text{for all } x \in [0,1]. \tag{4.2}$$

Now, to prove the coercivity of the functional $\Phi - \lambda\Psi$, first we assume that $\eta > 0$. So, for any fixed
$$\lambda \in \Big]\frac{3}{4}\frac{\|w\|^2}{\int_a^b F(x,w(x))dx},\ \frac{r}{\int_0^1 \sup_{t\in[-\frac{1}{\delta\pi}\sqrt{\frac{r}{2}},\frac{1}{\delta\pi}\sqrt{\frac{r}{2}}]} F(x,t)dx}\Big[,$$
since $\|u\|_{L_2([0,1])} \le \frac{1}{\delta\pi^2}\|u\|$ (see [5, pg 1168]), using (4.2), we have
$$\begin{aligned}
\Phi(u) - \lambda\Psi(u) &= \frac{\|u\|^2}{2} - \lambda\int_0^1 F(x,u(x))dx \\
&\ge \frac{1}{2}\|u\|^2 - \frac{\lambda\delta^2\pi^4}{2}(\gamma\int_0^1 |u(x)|^2dx + \tau) \\
&\ge \frac{1}{2}\Big(1 - \gamma\frac{r}{\int_0^1 \sup_{t\in[-\frac{1}{\delta\pi}\sqrt{\frac{r}{2}},\frac{1}{\delta\pi}\sqrt{\frac{r}{2}}]} F(x,t)dx}\Big)\|u\|^2 - \frac{\lambda\delta^2\pi^4}{2}\tau,
\end{aligned}$$
and thus
$$\lim_{\|u\|\to+\infty}(\Phi(u) - \lambda\Psi(u)) = +\infty.$$
On the other hand, if $\eta \le 0$, Clearly, we obtain $\lim_{\|u\|\to+\infty}(\Phi(u) - \lambda\Psi(u)) = +\infty$. Both cases lead to the coercivity of functional $\Phi - \lambda\Psi$.

So, the assumptions (a1) and (a2) in Theorem 2.2 are satisfied. Hence, by using Theorem 2.2, taking into account that the weak solutions of the problem (1.1) are exactly the solutions of the equation $\Phi'(u) - \lambda\Psi'(u) = 0$, then problem (1.1) admits at least three distinct weak solutions in $X$. □

*Proof of Theorem 3.3.* Let $\Phi$ and $\Psi$ be as in the proof of Theorem 3.1. Let us employ Theorem 2.3 to our functionals. Obviously, $\Phi$ and $\Psi$ satisfy the condition 1 of Theorem 2.3.

Now, we show that the functional $\Phi - \lambda\Psi$ satisfies the assumption 2 of Theorem 2.3. Let $u^*$ and $u^{**}$ be two local minima for $\Phi - \lambda\Psi$. Then $u^*$ and $u^{**}$ are critical points for $\Phi - \lambda\Psi$, and so, they are weak solutions for problem (1.1). Since $f(x,t) \ge 0$ for all $(x,t) \in [0,1]\times(\mathbb{R}^+\cup\{0\})$, from the Weak Maximum Principle (see for instance [13] ) we deduce $u^\star(x) \ge 0$ and $u^{\star\star}(x) \ge 0$ for every $x \in [0,1]$. So, it follows that $su^\star + (1-s)u^{\star\star} \ge 0$ for all $s \in [0,1]$, and that $f(su^\star + (1-s)u^{\star\star}, t) \ge 0$, and consequently, $\Psi(su^\star + (1-s)u^{\star\star}) \ge 0$ for all $s \in [0,1]$. Moreover, from the condition $4r_1 < \|w\|^2 < r_2$, we observe $2r_1 < \Phi(w) < \frac{r_2}{2}$. Note that $\|u\|_\infty \le \frac{1}{2\pi\delta}\|u\|$ for each $u \in X$, we observe
$$\Phi^{-1}(]-\infty, r_1[) = \big\{u \in X : |u(x)| < \frac{1}{\delta\pi}\sqrt{\frac{r_1}{2}} \text{ for all } x \in [0,1]\big\},$$

and it follows that

$$\begin{aligned}\sup_{(u_1,\dots,u_n)\in\Phi^{-1}(]-\infty,r_1[)}\Psi(u) &= \sup_{(u\in\Phi^{-1}(]-\infty,r_1[)}\int_0^1 F(x,u(x))dx\\ &\le \int_0^1 \sup_{t\in[-\frac{1}{\delta\pi}\sqrt{\frac{r}{2}},\frac{1}{\delta\pi}\sqrt{\frac{r}{2}}]} F(x,t)dx.\end{aligned}$$

Therefore, due to assumption (C1), we infer that

$$\begin{aligned}\frac{\sup_{u\in\Phi^{-1}(]-\infty,r_1[)}\Psi(u)}{r_1} &= \frac{\sup_{u\in\Phi^{-1}(]-\infty,r_1[)}\int_0^1 F(x,u(x))dx}{r_1}\\ &\le \frac{\int_0^1 \sup_{t\in[-\frac{1}{\delta\pi}\sqrt{\frac{r_1}{2}},\frac{1}{\delta\pi}\sqrt{\frac{r_1}{2}}]} F(x,t)dx}{r_1}\\ &< \frac{4}{3}\frac{\int_0^1 F(x,w(x))dx}{\|w\|^2} = \frac{2}{3}\frac{\Psi(w)}{\Phi(w)}.\end{aligned}$$

As above, bearing assumption (C2) in mind, we deduce that

$$\begin{aligned}\frac{\sup_{u\in\Phi^{-1}(]-\infty,r_2[)}\Psi(u)}{r_2} &= \frac{\sup_{(u_1,\dots,u_n)\in\Phi^{-1}(]-\infty,r_2[)}\int_0^1 F(x,u(x))dx}{r_2}\\ &\le \frac{\int_0^1 \sup_{t\in[-\frac{1}{\delta\pi}\sqrt{\frac{r_2}{2}},\frac{1}{\delta\pi}\sqrt{\frac{r_2}{2}}]} F(x,t)dx}{r_2}\\ &< \frac{2}{3}\frac{\int_0^1 F(x,w(x))dx}{\|w\|^2} = \frac{2}{3}\frac{\Psi(w)}{\Phi(w)}.\end{aligned}$$

So, the assumptions (b1) and (b2) in Theorem 2.3 are satisfied. Hence, by using Theorem 2.3, taking into account that the weak solutions of problem (1.1) are exactly the solutions of the equation $\Phi'(u)-\lambda\Psi'(u)=0$, problem (1.1) admits at least three distinct weak solutions in $X$. This completes the proof. □

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