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EXISTENCE OF SOLUTIONS FOR A FRACTIONAL NEUTRAL INTEGRO-DIFFERENTIAL EQUATION WITH UNBOUNDED DELAY

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Abstract. In this article, we study the existence of mild solutions for fractional neutral integro-differential equations with infinite delay.

1. Introduction

In this article, we study the existence of mild solutions for the neutral fractional integral evolutionary equation

$$D_t^\alpha(x(t)+f(t,x_t)) = Ax(t) + \int_0^t B(t-s)x(s)ds + g(t,x_t), \quad t>0, \tag{1.1}$$

$$x_0=\varphi, \quad x'(0)=x_1, \tag{1.2}$$

where $\alpha \in (1,2)$; $A, (B(t))_{t\geq 0}$ are closed linear operators defined on a common domain which is dense in a Banach space X, $D_t^\alpha h(t)$ represent the Caputo derivative of $\alpha > 0$ defined by

$$D_t^\alpha h(t) := \int_0^t g_{n-\alpha}(t-s)\frac{d^n}{ds^n}h(s)ds,$$

where n is the smallest integer greater than or equal to α and $g_\beta(t) := \frac{t^{\beta-1}}{\Gamma(\beta)}, t > 0, \beta \geq 0$. The history $x_t : (-\infty, 0] \to X$ given by $x_t(\theta) = x(t+\theta)$ belongs to some abstract phase space $\mathcal{B}$ defined axiomatically and $f, g : I \times \mathcal{B} \to X$ are appropriate functions.

The literature related to ordinary neutral functional differential equations is very extensive and we refer the reader to the Hale and Lunel book [8] and the references therein. Partial neutral differential equations arise, for instance, in the transmission line theory, see Wu and Xia [18] and the study of material with fanding memory, see [7, 16]. In the paper [9], Hernandez and Henriquez, study the existence of mild and strong solutions for the partial neutral system

$$\frac{d}{dt}\big(x(t)+g(t,x_t)\big) = Ax(t) + f(t,x_t), \quad t \in I = [0,a], \tag{1.3}$$

$$x_0 = \varphi, \tag{1.4}$$

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where $A : D(A) \subset X \to X$ is a generator of analytic semigroup and the history $x_t : (-\infty, 0] \to X$ given by $x_t(\theta) = x(t+\theta)$ belongs to some abstract phase space $\mathcal{B}$ defined axiomatically and $f, g : I \times \mathcal{B} \to X$ are appropriate functions. Very recently, Hernandez et al, [11], study the existence of mild, strong and classical solutions for the integro-differential neutral systems

$$\frac{d}{dt}(x(t) + f(t, x_t)) = Ax(t) + \int_0^t B(t-s)x(s)ds + g(t, x_t), \quad t \in I = [0, b], \tag{1.5}$$

$$x_0 = \varphi, \quad \varphi \in \mathcal{B}, \tag{1.6}$$

where $A : D(A) \subset X \to X$ and $B(t) : D(B(t)) \subset X \to X$, $t \geq 0$, are closed linear operators; $(X, \|\cdot\|)$ is a Banach space; the history $x_t : (-\infty, 0] \to X$, defined by $x_t(\theta) := x(t+\theta)$ belongs to an abstract phase space $\mathcal{B}$ defined axiomatically and $f, g : I \times \mathcal{B} \to X$ are appropriated functions. In the paper [1], Dos Santos et al. study the existence of mild and classical solutions for the partial neutral systems with unbounded delay

$$\frac{d}{dt}[x(t) + \int_{-\infty}^t N(t-s)x(s)ds] = Ax(t) + \int_{-\infty}^t B(t-s)x(s)ds + f(t, x_t), \ t \in [0, a], \tag{1.7}$$

$$x_0 = \varphi, \quad \varphi \in \mathcal{B}, \tag{1.8}$$

where $A, B(t)$ for $t \geq 0$ are closed linear operators defined on a common domain $D(A)$ which is dense in X, $N(t)$ $(t \geq 0)$ are bounded linear operators on X, without to use many of the strong restrictions considered in the literature. To the best of our knowledge, the existence of mild solutions for abstract fractional partial evolutionary integral equations with unbounded delay is an untreated topic in the literature and this fact is the main motivation of the present work.

2. Preliminaries

In what follows we recall some definitions, notation and results that we need in the sequel. Throughout this paper, $(X, \|\cdot\|)$ is a Banach space and $A, B(t)$, $t \geq 0$, are closed linear operators defined on a common domain $\mathcal{D} = D(A)$ which is dense in X. The notation $[D(A)]$ represents the domain of A endowed with the graph norm. Let $(Z, \|\cdot\|_Z)$ and $(W, \|\cdot\|_W)$ be Banach spaces. In this paper, the notation $\mathcal{L}(Z, W)$ stands for the Banach space of bounded linear operators from Z into W endowed with the uniform operator topology and we abbreviate this notation to $\mathcal{L}(Z)$ when $Z = W$. Furthermore, for appropriate functions $K : [0, \infty) \to Z$ the symbol $\widehat{K}$ denotes the Laplace transform of K. Thesymbol $B_r(x, Z)$ stands for the closed ball with center at x and radius $r > 0$ in Z. On the other hand, for a bounded function $\gamma : [0, a] \to Z$ and $t \in [0, a]$, the symbol $\|\gamma\|_{Z,t}$ is given by

$$\|\gamma\|_{Z,t} = \sup\{\|\gamma(s)\|_Z : s \in [0, t]\}, \tag{2.1}$$

and we simplify this notation to $\|\gamma\|_t$ when no confusion about the space Z arises.

To obtain our results, we assume that the abstract fractional integro-differential problem

$$D_t^\alpha x(t) = Ax(t) + \int_0^t B(t-s)x(s)ds, \tag{2.2}$$

$$x(0) = z \in X, \quad x'(0) = 0, \tag{2.3}$$

has an associated α-resolvent operator of bounded linear operators $(\mathcal{R}_\alpha(t))_{t\geq 0}$ on X.

Definition 2.1. A one parameter family of bounded linear operators $(\mathcal{R}_\alpha(t))_{t\geq 0}$ on X is called a α-resolvent operator of (2.2)-(2.3) if the following conditions are satisfied.

(a) The function $\mathcal{R}_\alpha(\cdot):[0,\infty)\to\mathcal{L}(X)$ is strongly continuous and $\mathcal{R}_\alpha(0)x=x$ for all $x\in X$ and $\alpha\in(1,2)$.
(b) For $x\in D(A)$, $\mathcal{R}_\alpha(\cdot)x\in C([0,\infty),[D(A)])\bigcap C^1((0,\infty),X)$, and

$$D_t^\alpha\mathcal{R}_\alpha(t)x=A\mathcal{R}_\alpha(t)x+\int_0^t B(t-s)\mathcal{R}_\alpha(s)xds, \tag{2.4}$$

$$D_t^\alpha\mathcal{R}_\alpha(t)x=\mathcal{R}_\alpha(t)Ax+\int_0^t \mathcal{R}_\alpha(t-s)B(s)xds, \tag{2.5}$$

for every $t\geq 0$.

The existence of a α-resolvent operator for problem (2.2)-(2.3) was studied in [4]. In this work we consider the following conditions.

(P1) The operator $A:D(A)\subseteq X\to X$ is a closed linear operator with $[D(A)]$ dense in X. Let $\alpha\in(1,2)$, for some $\phi_0\in(0,\frac{\pi}{2}]$ for each $\phi<\phi_0$ there is positive constants $C_0=C_0(\phi)$ such that $\lambda\in\rho(A)$ for each

$$\lambda\in\Sigma_{0,\alpha\vartheta}=\{\lambda\in\mathbb{C}:\lambda\neq 0,\ |\arg(\lambda)|<\alpha\vartheta\},$$

where $\vartheta=\phi+\frac{\pi}{2}$ and $\|R(\lambda,A)\|\leq\frac{C_0}{|\lambda|}$ for all $\lambda\in\Sigma_{0,\alpha\vartheta}$.

(P2) For all $t\geq 0$, $B(t):D(B(t))\subseteq X\to X$ is a closed linear operator, $D(A)\subseteq D(B(t))$ and $B(\cdot)x$ is strongly measurable on $(0,\infty)$ for each $x\in D(A)$. There exists $b(\cdot)\in L^1_{\rm loc}(\mathbb{R}^+)$ such that $\widehat{b}(\lambda)$ exists for $Re(\lambda)>0$ and $\|B(t)x\|\leq b(t)\|x\|_1$ for all $t>0$ and $x\in D(A)$. Moreover, the operator valued function $\widehat{B}:\Sigma_{0,\pi/2}\to\mathcal{L}([D(A)],X)$ has an analytical extension (still denoted by $\widehat{B}$) to $\Sigma_{0,\vartheta}$ such that $\|\widehat{B}(\lambda)x\|\leq\|\widehat{B}(\lambda)\|\,\|x\|_1$ for all $x\in D(A)$, and $\|\widehat{B}(\lambda)\|=O(\frac{1}{|\lambda|})$, as $|\lambda|\to\infty$.

(P3) There exists a subspace $D\subseteq D(A)$ dense in $[D(A)]$ and positive constants C_i, $i=1,2$, such that $A(D)\subseteq D(A)$, $\widehat{B}(\lambda)(D)\subseteq D(A)$, $\|A\widehat{B}(\lambda)x\|\leq C_1\|x\|$ for every $x\in D$ and all $\lambda\in\Sigma_{0,\vartheta}$.

In the sequel, for $r>0$ and $\theta\in(\frac{\pi}{2},\vartheta)$,

$$\Sigma_{r,\theta}=\{\lambda\in\mathbb{C}:\lambda\neq 0,|\lambda|>r,\ |arg(\lambda)|<\theta\},$$

for $\Gamma_{r,\theta},\Gamma^i_{r,\theta}$, $i=1,2,3$, are the paths

$$\begin{gathered}\Gamma^1_{r,\theta}=\{te^{i\theta}:t\geq r\},\\ \Gamma^2_{r,\theta}=\{re^{i\xi}:-\theta\leq\xi\leq\theta\},\\ \Gamma^3_{r,\theta}=\{te^{-i\theta}:t\geq r\},\end{gathered}$$

and $\Gamma_{r,\theta}=\bigcup_{i=1}^3\Gamma^i_{r,\theta}$ oriented counterclockwise. In addition, $\rho_\alpha(G_\alpha)$ are the sets

$$\rho_\alpha(G_\alpha)=\{\lambda\in\mathbb{C}:G_\alpha(\lambda):=\lambda^{\alpha-1}(\lambda^\alpha I-A-\widehat{B}(\lambda))^{-1}\in\mathcal{L}(X)\}.$$

We now define the operator family $(\mathcal{R}_\alpha(t))_{t\geq 0}$ by

$$\mathcal{R}_\alpha(t) = \begin{cases} \frac{1}{2\pi i}\int_{\Gamma_{r,\theta}} e^{\lambda t} G_\alpha(\lambda) d\lambda, & t>0, \\ I, & t=0. \end{cases} \tag{2.6}$$

Lemma 2.2 ([4, Lemma 2.2]). *There exists $r_1>0$ such that $\Sigma_{r_1,\vartheta} \subseteq \rho_\alpha(G_\alpha)$ and the function $G_\alpha : \Sigma_{r_1,\vartheta} \to \mathcal{L}(X)$ is analytic. Moreover,*

$$G_\alpha(\lambda) = \lambda^{\alpha-1} R(\lambda^\alpha, A)[I - \widehat{B}(\lambda)R(\lambda^\alpha, A)]^{-1}, \tag{2.7}$$

and there exist constants M_i for $i=1,2$ such that

$$\|G_\alpha(\lambda)\| \leq \frac{M_1}{|\lambda|}, \tag{2.8}$$

$$\|AG_\alpha(\lambda)x\| \leq \frac{M_2}{|\lambda|}\|x\|_1, \ x \in D(A), \tag{2.9}$$

$$\|AG_\alpha(\lambda)\| \leq \frac{M_4}{|\lambda|^{1-\alpha}}, \tag{2.10}$$

for every $\lambda \in \Sigma_{r_1,\vartheta}$.

The following result was established in [4, Theorem 2.1].

Theorem 2.3. *Assume that conditions* (P1)–(P3) *are fulfilled. Then there exists a unique α-resolvent operator for problem* (2.2)-(2.3).

Theorem 2.4 ([4, Lemma 2.5]). *The function $\mathcal{R}_\alpha : [0,\infty) \to \mathcal{L}(X)$ is strongly continuous and $\mathcal{R}_\alpha : (0,\infty) \to \mathcal{L}(X)$ is uniformly continuous.*

In what follows, we assume that the conditions (P1)–(P3) are satisfied. We consider now the non-homogeneous problem

$$D_t^\alpha x(t) = Ax(t) + \int_0^t B(t-s)x(s)ds + f(t), \quad t \in [0,a], \tag{2.11}$$

$$x(0) = x_0, \quad x'(0) = 0, \tag{2.12}$$

where $\alpha \in (1,2)$ and $f \in L^1([0,a],X)$. In the sequel, $\mathcal{R}_\alpha(\cdot)$ is the operator function defined by (2.6). We begin by introducing the following concept of classical solution.

Definition 2.5. A function $x : [0,a] \to X$, $0 < a$, is called a classical solution of (2.11)-(2.12) on $[0,a]$ if $x \in C([0,a],[D(A)]) \cap C([0,a],X), g_{n-\alpha} * x \in C^1([0,a],X), n = 1,2$, the condition (2.12) holds and the equations (2.11) is verified on $[0,a]$.

Definition 2.6. Let $\alpha \in (1,2)$, we define the family $(\mathcal{S}_\alpha(t))_{t\geq 0}$ by

$$\mathcal{S}_\alpha(t)x := \int_0^t g_{\alpha-1}(t-s)\mathcal{R}_\alpha(s)x ds,$$

for each $t \geq 0$.

Lemma 2.7 ([4, Lemma 3.9]). *If the function $\mathcal{R}_\alpha(\cdot)$ is exponentially bounded in $\mathcal{L}(X)$, then $\mathcal{S}_\alpha(\cdot)$ is exponentially bounded in $\mathcal{L}(X)$.*

Lemma 2.8 ([4, Lemma 3.10]). *If the function $\mathcal{R}_\alpha(\cdot)$ is exponentially bounded in $\mathcal{L}([D(A)])$, then $\mathcal{S}_\alpha(\cdot)$ is exponentially bounded in $\mathcal{L}([D(A)])$.*

We now establish a variation of constants formula for the solutions of (2.11)-(2.12).

Theorem 2.9 ([4, Theorem 3.2]). *Let $z \in D(A)$. Assume that $f \in C([0,a],X)$ and $u(\cdot)$ is a classical solution of* (2.11)–(2.12) *on* $[0,a]$. *Then*

$$u(t) = \mathcal{R}_\alpha(t)z + \int_0^t \mathcal{S}_\alpha(t-s)f(s)\,ds, \quad t \in [0,a]. \tag{2.13}$$

It is clear from the preceding definition that $\mathcal{R}_\alpha(\cdot)z$ is a solution of problem (2.2)-(2.3) on $(0,\infty)$ for $z \in D(A)$.

Definition 2.10. Let $f \in L^1([0,a],X)$. A function $u \in C([0,a],X)$ is called a mild solution of (2.11)-(2.12) if

$$u(t) = \mathcal{R}_\alpha(t)z + \int_0^t \mathcal{S}_\alpha(t-s)f(s)\,ds, \quad t \in [0,a].$$

The next results are proved in [4] and [5].

Theorem 2.11 ([4, Theorem 3.3]). *Let $z \in D(A)$ and $f \in C([0,a],X)$. If $f \in L^1([0,a],[D(A)])$ then the mild solution of* (2.11)-(2.12) *is a classical solution.*

Theorem 2.12 ([4, Theorem 3.4]). *Let $z \in D(A)$ and $f \in C([0,a],X)$. If $f \in W^{1,1}([0,a],X)$, then the mild solution of* (2.11)–(2.12) *is a classical solution.*

Lemma 2.13 ([5, Lemma 2.3]). *If $R(\lambda_0^\alpha, A)$ is compact for some $\lambda_0^\alpha \in \rho(A)$, then $\mathcal{R}_\alpha(t)$ and $\mathcal{S}_\alpha(t)$ are compact for all $t > 0$.*

We will herein define the phase space $\mathcal{B}$ axiomatically, using ideas and notations developed in [14]. More precisely, $\mathcal{B}$ will denote the vector space of functions defined from $(-\infty,0]$ into X endowed with a seminorm denoted $\|\cdot\|_{\mathcal{B}}$ and such that the following axioms hold:

- (A) If $x : (-\infty,\sigma+b) \to X$, $b > 0, \sigma \in \mathbb{R}$, is continuous on $[\sigma,\sigma+b)$ and $x_\sigma \in \mathcal{B}$, then for every $t \in [\sigma,\sigma+b)$ the following conditions hold:
  - (i) x_t is in $\mathcal{B}$.
  - (ii) $\|x(t)\| \le H\|x_t\|_{\mathcal{B}}$.
  - (iii) $\|x_t\|_{\mathcal{B}} \le K(t-\sigma)\sup\{\|x(s)\| : \sigma \le s \le t\} + M(t-\sigma)\|x_\sigma\|_{\mathcal{B}}$,

  where $H > 0$ is a constant; $K, M : [0,\infty) \to [1,\infty)$, $K(\cdot)$ is continuous, $M(\cdot)$ is locally bounded and H, K, M are independent of $x(\cdot)$.
- (A1) For the function $x(\cdot)$ in (**A**), the function $t \to x_t$ is continuous from $[\sigma,\sigma+b)$ into $\mathcal{B}$.
- (B) The space $\mathcal{B}$ is complete.

Example 2.14. (The phase space $C_r \times L^p(g,X)$) Let $r \ge 0$, $1 \le p < \infty$ and let $g : (-\infty,-r] \to \mathbb{R}$ be a nonnegative measurable function which satisfies the conditions (g-5), (g-6) in the terminology of [14]. Briefly, this means that g is locally integrable and there exists a non-negative, locally bounded function γ on $(-\infty,0]$ such that $g(\xi+\theta) \le \gamma(\xi)g(\theta)$, for all $\xi \le 0$ and $\theta \in (-\infty,-r) \setminus N_\xi$, where $N_\xi \subseteq (-\infty,-r)$ is a set with Lebesgue measure zero. The space $C_r \times L^p(g,X)$ consists of all classes of functions $\varphi : (-\infty,0] \to X$ such that φ is continuous on $[-r,0]$, Lebesgue-measurable, and $g\|\varphi\|^p$ is Lebesgue integrable on $(-\infty,-r)$. The seminorm in $C_r \times L^p(g,X)$ is defined by

$$\|\varphi\|_{\mathcal{B}} := \sup\{\|\varphi(\theta)\| : -r \le \theta \le 0\} + \Big(\int_{-\infty}^{-r} g(\theta)\|\varphi(\theta)\|^p d\theta\Big)^{1/p}.$$

The space $\mathcal{B} = C_r \times L^p(g;X)$ satisfies axioms (A), (A-1), (B). Moreover, when $r = 0$ and $p = 2$, we can take $H = 1$, $M(t) = \gamma(-t)^{1/2}$ and $K(t) = 1 + \left(\int_{-t}^{0} g(\theta)\,d\theta\right)^{1/2}$, for $t \geq 0$. (see [14, Theorem 1.3.8] for details).

For additional details concerning phase space we refer the reader to [14].

3. Neutral problem

In the next result we denote by $(-A)^{\vartheta}$ the fractional power of the operator $-A$, (see [17] for details).

Lemma 3.1. *Suppose that the conditions* (P1)–(P3) *are satisfied. Let* $\alpha \in (1,2)$ *and* $\vartheta \in (0,1)$ *such that* $\alpha\vartheta \in (0,1)$*, then there exists positive number* C *such that*

$$\|(-A)^{\vartheta}\mathcal{R}_\alpha(t)\| \leq Ce^{rt}t^{-\alpha\vartheta}, \tag{3.1}$$

$$\|(-A)^{\vartheta}\mathcal{S}_\alpha(t)\| \leq Ce^{rt}t^{\alpha(1-\vartheta)-1}, \tag{3.2}$$

for all $t > 0$.

Proof. Let $\vartheta \in (0,1)$. From [17, Theorem 6.10], there exist $C_\vartheta > 0$ such that

$$\|(-A)^{\vartheta}x\| \leq C_\vartheta \|Ax\|^{\vartheta}\|x\|^{1-\vartheta}, \quad x \in D(A).$$

From $G_\alpha(\cdot)$ is valued $D(A)$, for all $x \in X$

$$\begin{aligned}
\|(-A)^{\vartheta}G_\alpha(\lambda)x\| &\leq C_\vartheta \|AG_\alpha(\lambda)x\|^{\vartheta}\|G_\alpha(\lambda)x\|^{1-\vartheta} \\
&\leq C_\vartheta \frac{M_3^{\vartheta}}{|\lambda|^{\vartheta-\alpha\vartheta}}\|x\|^{\vartheta}\frac{M_1^{1-\vartheta}}{|\lambda|^{1-\vartheta}}\|x\|^{1-\vartheta} \\
&\leq \frac{M_\vartheta}{|\lambda|^{1-\alpha\vartheta}}\|x\|,
\end{aligned} \tag{3.3}$$

where M_ϑ is independent of λ. From (3.3), we obtain for $t \geq 1$, make the change $\lambda t = \gamma$. From the Cauchy's theorem we obtain that

$$\begin{aligned}
\|(-A)^{\vartheta}\mathcal{R}(t)\| &\leq \|\frac{1}{2\pi i}\int_{\Gamma_{r,\theta}} e^{\gamma}(-A)^{\vartheta}G(t^{-1}\gamma)t^{-1}d\gamma\| \\
&\leq \frac{M_\vartheta}{\pi}\int_r^\infty e^{s\cos\theta}\frac{t^{-1}ds}{(t^{-1}s)^{1-\alpha\vartheta}} + \frac{M_\vartheta}{2\pi}\int_{-\theta}^{\theta} e^{r\cos\xi}\frac{t^{-1}rd\xi}{(t^{-1}r)^{1-\alpha\vartheta}} \\
&\leq \Big(\frac{M_\vartheta}{\pi r^{1-\alpha\vartheta}|\cos\theta|} + \frac{M_\vartheta \theta r^{\alpha\vartheta}}{\pi}\Big)\frac{e^{rt}}{t^{\alpha\vartheta}} \\
&\leq C\frac{e^{rt}}{t^{\alpha\vartheta}}.
\end{aligned}$$

On the other hand, using that $G(\cdot)$ is analytic on $\Sigma_{r,\theta}$, for $t \in (0,1)$ we obtain

$$\begin{aligned}
\|(-A)^{\vartheta}\mathcal{R}(t)\| &= \|\frac{1}{2\pi i}\int_{\Gamma_{\frac{r}{t},\theta}} e^{\lambda t}(-A)^{\vartheta}G(\lambda)d\lambda\| \\
&\leq \frac{M_\vartheta}{\pi}\int_{\frac{r}{t}}^\infty e^{ts\cos\theta}\frac{ds}{s^{1-\alpha\vartheta}} + \frac{M_\vartheta}{2\pi}\int_{-\theta}^{\theta} e^{r\cos\xi}\frac{rt^{-1}d\xi}{r^{1-\alpha\vartheta}t^{\alpha\vartheta-1}} \\
&\leq \frac{M_\vartheta}{\pi}\int_r^\infty e^{u\cos\theta}\frac{t^{-1}du}{u^{1-\alpha\vartheta}t^{\alpha\vartheta-1}} + \frac{M_\vartheta}{2\pi}\int_{-\theta}^{\theta} e^{r\cos\xi}\frac{rt^{-1}d\xi}{r^{1-\alpha\vartheta}t^{\alpha\vartheta-1}}
\end{aligned}$$

$$
\begin{aligned}
&\le \Big(\frac{M_\vartheta}{\pi r^{1-\alpha\vartheta}|\cos\theta|}+\frac{M_\vartheta \theta r^{\alpha\vartheta}}{\pi}e^r\Big)\frac{1}{t^{\alpha\vartheta}}\\
&\le C\frac{e^{rt}}{t^{\alpha\vartheta}}.
\end{aligned}
$$

By the definition of $(\mathcal{S}_\alpha(t))_{t\ge 0}$, we obtain

$$
\begin{aligned}
\|(-A)^\vartheta \mathcal{S}_\alpha(t)\| &\le \int_0^t g_{\alpha-1}(t-s)\|(-A)^\vartheta \mathcal{R}_\alpha(s)\|ds\\
&\le \int_0^t g_{\alpha-1}(t-s)Ce^{rs}s^{-\alpha\vartheta}ds\\
&\le e^{rt}\int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}Cs^{-\alpha\vartheta}ds\\
&\le \frac{e^{rt}}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}Cs^{-\alpha\vartheta}ds\\
&\le \frac{e^{rt}}{\Gamma(\alpha-1)}\int_0^t (t-s)^{(\alpha-1)-1}Cs^{(1-\alpha\vartheta)-1}ds.
\end{aligned}
$$

From inequality [17, 6.24], we obtain

$$
\|(-A)^\vartheta \mathcal{S}_\alpha(t)\| \le \frac{e^{rt}\Gamma(1-\alpha\vartheta)}{\Gamma(\alpha-\alpha\vartheta)}Ct^{\alpha(1-\vartheta)-1} \le Ce^{rt}t^{\alpha(1-\vartheta)-1}.
$$

□

Remark 3.2. If $\widehat{B}(\lambda)(-A)^{-\vartheta}y=(-A)^{-\vartheta}\widehat{B}(\lambda)y$ for $y\in[D(A)]$. We can see that for $\vartheta\in(0,1)$ and $x\in[D((-A)^\vartheta)]$

$$
\begin{aligned}
(-A)^\vartheta G_\alpha(\lambda)x &= \lambda^{\alpha-1}(-A)^\vartheta R(\lambda^\alpha,A)[I-\widehat{B}(\lambda)R(\lambda^\alpha,A)]^{-1}x\\
&= \lambda^{\alpha-1}(-A)^\vartheta R(\lambda^\alpha,A)[I-\widehat{B}(\lambda)R(\lambda^\alpha,A)]^{-1}(-A)^{-\vartheta}(-A)^\vartheta x.
\end{aligned}
$$

Since

$$
\widehat{B}(\lambda)R(\lambda^\alpha,A)(-A)^{-\vartheta}(-A)^\vartheta x=(-A)^{-\vartheta}\widehat{B}(\lambda)R(\lambda^\alpha,A)(-A)^{-\vartheta}x,
$$

we obtain

$$
\begin{aligned}
(-A)^\vartheta G_\alpha(\lambda)x &= \lambda^{\alpha-1}(-A)^\vartheta R(\lambda^\alpha,A)(-A)^{-\vartheta}[I-\widehat{B}(\lambda)R(\lambda^\alpha,A)]^{-1}(-A)^\vartheta x\\
&= G_\alpha(\lambda)(-A)^\vartheta x.
\end{aligned}
$$

As consequences of before it is easy to see that

$$
(-A)^\vartheta\mathcal{R}_\alpha(t)x=\mathcal{R}_\alpha(t)(-A)^\vartheta x \quad\text{and}\quad (-A)^\vartheta\mathcal{S}_\alpha(t)x=\mathcal{S}_\alpha(t)(-A)^\vartheta x,
$$

if $x\in[D((-A)^\vartheta)]$.

If $x\in C(I;X)$ we define $\overline{x}:(-\infty,b]\to X$ is the extension of x to $(-\infty,b]$ such that $\overline{x}_0=\varphi$. In the sequel we introduce the following conditions:

(H1) The following conditions are satisfied.
- (a) $B(\cdot)x\in C(I,X)$ for every $x\in[D((-A)^{1-\vartheta})]$.
- (b) There is function $\mu(\cdot)\in L^1(I;\mathbb{R}^+)$, such that

$$
\|B(s)\mathcal{S}_\alpha(t)\|_{\mathcal{L}([D((-A)^\vartheta)],X)} \le M\mu(s)t^{\alpha\vartheta-1},\quad 0\le s<t\le b.
$$

(H2) The function $f(\cdot)$ is $(-A)^{\vartheta}$-valued, $f: I \times \mathcal{B} \to [D((-A)^{-\vartheta})]$ is continuous and there exists L_f such that for all $(t_i, \psi_j) \in I \times \mathcal{B}$,

$$\|(-A)^{\vartheta} f(t_1, \psi_1) - (-A)^{\vartheta} f(t_2, \psi_2)\| \le L_f(|t_1 - t_2| + \|\psi_1 - \psi_2\|_{\mathcal{B}}). \tag{3.4}$$

(H3) The function $g: I \times \mathcal{B} \to X$ satisfies the following properties.
   (a) The function $g(\cdot, \psi): I \to X$ is strongly measurable for every $\psi \in \mathcal{B}$.
   (b) The function $g(t, \cdot): \mathcal{B} \to X$ is continuous for each $t \in I$.
   (c) There exists an integrable function $m_g : I \to [0, \infty)$ and a continuous nondecreasing function $\Omega_g : [0, \infty) \to (0, \infty)$ such that

$$\|g(t, \psi)\| \le m_g(t)\Omega_g(\|\psi\|_{\mathcal{B}}), (t, \psi) \in I \times \mathcal{B}.$$

Remark 3.3. In the rest of this section, M_b and K_b are the constants $M_b = \sup_{s\in[0,b]} M(s)$ and $K_b = \sup_{s\in[0,b]} K(s)$.

Definition 3.4. A function $u : (-\infty, b] \to X$, $0 < b \le a$, is called a mild solution of (1.5) on $[0, b]$, if $u_0 = \varphi$; $u|_{[0,b]} \in C([0, b] : X)$; the function $\tau \to A\mathcal{S}_\alpha(t-\tau)f(\tau, u_\tau)$ and $\tau \to \int_0^\tau B(\tau - \xi)\mathcal{S}_\alpha(t - \tau)f(\xi, u_\xi)d\xi$ is integrable on $[0, t)$ for all $t \in (0, b]$ and for $t \in [0, b]$,

$$\begin{aligned} u(t) &= \mathcal{R}_\alpha(t)(\varphi(0) + f(0, \varphi)) - f(t, u_t) - \int_0^t A\mathcal{S}_\alpha(t-s)f(s, u_s)ds \\ &\quad - \int_0^t \int_0^s B(s-\xi)\mathcal{S}_\alpha(t-s)f(\xi, u_\xi)d\xi ds + \int_0^t \mathcal{S}_\alpha(t-s)g(s, u_s)ds. \end{aligned} \tag{3.5}$$

Theorem 3.5. *Let conditions* (H1), (H2), (H3) *hold. If*

$$K_b\Big[L_f\Big(\|(-A)^{-\vartheta}\| + \frac{Mb^{\alpha\vartheta}}{\alpha\vartheta} + \frac{Mb^{\alpha\vartheta}}{\alpha\vartheta}\int_0^b \mu(\xi)d\xi\Big) + M\liminf_{\xi\to\infty}\frac{\Omega_g(\xi)}{\xi}\int_0^b m_g(s)ds\Big] < 1,$$

then there exists a mild solution of (1.5) *on* $[0, b]$.

Proof. Consider the space $S(b) = \{u \in C(I; X) : u(0) = \varphi(0)\}$ endowed with the uniform convergence topology and define the operator $\Gamma : S(b) \to S(b)$ by

$$\begin{aligned} \Gamma x(t) &= \mathcal{R}_\alpha(t)(\varphi(0) + f(0, \varphi)) - f(t, \overline{x}_t) - \int_0^t A\mathcal{S}_\alpha(t-s)f(s, \overline{x}_s)ds \\ &\quad - \int_0^t \int_0^s B(s-\xi)\mathcal{S}_\alpha(t-s)f(\xi, \overline{x}_\xi)d\xi ds + \int_0^t \mathcal{S}_\alpha(t-s)g(s, \overline{x}_{\rho(t,\overline{x}_s)})ds, \end{aligned}$$

for $t \in [0, b]$. From our assumptions, it is easy to see that $\Gamma S(b) \subset S(b)$.

Let $\bar{\varphi} : (-\infty, b] \to X$ be the extension of φ to $(-\infty, b]$ such that $\bar{\varphi}(\theta) = \varphi(0)$ on I. We prove that there exists $r > 0$ such that $\Gamma(B_r(\bar{\varphi}|_I, S(b))) \subseteq B_r(\bar{\varphi}|_I, S(b))$. If this property is false, then for every $r > 0$ there exist $x^r \in B_r(\bar{\varphi}|_I, S(b))$ and $t^r \in I$ such that $r < \|\Gamma x^r(t^r) - \varphi(0)\|$. Then, we find that

$$\begin{aligned} &\|\Gamma x^r(t^r) - \varphi(0)\| \\ &\le \|\mathcal{R}_\alpha(t^r)(\varphi(0) + f(0, \varphi))\| + \|f(t^r, \overline{x^r}_{t^r})\| \\ &\quad + \int_0^{t^r} \|(-A)^{1-\vartheta}\mathcal{S}_\alpha(t^r - s)\|\|(-A)^{\vartheta} f(s, \overline{x^r}_s)\|ds \\ &\quad + \int_0^{t^r}\int_0^s \|B(s-\xi)\mathcal{S}_\alpha(t^r - s)f(\xi, \overline{x^r}_\xi)\|d\xi ds + \int_0^t \|\mathcal{S}_\alpha(t-s)\|\|g(s, \overline{x}_s)\|ds \\ &\le \|\mathcal{R}_\alpha(t^r)\varphi(0) - \varphi(0)\| + \|\mathcal{R}_\alpha(t^r)f(0, \varphi) - f(0, \varphi)\| \end{aligned}$$

$$
\begin{aligned}
&+ \|(-A)^{-\vartheta}\| \|(-A)^{\vartheta} f(t^r, \overline{(x^r)}_{t^r}) - (-A)^{\vartheta} f(0,\varphi)\| \\
&+ \int_0^{t^r} M(t^r-s)^{\alpha\vartheta-1} \|(-A)^{\vartheta} f(s^r, \overline{(x^r)}_s) - (-A)^{\vartheta} f(0,\varphi)\| ds \\
&+ \int_0^{t^r} M(t^r-s)^{\alpha\vartheta-1} \|(-A)^{\vartheta} f(0,\varphi)\| ds \\
&+ \int_0^{t^r} \int_0^s \mu(s-\xi) M(t^r-s)^{\alpha\vartheta-1} \|(-A)^{\vartheta} f(\xi^r, \overline{(x^r)}_{\xi}) - (-A)^{\vartheta} f(0,\varphi)\| d\xi ds \\
&+ \int_0^{t^r} \int_0^s \mu(s-\xi) M(t^r-s)^{\alpha\vartheta-1} \|(-A)^{\vartheta} f(0,\varphi)\| d\xi ds \\
&+ M \int_0^{t^r} m_g(t^r-s) \Omega_g(\|\overline{x^r}_s\|_{\mathcal{B}}) ds \\
&\le (M+1) H \|\varphi\|_{\mathcal{B}} + \|\mathcal{R}_\alpha(t^r) f(0,\varphi) - f(0,\varphi)\| \\
&+ \Big(\frac{M b^{\alpha\vartheta}}{\alpha\vartheta} + \frac{M b^{\alpha\vartheta}}{\alpha\vartheta} \int_0^b \mu(\xi) d\xi \Big) \|(-A)^{\vartheta} f(0,\varphi)\| \\
&+ \|(-A)^{-\vartheta}\| L_f \left(t^r + \|\overline{(x^r)}_{t^r} - \varphi\|_{\mathcal{B}} \right) \\
&+ \int_0^{t^r} M(t^r-s)^{\alpha\vartheta-1} L_f \left(s + \|\overline{(x^r)}_s - \varphi\|_{\mathcal{B}} \right) ds \\
&+ \int_0^{t^r} \int_0^s \mu(s-\xi) M(t^r-s)^{\alpha\vartheta-1} L_f \left(\xi + \|\overline{(x^r)}_{\xi^r} - \varphi\|_{\mathcal{B}} \right) d\xi ds \\
&+ \Omega_g \left(K_b r + (M_b + H K_b + 1) \|\varphi\|_{\mathcal{B}} \right) \int_0^b m_g(s) ds \\
&\le (M+1) H \|\varphi\|_{\mathcal{B}} + \|R(t^r) f(0,\varphi) - f(0,\varphi)\| \\
&+ \Big(\frac{M b^{\alpha\vartheta}}{\alpha\vartheta} + \frac{M b^{\alpha\vartheta}}{\alpha\vartheta} \int_0^b \mu(\xi) d\xi \Big) \|(-A)^{\vartheta} f(0,\varphi)\| \\
&+ \Big(\|(-A)^{-\vartheta}\| + \frac{M b^{\alpha\vartheta}}{\alpha\vartheta} + \frac{M b^{\alpha\vartheta}}{\alpha\vartheta} \int_0^b \mu(\xi) d\xi \Big) (L_f (b + (M_b + H K_b + 1) \|\varphi\|_{\mathcal{B}})) \\
&+ \Big(\|(-A)^{-\vartheta}\| + \frac{M b^{\alpha\vartheta}}{\alpha\vartheta} + \frac{M b^{\alpha\vartheta}}{\alpha\vartheta} \int_0^b \mu(\xi) d\xi \Big) L_f K_b r \\
&+ \Omega_g \left(K_b r + (M_b + H K_b + 1) \|\varphi\|_{\mathcal{B}} \right) \int_0^b m_g(s) ds,
\end{aligned}
$$

where $i_c : Y \to X$ represents the continuous inclusion of Y on X. Therefore

$$
1 \le K_b \Big[L_f \Big(\|(-A)^{-\vartheta}\| + \frac{M b^{\alpha\vartheta}}{\alpha\vartheta} + \frac{M b^{\alpha\vartheta}}{\alpha\vartheta} \int_0^b \mu(\xi) d\xi \Big) + M \liminf_{\xi \to \infty} \frac{\Omega_g(\xi)}{\xi} \int_0^b m_g(s) ds \Big],
$$

which contradicts our assumption.

Let $r > 0$ be such that $\Gamma(B_r(\bar{\varphi}|_I, S(b))) \subseteq B_r(\bar{\varphi}|_I, S(b))$. In the sequel, r^* and r^{**} are the numbers defined by $r^* := (K_b r + (M_b + H K_b + 1)\|\varphi\|_{\mathcal{B}})$ and $r^{**} := \Omega_g(r^*) \int_0^b m_g(s) ds$.

To prove that Γ is a condensing operator, we introduce the decomposition $\Gamma = \Gamma_1 + \Gamma_2$, where, for $t \in I$,

$$\begin{aligned}\Gamma_1 x(t) = \mathcal{R}_\alpha(t)(\varphi(0) + f(0,\varphi)) - f(t,\overline{x}_t) - \int_0^t A\mathcal{S}_\alpha(t-s) f(s,\overline{x}_s)ds \\ - \int_0^t \int_0^s B(s-\xi)\mathcal{S}_\alpha(t-s)f(\xi,\overline{x}_\xi)d\xi ds,\end{aligned}$$

$$\Gamma_2 x(t) = \int_0^t \mathcal{S}_\alpha(t-s) g(s,\overline{x}_s)ds\,.$$

On the other hand, for $u, v \in B_r(\bar{\varphi}|_I, S(b))$ and $t \in [0,b]$ we see that

$$\begin{aligned}&\|\Gamma_1 u(t) - \Gamma_1 v(t)\| \\ &\le \|(-A)^{-\vartheta}\| \|(-A)^{\vartheta} f(t,\overline{u}_t) - (-A)^{\vartheta} f(t,\overline{v}_t)\| \\ &\quad + \int_0^t \|(-A)^{1-\vartheta}\mathcal{S}_\alpha(t-s)\| \|(-A)^{\vartheta} f(s,\overline{u}_s) - (-A)^{\vartheta} f(s,\overline{v}_s)\|_Y ds \\ &\quad + \int_0^t \int_0^s \|B(s-\xi)\mathcal{S}_\alpha(t-s) f(\xi,\overline{u}_\xi) - f(\xi,\overline{v}_\xi)\| d\xi ds \\ &\le \|(-A)^{-\vartheta}\| L_f K_b \|u-v\|_b + L_f K_b \int_0^t M(t-s)^{\alpha v-1} ds \|u-v\|_b \\ &\quad + L_f K_b \int_0^t \int_0^s \mu(s-\xi) M(t-s)^{\alpha\vartheta-1} d\xi ds \|u-v\|_b, \\ &\le L_f K_b \Big(\|(-A)^{-\vartheta}\| + \frac{Mb^{\alpha\vartheta}}{\alpha\vartheta} + \frac{Mb^{\alpha\vartheta}}{\alpha\vartheta} \int_0^b \mu(\xi)d\xi \Big) \|u-v\|_b,\end{aligned}$$

which show that $\Gamma_1(\cdot)$ is a contraction on $B_r(\bar{\varphi}|_I, S(b))$.

Next we prove that $\Gamma_2(\cdot)$ is a completely continuous function from $B_r(\bar{\varphi}|_I, S(b))$ to $B_r(\bar{\varphi}|_I, S(b))$.

Step 1. The set $\Gamma_2(B_r(\bar{\varphi}|_I, S(b))(t)$ is relatively compact on X for every $t \in [0,b]$. The case $t = 0$ is trivial. Let $0 < \epsilon < t < b$. From the assumptions, we can fix numbers $0 = t_0 < t_1 < \cdots < t_n = t - \epsilon$ such that $\|\mathcal{S}_\alpha(t-s) - \mathcal{S}_\alpha(t-s')\| \le \epsilon$ if $s, s' \in [t_i, t_{i+1}]$, for some $i = 0, 1, 2, \cdots, n-1$. let $x \in B_r(\bar{\varphi}|_I, S(b))$. Under theses conditions, from the mean value theorem for the Bochner Integral (see [15, Lemma 2.1.3]) we see that

$$\begin{aligned}\Gamma_2 x(t) &= \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \mathcal{S}_\alpha(t-t_i) g(s,\overline{x}_s)ds \\ &\quad + \sum_{i=1}^n \int_{t_{i-1}}^{t_i} (\mathcal{S}_\alpha(t-s) - \mathcal{S}_\alpha(t-t_i)) g(s,\overline{x}_{\rho(t,\overline{x}_s)})ds \\ &\quad + \int_{t_n}^t \mathcal{S}_\alpha(t-s) g(s,\overline{x}_s)ds \\ &\in \sum_{i=1}^n (t_i - t_{i-1}) \overline{\mathrm{co}(\{\mathcal{S}_\alpha(t-t_i) g(s,\psi) : \psi \in B_{r^*}(0,\mathcal{B}), s \in [0,b]\})} \\ &\quad + \epsilon\, r^{**} + M\Omega_g(r^*) \int_{t-\epsilon}^t m_g(s)ds\end{aligned}$$

$$\in \sum_{i=1}^{n}(t_i - t_{i-1})\overline{\text{co}(\{W^2_{r^*}(t-t_i)\})} + \epsilon B_{r^{**}}(0,X) + C_\epsilon,$$

where $\text{diam}(C_\epsilon) \to 0$ when $\epsilon \to 0$. This prove that $\Gamma_2(B_q(0,S(b)))(t)$ is totally bounded and hence relatively compact in X for every $t \in [0,b]$.

Step 2. The set $\Gamma_2(B_r(\bar{\varphi}|_I, S(b)))$ is equicontinuous on $[0,b]$. Let $0 < \epsilon < t < b$ and $0 < \delta < \epsilon$ such that $\|\mathcal{S}_\alpha(s) - \mathcal{S}_\alpha(s')\| \le \epsilon$ for every $s, s' \in [\epsilon, b]$ with $|s - s'| \le \delta$. Under these conditions, for $x \in B_r(\bar{\varphi}|_I, S(b))$ and $0 < h \le \delta$ with $t + h \in [0,b]$, we obtain

$$\begin{aligned}
&\|\Gamma_2 x(t+h) - \Gamma_2 x(t)\| \\
&\le \int_0^{t-\epsilon} [\mathcal{S}_\alpha(t+h-s) - \mathcal{S}_\alpha(t-s)] g(s, \overline{x}_s) ds \\
&\quad + \int_{t-\epsilon}^{t} [\mathcal{S}_\alpha(t+h-s) - \mathcal{S}_\alpha(t-s)] g(s, \overline{x}_s) ds + \int_t^{t+h} \mathcal{S}_\alpha(t+h-s) g(s, \overline{x}_s) ds \\
&\le \epsilon r^{**} + 2M\Omega(r^*) \int_{t-\epsilon}^{t} m_g(s) ds + M\Omega(r^*) \int_t^{t+h} m_g(s) ds
\end{aligned}$$

which shows that the set of functions $\Gamma_2(B_r(\bar{\varphi}|_I, S(b)))$ is right equicontinuity at $t \in (0,b)$. A similar procedure permit to prove the right equicontinuity at zero and the left equicontinuity at $t \in (0,b]$. Thus, $\Gamma_2(B_r(\bar{\varphi}|_I, S(b)))$ is equicontinuous. By using a similar procedure to proof of [10, Theorem 2.3], we prove that that $\Gamma_2(\cdot)$ is continuous on $B_r(\bar{\varphi}|_I, S(b))$, which completes the proof that $\Gamma_2(\cdot)$ is completely continuous.

To complete the prove that $\Gamma_1(\cdot)$ is continuous, let $(x^n)_{n\in\mathbb{N}}$ be a sequence in $B_r(\bar{\varphi}|_I, S(b))$ and $x \in B_r(\bar{\varphi}|_I, S(b))$ such that $x^n \to x$ in $B_r(\bar{\varphi}|_I, S(b))$. From the phase space axioms we infer that $\overline{(x^n)}_s \to \overline{x}_s$ uniformly for $s \in I$ as $n \to \infty$. Consequently, from (3.4), $\|(-A)^{-\vartheta} f(s, \overline{(x^n)}_s) - (-A)^{-\vartheta} f(s, \overline{x}_s)\| \to 0$, uniformly on $[0,b]$ as $n \to \infty$. Now, a standard application of the Lebesgue dominated convergence Theorem permits to conclude that $\Gamma_1(\cdot)$ is continuous on $B_r(\bar{\varphi}|_I, S(b))$. The existence of a mild solution for (1.5) is now a consequence of [15, Theorem 4.3.2]. This completes the proof. □

4. Applications

To complete this paper, we discuss the existence of solutions for the partial integro-differential system

$$\begin{aligned}
&\frac{\partial^\alpha}{\partial t^\alpha}\Big(u(t,\xi) + \int_{-\infty}^{t}\int_0^\pi b(t-s,\eta,\xi) u(s,\eta) d\eta ds\Big) \\
&= \frac{\partial^2}{\partial \xi^2} u(t,\xi) + \int_0^t (t-s)^\delta e^{-\gamma(t-s)} \frac{\partial^2}{\partial \xi^2} u(s,\xi) ds + \int_{-\infty}^{t} a_0(s-t) u(s,\xi) ds, \\
&\quad (t,\xi) \in I \times [0,\pi],
\end{aligned} \tag{4.1}$$

$$u(t,0) = u(t,\pi) = 0, \quad t \in [0,b], \tag{4.2}$$

$$u(\theta,\xi) = \phi(\theta,\xi), \quad \theta \le 0,\ \xi \in [0,\pi]. \tag{4.3}$$

Where $\frac{\partial^\alpha}{\partial t^\alpha} = D_t^\alpha$, $\alpha \in (1,2)$. To treat this system in the abstract form (1.1)-(1.2), we choose the space $X = L^2([0,\pi])$, $\mathcal{B} = C_0 \times L^p(g,X)$ is the space introduced in Example 2.14 and $A : D(A) \subseteq X \to X$ is the operator defined by $Ax = x''$, with domain $D(A) = \{x \in X : x'' \in X, x(0) = x(\pi) = 0\}$. It is well known that A is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ on X. Moreover, A has a discrete spectrum with eigenvalues of the form $-n^2, n \in \mathbb{N}$, and corresponding normalized eigenfunctions given by $z_n(\xi) := (\frac{2}{\pi})^{1/2}\sin(n\xi)$ and the following properties hold

(a) $\{z_n : n \in \mathbb{N}\}$ is an orthonormal basis of X.
(b) For $x \in X, T(t)x = \sum_{n=1}^{\infty} e^{-n^2 t}\langle x, z_n\rangle z_n$.
(c) For $\alpha \in (0,1)$, the fractional power $(-A)^\alpha : D((-A)^\alpha) \subset X \to X$ of A is given by $(-A)^\alpha x = \sum_{n=1}^{\infty} n^{2\alpha}\langle x, z_n\rangle z_n$, where $D((-A)^\alpha) = \{x \in X : (-A)^\alpha x \in X\}$.

Hence, A is sectorial of type and the properties (P1) hold. We also consider the operator $B(t) : D(A) \subseteq X \to X$, $t \geq 0$, $B(t)x = t^\delta e^{-\gamma t}Ax$ for $x \in D(A)$. Moreover, it is easy to see that conditions (P2)-(P3) in Section 2 are satisfied with $b(t) = t^\delta e^{-\gamma t}$ and $D = C_0^\infty([0,\pi])$, where $C_0^\infty([0,\pi])$ is the space of infinitely differentiable functions that vanish at $\xi = 0$ and $\xi = \pi$. From the Lemma 3.1 it is easy to see that condition (H1) is satisfies.

In the sequel, we assume that $\varphi(\theta)(\xi) = \phi(\theta,\xi)$ is a function in $\mathcal{B}$ and that the following conditions are verified.

(i) The functions $a_0 : \mathbb{R} \to \mathbb{R}$ are continuous and $L_g := \Big(\int_{-\infty}^0 \frac{(a_0(s))^2}{g(s)} ds\Big)^{1/2} < \infty$.
(ii) The functions $\rho_i : [0,\infty) \to [0,\infty), i = 1,2$, are continuous.
(iii The functions $b(s,\eta,\xi)$, $\frac{\partial b(s,\eta,\xi)}{\partial \xi}$ are measurable, $b(s,\eta,\pi) = b(s,\eta,0) = 0$ for all (s,η) and

$$L_f := \max\{(\int_0^\pi \int_{-\infty}^0 \int_0^\pi g^{-1}(\theta)\Big(\frac{\partial^i}{\partial \xi^i} b(\theta,\eta,\xi)\Big)^2 d\eta d\theta d\xi)^{1/2} : i = 0,1\} < \infty.$$

Defining the operators $f, g : I \times \mathcal{B} \to X$ by

$$f(\psi)(\xi) = \int_{-\infty}^0 \int_0^\pi b(s,\eta,\xi)\psi(s,\eta)d\eta ds,$$
$$g(\psi)(\xi) = \int_{-\infty}^0 a_0(s)\psi(s,\xi)ds.$$

we can transform (4.1)-(4.3) into the abstract system (1.1)-(1.2). Moreover, f, g are bounded linear operators with $\|f(\cdot)\|_{\mathcal{L}(\mathcal{B},X)} \leq L_f$ and $\|g(\cdot)\|_{\mathcal{L}(\mathcal{B},X)} \leq L_g$. Moreover, a straightforward estimation using (ii) shows that $f(I \times \mathcal{B}) \subset D((-A)^{1/2})$ and $\|(-A)^{1/2} f\|_{\mathcal{L}(\mathcal{B},X)} \leq L_f$. The following result is a direct consequence of Theorem 3.5.

Proposition 4.1. *If*

$$\Big(1 + \int_{-b}^0 g(\theta)\,d\theta)\Big)\Big(L_f\Big(\|(-A)^{-1/2}\| + C_{1/2}\sqrt{b} + C_{1/2}\sqrt{b}\int_0^b a(s)ds\Big) + L_g\Big) < 1,$$

then there exist a mild solutions of (4.1)–(4.3).

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