Electronic Journal of Differential Equations, Vol. 2012 (2012), No. 97, pp. 1–11.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

EXISTENCE AND NONEXISTENCE OF PERIODIC SOLUTIONS OF N-VECTOR DIFFERENTIAL EQUATIONS OF ORDERS SIX AND SEVEN

VICTOR F. PAYNE, HAROON OLADIPO TEJUMOLA

Abstract. In this article, we extend our earlier results and establish new ones on the existence and non-existence of periodic solutions for n-vector non-dissipative, nonlinear ordinary differential equations. Our results involve both the homogeneous and non-homogeneous cases. The setting for non-existence results of periodic solutions involves a suitably defined scalar function endowed with appropriate properties relative to each equation. But the framework for proving existence results is via the standard Leray-Schauder fixed-point technique whose central theme is the verification of a-priori bounded periodic solutions for a parameter-dependent system of equations.

1. Introduction

The article by Ezeilo [2] on sixth-order equations marks the beginning of systematic study of the problem of existence, and nonexistence in the homogeneous case, of periodic solutions of non-dissipative, nonlinear ordinary differential equations of orders six and above. Bereketoglu [1] extended Ezeilo's work to equations of order seven, while Tejumola [3] widened the scope of these earlier investigations to more general class of equations and to situations, which hitherto were not considered. An n-vector analogue of the result of Bereketoglu [1, Theorem 1] was recently obtained by Tunç [4, 5, 6] in the seventh order homogeneous case

$$\begin{aligned} &x^{(6)} + a_1x^{(5)} + a_2x^{(4)} + a_3x^{(3)} + f(x')x'' + g(x)x' + h(x)\\ &= p(t, x, x', x'', x''', x^{(4)}, x^{(5)} \end{aligned}$$

and

$$\begin{aligned} &x^{(7)} + a_1x^{(6)} + a_2x^{(5)} + a_3x^{(4)} + a_4x''' + f(x')x'' + g(x)x' + h(x)\\ &= p(t, x, x', x'', x''', x^{(4)}, x^{(5)}, x^{(6)}), \end{aligned}$$

the problem of existence in the non-homogeneous case was however not considered. Our present investigation arose from our desire to provide results in the nonhomogeneous case and, more importantly, to extend our earlier results [3] to n-vector

---

2000 *Mathematics Subject Classification.* 34C25, 34K05, 34K13.
Key words and phrases. A priori bound; Leray-Schauder fixed-point teorem; parameter-dependent system; periodic solution.

Submitted April 12, 2012. Published June 10, 2012.

equations. The results we have been able to obtain, which include [1] and [4] as special cases, are stated in §2 and §3.

To end this section, we introduce some notation. $\mathbb{R}^n$ denotes the usual n-dimensional real Euclidean space with inner product $\langle X, Y\rangle = \sum_{i=1}^n x_i y_i$ and norm $\|X\| = \big(\sum_{i=1}^n x_i^2\big)^{1/2}$, for $X = (x_1, x_2, \dots, x_n)$, $Y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$. For a constant $n \times n$ matrix A, we define sign of A, $\operatorname{sgn} A$, by $\operatorname{sgn} A = 1$ or -1 according as A is positive definite or negative definite and we write $\gamma_A = \operatorname{sgn} A$. A is definite if A is positive or negative definite. For any function $h : \mathbb{R}^n \to \mathbb{R}^n$, $J(h(X))$ denotes the Jacobian of h, if it exists.

2. Statement of Results

We start with sixth order nonlinear differential equations of the form

$$
\begin{aligned}
&X^{(6)} + A_1 X^{(5)} + A_2 X^{(4)} + f_3(\dot{X}, \ddot{X})\dddot{X} + f_4(\dot{X})\ddot{X} + f_5(\dot{X}, \ddot{X})\dot{X} + f_6(X)\\
&= P_1(t, X, \dot{X}, \dots, X^{(5)})
\end{aligned} \tag{2.1}
$$

where A_1, A_2 are constant $n \times n$ symmetric matrices, f_3, f_5 are symmetric $n \times n$ continuous matrices, f_4 is an $n \times n$ continuous matrix, $f_6 : \mathbb{R}^n \to \mathbb{R}^n$, $P_1 : R \times \mathbb{R}^n \times \dots \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous functions and $P_1(t+\omega, X, \dot{X}, \dots, X^{(5)}) = P_1(t, X, \dot{X}, \dots, X^{(5)})$ for some $\omega > 0$. It will be assumed further that the Jacobians $J(f_4(\dot{X})\dot{X}), J(f_6(X))$ exist and are continuous.

Theorem 2.1. *Let A_1 be definite and let*

$$
f_6(0) = 0, \quad f_6(X) \neq 0 \quad \text{for } X \neq 0. \tag{2.2}
$$

Suppose that $(\gamma_{A_3} f_3)$ is negative semi-definite and that $(\gamma_{A_1} f_5)$ is positive definite. Suppose further that

$$
\inf_{X_2, X_3} \frac{\langle(\gamma_{A_1} f_5)X_2, X_2\rangle}{\|X_2\|} > \frac{1}{2a_1} \frac{\langle f_3^2 X_2, .X_3\rangle}{\|X_2\|^2}, \quad X_2 \neq 0, \tag{2.3}
$$

where $a_1 > 0$ is a constant such that

$$
\langle(\gamma_{A_1} A_1)Y, Y\rangle \geq a_1 \|Y\|^2 \quad \text{for all } Y \in \mathbb{R}^n. \tag{2.4}
$$

Then (2.1) with $P_1 \equiv 0$ has no nontrivial periodic solution of any period.

Theorem 2.2. *Let all the conditions of Theorem 2.1 hold, except for (2.2) and (2.3). Let*

$$
f_6(X) \operatorname{sgn} X \to +\infty(-\infty) \quad \text{as } \|X\| \to \infty. \tag{2.5}
$$

Suppose further that there exist constants $\beta_3 > 0$, $B_1 > 0$, $B_2 \geq 0$, with B_2 sufficiently small, such that

$$
\inf_{X_2, X_3} \frac{\langle(\gamma_{A_1} f_5)X_2, X_2\rangle}{\|X_2\|^2} > \frac{1}{2a_1}\beta_3^2, \quad {}_2 \neq 0, \tag{2.6}
$$

$$
\|P_1(t, X_1, X_2, \dots, X_6)\| \leq B_1 + B_2(\|X_2\| + \|X_3\|) \tag{2.7}
$$

for all $t \in R$ and $X_1, X_2, \dots, X_6 \in \mathbb{R}^n$, where $\beta_3 > 0$ is a constant such that $\|f_3(X_2, X_3)\| \leq \beta_3$. Then (2.1) has at least one periodic solution of period ω.

Note the absence of any restrictions on A_2 and f_6 in Theorem 2.1, similarly in Theorem 2.2 except for the additional condition (2.5) required to ensure uniform boundedness of X_1. Our results, which place restrictions on even-subscript terms (that is, f_6), concern a slightly different class of sixth order equations of the form

$$\begin{aligned} &X^{(6)} + A_1 X^{(5)} + A_2 X^{(4)} + A_3 \dddot{X} + g_4(\dot{X}, \ddot{X})\ddot{X} + g_5(X)\dot{X} + g_6(X, \dot{X}, \ddot{X}) \\ &= P_2(t, X, \ddot{X}, \dots, X^{(5)}). \end{aligned} \tag{2.8}$$

Here, A_1, A_2, A_3 are constant $n \times n$ symmetric matrices, g_4, g_5 are symmetric $n \times n$ continuous matrices, $J(g_5(X))$ exists and is continuous, g_6 and P_2 are continuous n-vector functions of their respective arguments and $P_2(t+\omega, X_1, X_2, \dots, X_6) = P_2(t, X_1, X_2, \dots, X_6)$ for some $\omega > 0$. The results are as follows.

Theorem 2.3. *Let A_2 be negative definite and let*

$$g_6(0, X_2, X_3) = 0 \quad \text{and} \quad g_6(X_1, X_2, X_3) \neq 0 \quad \text{if } X_1 \neq 0.$$

Suppose that

$$\sup_{X_1, X_2, X_3} \frac{\langle g_6(X_1, X_2, X_3), X_1\rangle}{\|X_1\|^2} < \frac{1}{4a_2} \frac{\langle g_4^2(X_2.X_3)X_1, X_1\rangle}{\|X_1\|^2}, \quad X_1 \neq 0 \tag{2.9}$$

where $a_2 < 0$ is a constant such that

$$\langle A_2 Y, Y\rangle < a_2 \langle Y, Y\rangle \quad \text{for all } Y \in \mathbb{R}^n. \tag{2.10}$$

Then (2.8) *with $P_2 \equiv 0$ has no nontrivial periodic solution of any period.*

Theorem 2.4. *Let A_2 be negative definite so that* (2.10) *holds, and let $\beta_4 > 0$ be a constant such that*

$$\beta_4 = \inf \|g_4(X_4, X_3)\|. \tag{2.11}$$

Suppose that

$$\sup_{X_1, X_2, X_3} \frac{\langle g_6(X_1, X_2, X_3), X_1\rangle}{\|X_1\|^2} < \frac{1}{4a_2}\beta_4^2, \quad X_1 \neq 0, \tag{2.12}$$

$$\|P_2(t, X_1, X_2, \dots, X_6\| \leq B_1^* + B_2^*(\|X_1\| + \|X_2\| + \|X_3\|) \tag{2.13}$$

where $B_1^* > 0$, $B_2^* \geq 0$ are constants, with B_2^* sufficiently small. Then (2.8) *has at least one ω-periodic solution.*

Theorems 2.1–2.4 are n-dimensional analogue of the results in [3, §2]. Note also that Theorem 2.3 holds true (as in [3, Theorem 3]) with g_4 and g_6 depending also on $\dddot{X}, X^{(4)}$ and $X^{(5)}$.

3. Further Results

We now state some parallel results in the seventh order case. The equations are of the form

$$\begin{aligned} &X^{(7)} + \sum_{k=1}^{4} A_k X^{(7-k)} + \varphi_5(\dot{X}, \ddot{X})\ddot{X} + \varphi_6(X)\dot{X} + \varphi_7(X, \dot{X}, \ddot{X}) \\ &= Q_1(t, X, \dot{X}, \dots, X^{(6)}) \end{aligned} \tag{3.1}$$

$$\begin{aligned} &X^{(7)} + \sum_{k=1}^{3} A_k X^{(7-k)} + \psi_4(\dot{X}, \ddot{X}, \dddot{X})\dddot{X} + \psi_5(\dot{X})\ddot{X} + \psi_6(\dot{X}, \ddot{X}, \dddot{X}) + \psi_7(X) \\ &= Q_2(t, X, \dot{X}, \dots, X^{(6)}) \end{aligned} \tag{3.2}$$

where $A_i, i = 1, 2, 3, 4$ are constant $n \times n$ symmetric matrices, $\varphi_5, \varphi_6, \psi_4$ and ψ_5 are symmetric $n \times n$ continuous matrices, φ_7, ψ_7, Q_1 and Q_2 are continuous n-vector functions of their respective arguments,

$$Q_i(t + \omega, X_1, X_2, \dots, X_7) = Q_i(t, X_1, X_2, \dots, X_7),$$

$i = 1, 2$, for some $\omega > 0$ and $J(\varphi_6(X))$, $J(\psi_5(\dot{X})\ddot{X})$ exist and are continuous.

Our first result concerns equation (3.1) with restrictions on terms with odd subscripts.

Theorem 3.1. *Let A_1, A_3 be definite matrices and let*

$$\gamma_{A_1}, \gamma_{A_3} = -1. \tag{3.3}$$

Suppose that $\varphi_7(0, X_2, X_3) = 0$, $\varphi_7(X_1, X_2, X_3) \neq 0$ $(X_1 \neq 0)$, and that

$$\inf_{X_1, X_2, X_3} \frac{\langle \gamma_{A_3} \varphi_7(X_1, X_2, X_3), X_1 \rangle}{\|X_1\|^2} > \frac{1}{4a_3} \frac{\langle \varphi_5(X_2, X_3) X_1, X_1 \rangle}{\|X_1\|^2}, \ X_1 \neq 0 \tag{3.4}$$

where $a_3 > 0$ is a constant such that

$$\langle (\gamma_{A_3} A_3) Y, Y \rangle \geq a_3 \langle Y, Y \rangle \quad \text{for all } Y \in \mathbb{R}^n. \tag{3.5}$$

Then (3.1) with $Q_1 \equiv 0$ has no nontrivial periodic solution of any period.

Theorem 3.1 extends the result in [4], and is an n-dimensional analogue of the nonexistence result [3, Theorem 2].

Theorem 3.2. *Let A_1, A_3 be definite matrices such that (3.3) holds. Let $\beta_5 > 0$ be a constant such that $\sup \|\varphi_5(X_2, X_3)\| \leq \beta_5$. Suppose that*

$$\inf_{X_1, X_2, X_3} \frac{\langle \gamma_{A_3} \varphi_7(X_1, X_2, X_3), X_1 \rangle}{\|X_1\|^2} > \frac{1}{4a_3} \beta_5^2, \quad X_1 \neq 0, \tag{3.6}$$

$$\|Q_1(t, X_1, X_2, \dots, X_7\| \leq C_1 + C_2(\|X_1\| + \|X_2\| + \|X_3\|) \tag{3.7}$$

where $a_3 > 0$ is a constant satisfying (3.5) and $C_1 > 0$, $C_2 \geq 0$ are constants, with C_2 sufficiently small. Then (3.1) has at least one periodic solution with period ω.

Our results in the other direction (that is, involving even subscripts) concern equation (3.2), and are as follows.

Theorem 3.3. *Let A_2 be negative definite and let*

$$\psi_7(0) = 0, \quad \psi_7(X_1) \neq 0 \quad \text{if } X_1 \neq 0, \quad \psi_6(0, X_3, X_4) = 0. \tag{3.8}$$

Suppose that

$$\sup_{X_2, X_3, X_4} \frac{\langle \psi_6(X_2, X_3, X_4), X_2 \rangle}{\|X_2\|^2} < \frac{1}{4a_2} \frac{\langle \psi_4^2(X_2, X_3, X_4) X_2, X_2 \rangle}{\|X_2\|^2}, \quad X_2 \neq 0, \tag{3.9}$$

where $a_2 < 0$ is a constant satisfying (2.10). Then (3.2), with $Q_2 \equiv 0$, has no nontrivial periodic solution of any period.

Theorem 3.4. *Let A_2 be negative definite so that (2.10) holds and let $\beta_4^* > 0$ be a constant such that $\inf \|\psi_4(X_2, X_3, X_4)\| \leq \beta_4^*$. Suppose that*

$$\sup_{X_2, X_3, X_4} \frac{\langle \psi_6(X_2, X_3, X_4), X_2 \rangle}{\|X_2\|^2} < \frac{1}{4a_2} \beta_4^{*2}, \ X_2 \neq 0, \tag{3.10}$$

$$\begin{gathered} \psi_7(X) \operatorname{sgn} X \to +\infty(-\infty) \quad \text{as} \quad \|X\| \to \infty \\ \|Q_2(t, X_1, X_2, \dots, X_7)\| \leq C_1^* + C_2^*(\|X_2\| + \|X_3\| + \|X_4\|) \end{gathered} \tag{3.11}$$

where $C_1^* > 0$, $C_2^* \geq 0$ *are constants, with* C_2^* *sufficiently small. Then* (3.2) *has at least one periodic solution of period* ω.

Theorems 3.2, 3.3, 3.4 are n-dimensional analogue of [3, Theorems 3, 6, 7].

The procedure for the proof the theorems is as in [1, 2, 3]. For nonexistence of periodic solutions, a suitably defined scalar function with appropriate properties relative to each equation is required; while for the existence of periodic solutions, the setting for each proof is the now standard Leray-Schauder fixed-point technique, the central problem of which is the verification of an a-priori bound for all possible ω-periodic solutions of a suitably defined parameter-dependent system of equations. We shall outline the salient points in the proof of each theorem in sections 4 and 5.

4. Proofs of Theorems 2.1, 2.2, 2.3, 2.4

Consider, instead of equation (2.1) with $P_1 \equiv 0$, the equivalent system

$$\begin{gathered} \dot{X}_i = X_{i+1}, \quad i = 1,2,3,4,5, \quad X_1 \equiv X, \\ \dot{X}_6 = -A_1X_6 - A_2X_5 - f_3(X_2, X_3)X_4 - f_4(X_2)X_3 - f_5(X_2, X_3)X_2 - f_6(X_1), \end{gathered} \tag{4.1}$$

together with the scalar function $W = W(X_1, X_2, \dots, X_6)$ defined by

$$W = \gamma_{A_1} V, \quad V = V_0 + V_1, \tag{4.2}$$

where

$$V_0 = -\int_0^1 \langle \sigma f_4(\sigma X_2)X_2, X_2\rangle d\sigma - \int_0^1 \langle f_6(\sigma X_1), X_1\rangle d\sigma, \tag{4.3}$$

$$V_1 = -\langle X_2, X_6 + A_1X_5 + A_2X_4\rangle + \langle X_3, X_5 + A_1X_4\rangle + \frac{1}{2}\langle A_2X_3, X_3\rangle - \frac{1}{2}\langle X_4, X_4\rangle. \tag{4.4}$$

Let $(X_1, X_2, \dots, X_6) \equiv (X_1(t), X_2(t), \dots, X_6(t))$ be an arbitrary nontrivial periodic solution of (4.1) of period α say. Then since

$$-\dot{V}_0 = \langle f_4(X_2)X_2, X_3\rangle + \langle f_6(X_1), X_2\rangle,$$

as can be verified as in [4, §2], we have from (4.2), (4.3), (4.4) and (4.1) that

$$\dot{V} = \langle A_1X_4, X_4\rangle + \langle f_5X_2, X_2\rangle + \langle f_3X_2, X_4\rangle$$

Thus, by (2.4),

$$\begin{aligned} \dot{W} &= \langle (\gamma_{A_1}A_1)X_4, X_4\rangle + \langle (\gamma_{A_1}f_5)X_2, X_2\rangle + \langle (\gamma_{A_1}f_3)X_2, X_4\rangle \\ &\geq \frac{1}{2}a_1\langle X_4, X_4\rangle + \frac{1}{2}a_1\|X_4 + \frac{1}{a_1}(\gamma_{A_1}f_3)X_2\|^2 + \langle (\gamma_{A_1}f_5)X_2, X_2\rangle \\ &\quad - \frac{1}{2a_1}\langle f_3^2X_2, X_2\rangle \\ &\geq \frac{1}{2}a_1\langle X_4, X_4\rangle + \langle (\gamma_{A_1}f_5)X_2, X_2\rangle - \frac{1}{2}a_1\langle f_3^2X_2, X_2\rangle. \end{aligned} \tag{4.5}$$

The hypothesis (2.3) now implies that $\dot{W} \geq 0$, so that W is monotone increasing. By (4.5) and the periodicity of $W(t)$, it will follow, as in [1, §3], that $X_1 = X_2 = X_3 = X_4 = X_5 = X_6 = 0$.

Turning now to Theorem 2.2, consider the parameter λ-dependent system

$$\begin{gathered}\dot{X}_i = X_{i+1}, \quad i = 1, 2, \dots, 5 \\ \begin{aligned}\dot{X}_6 = -A_1 X_6 - A_2 X_5 - \lambda f_3(X_2, X_3) X_4 - \lambda f_4(X_2) X_3 - (1-\lambda) a_5 \gamma_{A_1} X_2 \\ - \lambda f_5(X_2, X_3) X_2 - (1-\lambda) a_6 X_1 - \lambda f_6(X_1) + \lambda P_1(t, X_1, X_2, \dots, X_6),\end{aligned}\end{gathered} \tag{4.6}$$

where $0 \leq \lambda \leq 1$, a_6 is a constant chosen as positive or negative according as $f_6(X) \operatorname{sgn} X \to +\infty$ or $-\infty$ as $\|X\| \to \infty$, and a_5 is a constant chosen, in view (2.6), such that

$$\frac{\langle (\gamma_{A_1} f_5) X_2, X_2 \rangle}{\|X\|^2} \geq a_5 > \frac{1}{2a_1} \beta_3^2. \tag{4.7}$$

Clearly the system (4.6) with $\lambda = 0$, or equivalently, the equation

$$X^{(6)} + A_1 X^{(5)} + A_2 X^{(4)} + (a_5 \gamma_{A_1}) \dot{X} + a_6 X = 0.$$

has no nontrivial periodic solution of any period. Therefore, to prove the theorem, it suffices (by the Lerray-Schauder technique [1]) here to establish an a-priori bound

$$\max_{0 \leq t \leq \omega} (\|X_1(t)\| + \|X_2(t)\| + \dots + \|X_6(t)\|) \leq D \tag{4.8}$$

for all possible ω-periodic solutions $(X_1(t), X_2(t), \dots, X_6(t))$ of (4.6) with $D > 0$ a finite constant independent of λ and of solutions. Indeed, in view of the remark in [4, §4] and the form of system (4.6), (4.8) will follow once an estimate of the form

$$\max_{0 \leq t \leq \omega} (\|X_1(t)\| + \|X_2(t)\| + \|X_3(t)\| + \|X_4(t)\|) \leq D.$$

is obtained, with $D > 0$ as in (4.8).

To this end, consider the function $W_\lambda = W_\lambda(X_1, X_2, \dots, X_6)$ defined by

$$W_\lambda = \gamma_{A_1} V_\lambda, \quad V_\lambda = \lambda V_0 + V_1 \tag{4.9}$$

with V_0, V_1 given by (4.3) and (4.4). Let $(X_1, X_2, \dots, X_6)$ be an arbitrary ω-periodic solution of (4.6). Then on differentiating W_λ and using (4.9), (4.3) and (4.4) we have that

$$\begin{aligned}\dot{W}_\lambda = \langle (\gamma_{A_1} A_1) X_4, X_4 \rangle + \langle (1-\lambda) a_5 X_2 + \lambda (\gamma_{A_1} f_5) X_2, X_2 \rangle \\ + \lambda \langle (\gamma_{A_1} f_3) X_4, X_2 \rangle - \lambda \langle P_1, X_2 \rangle,\end{aligned}$$

so that by (2.4), (2.7) and (4.7)

$$\begin{aligned}\dot{W}_\lambda &\geq \frac{1}{2} a_1 \langle X_4, X_4 \rangle + a_4 \langle X_2, X_2 \rangle + \frac{1}{2} a_1 \|X_4 + \frac{1}{a_1} (\gamma_{A_1} f_3) X_2\|^2 \\ &\quad - \frac{1}{2a_1} \langle f_3^2 X_2, X_2 \rangle - \frac{3}{2} B_2 (\|X_2\|^2 + \|X_3\|^2) - B_2 \|X_2\| \\ &\geq \frac{1}{2} a_1 \langle X_4, X_4 \rangle + (a_5 - \frac{1}{2a_1} \beta_3^2) \langle X_2, X_2 \rangle - B_1 \|X_2\| \\ &\quad - \frac{3}{2} B_2 (\|X_2\|^2 + \|X_3\|^2) \\ &\geq D_1 (\|X_2\|^2 + \|X_4\|^2) + (D_1 \|X_2\|^2 - B_1 \|X_2\|) + [D_1 \|X_4\|^2 \\ &\quad - \frac{3}{2} (\|X_2\|^2 + \|X_4\|^2)]\end{aligned} \tag{4.10}$$

where $2D_1 = \min[\frac{1}{2}a_1, (a_5 - \frac{1}{2a_1}\beta_3^2)]$. But

$$\int_0^\omega \|X_{1+i}\|^2 dt \le \frac{\omega^2}{4\pi^2}\int_0^\omega \|X_{2+i}\|^2 dt, \quad i = 1, 2, \tag{4.11}$$

for any ω-periodic solution $(X_1, X_2, \dots, X_6)$ of (4.6). Thus, on integrating (4.10) and using the ω-periodicity of W_λ and (4.11), it will follow that

$$0 \ge D_1 \int_0^\omega (\|X_2\|^2 + \|X_4\|)^2) dt - D_2\omega$$

where $D_2 > 0$ is a constant chosen so that $D_1\|X_2\|^2 - B_1\|X_2\| \ge -D_2$, and B_2 is fixed such that

$$B_2 \le \frac{2}{3}\Big[\frac{\omega^2}{4\pi^2}(1 + \frac{\omega^2}{4\pi^2})\Big]^{-1} D_1.$$

Hence

$$\int_0^\omega \|X_2\|^2 dt \le D_1^{-1} D_2\omega, \quad \int_0^\omega \|X_4\|^2 dt \le D_1^{-1} D_2\omega, \tag{4.12}$$

and by periodicity of the solution, $\|X_2(t)\| \le D_3$, $\|X_3(t)\| \le D_3$ for some constant $D_3 > 0$. Multiplying (4.6) by $\operatorname{sgn} X_1$, and using the continuity of f_3, f_4, f_5 and (2.7), it can be readily shown, in view of (2.5), that $\|X_1(t_0)\| \le D_4$, $t_0 \in [0, \omega]$, and hence that $\|X_1(t)\| \le D_5$ for some constants $D_4 > 0$. The estimate for $\|X_4(t)\|$ can be obtained as in [2] and the desired estimate will follow.

We turn next to the proof of Theorems 2.3 and 2.4. Consider equation (2.8), with $P_2 \equiv 0$, in the equivalent system form

$$\begin{gathered} \dot X_i = X_{i+1}, \quad i = 1, 2, \dots, 5 \\ \dot X_6 = A_1X_6 - A_2X_5 - A_3X_4 - g_4(X_2, X_3)X_3 - g_5(X_1)X_2 - g_6(X_1, X_2, X_3), \end{gathered} \tag{4.13}$$

together with the scalar function $W = W(X_1, X_2, \dots, X_6)$ defined by $W_0 + W_1$, where

$$W_0 = \int_0^1 \langle \sigma g_5(\sigma X_1)X_1, X_1\rangle d\sigma, \tag{4.14}$$

$$\begin{aligned} W_1 &= \langle X_1, X_6 + A_1X_5 + A_2X_4 + A_3X_3\rangle - \langle X_2, X_5 + A_1X_4 + A_2X_3\rangle \\ &\quad + \langle X_3, X_4\rangle + \frac{1}{2}\langle A_1X_3, X_3\rangle - \frac{1}{2}\langle A_3X_2, X_2\rangle. \end{aligned} \tag{4.15}$$

For any nontrivial periodic solution $(X_1, X_2, \dots, X_6)$ of (4.13) of period α say, it is readily verified that

$$\dot W = \langle X_4, X_4\rangle - \langle A_2X_3, X_3\rangle - \langle g_4X_3, X_1\rangle - \langle g_6(X_1), X_1\rangle,$$

so that by (2.10) and (2.9),

$$\begin{aligned} \dot W &\ge \langle X_4, X_4\rangle - a_2\|X_3 + \frac{1}{2a_2}g_4X_1\|^2 - \langle g_6(X_1), X_1\rangle + \frac{1}{4a_2}\langle g_4^2X_1, X_1\rangle \\ &\ge \langle X_4, X_4\rangle - \langle g_6(X_1), X_1\rangle + \frac{1}{4a_2}\langle g_4^2X_1, X_1\rangle > 0. \end{aligned}$$

The conclusion of Theorem 2.3 now follows from the arguments in [1, §3].

For the proof of Theorem 2.4, consider the parameter λ-dependent system

$$\begin{aligned} \dot{X}_i &= X_{i+1}, \quad i = 1, 2, \dots, 5, \quad 0 \le \lambda \le 1, \\ \dot{X}_6 &= -A_1 X_6 - A_2 X_5 - A_3 X_4 - g_4^\lambda(X_2, X_3) X_3 \\ &\quad - \lambda g_5(X_1) X_2 - g_6^\lambda(X_1, X_2, X_3) + \lambda P_2, \end{aligned} \tag{4.16}$$

where

$$\begin{aligned} g_4^\lambda(X_2, X_3) &= (1-\lambda)\beta_4 I + \lambda g_4(X_2, X_3), \\ g_6^\lambda(X_1, X_2, X_3) &= (1-\lambda) a_6 X_1 + \lambda g_6(X_1, X_2, X_3), \end{aligned} \tag{4.17}$$

where I is the identity $n \times n$ matrix, $\beta_4 > 0$ is defined by (2.11) and $a_6 < 0$ is a constant chosen, in view of (2.12), such that

$$\frac{\langle g_6(X_1, X_2, X_3), X_1 \rangle}{\|X_1\|^2} < a_6 < \frac{1}{4a_2}\beta_4^2, \quad X_1 \neq 0 \tag{4.18}$$

The scalar function $W^\lambda = W^\lambda(X_1, X_2, \dots, X_6)$ is defined by $W^\lambda = \lambda W_0 + W_1$, with W_0, W_1 given by (4.14) and (4.15). By (4.16), (4.17), (2.10), (2.13) and (4.18) it can be verified that

$$\begin{aligned} \dot{W}^\lambda &= \langle X_4, X_4 \rangle - \langle A_2 X_3, X_3 \rangle - \langle g_4^\lambda X_3, X_1 \rangle - \langle g_6^\lambda(X_1, X_2, X_3), X_1 \rangle + \lambda \langle P_2, X_1 \rangle \\ &\ge \langle X_4, X_4 \rangle - a_2 \|X_3 + \frac{1}{2a_2} g_4^\lambda X_1\|^2 - \langle g_6^\lambda(X_1, X_2, X_3), X_1 \rangle \\ &\quad + \frac{1}{4a_2} \langle (g_4^\lambda)^2 X_1, X_1 \rangle - |\langle p_2 X_1 \rangle| \\ &\ge \|X_4\|^2 + D_6 \|X_1\|^2 - B_1^* \|X_1\| - B_2^* (\|X_1\|^2 + \|X_2\|^2 + \|X_2\|^2), \end{aligned} \tag{4.19}$$

where $D_6 \equiv \big(\frac{1}{4a_2}\beta_4^2 - a_6\big) > 0$. Now on integrating (4.19) and using the ω-periodicity of W^λ and (4.11), it will follow readily that

$$0 \ge \int_0^\omega \big(\frac{1}{2} D_6 - B_2^*\big) \|X_1\|^2 dt + \int_0^\omega \big\{1 - B_2^* \big(\frac{\omega^2}{4\pi^2} + \frac{\omega^4}{16\pi^4}\big)\big\} \|X_4\|^2 dt - D_7, \tag{4.20}$$

for some constant $D_7 > 0$ such that $\frac{1}{2} D_6 \|X_1\|^2 - B_1^* \|X_1\| \ge -D_7$. Fix B_2^* such that

$$B_2^* < \min\big[\frac{1}{2} D_6, \big(\frac{\omega}{2\pi}\big)^{-2} \big(1 + \frac{\omega^2}{4\pi^2}\big)^{-1}\big].$$

Then from (4.20),

$$\int_0^\omega \|X_1\|^2 dt \le D_7 D_8^{-1}, \quad \int_0^\omega \|X_4\|^2 dt \le D_4 D_9^{-1}, \tag{4.21}$$

where $D_8 = (\frac{1}{2} D_6 - B_2^*) > 0$, $D_9 = [1 - B_2^*(\frac{\omega^2}{4\pi^2} + \frac{\omega^4}{16\pi^4})] > 0$, and by (4.11)

$$\int_0^\omega \|X_2\|^2 dt \le D_{10}, \quad \int_0^\omega \|X_3(t)\|^2 dt \le D_{10} \tag{4.22}$$

for some $D_{10} > 0$. Since (4.21) implies the existence of a $t_0 \in [0, \omega]$ and a constant $D_{11} > 0$ such that $\|X(t_0)\| \le D_{11}$, it is clear, by periodicity, from (4.21) and (4.22), that

$$\|X_1(t)\| \le D_{12}, \quad \|X_2(t)\| \le D_{12}, \quad \|X_3(t)\| \le D_{12}.$$

for some constant $D_{12} > 0$. Using the arguments in [2], the estimate for $\|X_4(t)\|$ can be easily obtained.

5. Outline of Proof of Theorems 3.1, 3.2, 3.3, 3.4

Observe first that the results embodied in Theorems 3.1, 3.2, 3.3 and 3.4 for seventh order equations are essentially the same as those in Theorems 2.1, 2.2, 2.3 and 2.4 for sixth order equations. Since the proofs of Theorems 3.1–3.4 require the same arguments as those employed for Theorems 2.1–2.4 in §4, with some obvious modifications, we shall merely indicate here the appropriate equivalent system of equations and the scalar functions required in each case, and corresponding modifications in arguments.

We start with Theorem 3.1. The appropriate equivalent (to (3.1) with $Q_1 = 0$) system is

$$\begin{aligned} \dot{X}_i &= X_{i+1}, \quad i = 1, 2, \dots, 6 \\ \dot{X}_1 &= -A_1X_7 - A_2X_6 - A_3X_5 - A_4X_4 - \varphi_5(X_2, X_3)X_3 \\ &\quad - \varphi_6(X_1)X_2 - \varphi_7(X_1, X_2, X_3), \end{aligned} \tag{5.1}$$

and the scalar function is given by

$$V = \gamma_{A_3}U, \;\; U = U_0 + U_1 \tag{5.2}$$

where

$$\begin{aligned} U_1 &= -\langle X_1, X_7 + \sum_{k=1}^{4} A_kX_{7-k}\rangle + \langle X_2, X_6 + \sum_{k=1}^{3} A_kX_{6-k}\rangle \\ &\quad - \langle X_3, X_5 + A_1X_4\rangle + \frac{1}{2}\langle X_4, X_4\rangle + \frac{1}{2}\langle A_4X_2, X_2\rangle - \frac{1}{2}\langle A_2X_3, X_3\rangle \end{aligned} \tag{5.3}$$

$$U_0 = \int_0^1 \langle \sigma\varphi_6(\sigma X_1)X_1, X_1\rangle d\sigma \tag{5.4}$$

From (5.2), (5.3), (5.4) and (5.1) it will be clear, on proceeding as in §4, that $\dot{V} \geq 0$.

For the proof of Theorem 3.2, observe first from (3.6) that there exists a constant $a_7 > 0$ such that

$$\frac{\langle \gamma_{A_3}\varphi_7(X_1, X_2, X_3), X_1\rangle}{\|X_1\|^2} \geq a_7 > \frac{1}{4a_3}\beta_5^2, \quad X_1 \neq 0. \tag{5.5}$$

Set

$$\begin{gathered} \varphi_7^\lambda(X_1, X_2, X_3) = (1-\lambda)\gamma_{A_3}a_7X_1 + \lambda\varphi_7(X_1, X_2, X_3), \quad 0 \leq \lambda \leq 1, \\ \varphi_5^\lambda(X_2, X_3) = (1-\lambda)\beta_5 I + \lambda\varphi_5(X_2, X_3), \quad I \text{ the identity } n\times n \text{ matrix.} \end{gathered}$$

Then, by (5.5) and the fact that $\|\varphi_5(X_2, X_3)\| \leq \beta_5$, it will follow that

$$\|\varphi_5^\lambda(X_2, X_3)\| \leq \beta_5, \quad \frac{\langle \gamma_{A_3}\varphi_7^\lambda(X_1, X_2, X_3), X_1\rangle}{\|X_1\|^2} \geq a_7, \quad X_1 \neq 0. \tag{5.6}$$

With φ_5^λ, φ_7^λ defined as above and satisfying (5.6), the appropriate equivalent (to (3.1)) system to consider is

$$\begin{aligned} \dot{X}_i &= X_{i+1}, \quad i = 1, 2, \dots, 6 \\ \dot{X}_7 &= -A_1X_7 - A_2X_6 - A_3X_5 - A_4X_4 - \varphi_5^\lambda(X_2, X_3)X_3 - \lambda\varphi_6(X_1) \\ &\quad - \varphi_7^\lambda(X_1, X_2, X_3) + \lambda Q_1, \quad 0 \leq \lambda \leq 1, \end{aligned} \tag{5.7}$$

and the scalar function V^λ is defined by

$$V^\lambda = \gamma_{A_3} U, \quad U = \lambda U_0 + U_1, \tag{5.8}$$

with U_0, U_1 given by (5.4) and (5.3) respectively. Now, by proceeding as in §4, using obvious adaptations of the arguments in [3, §4], it can be readily shown that

$$\int_0^\infty \|X_1\|^2 dt \le D_{13}, \quad \int_0^\omega \|X_4\|^2 dt \le D_{13}$$

for some constant $D_{13} > 0$, and the desired a-priori bound will follow as in [2].

Turning next to Theorem 3.3, the appropriate equivalent system is

$$\begin{gathered} \dot{X}_i = X_{i+1}, \quad i = 1, 2, \dots, 6 \\ \dot{X}_7 = -A_1 X_7 - A_2 X_6 - A_3 X_5 - \psi_4(X_2, X_3, X_4) X_4 \\ - \psi_5(X_2) X_3 - \psi_6(X_2, X_3, X_4) - \psi_7(X_1) \end{gathered} \tag{5.9}$$

and the scalar function is defined by

$$V = U_0 + U_1, \tag{5.10}$$

where

$$U_0 = \int_0^1 \langle \psi_7(\sigma X_1), X_1 \rangle d\sigma + \int_0^1 \sigma \langle \psi_5(\sigma X_2) X_2, X_2 \rangle d\sigma, \tag{5.11}$$

$$\begin{aligned} U_1 = \langle X_2, X_7 + \sum_{k=1}^{3} A_k X_{7-k} \rangle - \langle X_3, X_6 + \sum_{k=1}^{2} A_k X_{6-k} \rangle + \langle X_4, X_5 \rangle \\ - \frac{1}{2} \langle A_3 X_3, X_3 \rangle + \frac{1}{2} \langle A_1 X_4, X_4 \rangle. \end{aligned} \tag{5.12}$$

It is readily shown that $\dot{V} \ge 0$.

Lastly for Theorem 3.4. Let $a_6 < 0$ be a constant chosen, in view of (3.9), such that

$$\frac{\langle \psi_6(X_2, X_3, X_4), X_2 \rangle}{\|X_2\|^2} \le a_6 < \frac{1}{4a_2} \beta_4^{2*}, \quad X_2 \ne 0, \tag{5.13}$$

and set

$$\begin{gathered} \psi_4^\lambda(X_2, X_3, X_4) = (1-\lambda) \beta_4^* I + \lambda \psi_4(X_2, X_3, X_4) \\ \psi_6^\lambda(X_2, X_3, X_4) = (1-\lambda) a_6 I + \lambda \psi_6(X_2, X_3, X_4), \quad 0 \le \lambda \le 1, \end{gathered} \tag{5.14}$$

where I the identity $n \times n$ matrix. The equivalent system is

$$\begin{gathered} X_i = X_{i+1}, \quad i = 1, 2, \dots, 6 \\ \dot{X}_7 = -A_1 X_7 - A_2 X_6 - A_3 X_5 - \psi_4^\lambda(X_2, X_3, X_4) X_4 - \lambda \psi_5(X_2) X_3 \\ - \psi_6^\lambda(X_2, X_3, X_4) - \lambda[\psi_7(X_1) - Q_2] \end{gathered} \tag{5.15}$$

and the scalar function $V^\lambda = V^\lambda(X_1, X_2, \dots, X_7)$ is defined by

$$V^\lambda = \lambda U_0 + U_1, \tag{5.16}$$

with U_0, U_1 given by (5.11) and (5.12) respectively. It can be readily shown from (5.11) to (5.16), that

$$\begin{aligned} \dot{V}^\lambda &\geq \langle X_5, X_5\rangle - \langle \psi_6^\lambda, X_2\rangle - \frac{1}{4a_2}\|\psi_4^\lambda X_2\|^2 - |\lambda\langle X_2, Q_2\rangle| \\ &\geq \|X_5\|^2 + \big(-a_6 - \frac{1}{4a_2}\beta_4^{*2}\big)\|X_2\|^2 - C_1^*(\|X_2\|) \\ &\quad - 2C_2^*(\|X_2\|^2 + \|X_3\|^2 + \|X_4\|^2), \end{aligned} \tag{5.17}$$

where $D_{12} = (-a_6 - \frac{1}{4a_2}\beta_4^{*2}) > 0$ by (5.13). Direct integration of (5.17), for any ω-periodic solution $(X_1, X_2, \dots, X_7)$ of (5.15), using the ω-periodicity of V^λ and (4.11), will yield, for some constants $D_{15} > 0$, $D_{16} > 0$,

$$\int_0^\omega \|X_2\|^2 dt \leq D_{15}, \quad \int_0^\omega \|X_5\|^2 dt \leq D_{16} \tag{5.18}$$

provided

$$C_2^* < \min[\frac{1}{4}D_{14}, \frac{2\pi}{\omega}(1 + \frac{\omega^2}{4\pi^2})^{-1}].$$

Clearly, the condition on ψ_7 in (3.11) implies the existence of a $t_0 \in [0, \omega]$ such that $\|X_1(t_0)\| \leq D_{17}$, for some constant $D_{17} > 0$. Thus, from (5.18), $\|X_1(t)\| \leq D_{16}$ for some constant $D_{18} > 0$. The rest of the proof follows from (5.18), in view of (4.11).

References

[1] H. Berektoglu; *On the periodic solutions of certain class of seventh-oder differential equations*, Periodical Mathematica Hungarica 24, 1(1992), 13-22.

[2] J. O .C. Ezeilo; *Periodic solutions of certain sixth order differential equations*, J. Nigeria Math. Soc. Vol. 1 (1982), 1-9.

[3] H. O. Tejumola; *Instability and periodic solutions of certain nonlinear differential equations of orders six and seven*, Proc. National Mathematical Centre, Abuja, Nigeria -Ordinary Differential Equations - Vol. 1, No. 1 (2000), 56-65.

[4] Ercan Tunç; *A result about periodic solutions of a class of seventh-order vector differential equations*, Analele Universitatii de Vest, Timişoara Seria Matematica-Informatica , Vol. XLIII (2005), fasc. 1, 143-150.

[5] Ercan Tunç; *Instability of solutions of certain nonlinear vector differential equations of third order*, Electronic Journal of Differential Equations, Vol. 2005 (2005), No. 51, pp. 1–6.

[6] Ercan Tunç; *Periodic solutions of a certain vector differential equation of sixth order*, The Arabian Journal for Science and Engineering, Volume 33 (2008), Number 1A, 107-112.

Victor F. Payne
Department of Mathematics, University of Ibadan
 Ibadan Nigeria
E-mail address: vpayne34@gmail.com

Haroon Oladipo Tejumola
Department of Mathematical Sciences, Redeemer's University, Mowe, Ogun State, Nigeria
E-mail address: haroontej@yahoo.co.uk