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STRICHARTZ ESTIMATES ON α-MODULATION SPACES

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ABSTRACT. In this article, we consider some dispersive equations, including Schrödinger equations, nonelliptic Schrödinger equations, and wave equations. We develop some Strichartz estimates in the frame of α-modulation spaces.

1. Introduction

We study the Cauchy problem for the Schrödinger type equation

$$\begin{gathered} iu_t + (-\Delta)^{\beta/2} u = F \\ u(0,x) = u_0, \end{gathered} \tag{1.1}$$

the Cauchy problem for nonelliptic Schrödinger equation

$$\begin{gathered} iv_t + \psi(D)v = F \\ v(0,x) = v_0 \end{gathered}, \tag{1.2}$$

where $\psi(\xi) = \Sigma_{l=1}^n \pm |\xi_l|^\beta$, and the Cauchy problem for wave equation

$$\begin{gathered} w_{tt} - \Delta w = F \\ w(0,x) = w_0, w_t(0,x) = w_1. \end{gathered} \tag{1.3}$$

The initial data belongs to the α-modulation space $M_{2,1}^{0,\alpha}$, and we use F to denote some nonlinear terms.

We recall Duhamel's formula for above three dispersive equations. The solution to (1.1) is

$$u(t,x) = e^{it(-\Delta)^{\beta/2}} u_0 - i \int_0^t e^{i(t-s)(-\Delta)^{\beta/2}} F(s) ds. \tag{1.4}$$

The solution to (1.2) is

$$v(t,x) = e^{it\psi(D)} v_0 - i \int_0^t e^{i(t-s)\psi(D)} F(s) ds. \tag{1.5}$$

The solution to (1.3) is

$$w(t,x) = \cos(t\sqrt{-\Delta}) w_0 + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}} w_1 + \int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}} F(s) ds. \tag{1.6}$$

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There are many publications about the theoretical and applied aspects of the Schrödinger equation and the wave equation; see for example Tao [14] and Sogge [11] for a nice introduction. We refer the reader to [1, 2, 3, 4, 10, 16, 17, 18] for the study of modulation space and dispersive equation.

In this paper, we are concerned mainly with the Strichartz estimates for the solutions of the above three equations. The original estimates are due to Strichartz [13], and they became fundamental and important tools in the study of dispersive equations. The theory of Strichartz estimates has also been studied by many authors. One is referred to [6], [7] and [9] for classical Strichartz estimates. We also refer the readers to [5] and [20] for the Strichartz estimates in the frame of Wiener amalgam spaces and modulation spaces. The following lemma is a basic Strchartz estimate, proved by Keel-Tao [9], that we will use it frequently in our proofs.

Definition 1.1. An exponent pair (r,p) is called σ-admissible if $r,p\geq 2$, $(r,p,n)\neq (2,\infty,2)$ and

$$\frac{1}{r}+\frac{\sigma}{p}\leq\frac{\sigma}{2}\,. \tag{1.7}$$

If the equality holds, we say that (r,p) is sharp σ-admissible, otherwise we say that (r,p) is nonsharp σ-admissible. If $\sigma>1$ we say the sharp σ-admissible pair

$$(2,\frac{2\sigma}{\sigma-1}) \tag{1.8}$$

is an endpoint.

Next we have the Strichartz estimates.

Lemma 1.2 ([9]). *Let $\{U(t)\}_{t\in\mathbb{R}}$ be a semigroup of operators that obey energy estimate*

$$\|U(t)f\|_{L_x^2}\lesssim\|f\|_{L_x^2} \tag{1.9}$$

and dispersive estimate

$$\|U(t)(U(s))^*g\|_{L_x^\infty}\lesssim|t-s|^{-\sigma}\|g\|_{L_x^1}. \tag{1.10}$$

Then the estimates

$$\|U(t)f\|_{L_t^rL_x^p}\lesssim\|f\|_{L_x^2}, \tag{1.11}$$

$$\|\int_{\mathbb{R}}(U(s))^*F(s)ds\|_{L_x^2}\lesssim\|F\|_{L_t^{r'}L_x^{p'}}, \tag{1.12}$$

$$\|\int_{s<t}U(t)(U(s))^*F(s)ds\|_{L_t^rL_x^p}\lesssim\|F\|_{L_t^{\tilde{r}'}L_x^{\tilde{p}'}} \tag{1.13}$$

hold for all sharp σ-admissible pairs (r,p) and $(\tilde{r},\tilde{p})$. If $U(t)$ satisfies stronger condition

$$\|U(t)(U(s))^*g\|_{L_x^\infty}\lesssim(1+|t-s|)^{-\sigma}\|g\|_{L_x^1}, \tag{1.14}$$

then the above estimates hold for all σ-admissible pairs.

In 2012, Zhang[20] established some Strichartz estimates in the frame of modulation spaces, here we will study the estimates in the frame of α-modulation spaces, our theorems will cover the estimates in [20]. First, we recall the definition of α-modulation space.

Definition 1.3. Let $\rho(\xi)$ be a smooth radial bump supported in the ball $|\xi|<2$, satisfying $\rho(\xi)=1$ as $|\xi|\le 1$. For any $k\in\mathbb{Z}^n$, we set

$$\rho_k^\alpha(\xi)=\rho\Big(\frac{\xi-\langle k\rangle^{\alpha/(1-\alpha)}k}{C\langle k\rangle^{\alpha/(1-\alpha)}}\Big), \tag{1.15}$$

and denote

$$\eta_k^\alpha(\xi)=\rho_k^\alpha(\xi)\Big(\sum_{l\in\mathbb{Z}^n}\rho_l^\alpha(\xi)\Big)^{-1}. \tag{1.16}$$

For any $k\in\mathbb{Z}^n$, we define

$$\Box_k^\alpha=\mathscr{F}^{-1}\eta_k^\alpha\mathscr{F}. \tag{1.17}$$

When $\alpha\in[0,1)$, the α-modulation space associated with the above decomposition is defined by

$$M_{p,q}^{s,\alpha}(\mathbb{R}^n)=\big\{f\in\mathcal{S}'(\mathbb{R}^n):\|f\|_{M_{p,q}^{s,\alpha}(\mathbb{R}^n)}=\Big(\sum_{k\in\mathbb{Z}^n}\langle k\rangle^{sq/(1-\alpha)}\|\Box_k^\alpha f\|_p^q\Big)^{1/q}<\infty\big\}$$

with the usual modifications when $q=\infty$.

The α-modulation space was introduced by Gröbner [8], and it is an intermediate space between modulation space and Besov space, when $\alpha=0$ it is the usual modulation space, and the Besov space can be regarded as the case $\alpha\to 1$. A comprehensive study of α-modulation space has been done in [19]. We also need some frequency decomposition spaces which are similar with the spaces defined in [16].

Definition 1.4. If $X=L_t^rL_x^p(\mathbb{R}\times\mathbb{R}^n)$ $(1\le r,p,q<\infty)$, we denote

$$l_{\Box^\alpha}^{s,q}(X)=\big\{u\in\mathcal{S}'(\mathbb{R}^{n+1}):\|u\|_{l_{\Box^\alpha}^{s,q}(X)}=\Big(\sum_{k\in\mathbb{Z}^n}\langle k\rangle^{sq/(1-\alpha)}\|\Box_k^\alpha u\|_X^q\Big)^{1/q}<\infty\big\}$$

with the usual modifications when $q=\infty$.

Using Minkowski's inequality, we can verify that for $r\ge q$,

$$\|f\|_{L_t^r(\mathbb{R},M_{p,q}^{\alpha,s})}\le\|f\|_{l_{\Box^\alpha}^{s,q}(L_t^rL_x^p)}. \tag{1.18}$$

In Section 2, we will give some basic definitions and properties associated with α-modulation spaces, and recall some basic estimates of oscillatory integral which are useful in our proof. We will give the proof of main theorems in Section 3. Standard techniques involving TT^* method and duality argument will be used to establish some Strichartz estimates. In the case when (r,p) and $(\tilde r,\tilde p)$ are sharp, we will use a dilation argument, based on some basic Strichartz estimates. Now, we present our main results. First the Strichartz estimates for Schrödinger equation:

Theorem 1.5. *Suppose $s\in\mathbb{R}$, $q\ge1$, $\alpha\in[0,1)$, $\beta\in(0,2]$ and $\beta\ne1$, (p,r) and $(\tilde p,\tilde r)$ are both $\frac{n}{2}$-admissible pairs, then the solution of (1.1) satisfies*

$$\|u(t,x)\|_{l_{\Box^\alpha}^{s,q}(L_t^rL_x^p)}\lesssim\|u_0\|_{M_{2,q}^{s+\delta(r,p),\alpha}}+\|F\|_{l_{\Box^\alpha}^{s+\delta(r,p)+\delta(\tilde r,\tilde p),q}(L_t^{\tilde r'}L_x^{\tilde p'})} \tag{1.19}$$

where $\delta(r,p)=\alpha(\frac{n}{2}-\frac{2}{r}-\frac{n}{p})+(2-\beta)\frac{1}{r}$. More precisely, we have

$$\|e^{it(-\Delta)^{\beta/2}}u_0\|_{l_{\Box^\alpha}^{s,q}(L_t^rL_x^p)}\lesssim\|u_0\|_{M_{2,q}^{s+\delta(r,p),\alpha}}, \tag{1.20}$$

$$\|\int_{s<t}e^{i(t-s)(-\Delta)^{\beta/2}}F(s)ds\|_{l_{\Box^\alpha}^{s,q}(L_t^rL_x^p)}\lesssim\|F\|_{l_{\Box^\alpha}^{s+\delta(r,p)+\delta(\tilde r,\tilde p),q}(L_t^{\tilde r'}L_x^{\tilde p'})}. \tag{1.21}$$

Next we have the Strichartz estimates for nonelliptic Schrödinger equation:

Theorem 1.6. *Suppose $s \in \mathbb{R}$, $q \geq 1$, $\alpha \in [0,1)$, $\beta \in (0,2]$ and $\beta \neq 1$, (p,r) and $(\tilde{p},\tilde{r})$ are both $\frac{n}{2}$-admissible pairs, then the solution of (1.2) satisfies*

$$\|v(t,x)\|_{l^{s,q}_{\square^\alpha}(L^r_t L^p_x)} \lesssim \|v_0\|_{M^{s+\delta(r,p),\alpha}_{2,q}} + \|F\|_{l^{s+\delta(r,p)+\delta(\tilde{r},\tilde{p}),q}_{\square^\alpha}(L^{\tilde{r}'}_t L^{\tilde{p}'}_x)} \tag{1.22}$$

where $\delta(r,p) = \alpha(\frac{n}{2} - \frac{2}{r} - \frac{n}{p}) + (2-\beta)\frac{1}{r}$. More precisely, we have

$$\|e^{it\psi(D)} v_0\|_{l^{s,q}_{\square^\alpha}(L^r_t L^p_x)} \lesssim \|v_0\|_{M^{s+\delta(r,p),\alpha}_{2,q}}, \tag{1.23}$$

$$\|\int_{s<t} e^{i(t-s)\psi(D)} F(s)ds\|_{l^{s,q}_{\square^\alpha}(L^r_t L^p_x)} \lesssim \|F\|_{l^{s+\delta(r,p)+\delta(\tilde{r},\tilde{p}),q}_{\square^\alpha}(L^{\tilde{r}'}_t L^{\tilde{p}'}_x)}. \tag{1.24}$$

Next we have the Strichartz estimates for wave equation:

Theorem 1.7. *Suppose $s \in \mathbb{R}$, $q \geq 1$, $\alpha \in [0,1)$, (p,r) and $(\tilde{p},\tilde{r})$ are both $\frac{n-1}{2}-$admissible pairs, if $\frac{n}{2} - \frac{n}{p} - \frac{1}{r} - 1 > 0$ and $n - 1 - \frac{n}{p} - \frac{1}{r} - \frac{n}{\tilde{p}} - \frac{1}{\tilde{r}} > 0$, then the solution of (1.3) satisfies*

$$\|w\|_{l^{s,q}_{\square^\alpha}(L^r_t L^p_x)} \lesssim \|w_0\|_{M^{s+\theta(r,p),\alpha}_{2,q}} + \|w_1\|_{M^{s+\theta(r,p)-1,\alpha}_{2,q}} + \|F\|_{l^{s+\theta(r,p)+\theta(\tilde{r},\tilde{p})-1,q}_{\square^\alpha}(L^{\tilde{r}'}_t L^{\tilde{p}'}_x)}$$

where $\theta(r,p) = \alpha\frac{n}{n-1}(\frac{n-1}{2} - \frac{2}{r} - \frac{n-1}{p}) + \frac{n+1}{n-1}\frac{1}{r}$. More precisely, we have

$$\|\cos(t\sqrt{-\Delta})w_0\|_{l^{s,q}_{\square^\alpha}(L^r_t L^p_x)} \lesssim \|w_0\|_{M^{s+\theta(r,p),\alpha}_{2,q}}, \tag{1.25}$$

$$\|\frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}} w_1\|_{l^{s,q}_{\square^\alpha}(L^r_t L^p_x)} \lesssim \|w_1\|_{M^{s+\theta(r,p)-1,\alpha}_{2,q}}, \tag{1.26}$$

$$\|\int_{s<t} \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}} F(s)ds\|_{l^{s,q}_{\square^\alpha}(L^r_t L^p_x)} \lesssim \|F\|_{l^{s+\theta(r,p)+\theta(\tilde{r},\tilde{p})-1,q}_{\square^\alpha}(L^{\tilde{r}'}_t L^{\tilde{p}'}_x)}. \tag{1.27}$$

We must point out that our results can't be deduced directly by a simple interpolation between the modulation space($\alpha = 0$) and the Besov space, since in [19], the authors have pointed out that $\alpha-$modulation space can't be reformulated by interpolations between modulation and Besov spaces at least for some special cases.

2. Preliminaries

We will often use the notation $X \lesssim Y$ whenever there exists some constant C so that $X \leq CY$, C can depend on $n, p, r, \tilde{p}, \tilde{r}, \alpha, \beta$. For $k = (k_1, k_2, \dots, k_n) \in \mathbb{Z}^n, \langle k \rangle := (1 + |k|^2)^{\frac{1}{2}}$. We denote by $\mathcal{S} := \mathcal{S}(\mathbb{R}^n)$ the Schwartz space and $\mathcal{S}' := \mathcal{S}'(\mathbb{R}^n)$ the tempered distribution space. We use $L^r_t(\mathbb{R})$ to denote the Banach space of functions $f : \mathbb{R} \to \mathbb{C}$ whose norm

$$\|f\|_{L^r_t(\mathbb{R})} := \Big(\int_{\mathbb{R}} |f(t)|^r dt\Big)^{1/r} \tag{2.1}$$

is finite, with the usual modifications when $r = \infty$. We use $L^r_t L^p_x(\mathbb{R} \times \mathbb{R}^n)$ to denote the spacetime norm

$$\|F\|_{L^r_t L^p_x(\mathbb{R}\times\mathbb{R}^n)} = \Big(\int_{\mathbb{R}} \|F\|^r_{L^p_x} dt\Big)^{1/r} \tag{2.2}$$

with the usual modifications when p, q or r is infinity.

In our proofs, the almost orthogonality property will be used frequently, this property is independent of α. Here we give a proof which is different from [19] and seems more regular. Firstly, we establish following position lemma.

Lemma 2.1. *For every $x, y \in \mathbb{R}^n$, we have the following*

$$\begin{aligned}\min(\langle x\rangle^{\alpha/(1-\alpha)}, \langle y\rangle^{\alpha/(1-\alpha)})|x-y| &\lesssim |\langle x\rangle^{\alpha/(1-\alpha)}x - \langle y\rangle^{\alpha/(1-\alpha)}y| \\ &\lesssim \max(\langle x\rangle^{\alpha/(1-\alpha)}, \langle y\rangle^{\alpha/(1-\alpha)})|x-y|.\end{aligned}$$

Particularly, if we choose $y = k$, $|x-y| = r$, then

$$\begin{aligned}\min(\langle x\rangle^{\alpha/(1-\alpha)}, \langle k\rangle^{\alpha/(1-\alpha)})r &\lesssim |\langle x\rangle^{\alpha/(1-\alpha)}x - \langle k\rangle^{\alpha/(1-\alpha)}k| \\ &\lesssim \max(\langle x\rangle^{\alpha/(1-\alpha)}, \langle k\rangle^{\alpha/(1-\alpha)})r,\end{aligned}$$

so we have

$$|\langle x\rangle^{\alpha/(1-\alpha)}x - \langle k\rangle^{\alpha/(1-\alpha)}k| \sim \langle k\rangle^{\alpha/(1-\alpha)} \quad \text{if } |k| \longrightarrow \infty \text{ and } |x-k| = r. \tag{2.3}$$

Proof. By the symmetry of x and y, we only prove the case $|x| \le |y|$. Let

$$h(t) = |x - ty|^2, \tag{2.4}$$

and derivative it, we have

$$h'(t) = \sum_{i=1}^{n} 2ty_i^2 - \sum_{i=1}^{n} 2x_iy_i. \tag{2.5}$$

Using the fact $|x| \le |y|$ and Cauchy-Schwartz inequality, we verify that

$$h'(t) \ge 0 \tag{2.6}$$

when $t \ge 1$. Using this inequality, we conclude

$$\begin{aligned}|\langle x\rangle^{\alpha/(1-\alpha)}x - \langle y\rangle^{\alpha/(1-\alpha)}y| &= \langle x\rangle^{\alpha/(1-\alpha)}|x - \frac{\langle y\rangle^{\alpha/(1-\alpha)}}{\langle x\rangle^{\alpha/(1-\alpha)}}y| \\ &\ge \langle x\rangle^{\alpha/(1-\alpha)}|x-y| \\ &= \min(\langle x\rangle^{\alpha/(1-\alpha)}, \langle y\rangle^{\alpha/(1-\alpha)})|x-y|.\end{aligned} \tag{2.7}$$

On the other hand,

$$|\langle x\rangle^{\alpha/(1-\alpha)}x - \langle y\rangle^{\alpha/(1-\alpha)}y| \le |\langle x\rangle^{\alpha/(1-\alpha)} - \langle y\rangle^{\alpha/(1-\alpha)}||x| + |\langle y\rangle^{\alpha/(1-\alpha)}||x-y|.$$

By mean-valued theorem and the fact $|x| \le |y|$, one can verify that

$$|\langle x\rangle^{\alpha/(1-\alpha)} - \langle y\rangle^{\alpha/(1-\alpha)}| \lesssim |x-y|\langle y\rangle^{\frac{\alpha}{1-\alpha}-1}. \tag{2.8}$$

Since $|x| \le |y| \le \langle y\rangle$, so we have

$$|\langle x\rangle^{\alpha/(1-\alpha)} - \langle y\rangle^{\alpha/(1-\alpha)}||x| \lesssim |x-y|\langle y\rangle^{\alpha/(1-\alpha)}, \tag{2.9}$$

so

$$\begin{aligned}|\langle x\rangle^{\alpha/(1-\alpha)}x - \langle y\rangle^{\alpha/(1-\alpha)}y| &\lesssim |\langle x\rangle^{\alpha/(1-\alpha)} - \langle y\rangle^{\alpha/(1-\alpha)}||x| + |\langle y\rangle^{\alpha/(1-\alpha)}||x-y| \\ &\lesssim |\langle y\rangle^{\alpha/(1-\alpha)}||x-y| \\ &= \max(\langle x\rangle^{\alpha/(1-\alpha)}, \langle y\rangle^{\alpha/(1-\alpha)})|x-y|.\end{aligned}$$

□

By the position lemma, we can conclude that there exists a positive constant c_1 independent of x and k, such that

$$c_1\langle k\rangle^{\alpha/(1-\alpha)}|x-k| \le |\langle x\rangle^{\alpha/(1-\alpha)}x - \langle k\rangle^{\alpha/(1-\alpha)}k|. \tag{2.10}$$

In fact, if $|k| > 2|x|$, we have

$$|x-k| \le |x| + |k| \le \frac{3}{2}|k|, \tag{2.11}$$

then

$$\begin{aligned} |\langle x\rangle^{\alpha/(1-\alpha)}x - \langle k\rangle^{\alpha/(1-\alpha)}k| &\ge |\langle k\rangle^{\alpha/(1-\alpha)}k| - |\langle x\rangle^{\alpha/(1-\alpha)}x| \\ &\ge |\langle k\rangle^{\alpha/(1-\alpha)}k| - \frac{1}{2}|\langle k\rangle^{\alpha/(1-\alpha)}k| = \frac{1}{2}|\langle k\rangle^{\alpha/(1-\alpha)}k| \\ &\gtrsim \frac{1}{2}\cdot\frac{2}{3}|\langle k\rangle^{\alpha/(1-\alpha)}||x-k| = \frac{1}{3}|\langle k\rangle^{\alpha/(1-\alpha)}||x-k|. \end{aligned}$$

If $|k| \le 2|x|$, then

$$\begin{aligned} |\langle x\rangle^{\alpha/(1-\alpha)}x - \langle k\rangle^{\alpha/(1-\alpha)}k| &\ge \min(\langle x\rangle^{\alpha/(1-\alpha)}, \langle k\rangle^{\alpha/(1-\alpha)})|x-k| \\ &\ge \min(c_\alpha\langle k\rangle^{\alpha/(1-\alpha)}, \langle k\rangle^{\alpha/(1-\alpha)})|x-k| \\ &\gtrsim \langle k\rangle^{\alpha/(1-\alpha)}|x-k|. \end{aligned} \tag{2.12}$$

Similarly, for a fixed constant $G > 0$, we can find a constant c_2 only depend on G, such that if $|x-k| < G$,

$$|\langle x\rangle^{\alpha/(1-\alpha)}x - \langle k\rangle^{\alpha/(1-\alpha)}k| \le c_2\langle k\rangle^{\alpha/(1-\alpha)}|x-k|. \tag{2.13}$$

This can be easily concluded by position lemma.

We set a map J_α from $\mathbb{R}^n$ to $\mathbb{R}^n$

$$J_\alpha(x) = \langle x\rangle^{\alpha/(1-\alpha)}x. \tag{2.14}$$

Using inequality (2.10), we can take R sufficiently large such that $Rc_1 > 2C$, thus

$$\operatorname{supp}\eta_k^\alpha \subset B(\langle k\rangle^{\alpha/(1-\alpha)}k, c_1\langle k\rangle^{\alpha/(1-\alpha)}R) \subset J_\alpha(B(k,R)). \tag{2.15}$$

Similarly, we can choose r small enough such that $rc_2 < C$, thus

$$J_\alpha(B(k,r)) \subset B(\langle k\rangle^{\alpha/(1-\alpha)}k, c_2\langle k\rangle^{\alpha/(1-\alpha)}r) \subset \operatorname{supp}\eta_k^\alpha. \tag{2.16}$$

So we have

$$J_\alpha(B(k,r)) \subset \operatorname{supp}\eta_k^\alpha \subset J_\alpha(B(k,R)), \tag{2.17}$$

and

$$\begin{aligned} \{(i,j) : \operatorname{supp}\eta_i^\alpha \cap \operatorname{supp}\eta_j^\alpha \ne \emptyset\} &\subset \{(i,j) : J_\alpha(B(i,R)) \cap J_\alpha(B(j,R)) \ne \emptyset\} \\ &= \{(i,j) : (B(i,R) \cap B(j,R)) \ne \emptyset\}. \end{aligned} \tag{2.18}$$

So, in some sense, the α-modulation space is as regular as modulation space up to a transform J_α. We recall some estimates of oscillatory integrals, which can be deduced by principle of stationary phase. One can find the methods in [12] and [11]. We use $\varphi(\xi)$ to denote the symbol of the Littlewood-Paley operator Δ_0.

Lemma 2.2. *If* $0 < \beta \ne 1$, *then*

$$\Big|\int_{\mathbb{R}^n} e^{it|\xi|^\beta}\varphi(\xi)e^{ix\cdot\xi}d\xi\Big| \lesssim |t|^{-n/2}. \tag{2.19}$$

If $\beta = 1$, *then*

$$\Big|\int_{\mathbb{R}^n} e^{it|\xi|}\varphi(\xi)e^{ix\cdot\xi}d\xi\Big| \lesssim |t|^{(n-1)/2}. \tag{2.20}$$

We omit the proof of the above lemma, and refer the reader to [20]. This lemma can be also concluded by a lemma by Littman [17]. Then we will show another inequality which will be used in the proof of Theorem 1.2, one can find the proof in [20] for the case that $\alpha = 0$.

Lemma 2.3. *If* $0 < \beta \le 2$ *and* $\beta \neq 1$*, then*

$$\Big|\int_{\mathbb{R}^n} e^{it\sum_{l=1}^n \pm|\xi_l|^\beta}\eta_k^\alpha(\xi)e^{ix\cdot\xi}d\xi\Big| \lesssim \langle k\rangle^{\frac{2-\beta}{1-\alpha}\frac{n}{2}}|t|^{-n/2}. \tag{2.21}$$

Proof. We only prove the case $\beta < 2$. One can easily find a one dimension smooth bump function $\phi(\xi)$ such that $\prod_{l=1}^n \phi(\frac{\xi_l-\langle k\rangle^{\alpha/(1-\alpha)}k_l}{\langle k\rangle^{\alpha/(1-\alpha)}})\eta_k^\alpha = \eta_k^\alpha$ for every $k \in \mathbb{Z}^n$. Then we have

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^n} e^{it\sum_{l=1}^n \pm|\xi_l|^\beta}\prod_{l=1}^n \phi(\frac{\xi_l-\langle k\rangle^{\alpha/(1-\alpha)}k_l}{\langle k\rangle^{\alpha/(1-\alpha)}})e^{ix\cdot\xi}d\xi_l\Big| \\
&= \prod_{l=1}^n\Big|\int_{\mathbb{R}} e^{i(x_l\cdot\xi_l \pm t|\xi_l|^\beta)}\phi(\frac{\xi_l-\langle k\rangle^{\alpha/(1-\alpha)}k_l}{\langle k\rangle^{\alpha/(1-\alpha)}})d\xi_l\Big|
\end{aligned} \tag{2.22}$$

and

$$\begin{aligned}
&\Big|\int_{\mathbb{R}} e^{i(x_l\cdot\xi_l \pm t|\xi_l|^\beta)}\phi(\frac{\xi_l-\langle k\rangle^{\alpha/(1-\alpha)}k_l}{\langle k\rangle^{\alpha/(1-\alpha)}})d\xi_l\Big| \\
&\lesssim \Big|\int_{\mathbb{R}^+} e^{iP_+(\xi_l)}\phi(\frac{\xi_l-\langle k\rangle^{\alpha/(1-\alpha)}k_l}{\langle k\rangle^{\alpha/(1-\alpha)}})d\xi_l\Big| + \Big|\int_{\mathbb{R}^+} e^{iP_-(\xi_l)}\phi(\frac{-\xi_l-\langle k\rangle^{\alpha/(1-\alpha)}k_l}{\langle k\rangle^{\alpha/(1-\alpha)}})d\xi_l\Big|.
\end{aligned}$$

When $k_l \neq 0$ (k_l is large), we have $|P_\pm''(\xi_l)| \gtrsim t\big(\langle k\rangle^{\alpha/(1-\alpha)}k_l\big)^{\beta-2}$, we use Van de Corputs lemma to deduce

$$\begin{aligned}
&\Big|\int_{\mathbb{R}} e^{i(x_l\cdot\xi_l \pm t|\xi_l|^\beta)}\phi(\frac{\xi_l-\langle k\rangle^{\alpha/(1-\alpha)}k_l}{\langle k\rangle^{\alpha/(1-\alpha)}})d\xi_l\Big| \\
&\lesssim \Big\|\Big(\phi(\frac{\xi_l-\langle k\rangle^{\alpha/(1-\alpha)}k_l}{\langle k\rangle^{\alpha/(1-\alpha)}})\Big)'\|_{L^1}\Big(\langle k\rangle^{\alpha/(1-\alpha)}k_l\Big)^{(2-\beta)/2}|t|^{-1/2} \\
&\lesssim \Big(\langle k\rangle^{\alpha/(1-\alpha)}k_l\Big)^{(2-\beta)/2}|t|^{-1/2} \lesssim \Big(\langle k\rangle^{\frac{1}{1-\alpha}}\Big)^{(2-\beta)/2}|t|^{-1/2}.
\end{aligned} \tag{2.23}$$

When $k_l = 0$ (k_l small), we have

$$\begin{aligned}
&\Big|\int_{\mathbb{R}} e^{i(x_l\cdot\xi_l \pm t|\xi_l|^\beta)}\phi(\frac{\xi_l}{\langle k\rangle^{\alpha/(1-\alpha)}})d\xi_l\Big| \\
&= \langle k\rangle^{\alpha/(1-\alpha)}\Big|\int_{\mathbb{R}} e^{i(\langle k\rangle^{\alpha/(1-\alpha)}x_l\cdot\xi_l \pm \langle k\rangle^{\frac{\alpha\beta}{1-\alpha}}t|\xi_l|^\beta)}\phi(\xi_l)d\xi_l\Big|.
\end{aligned}$$

Let $\varphi_j(\xi) = \varphi(\xi/2^j)$ be the symbol of one dimension Littlewood-Paley operator Δ_j, we use Lemma 2.2(one dimension version) and dilation to deduce that

$$\langle k\rangle^{\alpha/(1-\alpha)}\Big|\int_{\mathbb{R}} e^{i(\langle k\rangle^{\alpha/(1-\alpha)}x_l\cdot\xi_l \pm \langle k\rangle^{\frac{\alpha\beta}{1-\alpha}}t|\xi_l|^\beta)}\varphi_j(\xi_l)d\xi_l\Big| \lesssim 2^{j(1-\frac{\beta}{2})}\langle k\rangle^{\frac{\alpha}{1-\alpha}\frac{2-\beta}{2}}|t|^{-1/2},$$

then

$$\langle k\rangle^{\alpha/(1-\alpha)}\Big|\int_{\mathbb{R}} e^{i(\langle k\rangle^{\alpha/(1-\alpha)}x_l\cdot\xi_l \pm \langle k\rangle^{\frac{\alpha\beta}{1-\alpha}}t|\xi_l|^\beta)}\phi(\xi_l)d\xi_l\Big|$$

$$
\begin{aligned}
&\lesssim \sum_{j=-\infty}^{0} 2^{j(1-\frac{\beta}{2})} \langle k\rangle^{\frac{\alpha}{1-\alpha}\frac{2-\beta}{2}} |t|^{-1/2} \\
&\lesssim \langle k\rangle^{\frac{\alpha}{1-\alpha}\frac{2-\beta}{2}} |t|^{-1/2} \\
&\lesssim \Big(\langle k\rangle^{\frac{1}{1-\alpha}}\Big)^{(2-\beta)/2} |t|^{-1/2}.
\end{aligned}
$$

Using above estimates and the fact that $\|\mathscr{F}^{-1}\eta_k^\alpha\|_{L^1} \lesssim 1$, we complete the proof. □

3. Proof of main results

Proof of Theorem 1.5. We prove only the case that $\beta < 2$. Let $\varphi_j = \varphi(\frac{\xi}{2^j})$ be the symbol of the littlewood-paley operator Δ_j. Using dilation and Lemma 2.2, we can deduce that

$$
\Big|\int_{\mathbb{R}^n} e^{it|\xi|^\beta} \varphi(\frac{\xi}{2^j}) e^{ix\cdot\xi} d\xi\Big| \lesssim 2^{jn(1-\frac{\beta}{2})}|t|^{-n/2}, \tag{3.1}
$$

so we have

$$
\begin{aligned}
\|\Box_0^\alpha e^{it(-\Delta)^{\beta/2}} f\|_{L_x^\infty} &\lesssim \sum_{j\le c} \|\Delta_j \Box_0^\alpha e^{it(-\Delta)^{\beta/2}} f\|_{L_x^\infty} \\
&\lesssim \sum_{j\le c} 2^{jn(1-\frac{\beta}{2})}|t|^{-n/2}\|\Box_0^\alpha f\|_{L_x^1} \\
&\lesssim |t|^{-n/2}|\Box_0^\alpha f\|_{L_x^1} \lesssim |t|^{-n/2}\|f\|_{L_x^1}.
\end{aligned} \tag{3.2}
$$

For $k \ne 0$, we have

$$
\begin{aligned}
\|\Box_k^\alpha e^{it(-\Delta)^{\beta/2}} f\|_{L_x^\infty} &\lesssim \sum_{\operatorname{supp}\varphi_j \cap \operatorname{supp}\eta_k^\alpha \ne \emptyset} \|\Delta_j \Box_k^\alpha e^{it(-\Delta)^{\beta/2}} f\|_{L_x^\infty} \\
&\lesssim \sum_{\operatorname{supp}\varphi_j \cap \operatorname{supp}\eta_k^\alpha \ne \emptyset} 2^{jn(1-\frac{\beta}{2})}|t|^{-n/2}\|\Box_k^\alpha f\|_{L_x^1}.
\end{aligned} \tag{3.3}
$$

Using the almost orthogonality property and $2^j \sim \langle k\rangle^{\frac{1}{1-\alpha}}$, we can deduce that

$$
\|\Box_k^\alpha e^{it(-\Delta)^{\beta/2}} f\|_{L_x^\infty} \lesssim \Big(\langle k\rangle^{\frac{\beta-2}{1-\alpha}}|t|\Big)^{-n/2}\|f\|_{L_x^1}. \tag{3.4}
$$

On the other hand, we have

$$
\Big|\int_{\mathbb{R}^n} e^{it|\xi|^\beta} \eta_k^\alpha(\xi) e^{ix\cdot\xi} d\xi\Big| \lesssim \|\eta_k^\alpha\|_{L^1} \lesssim \langle k\rangle^{\frac{\alpha}{1-\alpha}n} = \Big(\langle k\rangle^{\frac{\alpha}{1-\alpha}(-2)}\Big)^{-n/2}. \tag{3.5}
$$

The above two estimates imply that

$$
\|\Box_k^\alpha e^{it(-\Delta)^{\beta/2}} f\|_{L_x^\infty} \lesssim \Big(\langle k\rangle^{\frac{\alpha}{1-\alpha}(-2)} + \langle k\rangle^{\frac{\beta-2}{1-\alpha}}|t|\Big)^{-n/2}\|f\|_{L_x^1}. \tag{3.6}
$$

When $k = 0$, we have

$$
\|\Box_0^\alpha e^{it(-\Delta)^{\beta/2}} f\|_{L_x^\infty} \lesssim (1+|t|)^{-n/2}\|f\|_{L_x^1} \tag{3.7}
$$

and the energy estimate

$$
\|\Box_0^\alpha e^{it(-\Delta)^{\beta/2}} f\|_{L_x^2} \lesssim \|f\|_{L_x^2}, \tag{3.8}
$$

we can use Lemma 1.2 to deduce that

$$
\|\Box_0^\alpha e^{it(-\Delta)^{\beta/2}} f\|_{L_t^r L_x^p} \lesssim \|f\|_{L_x^2},
$$

$$\|\Box_0^\alpha \int_{s<t} e^{i(t-s)(-\Delta)^{\beta/2}} F(s) ds\|_{L_t^r L_x^p} \lesssim \|F\|_{L_t^{\tilde{r}'} L_x^{\tilde{p}'}}.$$

Using Lemma 2.2, one can easily verify that

$$\|\Delta_0 e^{it(-\Delta)^{\beta/2}} f\|_{L_x^\infty} \lesssim (1+|t|)^{-n/2} \tag{3.9}$$

and deduce that

$$\|\Delta_0 e^{it(-\Delta)^{\beta/2}} f\|_{L_t^r L_x^p} \lesssim \|f\|_{L_x^2},$$
$$\|\Delta_0 \int_{s<t} e^{i(t-s)(-\Delta)^{\beta/2}} F(s) ds\|_{L_t^r L_x^p} \lesssim \|F\|_{L_t^{\tilde{r}'} L_x^{\tilde{p}'}}.$$

Using dilation, we obtain

$$\|\Delta_j e^{it(-\Delta)^{\beta/2}} f\|_{L_t^r L_x^p} \lesssim 2^{j\left((\frac{n}{2}-\frac{n}{p}-\frac{2}{r})+\frac{2-\beta}{r}\right)} \|f\|_{L_x^2}, \tag{3.10}$$

$$\begin{aligned} &\|\Delta_j \int_{s<t} e^{i(t-s)(-\Delta)^{\beta/2}} F(s) ds\|_{L_t^r L_x^p} \\ &\lesssim 2^{j\left((\frac{n}{2}-\frac{2}{r}-\frac{n}{p})+(\frac{n}{2}-\frac{2}{\tilde{r}}-\frac{n}{\tilde{p}})+(2-\beta)(\frac{1}{r}+\frac{1}{\tilde{r}})\right)} \|F\|_{L_t^{\tilde{r}'} L_x^{\tilde{p}'}}. \end{aligned} \tag{3.11}$$

When (r,p) is sharp $\frac{n}{2}$-admissible, we have

$$\begin{aligned} \|\Box_k^\alpha e^{it(-\Delta)^{\beta/2}} f\|_{L_t^r L_x^p} &\lesssim \sum_{\operatorname{supp} \varphi_j \cap \operatorname{supp} \eta_k^\alpha \neq \emptyset} \|\Delta_j e^{it(-\Delta)^{\beta/2}} f\|_{L_t^r L_x^p} \\ &\lesssim \sum_{\operatorname{supp} \varphi_j \cap \operatorname{supp} \eta_k^\alpha \neq \emptyset} 2^{j\frac{2-\beta}{r}} \|f\|_{L_x^2}. \end{aligned} \tag{3.12}$$

Using the almost orthogonality property and $2^j \sim \langle k\rangle^{\frac{1}{1-\alpha}}$, we can deduce that

$$\|\Box_k^\alpha e^{it(-\Delta)^{\beta/2}} f\|_{L_t^r L_x^p} \lesssim \langle k\rangle^{\frac{1}{1-\alpha}(\frac{2-\beta}{r})} \|f\|_{L_x^2}. \tag{3.13}$$

Similarly, we can deduce that

$$\|\Box_k^\alpha \int_{s<t} e^{i(t-s)(-\Delta)^{\beta/2}} F(s) ds\|_{L_t^r L_x^p} \lesssim \langle k\rangle^{\frac{1}{1-\alpha}(2-\beta)(\frac{1}{r}+\frac{1}{\tilde{r}})} \|F\|_{L_t^{\tilde{r}'} L_x^{\tilde{p}'}} \tag{3.14}$$

for all sharp $\frac{n}{2}$-admissible pairs (r,p). When (r,p) is nonsharp $\frac{n}{2}$-admissible; that is, $\frac{2}{r}+\frac{n}{p}<\frac{n}{2}$. Combining with 3.6 and energy estimate, we have

$$\|\Box_k^\alpha e^{it(-\Delta)^{\beta/2}} f\|_{L_x^p} \lesssim \left(\langle k\rangle^{\frac{\alpha}{1-\alpha}(-2)} + \langle k\rangle^{\frac{\beta-2}{1-\alpha}}|t|\right)^{-\frac{n}{2}(1-\frac{2}{p})} \|f\|_{L_x^{p'}}. \tag{3.15}$$

So

$$\begin{aligned} &\|\Box_k^\alpha \int_{\mathbb{R}} e^{i(t-s)(-\Delta)^{\beta/2}} F(s) ds\|_{L_t^r L_x^p} \\ &\lesssim \Big\| \int_{\mathbb{R}} \left(\langle k\rangle^{\frac{\alpha}{1-\alpha}(-2)} + \langle k\rangle^{\frac{\beta-2}{1-\alpha}}|t-s|\right)^{-\frac{n}{2}(1-\frac{2}{p})} \|F(s)\|_{L_x^{p'}} ds \Big\|_{L_t^r} \\ &\lesssim \Big\| \left(\langle k\rangle^{\frac{\alpha}{1-\alpha}(-2)} + \langle k\rangle^{\frac{\beta-2}{1-\alpha}}|t|\right)^{-\frac{n}{2}(1-\frac{2}{p})} \Big\|_{L_t^{r/2}} \|F(s)\|_{L_t^{r'} L_x^{p'}}. \end{aligned} \tag{3.16}$$

One can check that

$$\Big\| \left(\langle k\rangle^{\frac{\alpha}{1-\alpha}(-2)} + \langle k\rangle^{\frac{\beta-2}{1-\alpha}}|t|\right)^{-\frac{n}{2}(1-\frac{2}{p})} \Big\|_{L_t^{r/2}} \lesssim \langle k\rangle^{\frac{2\delta(r,p)}{1-\alpha}}, \tag{3.17}$$

then deduce that

$$\|\Box_k^\alpha \int_{\mathbb{R}} e^{i(t-s)(-\Delta)^{\beta/2}} F(s)ds\|_{L_t^r L_x^p} \lesssim \langle k\rangle^{\frac{2\delta(r,p)}{1-\alpha}} \|F(s)\|_{L_t^{r'} L_x^{p'}}. \tag{3.18}$$

Then homogeneous estimate(1.20) follows by using TT^* method, standard duality argument and the almost orthogonality property of α-modulation space. For the inhomogeneous part, for every $\frac{n}{2}$-admissible pairs (r,p) and $(\tilde r,\tilde p)$, there exist two constant p_1 and $\tilde p_1$ such that (r,p_1) and $(\tilde r,\tilde{p_1})$ are sharp. Combining the following inequalities

$$\|\Box_k^\alpha f\|_{L_x^p} \lesssim \langle k\rangle^{\frac{\alpha}{1-\alpha}n(\frac{1}{p_1}-\frac{1}{p})} \|\Box_k^\alpha f\|_{L_x^{p_1}},$$
$$\|\Box_k^\alpha f\|_{L_x^{\tilde{p_1}'}} \lesssim \langle k\rangle^{\frac{\alpha}{1-\alpha}n(\frac{1}{\tilde p_1}-\frac{1}{\tilde p})} \|\Box_k^\alpha f\|_{L_x^{\tilde p'}}$$

with (3.14), we have

$$\begin{aligned}
&\|\Box_k^\alpha \int_{s<t} e^{i(t-s)(-\Delta)^{\beta/2}} F(s)ds\|_{L_t^r L_x^p}\\
&\lesssim \langle k\rangle^{\frac{\alpha}{1-\alpha}n(\frac{1}{p_1}-\frac{1}{p})} \|\Box_k^\alpha \int_0^t e^{i(t-s)(-\Delta)^{\beta/2}} F(s)ds\|_{L_t^r L_x^{p_1}}\\
&\lesssim \langle k\rangle^{\frac{\alpha}{1-\alpha}n(\frac{1}{p_1}-\frac{1}{p})} \langle k\rangle^{\frac{1}{1-\alpha}(2-\beta)(\frac{1}{r}+\frac{1}{\tilde r})} \|F\|_{L_t^{\tilde r'} L_x^{\tilde{p_1}'}}\\
&\lesssim \langle k\rangle^{\frac{\alpha}{1-\alpha}n(\frac{1}{p_1}-\frac{1}{p})} \langle k\rangle^{\frac{1}{1-\alpha}(2-\beta)(\frac{1}{r}+\frac{1}{\tilde r})} \langle k\rangle^{\frac{\alpha}{1-\alpha}n(\frac{1}{\tilde p_1}-\frac{1}{\tilde p})} \|F\|_{L_t^{\tilde r'} L_x^{\tilde p'}}.
\end{aligned} \tag{3.19}$$

Recall that $\frac{n}{p_1} = \frac{n}{2} - \frac{2}{r}$ and $\frac{n}{\tilde{p_1}} = \frac{n}{2} - \frac{2}{\tilde r}$, so we have

$$\|\Box_k^\alpha \int_{s<t} e^{i(t-s)(-\Delta)^{\beta/2}} F(s)ds\|_{L_t^r L_x^p} \lesssim \langle k\rangle^{\frac{\delta(r,p)+\delta(\tilde r,\tilde p)}{1-\alpha}} \|F\|_{L_t^{\tilde r'} L_x^{\tilde p'}}. \tag{3.20}$$

Then the inhomogeneous estimate (1.21) follows by the definition and almost orthogonality property of α-modulation space.

Remark 3.1. In the case (r,p) is nonsharp $\frac{n}{2}$-admissible, one can also deduce the estimates of $\|\Box_k^\alpha e^{it(-\Delta)^{\beta/2}} f\|_{L_t^r L_x^p}$ and $\|\Box_k^\alpha \int_{s<t} e^{i(t-s)(-\Delta)^{\beta/2}} F(s)ds\|_{L_t^r L_x^p}$ by using (3.10) and (3.11) respectively, but it will lose more regularity.

Proof of Theorem 1.6. We prove only the case $\beta < 2$. Using Lemma 2.3 and the fact that

$$\Big| \int_{\mathbb{R}^n} e^{it\sum_{l=1}^n \pm|\xi_l|^\beta} \eta_k^\alpha(\xi) e^{ix\cdot\xi} d\xi\Big| \lesssim \|\eta_k^\alpha\|_{L^1} \lesssim \langle k\rangle^{\frac{\alpha}{1-\alpha}n} = \Big(\langle k\rangle^{\frac{\alpha}{1-\alpha}(-2)}\Big)^{-n/2}, \tag{3.21}$$

we have

$$\|\Box_k^\alpha e^{it\psi(D)} f\|_{L_x^\infty} \lesssim \Big(\langle k\rangle^{\frac{\alpha}{1-\alpha}(-2)} + \langle k\rangle^{\frac{\beta-2}{1-\alpha}} |t|\Big)^{-n/2} \|f\|_{L_x^1}. \tag{3.22}$$

When $k = 0$ we can deduce the estimates as the proof of Theorem 1.5. When $k \neq 0$, we can also obtain the homogeneous estimates (1.23) for nonsharp pair (r,p) by TT^* method and standard duality argument.

For the case that (r,p) and $(\tilde r,\tilde p)$ are sharp, we use dilation argument, but it's a little different from the proof Theorem 1.5. Let $S_j = \sum_{l\le j} \Delta_l$, $\varphi_0 = \sum_{j\le 0} \varphi_j$, using Lemma 2.3 with $k = 0$, we can deduce

$$\Big| \int_{\mathbb{R}^n} e^{it\sum_{l=1}^n \pm|\xi_l|^\beta} \varphi_0(\xi) e^{ix\cdot\xi} d\xi\Big| \lesssim |t|^{-n/2}, \tag{3.23}$$

so

$$\|S_0 e^{it\psi(D)} f\|_{L_x^\infty} \lesssim (1+|t|)^{-n/2}\|f\|_{L_x^1}. \tag{3.24}$$

Using Lemma 1.2, we can obtain the following estimates:

$$\|S_0 e^{it\psi(D)} f\|_{L_t^r L_x^p} \lesssim \|f\|_{L_x^2},$$
$$\|S_0 \int_{s<t} e^{i(t-s)\psi(D)} F(s) ds\|_{L_t^r L_x^p} \lesssim \|F\|_{L_t^{\tilde{r}'} L_x^{\tilde{p}'}}.$$

A dilation argument then yields

$$\|S_j e^{it\psi(D)} f\|_{L_t^r L_x^p} \lesssim 2^{j\frac{2-\beta}{r}}\|f\|_{L_x^2},$$
$$\|S_j \int_{s<t} e^{i(t-s)\psi(D)} F(s) ds\|_{L_t^r L_x^p} \lesssim 2^{j(2-\beta)(\frac{1}{r}+\frac{1}{\tilde{r}})}\|F\|_{L_t^{\tilde{r}'} L_x^{\tilde{p}'}}.$$

Then we use S_j to cover $\Box_k^\alpha$ to deduce

$$\|\Box_k^\alpha e^{it\psi(D)} f\|_{L_t^r L_x^p} \lesssim \langle k\rangle^{\frac{1}{1-\alpha}(\frac{2-\beta}{r})}\|f\|_{L_x^2},$$
$$\|\Box_k^\alpha \int_{s<t} e^{i(t-s)\psi(D)} F(s) ds\|_{L_t^r L_x^p} \lesssim \langle k\rangle^{\frac{1}{1-\alpha}(2-\beta)(\frac{1}{r}+\frac{1}{\tilde{r}})}\|F\|_{L_t^{\tilde{r}'} L_x^{\tilde{p}'}}$$

for any sharp $\frac{n}{2}$-admissible pairs (r,p) and $(\tilde{r},\tilde{p})$.

The remain case is that inhomogeneous estimate (1.24) for nonsharp admissible pairs, it can be deduced like the proof of Lemma 1.1, we omit the details.

Proof of Theorem 1.3. We need to prove only the following estimates:

$$\|\Box_k^\alpha e^{it\sqrt{-\Delta}} w_0\|_{L_t^r L_x^p} \lesssim \langle k\rangle^{\frac{\theta(r,p)}{1-\alpha}}\|w_0\|_{L_x^2}, \tag{3.25}$$
$$\|\Box_k^\alpha \frac{e^{it\sqrt{-\Delta}}}{\sqrt{-\Delta}} w_1\|_{L_t^r L_x^p} \lesssim \langle k\rangle^{\frac{\theta(r,p)-1}{1-\alpha}}\|w_1\|_{L_x^2}, \tag{3.26}$$
$$\|\Box_k^\alpha \int_{s<t} \frac{e^{i(t-s)\sqrt{-\Delta}}}{\sqrt{-\Delta}} F(s) ds\|_{L_t^r L_x^p} \lesssim \langle k\rangle^{\frac{\theta(r,p)+\theta(\tilde{r},\tilde{p})-1}{1-\alpha}}\|F\|_{L_t^{\tilde{r}'} L_x^{\tilde{p}'}}. \tag{3.27}$$

Using the same techniques as before, we can deduce (3.25) and

$$\|\Box_k^\alpha \int_{s<t} e^{i(t-s)\sqrt{-\Delta}} F(s) ds\|_{L_t^r L_x^p} \lesssim \langle k\rangle^{\frac{\theta(r,p)+\theta(\tilde{r},\tilde{p})}{1-\alpha}}\|F\|_{L_t^{\tilde{r}'} L_x^{\tilde{p}'}}.$$

If $k \neq 0$, then (3.26) and (3.27) will then follow by

$$\|\Box_k^\alpha (-\Delta)^{-1/2} f\|_{L_x^2} \lesssim \langle k\rangle^{-\frac{1}{1-\alpha}}\|f\|_{L_x^2}.$$

If $k=0$, we do not have energy estimate for the operator $\frac{e^{it\sqrt{-\Delta}}}{\sqrt{-\Delta}}$, so we can not use the TT^* argument. One can deduce

$$\Big|\int_{\mathbb{R}^n} e^{it|\xi|}|\xi|^{-1}\varphi(\xi)e^{ix\cdot\xi} d\xi\Big| \lesssim |t|^{(n-1)/2},$$
$$\Big|\int_{\mathbb{R}^n} e^{it|\xi|}|\xi|^{-1/2}\varphi(\xi)e^{ix\cdot\xi} d\xi\Big| \lesssim |t|^{(n-1)/2}$$

by principle of stationary phase as in Lemma 2.2, then Lemma 1.2 will yield

$$\|\Delta_0 \frac{e^{it\sqrt{-\Delta}}}{\sqrt{-\Delta}} f\|_{L_t^r L_x^p} \lesssim \|f\|_{L_x^2},$$
$$\|\Delta_0 \int_{s<t} \frac{e^{i(t-s)\sqrt{-\Delta}}}{\sqrt{-\Delta}} F ds\|_{L_t^r L_x^p} \lesssim \|F\|_{L_t^{\tilde{r}'} L_x^{\tilde{p}'}}.$$

Using a dilation argument, we obtain

$$\|\Delta_j \frac{e^{it\sqrt{-\Delta}}}{\sqrt{-\Delta}} f\|_{L_t^r L_x^p} \lesssim 2^{j(\frac{n}{2}-\frac{n}{p}-\frac{1}{r}-1)}\|f\|_{L_x^2},$$

$$\|\Delta_j \int_{s<t} \frac{e^{i(t-s)\sqrt{-\Delta}}}{\sqrt{-\Delta}} F ds\|_{L_t^r L_x^p} \lesssim 2^{j(n-1-\frac{n}{p}-\frac{1}{r}-\frac{n}{\tilde{p}}-\frac{1}{\tilde{r}})}\|F\|_{L_t^{\tilde{r}'} L_x^{\tilde{p}'}}.$$

If $\frac{n}{2}-\frac{n}{p}-\frac{1}{r}-1>0$ and $n-1-\frac{n}{p}-\frac{1}{r}-\frac{n}{\tilde{p}}-\frac{1}{\tilde{r}}>0$, we can use Δ_j to cover $\Box_0^\alpha$ and get the estimates (3.26) and (3.27) for $k=0$.

Remark 3.2. If we take $\alpha=0$ in Theorem $1.1-1.3$, we obtain the Strichartz estimates in the frame of modulation spaces. The Strichartz estimates of Schrödinger equation in the frame of α modulation spaces is the case that $\beta=2$.

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References

[1] A. Baenyi, K. Grochenig, K. A. Okoudjou, L. G. Rogers; *Unimodular Fourier multiplier for modulation spaces*, J. Funct. Anal., 246 (2007), 366-384.
[2] A. Baenyi, K. A. Okoudjou; *Local well-posedness of nonlinear dispersive equations on modulation spaces*, Bull. London Math. Soc., 41 (2009), 549-558.
[3] J. C. Chen, D. S. Fan; *Estimates for wave and Klein-Gordon equations on modulation spaces*, Sci. China Math. 55 (2012) 2109-2123.
[4] J. C. Chen, D. S. Fan, L. J. Sun; *Asymptotic estimates for unimodular Fourier multipliers on modulation spaces*. Discret Contin Dyn Syst, 2012, 32: 467-485.
[5] E. Cordero, D. Zucco; *Strichartz Estimates for the Schrödinger Equation*, [J]. Cubo (Temuco), 2010, 12(3): 213-239.
[6] J. Ginibre, G. Velo; *Smoothing properties and retarded estimates for some dispersive evolution equations*, Comm. Math. Phys. 123 (1989) 535-573.
[7] J. Ginibre, G. Velo; *Generalized Stnchartz Inequalities for the Wave Equation*, Jour. Func. Anal., 133 (1995), 50-68.
[8] P. Gröbner; *Banachräume Glatter Funktionen and Zerlegungsmethoden*, Doctoral thesis, University of Vienna, 1992.
[9] M. Keel, T. Tao; *Endpoint Strichartz Estimates*, Amer. Math. J. 120 (1998), 955-980.
[10] M. Ruzhansky, M. Sugimoto, B. X. Wang; *Modulation spaces and nonlinear evolution equations*, Progress in Mathematics Volume 301, 2012, pp 267-283.
[11] C. D. Sogge; *Lectures on Nonlinear Wave Equations*, International Press, Cambridge, MA, 1995.
[12] E. M. Stein; *Harmonic Analysis: Real-Variable Methods, Orthogonality and Oscillatory Integrals*, Princeton, NJ Princeton University Press, 1993.
[13] R. S. Strichartz; *Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations*, Duke Math. J. 44 (1977) 705-714.
[14] T. Tao; *Nonlinear Dispersive Equations: Local and Global Analysis*, CBMS Regional Conference Series in Mathematics 106.
[15] H. Triebel; *Theory of Function Spaces*, Birkhauser-Verlag, Basel, 1983.
[16] B.X. Wang, H. Hudzik; *The global Cauchy problem for the NLS and NLKG with small rough data*, J. Differential Equations, 232, 36-73, 2007.
[17] B. X. Wang, Z. H. Huo, C. C. Hao, Z. H. Guo; *Harmonic Analysis Method for Nonlinear Evolution Equations I.* Hackensack, NJ: World Scientfic, 2011.
[18] B. X. Wang, L. F. Zhao, B. L. Guo; *Isometric decomposition operators, function spaces $E^\lambda_{p,q}$ and applications to nonlinear evolution equations*, J. Funct. Anal. 233 (1) (2006) 1-39.
[19] J. S. Wang, B. X. Wang; *α-Modulation Spaces (I) Scaling*, Embedding and Algebraic Properties, arXiv:1108.0460.
[20] C.J. Zhang; *Strichartz estimates in the frame of modulation spaces*, [J]. Nonlinear Analysis, 2012.

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