Electronic Journal of Differential Equations, Vol. 2013 (2013), No. 124, pp. 1–3.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

EXISTENCE AND UNIQUENESS OF A LOCAL SOLUTION FOR $x' = f(t,x)$ USING INVERSE FUNCTIONS

JEFFREY T. HOAG

Abstract. A condition on the function f is given such that the scalar ordinary differential equation $x' = f(t,x)$ with initial condition $x(t_0) = x_0$ has a unique solution in a neighborhood of t_0. An example illustrates that this result can be used when other theorems which put conditions on the difference $f(t,x) - f(t,y)$ do not apply.

1. Introduction

Consider the differential equation with initial condition:

$$x'(t) = f(t,x(t)), \quad x(t_0) = x_0 \tag{1.1}$$

where f is a scalar-valued function which is continuous in a neighborhood N of (t_0,x_0). The continuity of f guarantees that there is at least one solution to this initial value problem. There are various other conditions that can be imposed on f which will ensure that (1.1) has a unique solution. Over twenty such uniqueness conditions are collected in [1]. Most of these, including results by Nagumo [3], Osgood [4] and Perron [5], rely on restrictions on $f(t,x) - f(t,y)$ and can be considered generalizations of the Lipschitz condition in the second argument.

In this article, a uniqueness theorem for (1.1) is given which instead puts the Lipschitz condition on the first argument of f. That is, the condition is on the difference $f(t,x) - f(s,x)$ for (t,x) and (s,x) in N. It is easy to see that this is possible when $f(t_0,x_0) \neq 0$ because in this case a solution of (1.1) is invertible in a neighborhood of (t_0,x_0) and so if $t(x)$ is the inverse of a solution to (1.1), it satisfies

$$t'(x) = g(x,t(x)), \quad t(x_0) = t_0 \tag{1.2}$$

where we define $g(x,t) = 1/f(t,x)$. If f is Lipschitz in its first argument in a neighborhood N of (t_0,x_0) then there is a neighborhood M of (x_0,t_0) where g is Lipschitz in its second argument. From this it follows that (1.2) has a unique solution in a neighborhood of (x_0,t_0) and therefore (1.1) has a unique solution in a neighborhood of (t_0,x_0).

The theorem that follows extends this approach to include cases when $f(t_0,x_0) = 0$. It will be followed by an example for which this theorem applies but other uniqueness theorems do not.

2000 *Mathematics Subject Classification.* 34A12.
Key words and phrases. Existence; uniqueness; ordinary differential equation.

Submitted January 13, 2013. Published May 20, 2013.

2. Main result

Theorem 2.1. *For $(t_0, x_0) \in \mathbb{R}^2$ and positive numbers a and b, define*

$$U = [t_0 - a, t_0 + a] \times [x_0 - b, x_0 + b].$$

Let $f : U \to \mathbb{R}$ be a continuous function satisfying the following three conditions:

(i) *there are constants $c > 0$ and $r \in (0, 1/2)$ such that*

$$|f(t, x)| \geq c|x - x_0|^r \quad \text{for all } (t, x) \in U;$$

(ii) *$f(t, x_0)$ is not identically zero on any interval $(t_0 - \varepsilon, t_0 + \varepsilon)$ for $0 < \epsilon < a$;*

(iii) *there is a number α such that for all (t, x) and (s, x) in U,*

$$|f(t, x) - f(s, x)| \leq \alpha|t - s|.$$

Then there is a unique solution to the initial value problem (1.1) in some interval $(t_0 - \nu, t_0 + \eta)$.

Proof. Let x be a solution to (1.1) where f satisfies the conditions of the theorem. Define the closed set $D = \{t \in [t_0, t_0 + a] : x'(t) = 0\}$. Suppose, for contradiction, that x is not strictly monotone in any interval $[t_0, t_0 + \varepsilon]$. Then D is infinite and $t_0 \in D$. Since D is closed, $\sup D \equiv t_1 \in D$. The set $(t_0, t_1) - D$ is open, and, by (ii), non-empty. Therefore, there is an interval $(u, v) \subseteq (t_0, t_1) - D$ with u and v both in D. Thus $x'(u) = x'(v) = 0$. Then by condition (i), $x(u) = x(v) = x_0$ and it follows by Rolle's Theorem that there is a $\xi \in (u, v)$ such that $x'(\xi) = 0$. But this leads to the contradiction that $\xi \in D \cap (u, v) = \emptyset$. Thus every solution of (1.1) is strictly monotone (and therefore invertible) on some interval $[t_0, t_0 + \delta_1)$. If $t(x)$ is the inverse of an increasing solution of (1.1) then $t(x)$ satisfies

$$t'(x) = \frac{1}{f(t, x)}, \quad t(x_0) = t_0 \tag{2.1}$$

for $x > x_0$.

Now let x and $\tilde{x}$ be any two increasing solutions of (1.1) with inverses t and $\tilde{t}$. Since t and $\tilde{t}$ are both solutions to (2.1),

$$|t(x) - \tilde{t}(x)| \leq |t(y) - \tilde{t}(y)| + \int_y^x \frac{|f(t(s), s) - f(\tilde{t}(s), s)|}{|f(t(s), s)|\,|f(\tilde{t}(s), s)|} ds$$

for $x \geqslant y > x_0$. Then, using conditions (i) and (iii),

$$|t(x) - \tilde{t}(x)| \leq |t(y) - \tilde{t}(y)| + \frac{\alpha}{c^2} \int_y^x \frac{|t(s) - \tilde{t}(s)|}{|s - x_0|^{2r}} ds\,.$$

Applying the Gronwall-Reid Lemma to this inequality yields

$$|t(x) - \tilde{t}(x)| \leq |t(y) - \tilde{t}(y)| \exp\Big\{\frac{\alpha}{c^2} \int_y^x \frac{1}{|s - x_0|^{2r}} ds\Big\}.$$

Now take the limit as $y \to x_0+$. Since $2r < 1$, the improper integral converges. Also $|t(y) - \tilde{t}(y)| \to |t(x_0) - \tilde{t}(x_0)| = 0$. Therefore, $t(x) = \tilde{t}(x)$ in some interval $[x_0, x(t_0 + \delta_1)]$ and so $x(t) = \tilde{x}(t)$ for $t \in [t_0, t_0 + \delta_1)$.

Thus there is at most one increasing solution to (1.1) on an interval $[t_0, t_0 + \delta_1)$. A similar arguments shows that there is at most one decreasing solution to (1.1) on an interval $[t_0, t_0 + \delta_2)$. Since it is well-known that (1.1) has either one solution or infinitely many solutions, and since every solution of (1.1) is monotone, it follows that (1.1) has a unique solution on some interval $[t_0, t_0 + \eta)$. A similar argument shows that there is also a unique solution on some interval $(t_0 - \nu, t_0]$. $\square$

Examples. Consider the initial-value problem

$$x'(t) = g(t) + h(t)|x(t)|^r, \quad x(0) = 0 \tag{2.2}$$

where g and h are non-negative Lipschitz continuous functions and $0 < r < 1$. The theorem given here can be applied to show that (2.2) has a unique solution in a neighborhood of 0 provided $h(0) \neq 0$ and $0 < r < 1/2$. However, any theorem which relies only on the difference $f(t,x) - f(t,y)$ –such as those mentioned in the introduction– would evidently not apply to (2.2). For if such a theorem did apply to (2.2) it would also have to apply to the example $x'(t) = h(t)|x(t)|^r$, $x(0) = 0$, since $f(t,x) - f(t,y)$ is the same in this example as in (2.2). But this example has the two solutions $x(t) = [((1-r)\int_0^t h(s)ds)]^{1/(1-r)}$ and $x(t) \equiv 0$.

Example (2.2) is a generalization of an example which appears in [2], where a theorem is given which also does not rely on the difference $f(t,x) - f(t,y)$. But the theorem in [2] does not apply to (2.2) unless $g(0) \neq 0$.

Remarks. Conditions (i) and (ii) in Theorem 2.1 replace the stronger condition that $f(t_0, x_0) \neq 0$ as discussed in the Introduction. Neither of these conditions can be dropped. Consider these two choices for f:

(a) $f(t,x) = x^{3/5} + \frac{1}{100} t^{3/2}$
(b) $f(t,x) = x^{1/3}$.

With $t_0 = x_0 = 0$, both of these functions satisfy the conditions of Theorem 2.1, except that example (a) does not satisfy condition (i), and example (b) does not satisfy condition (ii). The non-uniqueness of solutions of the corresponding initial value problem (1.1) is shown below.

(a) $x(t) = kt^{5/2}$ is a solution where k is any of the three real numbers which satisfy the equation $(\frac{5k}{2} - \frac{1}{100})^5 = k^3$.
(b) $x(t) \equiv 0$ and $x(t) = (\frac{2t}{3})^{3/2}$ are both solutions.

References

[1] R. P. Agarwal, V. Lakshmikantham; *Uniqueness and nonuniqueness criteria for ordinary differential equations*, World Scientific Publishing, 1993.

[2] J. M. Bownds; *A uniqueness theorem for $y'(x) = f(x,y)$* using a certain factorization of f, Journal of Differential Equations, Vol 7 (1970), 227-231

[3] M. Nagumo; *Eine hinreichende bedingung für die unität der lösung von differentialgleichungen erster ordnung*, Japan J. Math. 3 (1926), 107-112.

[4] W. Osgood; *Beweis der existenz einer lösung der differentialgle ichung $dy/dx = f(x,y)$ ohne hinzunahme der Cauchy-Lipshitzchen bedingung*, Monatah. Math. Phys., Vol 9 (1898), 331-345.

[5] O. Perron; *Eine hinreichende bedingung für die unität der lösung von Differentialgleichungen erster ordnung*, Math. Z., Vol 28 (1928), 216-219.

Jeffrey T. Hoag
Mathematics Department, Providence College, Providence, RI 02918, USA
E-mail address: jhoag@providence.edu