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PROPERTIES OF MEROMORPHIC SOLUTIONS TO CERTAIN DIFFERENTIAL-DIFFERENCE EQUATIONS

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Abstract. We consider the properties of meromorphic solutions to certain type of non-linear difference equations. Also we show the existence of meromorphic solutions with finite order for differential-difference equations related to the Fermat type functional equation. This article extends earlier results by Liu et al [12].

1. Introduction

In this article, we assume that the reader is familiar with standard symbols and fundamental results of Nevanlinna Theory [7, 15]. A meromorphic function will mean meromorphic in the whole complex plane. In particular, we denote the order of growth of a meromorphic function $f(z)$ by $\sigma(f)$. The values $m(r,f)$, $N(r,f)$, $\overline{N}(r,f)$ and $T(r,f)$ denote the proximity function, the counting function, the reduced counting function and the characteristic function of $f(z)$, respectively:

$$\begin{gathered}
m(r,f) := \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})|\, d\theta, \\
N(r,f) := \int_0^r \frac{n(t,f)-n(0,f)}{t}\, dt + n(0,f)\log r, \\
\overline{N}(r,f) := \int_0^r \frac{\overline{n}(t,f)-\overline{n}(0,f)}{t}\, dt + \overline{n}(0,f)\log r, \\
T(r,f) := m(r,f) + N(r,f),
\end{gathered}$$

where $\log^+ x = \max(\log x, 0)$ for all $x \geq 0$, $n(t,f)$ denotes the number of poles of $f(z)$ in the disc $|z| \leq t$, counting multiplicities; and $\overline{n}(t,f)$ denotes the number of poles of $f(z)$ in the disc $|z| \leq t$, ignoring multiplicities.

Nevanlinna's value distribution theory of meromorphic functions has been used to study the growth, oscillation and existence of entire or meromorphic solutions of differential equations. In 2001, Yang [14] started to study the existence and uniqueness of finite order entire solutions of the following type of non-linear differential equation

$$L(f) - p(z)f(z)^n = H(z). \tag{1.1}$$

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Subsequently, several papers have appeared in which the solutions of equation (1.1) are studied. The reader is referred to [8, 10, 11].

Recently, many articles focused on complex difference equations. The background for these studies lies in the recent difference counterparts of Nevanlinna theory. The key result here is the difference analogue of the lemma on the logarithmic derivative obtained by Halburd-Korhonen [5, 6] and Chiang-Feng [2], independently.

Yang and Laine [16] gave difference, resp. differential-difference, analogues of previous results concerning the equation (1.1). In fact, they proved the following theorem.

Theorem 1.1 ([16, Theorem 2.6]). *Let $n \geq 4$ be an integer, $L(z,f)$ be a linear differential-difference polynomial of $f(z)$, with small meromorphic coefficients, and $H(z)$ be a meromorphic function of finite order. Then the differential-difference equation*

$$f(z)^n + L(z,f) = H(z)$$

possesses at most one transcendental entire solution of finite order, unless $L(z,f)$ vanishes identically. If such a solution $f(z)$ exists, then $f(z)$ is of the same order as $H(z)$.

Using Theorem 1.1, the authors investigate the existence and the growth of meromorphic solutions with a few poles of the difference equation

$$f(z)^n + L(z,f) = H(z), \tag{1.2}$$

where $L(z,f) = a_0 f + a_1 f(z+c_1) + \dots a_k f(z+c_k)$ is a linear difference polynomials in $f(z)$ with small meromorphic functions as the coefficients, and c_i are constants, $i = 1, 2, \dots k$. Here, $H(z)$ is meromorphic of finite order, and n is an integer such that $n \geq 2$. In fact, if $n = 0$ or $n = 1$, then (1.2) reduces to a linear difference equation, which has been considered in [1, 2, 9, 17].

AUTHORS: Please define $N(r,f)$ and $S(r,f)$ $S(r,f)$

Theorem 1.2. *Given $L(z,f)$ and $H(z)$ as above. If $f(z)$ is a finite order meromorphic solution of (1.2) satisfying $N(r,f) = S(r,f)$ and $n \geq 4$, then one of the following statements hold:*

(1) Equation (1.2) has $f(z)$ as its unique transcendental meromorphic solution with finite order such that $N(r,f) = S(r,f)$.

(2) Equation (1.2) has exactly n transcendental meromorphic solutions, f_j ($j = 1, 2, 3 \dots n$), with finite order such that $N(r,f_j) = S(r,f_j)$.

Next, we consider the growth of meromorphic solutions of (1.2). In fact, using the same method as Theorem 1.2, we prove the following result.

Theorem 1.3. *Given $L(z,f)$ and $H(z)$ as above. Let $f_1(z)$ and $f_2(z)$ be two distinct arbitrary solutions such that $N(r,f_i) = S(r,f_i)$ ($i = 1, 2$). Then*

$$T(r,f_1) = T(r,f_2) + S(r,f_1).$$

Theorem 1.4. *Given $L(z,f)$ and $H(z)$ as above, assume that $f(z)$ is a meromorphic solution of for (1.2) with finite order. Then $\sigma(f) \leq \sigma(H)$. Furthermore, if $f(z)$ satisfies any one of the following two conditions*

(1) $n \geq k+2$, *or*
(2) $N(r,f) = S(r,f)$,

then $\sigma(f) = \sigma(H)$.

Remark. It seems that replacing $L(z, f)$ with differential-difference polynomial in $f(z)$, the same conclusions of Theorem 1.2–Theorem 1.4 can be proved.

In a recent publication, Liu et al [12, 13] discussed the existence of entire solutions with finite order of the Fermat type differential-difference equation

$$(f'(z))^n + f(z+c)^m = 1. \tag{1.3}$$

They showed that the above equation has no transcendental entire solutions with finite order, provided that $m \neq n$, where n, m are positive integers. Here and in the following, c is a non-zero constant, unless otherwise specified. It is natural to ask what happens if the right side of (1.3) is a meromorphic function $H(z)$. Corresponding to this question, we give the following results:

Theorem 1.5. *Let $f(z)$ be a transcendental meromorphic function with finite order, m and n be two positive integers such that $m \geq 2n + 4$ and $H(z)$ be a meromorphic function satisfying $\overline{N}(r, 1/H) = S(r, f)$. Then $f(z)$ is not a solution of the equation*

$$(f'(z))^n + f(z+c)^m = H(z). \tag{1.4}$$

Using a similar reasoning as in Theorem 1.5, we conclude have the following result.

Corollary 1.6. *Let $f(z)$ be a transcendental entire function with finite order, m and n be two positive integers such that $m \geq n + 2$ and $H(z)$ be a meromorphic function satisfying $\overline{N}(r, 1/H) = S(r, f)$. Then $f(z)$ is not a solution of* (1.4).

Remarks (1) Corollary 1.6 does not hold when $n = m$. In particular,

$$f'(z) + f(z + 2\pi i) = 2e^z$$

admits a transcendental entire solution, e^z. This implies that the restriction that $m \geq n + 2$ is necessary. Meanwhile, we considered Corollary 1.6 for $m = n + 1$. Unfortunately, we have not succeed.

(2) Let $f(z) = \cos z$, then $f(z)$ is a transcendental entire solution of the equation

$$f'(z) + f(z - \frac{\pi}{2})^3 = \sin z(\sin^2 z - 1).$$

Indeed, this example shows Corollary 1.6 cannot hold when $\overline{N}(r, 1/H) \neq S(r, f)$, (If $\overline{N}(r, 1/H) = \overline{N}(r, \frac{1}{\sin z(\sin^2 z - 1)}) = S(r, f)$, then we can have a contradiction by the second main theorem.) which means the assumption $\overline{N}(r, 1/H) = S(r, f)$ in Corollary 1.6 is sharp.

(3) If we omit the restriction of the order of the solutions, then (1.4) may have an infinite order entire solution. Indeed, $f(z) = e^{e^z}$ is an entire function with infinite order and solves the equation

$$f'(z) + f(z + \ln \frac{1}{3})^3 = (e^z + 1)e^{e^z}.$$

(4) Some ideas in this paper are based on [3, 8].

Theorem 1.7. *Let $f(z)$ be a transcendental entire function with finite order, m and n be two positive integers such that $m \neq n$ and $H(z)$ be a small function of $f(z)$. Then $f(z)$ is not a solution of equation* (1.4).

The proof of Theorem 1.7 is similar to the proof of [12, Theorem 1.2]. One can apply the the second main theorem for small target functions, instead of the classical second main theorem, and use an elementary computation. Therefore, we omit the proof here.

2. Preliminaries

Lemma 2.1 ([2, Theorem 2.1]). *Let $f(z)$ be a finite order meromorphic function, then for each $\varepsilon > 0$,*

$$T(r, f(z+c)) = T(r, f(z)) + O(r^{\sigma(f)-1+\varepsilon}) + O(\log r)$$

and

$$\sigma(f(z+c)) = \sigma(f(z)).$$

Thus, if $f(z)$ is a transcendental meromorphic function with finite order, then

$$T(r, f(z+c)) = T(r, f) + S(r, f).$$

Lemma 2.2 ([6, Theorem 2.1]). *Let $f(z)$ be a meromorphic function with finite order, and let $c \in \mathbb{C}$ and $\delta \in (0,1)$. Then*

$$m\Big(r, \frac{f(z+c)}{f(z)}\Big) + m\Big(r, \frac{f(z)}{f(z+c)}\Big) = o\Big(\frac{T(r,f)}{r^{\delta}}\Big) = S(r,f),$$

outside of a possible set E with finite logarithmic measure.

Lemma 2.3 ([4, Lemma 5]). *Let F and G be non-decreasing functions on $(0, +\infty)$. If $F(r) \le G(r)$ for $r \notin E \cup [0,1]$, where the set $E \subset (1, +\infty)$ has finite logarithmic measure, then, for any constant $\alpha > 1$, there exists a value $r_0 > 0$, such that $F(r) \le G(\alpha r)$ for $r > r_0$.*

Lemma 2.4. *Let $f(z)$ be a meromorphic solution of (1.4), and*

$$G(z) = \frac{(f^m(z+c))'}{f^m(z+c)} - \frac{H'}{H}. \tag{2.1}$$

Then

$$N(r, G) \le \overline{N}(r, \frac{1}{H}) + \overline{N}(r, f) + \overline{N}(r, \frac{1}{f(z+c)}) + S(r,f).$$

Remark. In the following proof, first impression of the reader is that the poles of $G(z)$ are at the poles of $H(z)$ as well. But looking at the equation (1.4), one realizes that the poles of $H(z)$ should be at the poles of $f(z)$ and $f(z+c)$. Hence, it is sufficient to discuss the poles of $f(z)$ and $f(z+c)$ here.

Proof. Observe that, the poles of $G(z)$ are at the zeros of $H(z)$ and $f(z+c)$, and at the poles of $f(z)$, $f(z+c)$ from (1.4) and (2.1). If z_0 is a zero of $H(z)$, zero of $f(z+c)$, or pole of $f(z)$, then z_0 is at most a simple pole of $G(z)$ by (1.4) and (2.1). If z_0 is a pole of $f(z+c)$ but not a pole of $f(z)$, then by the Laurent expansion of $G(z)$ at z_0, we obtain that $G(z)$ is analytic at z_0. Hence, from the discussions above, we can conclude that

$$N(r, G) \le \overline{N}(r, \frac{1}{H}) + \overline{N}(r, f) + \overline{N}(r, \frac{1}{f(z+c)}) + S(r,f).$$

□

3. Proof of main results

Proof of Theorem 1.2. Suppose $f_1(z)$, $f_2(z)$ are two distinct finite order meromorphic solutions of (1.2) such that $N(r, f_i) = S(r, f_i)$ $(i = 1, 2)$. From (1.2), we know that

$$\frac{L(z, f_1) - L(z, f_2)}{f_2 - f_1} = \frac{f_1^n - f_2^n}{f_1 - f_2} = F(z) = \frac{L(z, f_1 - f_2)}{f_2 - f_1}, \tag{3.1}$$

where $F(z) = (f_1 - t_1 f_2)(f_1 - t_2 f_2)\dots(f_1 - t_{n-1} f_2)$. Here $t_j \neq 1$ $(j = 1, 2, \dots n-1)$ are the distinct n-th roots of the unity. From Lemma 2.2 to (3.1), we obtain

$$m(r, F) = S(r, f_1) + S(r, f_2).$$

Since $N(r, f_i) = S(r, f_i)$, it follows that $N(r, F) = S(r, f_1) + S(r, f_2)$. Hence

$$T(r, F) = S(r, f_1) + S(r, f_2). \tag{3.2}$$

We will discuss the following two cases.

Case 1. If $F(z) \equiv 0$, then we conclude that $f_1^n = f_2^n$, that is, $f_2 = t_j f_1$. Substituting $f_2 = t_j f_1$ into (3.1), we have $L(z, f_1) - L(z, t_j f_1) = (1 - t_j)L(z, f_1) = 0$. Hence, $L(z, f_1) = 0$ and $L(z, t_j f_1) = 0$. This means f_1 and $t_j f_1$ $(j = 1, 2, \dots n-1)$ are the solutions of (1.2), as asserted in part (2).

Case 2. If $F(z) \not\equiv 0$, then

$$F(z) = f_2^{n-1}\big(\frac{f_1}{f_2} - t_1\big)\big(\frac{f_1}{f_2} - t_2\big)\dots\big(\frac{f_1}{f_2} - t_{n-1}\big). \tag{3.3}$$

Then equation (3.3) gives

$$\frac{F(z)}{f_2^{n-1}} = P(\frac{f_1}{f_2}),$$

where P is a polynomial in f_1/f_2 of degree $n-1$ with constant coefficients. Applying Valiron-Mohon'ko theorem and (3.2) to the above equation, we have

$$(n-1)T(r, \frac{f_1}{f_2}) = T(r, \frac{F(z)}{f_2^{n-1}}) = (n-1)T(r, f_2) + S(r, f_1) + S(r, f_2). \tag{3.4}$$

Using the same way, we obtain

$$(n-1)T(r, \frac{f_2}{f_1}) = T(r, \frac{F(z)}{f_1^{n-1}}) = (n-1)T(r, f_1) + S(r, f_1) + S(r, f_2) \tag{3.5}$$

as well. Combining (3.4) and (3.5), we have

$$T(r, f_1) + S(r, f_1) = T(r, f_2) + S(r, f_2).$$

Thus, $S(r, f_1) = S(r, f_2)$. Moreover, substituting $S(r, f_1) = S(r, f_2)$ into (3.4), we see

$$T(r, \frac{f_1}{f_2}) = T(r, f_2) + S(r, f_2),$$

hence $S(r, \frac{f_1}{f_2}) = S(r, f_2)$. Assume now that z_0 such that $\frac{f_1(z_0)}{f_2(z_0)} = t_j$, then either $F(z_0) = 0$ or $f_2(z_0) = \infty$ by (3.1). That means that

$$N(r, \frac{1}{\frac{f_1}{f_2} - t_j}) = S(r, f_2)$$

by the assumption and equation (3.2). From the arguments above and the second main theorem, we obtain

$$(n-3)T(r,\frac{f_1}{f_2}) \le \sum_{j=1}^{n-1} N\big(r,\frac{1}{\frac{f_1}{f_2}-t_j}\big) = S(r,f_2) = S\big(r,\frac{f_1}{f_2}\big),$$

which contradicts the assumption that $n \ge 4$. Completing the proof of the part (1). $\square$

Proof of Theorem 1.4. If $L(z,f) \equiv 0$, then the conclusion follows. In the following, we suppose $L(z,f) \not\equiv 0$. Since $f(z)$ is a meromorphic solution of (1.2), with finite order, it follows from Lemma 2.1 that

$$T(r,L(z,f)) \le (k+1)T(r,f) + S(r,f). \tag{3.6}$$

From (1.2), we obtain

$$T(r,H) \le T(r,f^n) + T(r,L(z,f)) + S(r,f). \tag{3.7}$$

Combining (3.6) and (3.7), and applying Lemma 2.3, we know that, for $\alpha > 1$, there exists a value $r_0 > 0$, such that

$$T(r,H) \le (n+k+1)T(\alpha r,f) + S(r,f)$$

for $r > r_0$. By the definition of $\sigma(f)$, we conclude that $\sigma(H) \le \sigma(f)$. Next, we investigate the special cases. By the conclusion above, it suffices to show that $\sigma(H) \ge \sigma(f)$.

Case 1. If $n \ge k+2$, then (1.2) gives

$$T(r,f^n) \le T(r,H) + T(r,L(z,f)) + S(r,f). \tag{3.8}$$

Substituting (3.6) into (3.8), and from Lemma 2.3, we obtain that for $\alpha > 1$ there exists a value $r_0 > 0$, such that

$$(n-k-1)T(r,f) \le T(\alpha r,H) + S(r,f)$$

for $r > r_0$. By the assumption that $n \ge k+2$ and the definition of $\sigma(f)$, it follows that $\sigma(H) \ge \sigma(f)$.

Case 2. If $N(r,f) = S(r,f)$, then by Lemma 2.2, we obtain

$$\begin{aligned} T(r,L(z,f)) = m(r,L(z,f)) &\le m(r,\frac{L(z,f)}{f}) + m(r,f) + S(r,f) \\ &\le T(r,f) + S(r,f). \end{aligned} \tag{3.9}$$

Substituting (3.9) into (3.8), and using the same way as in Case 1, we have

$$(n-1)T(r,f) \le T(\alpha r,H) + S(r,f)$$

for $r > r_0$. The conclusion follows. $\square$

Proof of Theorem 1.5. If $H(z)$ is infinite order, then (1.4) has no meromorphic solution with finite order, by comparing the growth of both sides of the equation. It remains to consider that $H(z)$ is finite order. Suppose, contrary to the assertion, that $f(z)$ is a transcendental meromorphic function with finite order satisfying (1.4). Then we will distinguish two cases:

Case 1. If $T(r,H) \ne S(r,f)$. Then from (1.4), we obtain

$$f^m(z+c) = \frac{\frac{H'}{H}(f'(z))^n - ((f'(z))^n)'}{\frac{(f^m(z+c))'}{f^m(z+c)} - \frac{H'}{H}}. \tag{3.10}$$

First of all, we affirm that $\frac{(f^m(z+c))'}{f^m(z+c)} - \frac{H'}{H}$ cannot vanish identically. Indeed, if $\frac{(f^m(z+c))'}{f^m(z+c)} - \frac{H'}{H} \equiv 0$, then we see

$$H(z) = Af^m(z+c),$$

where A is a non-zero constant. Combining the above equality and equation (1.4),

$$(f'(z))^n = (A-1)f^m(z+c)$$

follows. By Lemma 2.1 and the above equation, we obtain

$$mT(r,f) \leq 2nT(r,f) + S(r,f),$$

or $f'(z) \equiv 0$, which contradicts the assumptions.

From equation (3.10), we obtain that

$$\begin{aligned} T(r, f^m(z+c)) = mT(r,f) + S(r,f) &\leq m(r,(f'(z))^n) + m\Big(r, \frac{H'}{H} - \frac{((f'(z))^n)'}{(f'(z))^n}\Big) \\ &\quad + N\Big(r, \frac{H'}{H}(f'(z))^n - ((f'(z))^n)'\Big) + m\Big(r, \frac{(f^m(z+c))'}{f^m(z+c)} - \frac{H'}{H}\Big) \\ &\quad + N\Big(r, \frac{(f^m(z+c))'}{f^m(z+c)} - \frac{H'}{H}\Big) + S(r,f). \end{aligned} \tag{3.11}$$

Then, Lemma 2.1 together with equation (1.4), implies that

$$T(r,H) \leq (m+2n)T(r,f) + S(r,f),$$

which means all meromorphic functions $a(z)$ that satisfy $T(r,a) = S(r,H)$ must be $S(r,f)$. To apply Lemma 2.1, Lemma 2.2 and the Lemma on logarithmic derivative to equation (3.11), we obtain that

$$\begin{aligned} mT(r,f) &\leq nm(r,f) + N\Big(r, \frac{H'}{H}(f'(z))^n - ((f'(z))^n)'\Big) \\ &\quad + N\Big(r, \frac{(f^m(z+c))'}{f^m(z+c)} - \frac{H'}{H}\Big) + S(r,f). \end{aligned} \tag{3.12}$$

We will estimate $N\big(r, \frac{H'}{H}(f'(z))^n - ((f'(z))^n)'\big)$ and $N\big(r, \frac{(f^m(z+c))'}{f^m(z+c)} - \frac{H'}{H}\big)$ next. Set

$$M(z) = \frac{H'}{H}(f'(z))^n - ((f'(z))^n)', \tag{3.13}$$

$$G(z) = \frac{(f^m(z+c))'}{f^m(z+c)} - \frac{H'}{H}. \tag{3.14}$$

From (1.4) and (3.13), we know the poles of $M(z)$ are at the zeros of $H(z)$, and at the poles of $f(z)$, $f(z+c)$. If z_0 is a zero of $H(z)$ or z_0 is a pole of $f(z+c)$ but not a pole of $f(z)$, then z_0 is at most a simple pole of $M(z)$ by (3.13). If z_0 is a pole of $f(z)$ but not a pole of $f(z+c)$, then z_0 is at most a simple pole of $M(z)$ by (3.10). If z_0 is a pole of $f(z)$ with multiplicity p and a pole of $f(z+c)$ with multiplicity q, then z_0 is a pole of $M(z)$ with the multiplicity no more than $n(p+1)+1$ by (3.13). From above arguments and our assumption, we conclude that

$$\begin{aligned} N(r,M) &\leq \overline{N}(r, \frac{1}{H}) + N(r,(f'(z))^n) + \overline{N}(r, f(z+c)) + S(r,f) \\ &\leq nN(r, f'(z)) + \overline{N}(r, f(z+c)) + S(r,f). \end{aligned} \tag{3.15}$$

On the other hand, by Lemma 2.4 and our assumption, it follows that

$$\begin{aligned} N(r,G) &\le \overline{N}(r,\frac{1}{H}) + \overline{N}(r,f) + \overline{N}(r,\frac{1}{f(z+c)}) + S(r,f) \\ &\le \overline{N}(r,f) + \overline{N}(r,\frac{1}{f(z+c)}) + S(r,f). \end{aligned} \tag{3.16}$$

From equations (3.12), (3.15) and (3.16), we have

$$\begin{aligned} mT(r,f) &\le nm(r,f) + n(N(r,f) + \overline{N}(r,f)) + \overline{N}(r,f) \\ &\quad + \overline{N}(r,f(z+c)) + \overline{N}(r,\frac{1}{f(z+c)}) + S(r,f) \\ &\le (2n+3)T(r,f) + S(r,f), \end{aligned}$$

which contradicts the assumption that $m \ge 2n+4$.

Case 2. If $T(r,H) = S(r,f)$, then applying Lemma 2.1 to equation (1.4), we have

$$mT(r,f) \le 2nT(r,f) + S(r,f),$$

which contradicts the assumption that $m \ge 2n+4$. We get a conclusion as well, completing the proof of Theorem 1.5. □

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