STEPANOV-LIKE PSEUDO-ALMOST AUTOMORPHIC FUNCTIONS IN LEBESGUE SPACES WITH VARIABLE EXPONENTS $L^{p(x)}$

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Abstract. In this article we introduce and study a new class of functions called Stepanov-like pseudo-almost automorphic functions with variable exponents, which generalizes in a natural way classical Stepanov-like pseudo-almost automorphic spaces. Basic properties of these new spaces are investigated. The existence of pseudo-almost automorphic solutions to some first-order differential equations with $S^{p,q(x)}$-pseudo-almost automorphic coefficients will also be studied.

1. Introduction

The impetus of this article comes from three main sources. The first one is a series of papers by Liang et al [16, 22, 23] in which the concept of pseudo-almost automorphy was introduced and intensively studied. Pseudo-almost automorphic functions are natural generalizations to various classes of functions including almost periodic functions, almost automorphic functions, and pseudo-almost periodic functions.

The second source is a paper by Diagana [7] in which the concept of $S^{p}$-pseudo-almost automorphy ($p \geq 1$ being a constant) was introduced and studied. Note that $S^{p}$-pseudo-almost automorphic functions (or Stepanov-like pseudo-almost automorphic functions) are natural generalizations of pseudo-almost automorphic functions. The spaces of Stepanov-like pseudo-almost automorphic functions are now fairly well-understood as most of their fundamental properties have recently been established through the combined efforts of several mathematicians. Some of the recent developments on these functions can be found in [6, 9, 12, 13, 15].

The third and last source is a paper by Diagana and Zitane [11] in which the class of $S^{p,q(x)}$-pseudo-almost periodic functions was introduced and studied, where $q : \mathbb{R} \mapsto \mathbb{R}$ is a measurable function satisfying some additional conditions. The construction of these new spaces makes extensive use of basic properties of the Lebesgue spaces with variable exponents $L^{p(x)}$ (see [5, 14, 21]).

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In this article we extend $S^p$-pseudo-almost automorphic spaces by introducing $S^{p,q(x)}$-pseudo-almost automorphic spaces (or Stepanov-like pseudo-almost automorphic spaces with variable exponents). Basic properties as well as some composition results for these new spaces are established (see Theorems 4.18 and 4.20).

To illustrate our above-mentioned findings, we will make extensive use of the newly-introduced functions to investigate the existence of pseudo-almost automorphic solutions to the first-order differential equations

\[ u'(t) = A(t)u(t) + f(t), \quad t \in \mathbb{R}, \quad (1.1) \]

and

\[ u'(t) = A(t)u(t) + F(t, Bu(t)), \quad t \in \mathbb{R}, \quad (1.2) \]

where $A(t) : D(A(t)) \subset X \mapsto X$ is a family of closed linear operators on a Banach space $X$, satisfying the well-known Acquistapace–Terreni conditions, the forcing terms $f : \mathbb{R} \rightarrow X$ is an $S^{p,q}$-pseudo-almost automorphic function and $F : \mathbb{R} \times X \rightarrow X$ is an $S^{p,q}$-pseudo-almost automorphic function, satisfying some additional conditions, and $B : X \mapsto X$ is a bounded linear operator. Such result (Theorems 5.3 and 5.4) generalize most of the known results encountered in the literature on the existence and uniqueness of pseudo-almost automorphic solutions to Equations (1.1)-(1.2).

2. Preliminaries

Let $(X, \| \cdot \|), (Y, \| \cdot \|_Y)$ be two Banach spaces. Let $BC(\mathbb{R}, X)$ (respectively, $BC(\mathbb{R} \times Y, X)$) denote the collection of all bounded continuous functions from $\mathbb{R}$ into $X$ (respectively, the class of jointly bounded continuous functions $F : \mathbb{R} \times Y \rightarrow X$). The space $BC(\mathbb{R}, X)$ equipped with the sup norm $\| \cdot \|_\infty$ is a Banach space. Furthermore, $C(\mathbb{R}, Y)$ (respectively, $C(\mathbb{R} \times Y, X)$) denotes the class of continuous functions from $\mathbb{R}$ into $Y$ (respectively, the class of jointly continuous functions $F : \mathbb{R} \times Y \rightarrow X$). Let $B(X, Y)$ stand for the Banach space of bounded linear operators from $X$ into $Y$ equipped with its natural operator topology $\| \cdot \|_{B(X, Y)}$ with $B(X, Y) := B(\mathbb{R})$.

2.1. Pseudo-almost automorphic functions.

**Definition 2.1** ([4] [6] [20]). A function $f \in C(\mathbb{R}, X)$ is said to be almost automorphic if for every sequence of real numbers $(s'_n)_{n \in \mathbb{N}}$ there exists a subsequence $(s_n)_{n \in \mathbb{N}}$ such that

\[ g(t) := \lim_{n \rightarrow \infty} f(t + s_n) \]

is well defined for each $t \in \mathbb{R}$ and

\[ f(t) = \lim_{n \rightarrow \infty} g(t - s_n) \]

for each $t \in \mathbb{R}$.

The collection of all such functions will be denoted by $AA(X)$, which turns out to be a Banach space when it is equipped with the sup-norm.

**Definition 2.2** ([5] [16]). A function $F \in C(\mathbb{R} \times Y, X)$ is said to be almost automorphic if $F(t, u)$ is almost automorphic in $t \in \mathbb{R}$ uniformly for all $u \in K$, where $K \subset Y$ is an arbitrary bounded subset. The collection of all such functions will be denoted by $AA(\mathbb{R} \times X)$. 
Definition 2.3 ([15]). A function \( L \in C(\mathbb{R} \times \mathbb{R}, X) \) is called bi-almost automorphic if for every sequence of real numbers \((s'_n)_n\) we can extract a subsequence \((s_n)_n\) such that
\[
H(t, s) := \lim_{n \to \infty} L(t + s_n, s + s_n)
\]
is well defined for each \( t, s \in \mathbb{R} \), and
\[
L(t, s) = \lim_{n \to \infty} H(t - s_n, s - s_n)
\]
for each \( t, s \in \mathbb{R} \). The collection of all such functions will be denoted by \( bAA(\mathbb{R} \times \mathbb{R}, X) \).

Proposition 2.4 ([20]). Assume \( f, g : \mathbb{R} \to X \) are almost automorphic and \( \lambda \) is any scalar. Then the following hold

(a) \( f + g, \lambda f, f_{\tau}(t) := f(t + \tau) \) and \( \hat{f}(t) := f(-t) \) are almost automorphic;

(b) The range \( R_f \) of \( f \) is precompact, so \( f \) is bounded;

(c) If \( \{f_n\} \) is a sequence of almost automorphic functions and \( f_n \to f \) uniformly on \( \mathbb{R} \), then \( f \) is almost automorphic.

Define
\[
PAA_0(X) := \left\{ f \in BC(\mathbb{R}, X) : \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \|f(\sigma)\|d\sigma = 0 \right\}.
\]

Similarly, define \( PAA_0(\mathbb{R} \times X) \) as the collection of jointly continuous functions \( F : \mathbb{R} \times Y \to X \) such that \( F(\cdot, y) \) is bounded for each \( y \in Y \) and
\[
\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \|F(s, y)\|ds = 0
\]
uniformly in \( y \in Y \).

Definition 2.5 ([4]). A function \( f \in BC(\mathbb{R}, X) \) is said to be pseudo-almost automorphic if it can be decomposed as \( f = g + \varphi \) where \( g \in AA(X) \) and \( \varphi \in PAA_0(X) \). The set of all such functions will be denoted by \( PAA(X) \).

Definition 2.6 ([16]). A function \( F \in C(\mathbb{R} \times Y, X) \) is said to be pseudo-almost automorphic if it can be decomposed as \( f = G + \Phi \) where \( G \in AA(\mathbb{R} \times X) \) and \( \Phi \in AA_0(\mathbb{R} \times X) \). The collection of such functions will be denoted by \( PAA(\mathbb{R} \times X) \).

Theorem 2.7 ([22]). The space \( PAA(X) \) equipped with the sup-norm is a Banach space.

Theorem 2.8 ([15]). If \( u \in PAA(X) \) and if \( C \in B(\mathbb{R}), X \), then the function \( t \mapsto Cu(t) \) belongs to \( PAA(X) \).

Theorem 2.9 ([7, 15]). Assume \( F \in PAA(\mathbb{R} \times X) \). Suppose that \( u \mapsto F(t, u) \) is Lipschitz uniformly in \( t \in \mathbb{R} \), in the sense that there exists \( L > 0 \) such that
\[
\|F(t, u) - F(t, v)\| \leq L\|u - v\| \quad \text{for all } t \in \mathbb{R}, u, v \in X
\]
(2.1)

If \( \Phi \in PAA(X) \), then \( F(\cdot, \Phi(\cdot)) \in PAA(X) \).
2.2. Evolution family and exponential dichotomy.

**Definition 2.10** ([6] [18]). A family of bounded linear operators \((U(t, s))_{t \geq s}\) on a Banach space \(X\) is called a strongly continuous evolution family if

(i) \(U(t, t) = I\) for all \(t \in \mathbb{R}\);
(ii) \(U(t, s) = U(t, r)U(r, s)\) for all \(t \geq r \geq s\) and \(t, r, s \in \mathbb{R}\); and
(iii) the map \((t, s) \mapsto U(t, s)x\) is continuous for all \(x \in X, t \geq s\) and \(t, s \in \mathbb{R}\).

**Definition 2.11** ([6] [18]). An evolution family \((U(t, s))_{t \geq s}\) on a Banach space \(X\) is called hyperbolic (or has exponential dichotomy) if there exist projections \(Q, P\) such that

(i) \(U(t, s)P(s) = P(t)U(t, s)\) for \(t \geq s\) and \(t, s \in \mathbb{R}\);
(ii) The restriction \(U_Q(t, s) : Q(s)X \mapsto Q(t)X\) of \(U(t, s)\) is invertible for \(t \geq s\) (and we set \(U_Q(s, t) := U(t, s)^{-1}\));
(iii) \(\|U(t, s)P(s)\| \leq Me^{-\delta(t-s)}\), \(\|U_Q(s, t)Q(t)\| \leq Me^{-\delta(t-s)}\) for \(t \geq s\) and \(t, s \in \mathbb{R}\),

where \(Q(t) := I - P(t)\) for all \(t \in \mathbb{R}\).

**Definition 2.12** ([18]). Given a hyperbolic evolution family \(U(t, s)\), we define its so-called Green’s function by

\[
\Gamma(t, s) := \begin{cases} 
U(t, s)P(s), & \text{for } t \geq s, \ t, s \in \mathbb{R}, \\
U_Q(t, s)Q(s), & \text{for } t < s, \ t, s \in \mathbb{R}.
\end{cases}
\] (2.2)

3. LEBESGUE SPACES WITH VARIABLE EXPONENTS \(L^{p(x)}\)

The setting of this section follows that of Diagana and Zitane [11]. This section is mainly devoted to the so-called Lebesgue spaces with variable exponents \(L^{p(x)}(\mathbb{R}, X)\). Various basic properties of these functions are reviewed. For more on these spaces and related issues we refer to Diening et al [6].

Let \((X, \| \cdot \|)\) be a Banach space and let \(\Omega \subseteq \mathbb{R}\) be a subset. Let \(M(\Omega, X)\) denote the collection of all measurable functions \(f : \Omega \mapsto X\). Let us recall that two functions \(f\) and \(g\) of \(M(\Omega, X)\) are equal whether they are equal almost everywhere. Set \(m(\Omega) := M(\Omega, \mathbb{R})\) and fix \(p \in m(\Omega)\). Let \(\varphi(x, t) = \theta^{p(x)}\) for all \(x \in \Omega\) and \(t \geq 0\), and define

\[
\rho(u) = \rho_{p(x)}(u) = \int_{\Omega} \varphi(x, \|u(x)\|)dx = \int_{\Omega} \|u(x)\|^{p(x)}dx,
\]

\[
L^{p(x)}(\Omega, X) = \left\{ u \in M(\Omega, X) : \lim_{\lambda \to 0^+} \rho(\lambda u) = 0 \right\},
\]

\[
L^{p(x)}_{OC}(\Omega, X) = \left\{ u \in L^{p(x)}(\Omega, X) : \rho(u) < \infty \right\}, \text{ and}
\]

\[
E^{p(x)}(\Omega, X) = \left\{ u \in L^{p(x)}(\Omega, X) : \text{ for all } \lambda > 0, \rho(\lambda u) < \infty \right\}.
\]

Note that the space \(L^{p(x)}(\Omega, X)\) defined above is a Musielak-Orlicz type space while \(L^{p(x)}_{OC}(\Omega, X)\) is a generalized Orlicz type space. Further, the sets \(E^{p(x)}(\Omega, X)\) and \(L^{p(x)}(\Omega, X)\) are vector subspaces of \(M(\Omega, X)\). In addition, \(L^{p(x)}_{OC}(\Omega, X)\) is a convex subset of \(L^{p(x)}(\Omega, X)\), and the following inclusions hold

\[
E^{p(x)}(\Omega, X) \subset L^{p(x)}_{OC}(\Omega, X) \subset L^{p(x)}(\Omega, X).
\]
Definition 3.1. A convex and left-continuous function \( \psi : [0, \infty) \to [0, \infty] \) is called a \( \Phi \)-function if it satisfies the following conditions:

(a) \( \psi(0) = 0 \);
(b) \( \lim_{t \to 0^+} \psi(t) = 0 \); and
(c) \( \lim_{t \to \infty} \psi(t) = \infty \).

Moreover, \( \psi \) is said to be positive whether \( \psi(t) > 0 \) for all \( t > 0 \).

Let us mention that if \( \psi \) is a \( \Phi \)-function, then on the set \( \{ t > 0 : \psi(t) < \infty \} \), the function \( \psi \) is of the form

\[
\psi(t) = \int_0^t k(t) dt,
\]

where \( k(\cdot) \) is the right-derivative of \( \psi(t) \). Moreover, \( k \) is a non-increasing and right-continuous function. For more on these functions and related issues we refer to [5].

Example 3.2. (a) Consider the function \( \varphi_p(t) = p^{-1} t^p \) for \( 1 \leq p < \infty \). It can be shown that \( \varphi_p \) is a \( \Phi \)-function. Furthermore, the function \( \varphi_p \) is continuous and positive.

(b) It can be shown that the function \( \varphi \) defined above; that is, \( \varphi(x, t) = t^{p(x)} \) for all \( x \in \mathbb{R} \) and \( t \geq 0 \) is a \( \Phi \)-function.

For any \( p \in m(\Omega) \), we define

\[
 p^- := \text{ess inf}_{x \in \Omega} p(x), \quad p^+ := \text{ess sup}_{x \in \Omega} p(x).
\]

Define

\[
 C_+(\Omega) := \left\{ p \in m(\Omega) : 1 < p^- \leq p(x) \leq p^+ < \infty, \text{ for each } x \in \Omega \right\}.
\]

Let \( p \in C_+(\Omega) \). Using similar argument as in [5 Theorem 3.4.1], it can be shown that

\[
 E^{p(x)}(\Omega, \mathcal{X}) = L^{p(x)}_{\text{loc}}(\Omega, \mathcal{X}) = L^{p(x)}(\Omega, \mathcal{X}).
\]

In view of the above, we define the Lebesgue space \( L^{p(x)}(\Omega, \mathcal{X}) \) with variable exponents \( p \in C_+(\Omega) \), by

\[
 L^{p(x)}(\Omega, \mathcal{X}) := \left\{ u \in M(\Omega, \mathcal{X}) : \int_{\Omega} \| u(x) \|^{p(x)} dx < \infty \right\}.
\]

Define, for each \( u \in L^{p(x)}(\Omega, \mathcal{X}) \),

\[
\| u \|_{p(x)} := \inf \left\{ \lambda > 0 : \int_{\Omega} \frac{u(x)}{\lambda} \| u(x) \|^{p(x)} dx \leq 1 \right\}.
\]

It can be shown that \( \| \cdot \|_{p(x)} \) is a norm upon \( L^{p(x)}(\Omega, \mathcal{X}) \), which is referred to as the Luxemburg norm.

Remark 3.3. Let \( p \in C_+(\Omega) \). If \( p \) is constant, then the space \( L^{p(\cdot)}(\Omega, \mathcal{X}) \), as defined above, coincides with the usual space \( L^p(\Omega, \mathcal{X}) \).

We now establish some basic properties for these spaces. For more on these functions and related issues we refer to [5].

Proposition 3.4. Let \( p \in C_+(\Omega) \) and let \( u, u_k, v \in M(\Omega, \mathcal{X}) \) for \( k = 1, 2, \ldots \).

Then the following statements hold:

(a) If \( u_k \to u \) a.e., then \( \rho_p(u) \leq \lim_{k \to \infty} \inf(\rho_p(u_k)) \);
(b) If \( \|u_k\| \to \|u\| \) a.e., then \( \rho_p(u) = \lim_{k \to \infty} \rho_p(u_k) \);
(c) If \( u_k \to u \) a.e., \( \|u_k\| \leq \|v\| \) and \( v \in L^{p(x)}(\Omega, \mathbb{X}) \), then \( u_k \to u \) in the space \( L^{p(x)}(\Omega, \mathbb{X}) \).

**Proposition 3.5** ([5] [21]). Let \( p \in C_+(\Omega) \). If \( u, v \in L^{p(x)}(\Omega, \mathbb{X}) \), then the following properties hold,

(a) \( \|u\|_{p(x)} \geq 0 \), with equality if and only if \( u = 0 \);
(b) \( \rho_p(u) \leq \rho_p(v) \) and \( \|u\|_{p(x)} \leq \|v\|_{p(x)} \) if \( \|u\| \leq \|v\| \);
(c) \( \rho_p(u\|u\|_{p(x)}^{-1}) = 1 \) if \( u \neq 0 \);
(d) \( \rho_p(u) \leq 1 \) if and only if \( \|u\|_{p(x)} \leq 1 \);
(e) If \( \|u\|_{p(x)} \leq 1 \), then
   \[
   \left[ \rho_p(u) \right]^{1/p^-} \leq \|u\|_{p(x)} \leq \left[ \rho_p(u) \right]^{1/p^+}.
   \]
(f) If \( \|u\|_{p(x)} \geq 1 \), then
   \[
   \left[ \rho_p(u) \right]^{1/p^+} \leq \|u\|_{p(x)} \leq \left[ \rho_p(u) \right]^{1/p^-}.
   \]

**Proposition 3.6** ([5]). Let \( p \in C_+(\Omega) \) and let \( u, u_k, v \in M(\Omega, \mathbb{X}) \) for \( k = 1, 2, \ldots \). Then the following statements hold:

(a) If \( u \in L^{p(x)}(\Omega, \mathbb{X}) \) and \( 0 \leq \|v\| \leq \|u\| \), then \( v \in L^{p(x)}(\Omega, \mathbb{X}) \) and \( \|v\|_{p(x)} \leq \|u\|_{p(x)} \);
(b) If \( u_k \to u \) a.e., then \( \|u\|_{p(x)} \leq \lim_{k \to -\infty} \inf(\|u_k\|_{p(x)}) \).
(c) If \( \|u_k\| \to \|u\| \) a.e. with \( u_k \in L^{p(x)}(\Omega, \mathbb{X}) \) and \( \sup_k \|u_k\|_{p(x)} < \infty \), then \( u \in L^{p(x)}(\Omega, \mathbb{X}) \) and \( \|u_k\|_{p(x)} \to \|u\|_{p(x)} \).

Using similar arguments as in Fan et al [14], we obtain the following result.

**Proposition 3.7.** If \( u, u_n \in L^{p(x)}(\Omega, \mathbb{X}) \) for \( k = 1, 2, \ldots \), then the following statements are equivalent:

(a) \( \lim_{k \to -\infty} \|u_k - u\|_{p(x)} = 0 \);
(b) \( \lim_{k \to -\infty} \rho_p(u_k - u) = 0 \);
(c) \( u_k \to u \) a.e. and \( \lim_{k \to -\infty} \rho_p(u_k) = \rho_p(u) \).

**Theorem 3.8** ([5] [14]). Let \( p \in C_+(\Omega) \). The space \( (L^{p(x)}(\Omega, \mathbb{X}), \| \cdot \|_{p(x)}) \) is a Banach space that is separable and uniform convex. Its topological dual is \( L^{q(x)}(\Omega, \mathbb{X}) \), where \( p^{-1}(x) + q^{-1}(x) = 1 \). Moreover, for any \( u \in L^{p(x)}(\Omega, \mathbb{X}) \) and \( v \in L^{q(x)}(\Omega, \mathbb{R}) \), we have

\[
\| \int_{\Omega} u v \| \leq \left( \frac{1}{p^-} + \frac{1}{q^-} \right) \|u\|_{p(x)} \|v\|_{q(x)}.
\]

(3.1)

Define

\[
D_+(\Omega) := \left\{ p \in m(\Omega) : 1 \leq p^- \leq p(x) \leq p^+ < \infty, \text{ for each } x \in \Omega \right\}.
\]

**Corollary 3.9** ([21]). Let \( p, r \in D_+(\Omega) \). If the function \( q \) defined by the equation

\[
\frac{1}{q(x)} = \frac{1}{p(x)} + \frac{1}{r(x)}
\]

is in \( D_+(\Omega) \), then there exists a constant \( C = C(p, r) \in [1, 5] \) such that

\[
\|uv\|_{q(x)} \leq C \|u\|_{p(x)} \|v\|_{r(x)},
\]

for every \( u \in L^{p(x)}(\Omega, \mathbb{X}) \) and \( v \in L^{r(x)}(\Omega, \mathbb{R}) \).
Corollary 3.10 ([5]). Let \( \text{meas}(\Omega) < \infty \) where \( \text{meas}(\cdot) \) stands for the Lebesgue measure and \( p, q \in D_+(\Omega) \). If \( q(\cdot) \leq p(\cdot) \) almost everywhere in \( \Omega \), then the embedding \( L^{p(x)}(\Omega, X) \hookrightarrow L^{q(x)}(\Omega, X) \) is continuous whose norm does not exceed 2(meas(\Omega) + 1).

4. **Stepanov-like pseudo-almost automorphic functions with variable exponents**

**Definition 4.1.** The Bochner transform \( f^b(t,s), t \in \mathbb{R}, s \in [0,1] \) of a function \( f : \mathbb{R} \to X \) is defined by \( f^b(t,s) := f(t+s) \).

**Remark 4.2.** A function \( \varphi(t,s), t \in \mathbb{R}, s \in [0,1] \), is the Bochner transform of a certain function \( f \), \( \varphi(t,s) = f^b(t,s) \), if and only if \( \varphi(t+\tau, s-\tau) = \varphi(s, t) \) for all \( t \in \mathbb{R}, s \in [0,1] \) and \( \tau \in [s-1, s] \). Moreover, if \( f = h + \varphi \), then \( f^b = h^b + \varphi^b \). Moreover, \( (\lambda f)^b = \lambda f^b \) for each scalar \( \lambda \).

**Definition 4.3.** The Bochner transform \( F^b(t,s,u), t \in \mathbb{R}, s \in [0,1], u \in X \) of a function \( F : \mathbb{R} \times \mathbb{R} \times X \to X \), is defined by \( F^b(t,s,u) := F(t+s,u) \) for each \( u \in X \).

**Definition 4.4.** Let \( p \in [1,\infty) \). The space \( BS^p(X) \) of all Stepanov bounded functions, with the exponent \( p \), consists of all measurable functions \( f \) on \( \mathbb{R} \) with values in \( X \) such that \( f^b \in L^\infty(\mathbb{R}, L^p(0,1), X) \). This is a Banach space with the norm

\[
\|f\|_{S^p} = \|f^b\|_{L^\infty(\mathbb{R}, L^p)} = \sup_{t \in \mathbb{R}} \left( \int_t^{t+1} \|f(\tau)\|^p \, d\tau \right)^{1/p}.
\]

Note that for each \( p \geq 1 \), we have the following continuous inclusion:

\( (BC(X), \| \cdot \|_\infty) \hookrightarrow (BS^p(X), \| \cdot \|_{S^p}) \).

**Definition 4.5** (Diagana and Zitane [11]). Let \( p \in C_+(\mathbb{R}) \). The space \( BS^{p(x)}(X) \) consists of all functions \( f \in M(\mathbb{R}, X) \) such that \( \|f\|_{S^{p(x)}} < \infty \), where

\[
\|f\|_{S^{p(x)}} = \sup_{t \in \mathbb{R}} \left[ \inf_{\lambda > 0} \left\{ \frac{1}{\lambda} \int_0^1 \|f(x+t)/\lambda\|^{p(x+t)} \, dx \leq 1 \right\} \right] = \sup_{t \in \mathbb{R}} \left[ \inf_{\lambda > 0} \left\{ \frac{1}{\lambda} \int_t^{t+1} \|f(x)/\lambda\|^{p(x)} \, dx \leq 1 \right\} \right].
\]

Note that the space \( (BS^{p(x)}(X), \| \cdot \|_{S^{p(x)}}) \) is a Banach space, which, depending on \( p(\cdot) \), may or may not be translation-invariant.

**Definition 4.6** (Diagana and Zitane [11]). If \( p, q \in C_+(\mathbb{R}) \), we then define the space \( BS^{p(x), q(x)}(X) \) as follows

\[
BS^{p(x), q(x)}(X) := BS^{p(x)}(X) + BS^{q(x)}(X)
\]

\[
= \left\{ f = h + \varphi \in M(\mathbb{R}, X) : h \in BS^{p(x)}(X) \text{ and } \varphi \in BS^{q(x)}(X) \right\}.
\]

We equip \( BS^{p(x), q(x)}(X) \) with the norm \( \| \cdot \|_{S^{p(x), q(x)}} \) defined by

\[
\|f\|_{S^{p(x), q(x)}} := \inf \left\{ \|h\|_{S^{p(x)}} + \|\varphi\|_{S^{q(x)}} : f = h + \varphi \right\}.
\]

Clearly, \( (BS^{p(x), q(x)}(X), \| \cdot \|_{S^{p(x), q(x)}}) \) is a Banach space, which, depending on both \( p(\cdot) \) and \( q(\cdot) \), may or may not be translation-invariant.
Lemma 4.7 (Diagana and Zitane [11]). Let \( p, q \in C_+(\mathbb{R}) \). Then the following continuous inclusion holds,
\[
\left( BC(\mathbb{R}, \mathbb{X}), \| \cdot \|_\infty \right) \hookrightarrow \left( BS^{p(x)}(\mathbb{X}), \| \cdot \|_{S^{p(x)}} \right) \hookrightarrow \left( BS^{p(x), q(x)}(\mathbb{X}), \| \cdot \|_{S^{p(x), q(x)}} \right).
\]

Proof. The fact that \( (BS^{p(x)}(\mathbb{X}), \| \cdot \|_{S^{p(x)}}) \hookrightarrow (BS^{p(x), q(x)}(\mathbb{X}), \| \cdot \|_{S^{p(x), q(x)}}) \) is obvious. Thus we will only show that \( (BC(\mathbb{R}, \mathbb{X}), \| \cdot \|_\infty) \hookrightarrow (BS^{p(x)}(\mathbb{X}), \| \cdot \|_{S^{p(x)}}) \) is continuous. Indeed, let \( f \in BC(\mathbb{R}, \mathbb{X}) \subset M(\mathbb{R}, \mathbb{X}) \). If \( \| f \|_\infty = 0 \), which yields \( f = 0 \), then there is nothing to prove. Now suppose that \( \| f \|_\infty \neq 0 \). Using the facts that
\[
0 < \frac{\| f(x) \|}{\| f \|_\infty} \leq 1 \quad \text{and that} \quad p \in C_+(\mathbb{R}) \quad \text{it follows that for every} \quad t \in \mathbb{R},
\]
\[
\int_t^{t+1} \frac{\| f(x) \|^{p(x)}}{\| f \|_\infty} \, dx \leq \int_t^{t+1} 1^{p(x)} \, dx = 1,
\]
and hence \( \| f \|_\infty \in \left\{ \lambda > 0 : \int_t^{t+1} \frac{\| f(x) \|^{p(x)}}{\lambda} \, dx \leq 1 \right\} \), which yields
\[
\inf \left\{ \lambda > 0 : \int_t^{t+1} \frac{\| f(x) \|^{p(x)}}{\lambda} \, dx \leq 1 \right\} \leq \| f \|_\infty.
\]
Therefore, \( \| f \|_{S^{p(x)}} \leq \| f \|_\infty < \infty \). This shows that not only \( f \in (BS^{p(x)}(\mathbb{X}), \| \cdot \|_{S^{p(x)}}) \) but also the injection \( (BC(\mathbb{R}, \mathbb{X}), \| \cdot \|_\infty) \hookrightarrow (BS^{p(x)}(\mathbb{X}), \| \cdot \|_{S^{p(x)}}) \) is continuous. \( \square \)

Definition 4.8. Let \( p \geq 1 \) be a constant. A function \( f \in BS^p(\mathbb{X}) \) is said to be \( BS^p \)-almost automorphic (or Stepanov-like almost automorphic function) if \( f^b \in AA(L^p([0, 1), \mathbb{X})) \). That is, a function \( f \in L^p_{loc}(\mathbb{R}, \mathbb{X}) \) is said to be Stepanov-like almost automorphic if its Bochner transform \( f^b : \mathbb{R} \to L^p([0, 1); \mathbb{X}) \) is almost automorphic in the sense that for every sequence of real numbers \((s_n)\), there exists a subsequence \((s'_n)\) and a function \( g \in L^p_{loc}(\mathbb{R}, \mathbb{X}) \) such that
\[
\left( \int_0^1 \| f(t+s+s_n) - g(t+s) \|^{p} \, ds \right)^{1/p} \to 0, \quad \left( \int_0^1 \| g(t+s-s_n) - f(t+s) \|^{p} \, ds \right)^{1/p} \to 0
\]
as \( n \to \infty \) pointwise on \( \mathbb{R} \). The collection of such functions will be denoted by \( S^{p}_{aa}(\mathbb{X}) \).

Remark 4.9. There are some difficulties in defining \( S^{p(x)}_{aa}(\mathbb{X}) \) for a function \( p \in C_+(\mathbb{R}) \) that is not necessarily constant. This is mainly due to the fact that the space \( BS^{p(x)}(\mathbb{X}) \) is not always translation-invariant. In other words, the quantities \( f^b(t+\tau, s) \) and \( f^b(t, s) \) (for \( t \in \mathbb{R}, s \in [0, 1] \)) that are used in the definition of \( BS^{p(x)} \)-almost automorphy, do not belong to the same space, unless \( p \) is constant.

Remark 4.10. It is clear that if \( 1 \leq p < q < \infty \) and \( f \in L^p_{loc}(\mathbb{R}, \mathbb{X}) \) is \( S^q \)-almost automorphic, then \( f \) is \( S^p \)-almost automorphic. Also if \( f \in AA(\mathbb{X}) \), then \( f \) is \( S^p \)-almost automorphic for any \( 1 \leq p < \infty \).

Taking into account Remark [4.9] we introduce the concept of \( S^{p(x), q(x)} \)-pseudo-almost automorphy as follows, which obviously generalizes the notion of \( S^p \)-pseudo-almost automorphy.

Definition 4.11. Let \( p \geq 1 \) be a constant and let \( q \in C_+(\mathbb{R}) \). A function \( f \in BS^{p(x), q(x)}(\mathbb{X}) \) is said to be \( S^{p(x), q(x)} \)-pseudo-almost automorphic (or Stepanov-like pseudo-almost automorphic with variable exponents \( p, q(x) \)) if it can be decomposed as
\[
f = h + \varphi,
\]
where \( h \in S^{p}_{aa}(\mathbb{X}) \) and \( \varphi \in S^{q}_{paa}(\mathbb{X}) \) with \( S^{q}_{paa}(\mathbb{X}) \) being the space of all \( \psi \in BS^{q}(\mathbb{X}) \) such that
\[
\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \inf \left\{ \lambda > 0 : \int_{t}^{t+1} \frac{\| \varphi(x) \|}{\lambda} \, dx \leq 1 \right\} dt = 0.
\]
The collection of \( S^{p,q}(\mathbb{X}) \)-pseudo-almost automorphic functions will be denoted by \( S^{p,q}(\mathbb{X}) \).

**Lemma 4.12.** Let \( r,s \geq 1, p,q \in D_{+}(\mathbb{R}) \). If \( s < r, q^{-} < p^{-} \) and \( f \in BS^{r,p}(\mathbb{X}) \) is \( S^{p,q}(\mathbb{X}) \)-pseudo-almost automorphic, then \( f \) is \( S^{s,q}(\mathbb{X}) \)-pseudo-almost automorphic.

**Proof.** Suppose that \( f \in BS^{r,p}(\mathbb{X}) \) is \( S^{r,p}(\mathbb{X}) \)-pseudo-almost automorphic. Thus there exist two functions \( h, \varphi : \mathbb{R} \to \mathbb{X} \) such that
\[
f = h + \varphi,
\]
where \( h \in S^{s}_{aa}(\mathbb{X}) \) and \( \varphi \in S^{q}(\mathbb{X}) \). From remark 4.10, \( h \) is \( S^{s} \)-almost automorphic.

In view of \( q(\cdot) \leq q^{+} < p^{-} \leq p(\cdot) \), it follows from Corollary 3.10 that,
\[
\left[ \inf \left\{ \lambda > 0 : \int_{t}^{t+1} \frac{\| \varphi(x) \|}{\lambda} \, dx \leq 1 \right\} \right] 
\leq 4 \left[ \inf \left\{ \lambda > 0 : \int_{t}^{t+1} \frac{\| \varphi(x) \|}{\lambda} \, dx \leq 1 \right\} \right].
\]
Using the fact that \( \varphi \in S^{q}_{paa}(\mathbb{X}) \) and the previous inequality it follows that
\[
\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \inf \left\{ \lambda > 0 : \int_{t}^{t+1} \frac{\| \varphi(x) \|}{\lambda} \, dx \leq 1 \right\} dt = 0;
\]
that is, \( \varphi \in S^{q}(\mathbb{X}) \). Therefore, \( f \in S^{s,q}(\mathbb{X}) \). \( \square \)

**Proposition 4.13.** Let \( p \geq 1 \) be a constant and let \( q \in C_{+}(\mathbb{R}) \). If \( f \in PAA(\mathbb{X}) \), then \( f \) is \( S^{p,q}(\mathbb{X}) \)-pseudo-almost automorphic.

**Proof.** Let \( f \in PAA(\mathbb{X}) \), that is, there exist two functions \( h, \varphi : \mathbb{R} \to \mathbb{X} \) such that \( f = h + \varphi \) where \( h \in AA(\mathbb{X}) \) and \( \varphi \in PAA_{0}(\mathbb{X}) \). Now from remark 4.10 \( h \in AA(\mathbb{X}) \subset S^{p}_{aa}(\mathbb{X}) \). The proof of \( \varphi \in S^{q}_{paa}(\mathbb{X}) \) was given in [11]. However for the sake of clarity, we reproduce it here. Using (e)-(f) of Proposition 3.5 and the usual Hölder inequality, it follows that
\[
\int_{-T}^{T} \inf \left\{ \lambda > 0 : \int_{0}^{1} \frac{\| \varphi(t + x) \|}{\lambda} \, dx \leq 1 \right\} dt
\leq \left( \int_{-T}^{T} \left( \int_{0}^{1} \| \varphi(t + x) \|^{q(t + x)} \, dx \right)^{\gamma} \right) dt
\leq (2T)^{1-\gamma} \left[ \int_{-T}^{T} \left( \int_{0}^{1} \| \varphi(t + x) \|^{q(t + x)} \, dx \right) dt \right]^{\gamma}
\leq (2T)^{1-\gamma} \left[ \int_{-T}^{T} \left( \int_{0}^{1} \| \varphi(t + x) \|^{q(t + x) - 1} \, dx \right) dt \right]^{\gamma}
\leq (2T)^{1-\gamma} \left( \| \varphi \|_{\infty} + 1 \right)^{\frac{q-1}{\gamma}} \left[ \int_{-T}^{T} \left( \int_{0}^{1} \| \varphi(t + x) \| \, dx \right) dt \right]^{\gamma}
\]
\[
= (2T)^{1-\gamma} \left( \|\varphi\|_{\infty} + 1 \right)^{\frac{q+1}{q}} \left[ \int_0^1 \left( \int_{-T}^T \|\varphi(t + x)\| \, dx \right) \, dt \right]^\gamma
\]

\[
= (2T) \left( \|\varphi\|_{\infty} + 1 \right)^{\frac{q+1}{q}} \left[ \int_0^1 \left( \frac{1}{2T} \int_{-T}^T \|\varphi(t + x)\| \, dt \right) \, dx \right]^\gamma,
\]

where

\[
\gamma = \begin{cases} 
\frac{1}{q} & \text{if } \|\varphi\| < 1, \\
\frac{1}{q} & \text{if } \|\varphi\| \geq 1.
\end{cases}
\]

Using the fact that \( PAA_0(X) \) is translation invariant and the (usual) Dominated Convergence Theorem, it follows that

\[
\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^T \inf \left\{ \lambda > 0 : \int_0^1 \left| \frac{\varphi(x + t)}{\lambda} \right|^q(\varphi(x+t)) \, dx \leq 1 \right\} dt
\]

\[
\leq \left( \|\varphi\|_{\infty} + 1 \right)^{\frac{q+1}{q}} \left[ \int_0^1 \left( \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^T \|\varphi(t + x)\| \, dx \right) \, dt \right]^\gamma = 0.
\]

\[\square\]

Using similar argument as in [22], the following Lemma can be established.

**Lemma 4.14.** Let \( p, q \geq 1 \) be a constants. If \( f = h + \varphi \in S_{p,\text{para}}^p(X) \) such that \( h^b \in AA(L^p((0, 1), X)) \) and \( \varphi^b \in PAA_0(L^q((0, 1), X)) \), then

\[
\{ h(t + .) : t \in \mathbb{R} \} \subset \{ f(t + .) : t \in \mathbb{R} \}, \quad n a \, S_{p,\text{para}}^p(X).
\]

**Proof.** We prove it by contradiction. Indeed, if this is not true, then there exist a \( t_0 \in \mathbb{R} \) and an \( \varepsilon > 0 \) such that

\[
\|h(t_0 + \cdot) - f(t + \cdot)\|_{S_{p,\text{para}}^p} \geq 2\varepsilon, \quad t \in \mathbb{R}.
\]

Since \( h^b \in AA(L^p((0, 1), X)) \) and \( \{BS_{X,h}^p\| \cdot \|_{S_{p,\text{para}}^p} \} \rightarrow \{BS_{X,h}^p\| \cdot \|_{S_{p,\text{para}}^p} \} \), fix \( t_0 \in \mathbb{R}, \varepsilon > 0 \) and write, \( B_\varepsilon := \{ \tau \in \mathbb{R} : \|h(t_0 + \tau + \cdot) - g(t_0 + \cdot)\|_{S_{p,\text{para}}^p} < \varepsilon \} \). By [22] Lemma 2.1], there exist \( s_1, \ldots, s_m \in \mathbb{R} \) such that

\[
\bigcup_{i=1}^m (s_i + B_\varepsilon) = \mathbb{R}.
\]

Write

\[
\hat{s}_i = s_i - t_0 \quad (1 \leq i \leq m), \quad \eta = \max_{1 \leq i \leq m} |\hat{s}_i|.
\]

For \( T \in \mathbb{R} \) with \( |T| > \eta \); we put

\[
B_{\varepsilon,T}^{(i)} = [-T + \eta - \hat{s}_i, T - \eta - \hat{s}_i] \cap (t_0 + B_\varepsilon), \quad 1 \leq i \leq m,
\]

one has \( \bigcup_{i=1}^m (\hat{s}_i + B_{\varepsilon,T}^{(i)}) = [-T + \eta, T - \eta] \).

Using the fact that \( B_{\varepsilon,T}^{(i)} \subset [-T, T] \cap (t_0 + B_\varepsilon), i = 1, \ldots, m, \) we obtain

\[
2(T - \eta) = \text{meas}([-T + \eta, T - \eta])
\]

\[
\leq \sum_{i=1}^m \text{meas}(\hat{s}_i + B_{\varepsilon,T}^{(i)})
\]

\[
= \sum_{i=1}^m \text{meas}(B_{\varepsilon,T}^{(i)})
\]

\[
\leq m \max_{1 \leq i \leq m} \{ \text{meas}(B_{\varepsilon,T}^{(i)}) \}.
\]
Proof.\) Let \(p, q \geq 1\) be constants. The space \(S^{p,q}_{\text{paa}}(X)\) equipped with the norm \(\| \cdot \|_{S^{p,q}}\) is a Banach space.

Theorem 4.15. Let \(p, q \geq 1\) be constants. The space \(S^{p,q}_{\text{paa}}(X)\) is a closed subspace of \(BS^{p,q}(X)\). Let \(f_n = h_n + \varphi_n\) be a Cauchy sequence in \(S^{p,q}_{\text{paa}}(X)\) with \((h_n^b)_{n \in \mathbb{N}} \subset AA(L^p((0, 1), X))\) and \((\varphi_n^b)_{n \in \mathbb{N}} \subset PAA_0(L^q((0, 1), X))\) such that \(\|f_n - f\|_{S^{p,q}} \to 0\) as \(n \to \infty\). By Lemma 4.14, one has

\[
\{h_n(t + \cdot): t \in \mathbb{R}\} \subset \{f_n(t + \cdot): t \in \mathbb{R}\},
\]

and hence

\[
\|h_n\|_{S^{p,q}} = \|h_n\|_{S^{p,q}} \leq \|f_n\|_{S^{p,q}} \quad \text{for all } n \in \mathbb{N}.
\]

Consequently, there exists a function \(h \in S^{p,q}_{\text{paa}}(X)\) such that \(\|h_n - h\|_{S^{p,q}} \to 0\) as \(n \to \infty\). Using the previous fact, it easily follows that the function \(\varphi := f - h \in BS^q(X)\) and that \(\|\varphi_n - \varphi\|_{S^q} = \|(f_n - h_n) - (f - h)\|_{S^q} \to 0\) as \(n \to \infty\). Using the fact that \(\varphi = (\varphi - \varphi_n) + \varphi_n\) it follows that

\[
\frac{1}{2T} \int_{-T}^{T} \left( \int_{0}^{1} \|\varphi(t + \cdot)\|^q d\tau \right)^{1/q} dt \leq \frac{1}{2T} \int_{-T}^{T} \left( \int_{0}^{1} \|\varphi(t + \cdot) - \varphi_n(t + \cdot)\|^q d\tau \right)^{1/q} dt
\]

\[
+ \left( \int_{0}^{1} \|\varphi_n(t + \cdot)\|^q d\tau \right)^{1/q} dt
\]

\[
\leq \|\varphi_n - \varphi\|_{S^q} + \frac{1}{2T} \int_{-T}^{T} \left( \int_{0}^{1} \|\varphi_n(t + \cdot)\|^q d\tau \right)^{1/q} dt.
\]

Letting \(T \to \infty\) and then \(n \to \infty\) in the previous inequality, we obtain that \(\varphi^b \in PAA_0(L^q((0, 1), X))\); that is, \(f = h + \varphi \in S^{p,q}_{\text{paa}}(X)\). \(\square\)

Using similar arguments as in the proof of [15] Theorem 3.4, we obtain the next theorem.

Theorem 4.16. If \(u \in S^{p,q}_{\text{paa}}(Y)\) and if \(C \in B(Y, X)\), then the function \(t \mapsto Cu(t)\) belongs to \(S^{p,q}_{\text{paa}}(X)\).
Definition 4.17. Let \( p \geq 1 \) and \( q \in C_+(\mathbb{R}) \). A function \( F : \mathbb{R} \times \mathbb{Y} \rightarrow \mathbb{X} \) with \( F(\cdot, u) \in BS^{p,q}(\mathbb{X}) \) for each \( u \in \mathbb{Y} \), is said to be \( S^{p,q}(\mathbb{X}) \)-pseudo-almost automorphic in \( t \in \mathbb{R} \) uniformly in \( u \in \mathbb{Y} \) if \( t \mapsto F(t,u) \) is \( S^{p,q}(\mathbb{X}) \)-pseudo-almost automorphic for each \( u \in B \) where \( B \subseteq \mathbb{Y} \) is an arbitrary bounded set. This means, there exist two functions \( G, H : \mathbb{R} \times \mathbb{Y} \rightarrow \mathbb{X} \) such that \( F = G + H \), where \( G^b \in AA(\mathbb{Y}, L^p((0,1), \mathbb{X})) \) and \( H^b \in PAA_0(\mathbb{Y}, L^{p/q}(0,1), \mathbb{X})) \); that is,

\[
\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \inf \left\{ \lambda > 0 : \int_{0}^{1} \left\| \frac{H(x+t,u)}{\lambda} \right\|^{p/(p+q)} \, dx \leq 1 \right\} \, dt = 0,
\]

uniformly in \( u \in B \) where \( B \subset \mathbb{Y} \) is an arbitrary bounded set. The collection of such functions will be denoted by \( S^{p,q}(\mathbb{Y}, \mathbb{X}) \).

Let \( Lip^r(\mathbb{Y}, \mathbb{X}) \) denote the collection of functions \( f : \mathbb{R} \times \mathbb{Y} \rightarrow \mathbb{X} \) satisfying: there exists a nonnegative function \( L_f \in L^r(\mathbb{R}) \) such that

\[
\| f(t,u) - f(t,v)\| \leq L_f(t)\| u - v\|_{\mathbb{Y}} \quad \text{for all } u, v \in \mathbb{Y}, \ t \in \mathbb{R}.
\]

Now, we recall the following composition theorem for \( S^{p,q} \) functions.

**Theorem 4.18 (I7).** Let \( p > 1 \) be a constant. We suppose that the following conditions hold:

(a) \( f \in S^{p,q}_{\text{PAA}}(\mathbb{Y}, \mathbb{X}) \) \( \cap \ Lip^r(\mathbb{Y}, \mathbb{X}) \) with \( r \geq \frac{p}{p-1} \).

(b) \( \phi \in S^{p,q}_{\text{PAA}}(\mathbb{X}) \) and there exists a set \( E \subset \mathbb{R} \) such that \( K := \{ \phi(t) : t \in \mathbb{R} \setminus E \} \) is compact in \( \mathbb{X} \).

Then there exists \( m \in [1, p] \) such that \( f(\cdot, \phi(\cdot)) \in S^{m}_{\text{PAA}}(\mathbb{X}) \).

To obtain a composition theorem for \( S^{p,q}_{\text{PAA}} \) functions, we need the following lemma.

**Lemma 4.19.** Let \( p, q > 1 \) be a constants. Assume that \( f = g + h \in S^{p,q}_{\text{PAA}}(\mathbb{R} \times \mathbb{X}) \) with \( g^b \in AA(\mathbb{R} \times L^p((0,1), \mathbb{X})) \) and \( h^b \in PAA_0(\mathbb{R} \times L^{p/q}(0,1), \mathbb{X})) \). If \( f \in Lip^r(\mathbb{R}, \mathbb{X}) \), then \( g \) satisfies

\[
\left( \int_{0}^{1} \| g(t+s, u(s)) - g(t+s, v(s)) \|^p \, ds \right)^{1/p} \leq c \| L_f \|_{S^r} \| u - v\|_{\mathbb{Y}}.
\]

for all \( u, v \in \mathbb{Y} \) and \( t \in \mathbb{R} \), where \( c \) is a nonnegative constant.

**Proof.** Let \( f = g + h \in S^{p,q}_{\text{PAA}}(\mathbb{R} \times \mathbb{X}) \) with \( g^b(\cdot, u) \in AA(L^p((0,1), \mathbb{X})) \) and \( h^b(\cdot, u) \in PAA_0(L^{p/q}(0,1), \mathbb{X})) \) for each \( u \in \mathbb{Y} \). Using Lemma 4.14 it follows that

\[
\{ g(t+, u) : t \in \mathbb{R} \} \subset \{ f(t+, u) : t \in \mathbb{R} \} \quad \text{in } S^{p,q} \text{ for each } u \in \mathbb{Y}.
\]

Since \( f \in Lip^r(\mathbb{R}, \mathbb{X}) \) and \( \left( BS^{p}(\mathbb{X}), \| \cdot \|_{S^p} \right) \hookrightarrow \left( BS^{p,q}(\mathbb{X}), \| \cdot \|_{S^{p,q}} \right) \), it follows that

\[
\left( \int_{0}^{1} \| g(t+s, u(s)) - g(t+s, v(s)) \|^p \, ds \right)^{1/p} \leq \| g(\cdot, u) - g(\cdot, v) \|_{S^p}
\]

\[
= \| g(\cdot, u) - g(\cdot, v) \|_{S^{p,q}}
\]

\[
\leq \| f(\cdot, u) - f(\cdot, v) \|_{S^{p,q}}
\]

\[
\leq c \| f(\cdot, u) - f(\cdot, v) \|_{S^r}
\]

\[
\leq c \| L_f \|_{S^r} \| u - v\|_{\mathbb{Y}}.
\]

for all \( u, v \in \mathbb{Y} \) and \( t \in \mathbb{R} \). \( \square \)
Theorem 4.20. Let $p, q > 1$ be a constants such that $p \leq q$. Assume that the following conditions hold:

(a) $f = g + h \in S^p_{\text{paa}}(\mathbb{R} \times X)$ with $g \in S^p_{\text{aa}}(\mathbb{R} \times X)$ and $h \in S^q_{\text{paa}}(\mathbb{R} \times X)$. Moreover, $f, g \in \text{Lip}^r(\mathbb{R}, X)$ with $r \geq \max\{p, \frac{p}{q-1}\}$;

(b) $\phi = \alpha + \beta \in S^p_{\text{paa}}(Y)$ with $\alpha \in S^p_{\text{aa}}(Y)$ and $\beta \in S^q_{\text{paa}}(Y)$, and $K := \{\alpha(t) : t \in \mathbb{R}\}$ is compact in $Y$.

Then there exists $m \in [1, p)$ such that $f(\cdot, \phi(\cdot)) \in S^m_{\text{paa}}(\mathbb{R} \times X)$.

Proof. First of all, write

$$f^b(\cdot, \phi^b(\cdot)) = g^b(\cdot, \alpha^b(\cdot)) + f^b(\cdot, \phi^b(\cdot)) - f^b(\cdot, \alpha^b(\cdot)) + h^b(\cdot, \alpha^b(\cdot)).$$

From Lemma 4.19, one has $g \in S^p_{\text{aa}}(\mathbb{R} \times X)$. Now using the theorem of composition of $S^p$-almost automorphic functions (Theorem 4.18), it is easy to see that there exists $m \in [1, p)$ with $\frac{1}{m} = \frac{1}{p} + \frac{1}{q} \leq \frac{1}{q}$ such that $g^b(\cdot, \alpha^b(\cdot)) \in A\text{A}(\mathbb{R} \times L^m((0, 1), X))$.

Set $\Phi^b(\cdot) = f^b(\cdot, \phi^b(\cdot)) - f^b(\cdot, \alpha^b(\cdot))$. Clearly, $\Phi^b \in P\text{AA}_0(\mathbb{R} \times L^m((0, 1), X))$.

Now, for $T > 0$,

$$\frac{1}{2T} \int_{-T}^{T} \left( \int_t^{t+1} \|\Phi^b(s)\|^m ds \right)^{1/m} dt$$

$$= \frac{1}{2T} \int_{-T}^{T} \left( \int_t^{t+1} \|f^b(s, \phi^b(s)) - f^b(s, \alpha^b(s))\|^m ds \right)^{1/m} dt$$

$$\leq \frac{1}{2T} \int_{-T}^{T} \left( \int_t^{t+1} \left( L_f^b(s) \|\beta^b(s)\|_Y \right)^m ds \right)^{1/m} dt$$

$$\leq \|L_f^b\|_{s^r} \left[ \frac{1}{2T} \int_{-T}^{T} \left( \int_t^{t+1} \|\beta^b(s)\|^p ds \right)^{1/p} dt \right]$$

$$\leq \|L_f^b\|_{s^r} \left[ \frac{1}{2T} \int_{-T}^{T} \left( \int_t^{t+1} \|\beta^b(s)\|^q ds \right)^{1/q} dt \right].$$

Using the fact that $\beta^b \in P\text{AA}_0(L^q((0, 1), Y))$, it follows that $\Phi^b \in P\text{AA}_0(\mathbb{R} \times L^m((0, 1), X))$.

On the other hand, since $f, g \in \text{Lip}^r(\mathbb{R}, X) \subset \text{Lip}^p(\mathbb{R}, X)$, one has

$$\left( \int_0^1 \|h(t + s, u(s)) - h(t + s, v(s))\|^m ds \right)^{1/m} \leq \left( \int_0^1 \|f(t + s, u(s)) - f(t + s, v(s))\|^m ds \right)^{1/m}$$

$$+ \left( \int_0^1 \|g(t + s, u(s)) - g(t + s, v(s))\|^m ds \right)^{1/m}$$

$$\leq \left( \int_0^1 \left( L_f(t + s) \|u(s) - v(s)\|_Y \right)^m ds \right)^{1/m}$$

$$+ \left( \int_0^1 \left( L_g(t + s) \|u(s) - v(s)\|_Y \right)^m ds \right)^{1/m}$$

$$\leq \left( \|L_f\|_{s^r} + \|L_g\|_{s^r} \right) \|u(s) - v(s)\|_p.$$
that
\[ \{ \alpha(t) : t \in \mathbb{R} \} \subset \bigcup_{k=1}^{m} B_k. \]

Therefore, for \( 1 \leq k \leq m \), the set \( U_k = \{ t \in \mathbb{R} : \alpha \in B_k \} \) is open and \( \mathbb{R} = \bigcup_{k=1}^{m} U_k. \)

Now, for \( 2 \leq k \leq m \), set \( V_k = U_k - \bigcup_{i=1}^{k-1} U_i \) and \( V_1 = U_1 \). Clearly, \( V_i \cap V_j = \emptyset \) for all \( i \neq j \). Define the step function \( \varpi : \mathbb{R} \to \mathbb{Y} \) by \( \varpi(t) = x_k, t \in V_k, k = 1, 2, \ldots, m. \)

It easy to see that
\[ \| \alpha(s) - \varpi(s) \|_Y \leq \varepsilon, \quad \text{for all } s \in \mathbb{R}. \]

which yields
\[
\frac{1}{2T} \int_{-T}^{T} \left( \int_{t}^{t+1} \| h(s, \alpha(s)) \|_Y \, ds \right)^{1/m} \, dt \\
\leq \frac{1}{2T} \int_{-T}^{T} \left( \int_{t}^{t+1} \| h(s, \alpha(s)) - h(s, \varpi(s)) \|_Y \, ds \right)^{1/m} \, dt \\
+ \frac{1}{2T} \int_{-T}^{T} \left( \int_{t}^{t+1} \| h(s, \varpi(s)) \|_Y \, ds \right)^{1/m} \, dt \\
\leq \left( \| L_f \|_{S^r} + \| L_g \|_{S^r} \right) \varepsilon + \frac{1}{2T} \int_{-T}^{T} \left( \sum_{k=1}^{m} \int_{V_k \cap [t,t+1]} \| h(s, \varpi(s)) \|_Y \, ds \right)^{1/m} \, dt \\
\leq \left( \| L_f \|_{S^r} + \| L_g \|_{S^r} \right) \varepsilon + \frac{1}{2T} \int_{-T}^{T} \left( \sum_{k=1}^{m} \int_{V_k \cap [t,t+1]} \| h(s, \varpi(s)) \|_Y^{q} \, ds \right)^{1/q} \, dt.
\]

Since \( \varepsilon \) is arbitrary and \( h^b \in PAA_0(\mathbb{R} \times L^q((0, 1), \mathbb{X})) \), it follows that the function \( h^b(\cdot, \alpha^b(\cdot)) \) belongs to \( PAA_0(\mathbb{R} \times L^m((0, 1), \mathbb{X})) \).

\[ \square \]

**Remark 4.21.** A general composition theorem in \( S^{p,q(x)}_{pa}(\mathbb{R} \times \mathbb{X}) \) is unlikely as compositions of elements of \( S^{p,q(x)}_{pa}(\mathbb{R} \times \mathbb{X}) \) may not be well-defined unless \( q(\cdot) \) is the constant function.

5. **Existence of pseudo-almost automorphic solutions**

Let \( p, q > 1 \) be constants such that \( p \leq q \). In this section, we discuss the existence and uniqueness of pseudo-almost automorphic solutions to the first-order linear differential equation \([1.1]\) and to the semilinear equation \([1.2]\). For that, we make the following assumptions:

(H1) The family of closed linear operators \( A(t) \) satisfy Acquistapace–Terreni conditions.

(H2) The evolution family \( (U(t,s))_{t \geq s} \) generated by \( A(t) \) has an exponential dichotomy with constants \( M > 0, \delta > 0 \), dichotomy projections \( P(t) \), \( t \in \mathbb{R} \), and Green’s function \( \Gamma(t,s) \).

(H3) \( \Gamma(t,s) \in bAA(\mathbb{R} \times \mathbb{X}, B(\mathbb{X})). \)

(H4) \( B : \mathbb{X} \to \mathbb{X} \) is a bounded linear operator and let \( \| B \|_{B(\mathbb{X})} = c. \)

(H5) \( F = G + H \in S_{pa}^{p,q(x)}(\mathbb{R} \times \mathbb{X}) \cap C(\mathbb{R} \times \mathbb{X}, \mathbb{X}) \) with \( G^b \in AA(\mathbb{R} \times L^p((0, 1), \mathbb{X})) \) and \( H^b \in PAA_0(\mathbb{R} \times L^q((0, 1), \mathbb{X})) \). Moreover, \( F, G \in Lip^r(\mathbb{R}, \mathbb{X}) \) with
\[
r \geq \max \left\{ p, \frac{p}{p-1} \right\}.
\]
Let us also mention that (H1) was introduced in the literature by Acquistapace and Terreni in [2, 3]. Among other things, from [1, Theorem 2.3] (see also [3, 24, 25]), assumption (H1) does ensure that the family of operators $A(t)$ generates a unique strongly continuous evolution family on $X$, which we will denote by $\{U(t,s)\}_{t \geq s}$.

**Definition 5.1.** Under (H1), if $f : \mathbb{R} \to X$ is a bounded continuous function, then a mild solution to (1.1) is a continuous function $u : \mathbb{R} \to X$ satisfying

$$u(t) = U(t, s)u(s) + \int_s^t U(t, \sigma)f(\sigma)d\sigma$$

(5.1)

for all $(t, s) \in T := \{(t, s) \in \mathbb{R} \times \mathbb{R} : \ t \geq s\}$.

**Definition 5.2.** Suppose (H1) and (H4) hold. If $F : \mathbb{R} \times X \to X$ is a bounded continuous function, then a mild solution to (1.2) is a continuous function $u : \mathbb{R} \to X$ satisfying

$$u(t) = U(t, s)u(s) + \int_s^t U(t, \sigma)F(\sigma, Bu(\sigma))d\sigma$$

(5.2)

for all $(t, s) \in T$.

**Theorem 5.3.** Let $p > 1$ be a constant and let $q \in C_+(\mathbb{R})$. Suppose that (H1)–(H3) hold. If \( F \in S^b_{paa}(X) \cap C(\mathbb{R}, X) \), then the (1.1) has a unique pseudo-almost automorphic solution given by

$$u(t) = \int_{-\infty}^{+\infty} \Gamma(t, \sigma)f(\sigma)d\sigma, \quad t \in \mathbb{R}.$$  

(5.3)

**Proof.** Define the function $u : \mathbb{R} \mapsto X$ by

$$u(t) := \int_t^\infty U(t, \sigma)P(\sigma)f(\sigma)d\sigma - \int_t^{+\infty} Q(t, \sigma)Q(\sigma)f(\sigma)d\sigma, \quad t \in \mathbb{R}.$$  

Let us show that $u$ satisfies (5.1) for all $t \geq s$, all $t, s \in \mathbb{R}$. Indeed, applying $U(t, s)$ for all $t \geq s$, to both sides of the expression of $u$, we obtain,

$$U(t, s)u(s) = \int_{-\infty}^s U(t, \sigma)P(\sigma)f(\sigma)d\sigma - \int_s^{+\infty} Q(t, \sigma)Q(\sigma)f(\sigma)d\sigma$$

$$= \int_t^s U(t, \sigma)P(\sigma)f(\sigma)d\sigma - \int_s^t U(t, \sigma)P(\sigma)f(\sigma)d\sigma$$

$$- \int_s^{+\infty} Q(t, \sigma)Q(\sigma)f(\sigma)d\sigma - \int_s^t U(t, \sigma)Q(\sigma)f(\sigma)d\sigma$$

$$= u(t) - \int_s^t U(t, \sigma)f(\sigma)d\sigma$$

and hence $u$ is a mild solution to (1.1).

Let us show that $u \in PA(A(X))$. Indeed, since $f \in S^b_{paa}(X) \cap C(\mathbb{R}, X)$, then $f = g + \varphi$, where $g^b \in AA(L^p((0, 1), X))$ and $\varphi^b \in PA(A_0(L^p((0, 1), X)))$. Then $u$ can be decomposed as $u(t) = X(t) + Y(t)$, where

$$X(t) = \int_{-\infty}^t U(t, s)P(s)g(s)ds + \int_t^{+\infty} Q(t, s)Q(s)g(s)ds,$$
Y(t) = \int_{-\infty}^{t} U(t, s)P(s)\varphi(s)ds + \int_{+\infty}^{t} U(t, s)Q(s)\varphi(s)ds.

The proof that \( X \in AA(\mathbb{X}) \) is obvious and hence is omitted. To prove that \( Y \in PA\mathbb{A}_0(\mathbb{X}) \), we define for all \( n = 1, 2, \ldots \), the sequence of integral operators

\[
Y_n(t) := \int_{t-n}^{t-n+1} U(t, s)P(s)\varphi(s)ds + \int_{t+n-1}^{t+n} U(t, s)Q(s)\varphi(s)ds
\]

for each \( t \in \mathbb{R} \).

Let \( d \in m(\mathbb{R}) \) such that \( q^{-1}(x) + d^{-1}(x) = 1 \). From exponential dichotomy of \((U(t, s))_{t \geq s}\) and Hölder’s inequality (Theorem 3.8), it follows that

\[
\|Y_n(t)\| \leq M \int_{t-n}^{t-n+1} e^{-\delta(t-s)}\|\varphi(s)\|ds + M \int_{t+n-1}^{t+n} e^{\delta(t-s)}\|\varphi(s)\|ds
\]

\[
\leq M \left( \frac{1}{d} + \frac{1}{q} \right) \left[ \inf \left\{ \lambda > 0 : \int_{t-n}^{t-n+1} \left( \frac{e^{-\delta(t-s)}}{\lambda} \right) ds \leq 1 \right\} \right]
\times \left[ \inf \left\{ \lambda > 0 : \int_{t-n}^{t-n+1} \left( \frac{\varphi(s)}{\lambda} \right) d\|q(s)\|ds \leq 1 \right\} \right]

+ M \left( \frac{1}{d} + \frac{1}{q} \right) \left[ \inf \left\{ \lambda > 0 : \int_{t+n-1}^{t+n} \left( \frac{e^{\delta(t-s)}}{\lambda} \right) d\|s\|ds \leq 1 \right\} \right]

\times \left[ \inf \left\{ \lambda > 0 : \int_{t+n-1}^{t+n} \left( \frac{\varphi(s)}{\lambda} \right) d\|q(s)\|ds \leq 1 \right\} \right].

Now since

\[
\int_{t-n}^{t-n+1} \left[ \frac{e^{-\delta(t-s)}}{e^{-\delta(n-1)}} \right] d\|s\|ds = \int_{t-n}^{t-n+1} \left[ e^{\delta(t-s-n+1)} \right] d\|s\|ds
\]

\[
\leq \int_{t-n}^{t-n+1} \left[ 1 \right] d\|s\|ds \leq 1
\]

it follows that

\[
e^{-\delta(n-1)} \in \left\{ \lambda > 0 : \int_{t-n}^{t-n+1} \left( \frac{e^{-\delta(t-s)}}{\lambda} \right) d\|s\|ds \leq 1 \right\},
\]

which shows that

\[
\left[ \inf \left\{ \lambda > 0 : \int_{t-n}^{t-n+1} \left( \frac{e^{-\delta(t-s)}}{\lambda} \right) d\|s\|ds \leq 1 \right\} \right] \leq e^{-\delta(n-1)}.
\]

Consequently,

\[
\|Y_n(t)\| \leq M \left( \frac{1}{d} + \frac{1}{q} \right) e^{-\delta(n-1)}\|\varphi\|_{S_q(x)} + M \left( \frac{1}{d} + \frac{1}{q} \right) e^{\delta(1-n)}\|\varphi\|_{S_q(x)}
\]

\[
\leq 2M \left( \frac{1}{d} + \frac{1}{q} \right) e^{-\delta(n-1)}\|\varphi\|_{S_q(x)}.
\]
Since the series $\sum_{n=1}^{\infty} e^{-\delta (n-1)}$ converges, we deduce from the well-known Weierstrass test that the series $\sum_{n=1}^{\infty} Y_n(t)$ is uniformly convergent on $\mathbb{R}$. Furthermore,

$$Y(t) = \int_{-\infty}^{t} U(t, s)P(s)\varphi(s)ds + \int_{t}^{+\infty} U_Q(t, s)Q(s)\varphi(s)ds = \sum_{n=1}^{\infty} Y_n(t),$$

$Y \in C(\mathbb{R}, \mathfrak{X})$, and

$$\|Y(t)\| \leq \sum_{n=1}^{\infty} \|Y_n(t)\| \leq 2M \left( \frac{1}{d} + \frac{1}{q} \right) \sum_{n=1}^{\infty} e^{-\delta (n-1)} \|\varphi\|_{S_Y(x)}.$$

Next, we will show that

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \|Y(s)\| ds = 0.$$

Indeed,

$$\frac{1}{2T} \int_{-T}^{T} \|Y_n(t)\| dt \leq 2M \left( \frac{1}{d} + \frac{1}{q} \right) e^{-\delta (n-1)} \left[ \frac{1}{2T} \int_{-T}^{T} \inf \left\{ \lambda > 0 : \int_{t+n-1}^{t+n} \left\| \frac{\varphi(s)}{\lambda} \right\| ds \leq 1 \right\} \right].$$

Since $\varphi \in PAA_0(L^{b}(\mathfrak{X})), the above inequality leads to $Y_n \in PAA_0(\mathfrak{X}).$

Using the following inequality

$$\frac{1}{2T} \int_{-T}^{T} \|Y(s)\| ds \leq \frac{1}{2T} \int_{-T}^{T} \|Y(s) - \sum_{n=1}^{\infty} Y_n(s)\| dt + \sum_{n=1}^{\infty} \frac{1}{2T} \int_{-T}^{T} \|Y_n(s)\| ds,$$

we deduce that the uniform limit $Y(\cdot) = \sum_{n=1}^{\infty} Y_n(\cdot) \in PAA_0(\mathfrak{X}).$ Therefore $u \in PAA(\mathfrak{X}).$

It remains to prove the uniqueness of $u$ as a mild solution. This has already been done by Diagana [6, 10]. However, for the sake of clarity let us reproduce it here. Let $u, v$ be two bounded mild solutions to (1.1). Setting $w = u - v$, one can easily see that $w$ is bounded and that $w(t) = U(t, s)w(s)$ for all $(t, s) \in \mathfrak{T}$. Now using property (i) from exponential dichotomy (Definition 2.11) it follows that $P(t)w(t) = P(t)U(t, s)w(s) = U(t, s)P(s)w(s),$ and hence

$$\|P(t)w(t)\| = \|U(t, s)P(s)w(s)\| \leq Me^{-\delta(t-s)}\|w(s)\| \leq Me^{-\delta(t-s)}\|w\|_{\infty}.$$

for all $(t, s) \in \mathfrak{T}.$

Now, given $t \in \mathbb{R}$ with $t \geq s$, if we let $s \to -\infty$, we then obtain that $P(t)w(t) = 0$, that is, $P(t)u(t) = P(t)v(t).$ Since $t$ is arbitrary it follows that $P(t)w(t) = 0$ for all $t \geq s$. Similarly, from $w(t) = U(t, s)w(s)$ for all $t \geq s$ and property (i) from exponential dichotomy (Definition 2.11) it follows that $Q(t)w(t) = Q(t)U(t, s)w(s) = U(t, s)Q(s)w(s),$ and hence $UQ(s, t)Q(t)w(t) = Q(s)w(s)$ for all $t \geq s$. Moreover,

$$\|Q(s)w(s)\| = \|UQ(s, t)Q(t)w(t)\| \leq Me^{-\delta(t-s)}\|w\|_{\infty}.$$

for all $t \geq s.$

Now, given $s \in \mathbb{R}$ with $t \geq s$, if we let $t \to +\infty$, we then obtain that $Q(t)w(t) = 0$, that is, $Q(s)u(s) = Q(s)v(s).$ Since $s$ is arbitrary it follows that $Q(s)w(s) = 0$ for all $t \geq s.$ \hfill \square

Using Theorem 3.3 one easily proves the following theorem.
Theorem 5.4. Let $p, q > 1$ be constants such that $p \leq q$. Under assumptions (H1)-(H5), then $\text{(1.2)}$ has a unique solution whenever $\|L_F\|_{S^r}$ is small enough. And the solution satisfies the integral equation

$$u(t) = \int_{-\infty}^{t} U(t, \sigma) P(\sigma) F(\sigma, Bu(\sigma)) d\sigma - \int_{t}^{+\infty} U_Q(t, \sigma) Q(\sigma) F(\sigma, Bu(\sigma)) d\sigma, \; t \in \mathbb{R}.$$  

Proof. Define $\Xi : \text{PAA}(X) \to \text{PAA}(\mathbb{X})$ as

$$(\Xi u)(t) = \int_{-\infty}^{t} U(t, \sigma) P(\sigma) F(\sigma, Bu(\sigma)) d\sigma - \int_{t}^{+\infty} U_Q(t, \sigma) Q(\sigma) F(\sigma, Bu(\sigma)) d\sigma.$$  

Let $u \in \text{PAA}(\mathbb{X}) \subset S^p_{\text{paa}}(\mathbb{X})$. From (H4) and Theorem 4.16 it is clear that $Bu(.) \in S^p_{\text{paa}}(\mathbb{X})$. Using the composition theorem for $S^p_{\text{paa}}$ functions, we deduce that there exists $m \in [1, p)$ such that $F(., Bu(.)) \in S^{m,m}_{\text{paa}}(\mathbb{X})$. Applying the proof of Theorem 5.3 to $f(.) = F(., Bu(.))$, one can easily see that the operator $\Xi$ maps $\text{PAA}(\mathbb{X})$ into itself. Moreover, for all $u, v \in \text{PAA}(\mathbb{X})$, it is easy to see that

$$\| (\Xi u)(t) - (\Xi v)(t) \|$$

$$\leq \int_{\mathbb{R}} \| \Gamma(t-s) \| \| F(s, Bu(s)) - F(s, Bv(s)) \| ds$$

$$\leq \int_{-\infty}^{t} c M e^{-\delta(t-s)} L_F(s) ds \| u - v \|_{\infty} + \int_{t}^{+\infty} c M e^{\delta(t-s)} L_F(s) ds \| u - v \|_{\infty}$$

$$\leq \sum_{n=1}^{\infty} \int_{t-n}^{t-n+1} c M e^{-\delta(t-s)} L_F(s) ds \| u - v \|_{\infty}$$

$$+ \sum_{n=1}^{\infty} \int_{t-n}^{t+n} c M e^{\delta(t-s)} L_F(s) ds \| u - v \|_{\infty}$$

$$\leq c M \sum_{n=1}^{\infty} \left( \int_{t-n}^{t-n+1} e^{-r_0 \delta(t-s)} ds \right)^{\frac{1}{\alpha_0}} \| L_F \|_{S^r} \| u - v \|_{\infty}$$

$$+ c M \sum_{n=1}^{\infty} \left( \int_{t-n}^{t+n} e^{r_0 \delta(t-s)} ds \right)^{\frac{1}{\alpha_0}} \| L_F \|_{S^r} \| u - v \|_{\infty}$$

$$\leq 2c M \sum_{n=1}^{\infty} \left( e^{-r_0(n-2)\delta} - e^{-r_0 n\delta} \right)^{\frac{1}{\alpha_0}} \| L_F \|_{S^r} \| u - v \|_{\infty}$$

$$\leq 2c M \sum_{n=1}^{\infty} \left( e^{-r_0(n-2)\delta} - e^{-r_0 n\delta} \right)^{\frac{1}{\alpha_0}} \| L_F \|_{S^r} \| u - v \|_{\infty},$$

for each $t \in \mathbb{R}$, where $\frac{1}{r} + \frac{1}{r_0} = 1$. Hence whenever $\|L_F\|_{S^r}$ is small enough, that is,

$$2c M \sum_{n=1}^{\infty} \left( e^{-r_0(n-2)\delta} - e^{-r_0 n\delta} \right)^{\frac{1}{\alpha_0}} \| L_F \|_{S^r} \| u - v \|_{\infty} < 1,$$

then $\Xi$ has a unique fixed point, which obviously is the unique pseudo-almost automorphic solution to $\text{(1.2)}$. □

References


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