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POSITIVE SOLUTIONS FOR CLASSES OF POSITONE/SEMIPOSITONE SYSTEMS WITH MULTIPARAMETERS

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Abstract. We study the existence and nonexistence of solution for a system involving p,q-Laplacian and nonlinearity with multiple parameteres. We use the method of lower and upper solutions for prove the existence of solutions.

1. Introduction

We study the existence of solutions for the positone/semipositone system involving p,q-Laplacian

$$\begin{gathered} -\Delta_p u = \lambda f_1(x,u,v) + \mu g_1(x,u,v) \quad \text{in } \Omega, \\ -\Delta_q v = \lambda f_2(x,u,v) + \mu g_2(x,u,v) \quad \text{in } \Omega, \\ u = v = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$, $n \geq 1$, is a bounded domain with boundary C^2, and $f_i, g_i : \Omega \times (0,+\infty) \times (0,+\infty) \to \mathbb{R}$, $i = 1,2$, are Carathéodory functions, g_i, $i = 1,2$, are bounded on bounded sets. Moreover, we assume that there exists $h_i : \mathbb{R} \to \mathbb{R}$ continuous and nondecreasing such that $h_i(0) = 0$, $0 \leq h_i(s) \leq C(1 + |s|^{r-1})$, for all $s \in \mathbb{R}$, $r = \min\{p,q\}$, $C > 0$, $i = 1,2$, and the maps

$$\begin{gathered} s \mapsto f_1(x,s,t) + h_1(s), \quad t \mapsto f_2(x,s,t) + h_1(t), \\ s \mapsto g_1(x,s,t) + h_2(s), \quad t \mapsto g_2(x,s,t) + h_2(t), \end{gathered} \tag{1.2}$$

are nondecreasing for almost everywhere $x \in \Omega$. Also, we will prove the nonexistence of nontrivial solution for system (1.1) in the positone case.

In the scalar case, Castro, Hassanpour, and Shivaji in [4], using the lower and upper solutions method, focused their attention on a class of problems, so called semipositione problems, of the form

$$\begin{gathered} -\Delta u = \lambda f(u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered}$$

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where Ω is a smooth bounded domain in $\mathbb{R}^N$, λ is a positive parameter, and $f : [0,\infty) \to \mathbb{R}$ is a monotone and continuous function satisfying the conditions $f(0) < 0$, $\lim_{s\to\infty} f(s) = +\infty$, and with the sublinear condition at infinity, $\lim_{s\to\infty} f(s)/s = 0$. In 2008, Perera and Shivaji [11] proved the existence of solutions for the problem

$$\begin{gathered} -\Delta_p u = \lambda f(x,u) + \mu g(x,u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered}$$

where $\Omega \subset \mathbb{R}^n$, $n \geq 1$, is a bounded domain with boundary C^2, and $f, g : \Omega \times (0,+\infty) \times (0,+\infty) \to \mathbb{R}$ are Carathéodory functions, g is bounded on bounded sets and $|f(x,t)| \geq a_0$ for all $t \geq t_0$, where a_0, t_0 are positive constants. Moreover, the existence of solutions is assured for $\lambda \geq \lambda_0$ and small $0 < |\mu| \leq \mu_0$, for some $\lambda_0 > 0$ and $\mu_0 = \mu(\lambda_0) > 0$.

Many authors have studied the existence of positive solutions for elliptic systems, due to the great number of applications in reaction-diffusion problems, in fluid mechanics, in newtonian fluids, glaciology, population dynamics, etc; see [3, 8] and references therein.

Hai and Shivaji [9] applied the lower and upper solutions method for obtaining the existence of solution for the semipositone system

$$\begin{gathered} -\Delta_p u = \lambda f_1(v) \quad \text{in } \Omega, \\ -\Delta_p v = \lambda f_2(u) \quad \text{in } \Omega, \\ u = v = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.3}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ with smooth boundary, λ is a positive parameter, and $f_1, f_2 : [0,\infty) \to \mathbb{R}$ are monotone and continuous functions satisfying conditions $f_i(0) < 0$, $\lim_{s\to+\infty} f_i(s) = +\infty$, $i = 1, 2$, and

$$\lim_{s\to+\infty} \frac{f_1(M(f_2(s))^{1/(p-1)})}{s^{p-1}} = 0 \quad \text{for all } M > 0. \tag{1.4}$$

While, Chhetri, Hai, and Shivaji [6] proved an existence result for system (1.3) with the condition

$$\lim_{s\to+\infty} \frac{\max\{f_1(s), f_2(s)\}}{s^{p-1}} = 0, \tag{1.5}$$

instead of (1.4).

In 2007, Ali and Shivaji [1] obtained a positive solution for the system

$$\begin{gathered} -\Delta_p u = \lambda_1 f_1(v) + \mu_1 g_1(u) \quad \text{in } \Omega, \\ -\Delta_q v = \lambda_2 f_2(u) + \mu_2 g_2(v) \quad \text{in } \Omega, \\ u = v = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.6}$$

when Ω is a smooth bounded domain in $\mathbb{R}^N$, λ_i, μ_i, $i = 1, 2$, are nonnegative parameters with $\lambda_1 + \mu_1$ and $\lambda_2 + \mu_2$ large and

$$\lim_{x\to+\infty} \frac{f_1(M[f_2(x)]^{1/q-1})}{x^{p-1}} = 0,$$

for all $M > 0$, $\lim_{x\to+\infty} \frac{g_1(x)}{x^{p-1}} = 0$, and $\lim_{x\to+\infty} \frac{g_2(x)}{x^{q-1}} = 0$.

Our first result deal with the existence of solution for (1.1) which has p,q-Laplacian operators and nonautonomous nonlinearity with multiple parameters. Note that, we make no suppositions about the signs of $g_1(x,0,0)$ and $g_2(x,0,0)$, and hence can occur the positone case: $\lambda f_i(x,0,0)+\mu g_i(x,0,0)\geq 0$, $i=1,2$; the semipositone case: $\lambda f_i(x,0,0)+\mu g_i(x,0,0)<0$, $i=1,2$; the case $\lambda f_1(x,0,0)+\mu g_1(x,0,0)\geq 0$ and $\lambda f_2(x,0,0)+\mu g_2(x,0,0)<0$; or the case $\lambda f_1(x,0,0)+\mu g_1(x,0,0)<0$ and $\lambda f_2(x,0,0)+\mu g_2(x,0,0)\geq 0$; for almost everywhere $x\in\Omega$.

Theorem 1.1. *Consider the system* (1.1) *assuming* (1.2)*, and that there exist* $a_0,\gamma,\delta>0$ *and* $\alpha,\beta\geq 0$ *such that* $0\leq\alpha<p-1$, $0\leq\beta<q-1$, $(p-1-\alpha)(q-1-\beta)-\gamma\delta>0$, *and*

$$|f_1(x,s,t)|\leq a_0|s|^{\alpha}|t|^{\gamma},\quad |f_2(x,s,t)|\leq a_0|s|^{\delta}|t|^{\beta}, \tag{1.7}$$

for all $s,t\in(0,+\infty)$ *and* $x\in\Omega$. *In addition, suppose there exist* $a_1>0$, $a_2>0$, *and* $R>0$ *such that*

$$f_i(x,s,t)\geq a_1,\quad \text{for } i=1,2, \text{ and all } s>R,\ t>R, \tag{1.8}$$

and

$$f_i(x,s,t)\geq -a_2,\quad \text{for } i=1,2, \text{ and all } s,t\in(0,+\infty), \tag{1.9}$$

uniformly in $x\in\Omega$. *Then, there exists* $\lambda_0>0$ *such that for each* $\lambda>\lambda_0$, *there exists* $\mu_0=\mu_0(\lambda)>0$ *for which system* (1.1) *has a solution* $(u,v)\in C^{1,\rho_1}(\Omega)\times C^{1,\rho_2}(\Omega)$ *for some* $\rho_1,\rho_2>0$, *where each component is positive, whenever* $|\mu|\leq\mu_0$.

Let $\lambda_p>0$ and $\lambda_q>0$ be the first eigenvalue of p-Laplacian and q-Laplacian, respectively, where $\phi_p\in C^{1,\alpha_p}(\Omega)$ and $\phi_q\in C^{1,\alpha_q}(\Omega)$ are the respective positive eigenfunctions (see [7]).

Chen [5] proved the nonexistence of nontrivial solution for the system

$$\begin{gathered}
-\Delta_p u=\lambda u^{\alpha}v^{\gamma},\quad \text{in } \Omega,\\
-\Delta_q v=\lambda u^{\delta}v^{\beta},\quad \text{in } \Omega,\\
u=v=0\quad \text{on } \Omega,
\end{gathered}$$

when Ω is a smooth bounded domain in $\mathbb{R}^N$, $p\gamma=q(p-1-\alpha)$, $(p-1-\alpha)(q-1-\beta)-\gamma\delta=0$, and $0<\lambda<\lambda_0$ where $\lambda_0=\min\{\lambda_p,\lambda_q\}$ (see also [10]). We note that due to Young's inequality we have

$$u^{\alpha+1}v^{\gamma}\leq\frac{1+\alpha}{p}u^p+\frac{p-1-\alpha}{p}v^q,\quad u^{\delta}v^{\beta+1}\leq\frac{q-1-\beta}{q}u^p+\frac{\beta+1}{q}v^q.$$

Now, we will enunciated the nonexistence theorem for the system (1.1), improving the result proved by Chen in [5].

Theorem 1.2. *Suppose that there exist* $k_i>0$, $i=1,\dots,8$, *such that*

$$\begin{aligned}
&|f_1(x,s,t)s|\leq\left(k_1|s|^p+k_2|t|^q\right),\quad |f_2(x,s,t)t|\leq\left(k_3|s|^p+k_4|t|^q\right),\\
&|g_1(x,s,t)s|\leq\left(k_5|s|^p+k_6|t|^q\right),\quad |g_2(x,s,t)t|\leq\left(k_7|s|^p+k_8|t|^q\right),
\end{aligned} \tag{1.10}$$

for all $x\in\Omega$ *and* $s,t\in(0,+\infty)$. *Then* (1.1) *does not possess nontrivial solutions, for all* λ,μ *satisfying*

$$|\lambda|(k_1+k_3)+|\mu|(k_5+k_7)<\lambda_p,\quad |\lambda|(k_2+k_4)+|\mu|(k_6+k_8)<\lambda_q. \tag{1.11}$$

Remark 1.3. The typical functions considered in Theorem 1.1 are as follows:

$$f_1(x,s,t)=A(x)s^{\alpha}t^{\gamma},\quad f_2(x,s,t)=B(x)s^{\delta}t^{\beta},$$

where $A(x),B(x)$ are continuous functions on Ω satisfying $\inf_{x\in\Omega}A(x)>0$ and $\sup_{x\in\Omega}A(x)<+\infty$, $\inf_{x\in\Omega}B(x)>0$, and $\sup_{x\in\Omega}B(x)<+\infty$ for all $x\in\Omega$, $0\leq\alpha<p-1$, $0\leq\beta<q-1$, $(p-1-\alpha)(q-1-\beta)-\gamma\delta>0$, and $g_1(x,s,t)$ and

$g_2(x,s,t)$ are any continuous functions on $\overline{\Omega}\times[0,+\infty)\times[0,+\infty)$ with $g_1(x,s,t)$ nondecreasing in variable s and $g_2(x,s,t)$ nondecreasing in variable t.

Remark 1.4. Theorem 1.2 can be applied for functions of the form

$$f_1(x,s,t)=\sum_{i=1}^{m}a_is^{\alpha_{1,i}}t^{\gamma_{1,i}},\quad f_2(x,s,t)=\sum_{i=1}^{m}b_is^{\delta_{1,i}}t^{\beta_{1,i}}$$
$$g_1(x,s,t)=\sum_{i=1}^{m}c_is^{\alpha_{2,i}}t^{\gamma_{2,i}},\quad g_2(x,s,t)=\sum_{i=1}^{m}d_is^{\delta_{2,i}}t^{\beta_{2,i}},$$

with $a_i,b_i,c_i,d_i\ge 0$, $p\gamma_{j,i}=q(p-1-\alpha_{j,i})$, and $(p-1-\alpha_{j,i})(q-1-\beta_{j,i})=\gamma_{j,i}\delta_{j,i}$, for $j=1,2$ and $i=1,\cdots,m$.

Theorems 1.1 and Theorem 1.2 will be proved in the next sections.

2. Proof of Theorem 1.1

We prove Theorem 1.1 by using a general method of lower and upper-solutions. This method, in the scalar situation, has been used by many authors, for instance [2] and [3]. The proof for the system case can be found in [10].

2.1. **Upper-solution.** First of all, we will prove that (1.1) possesses a upper-solution. Consider $e_i\in C^{1,\alpha_i}(\overline{\Omega})$, with $\alpha_i>0$, $i=1,2$, where (e_1,e_2) is a solution of (1.1) with $f_1(x,u,v)=\frac{1}{\lambda}$, $f_2(x,u,v)=\frac{1}{\lambda}$, and $g_1(x,u,v)=g_2(x,u,v)=0$, and each component is positive.

Claim. Since $\delta>0$, $\gamma>0$, $0\le\alpha<p-1$, $0\le\beta<q-1$, and $(p-1-\alpha)(q-1-\beta)-\gamma\delta>0$, there exist s_1 and s_2 such that

$$s_1>\frac{1}{p-1},\quad s_2>\frac{1}{q-1},\quad \frac{\delta}{q-1-\beta}<\frac{s_2}{s_1}<\frac{p-1-\alpha}{\gamma}\,. \tag{2.1}$$

In fact, since

$$0<\frac{\delta}{q-1-\beta}<\frac{p-1-\alpha}{\gamma},$$

there exist $k>0$ such that

$$\frac{\delta}{q-1-\beta}<k<\frac{p-1-\alpha}{\gamma}.$$

Define $\vartheta:(0,+\infty)\to\mathbb{R}$ by $\vartheta(\epsilon)=k(\frac{1}{p-1}+\epsilon)$. Evidently, we have

$$\lim_{\epsilon\to+\infty}\vartheta(\epsilon)=+\infty,$$

therefore, there exists $\epsilon_0>0$ satisfying $\vartheta(\epsilon)>\frac{1}{q-1}$ for all $\epsilon>\epsilon_0$. Fixed $\epsilon>\epsilon_0$, we define $s_1=\frac{1}{p-1}+\epsilon$ and $s_2=\vartheta(\epsilon)=ks_1$. Then, $s_1>\frac{1}{p-1}$, $s_2>\frac{1}{q-1}$, and $\frac{s_1}{s_2}=k$, which proves the claim.

Then, by using (2.1), we obtain $\lambda_0>0$ such that

$$a_\lambda:=\max\{a_0\lambda^{s_1(\alpha-p+1)+s_2\gamma},\ a_0\lambda^{s_1\delta+s_2(\beta-q+1)}\}<1, \tag{2.2}$$

for all $\lambda>\lambda_0$. Moreover, there exist A and B positive constants satisfying

$$A^{p-1}=\lambda A^\alpha l^\alpha B^\gamma L^\gamma \text{ and } B^{q-1}=\lambda A^\delta l^\delta B^\beta L^\beta, \tag{2.3}$$

where $l=\|e_1\|_\infty$ and $L=\|e_2\|_\infty$.

For a fixed $\lambda>\lambda_0$, we define

$$(\bar u(x),\bar v(x)):=(\lambda^{s_1}Ae_1(x),\lambda^{s_2}Be_2(x))\,.$$

Note that $\bar{u} \in C^{1,\alpha_1}(\overline{\Omega})$ and $\bar{v} \in C^{1,\alpha_2}(\overline{\Omega})$. Let $w \in W_0^{1,p}(\Omega)$ with $w(x) \geq 0$ for a.e. (almost everywhere) $x \in \Omega$. Then

$$\int_\Omega |\nabla \bar{u}|^{p-2}\nabla \bar{u}\nabla w\,dx = \lambda^{s_1(p-1)}A^{p-1}\int_\Omega w\,dx \tag{2.4}$$

and, for $z \in W_0^{1,q}(\Omega)$ with $z(x) \geq 0$ for a.e. $x \in \Omega$,

$$\int_\Omega |\nabla \bar{v}|^{q-2}\nabla \bar{v}\nabla z\,dx = \lambda^{s_2(q-1)}B^{q-1}\int_\Omega z\,dx. \tag{2.5}$$

On the other hand, by using (1.7), (2.2), and (2.3), we have

$$\begin{aligned}
\lambda f_1(x,\bar{u}(x),\bar{v}(x)) &\leq \lambda a_0\lambda^{s_1\alpha}A^\alpha l^\alpha \lambda^{s_2\gamma}B^\gamma L^\gamma \\
&= \lambda a_0 \lambda^{s_1(\alpha-p+1)+s_2\gamma}\lambda^{s_1(p-1)}A^\alpha l^\alpha B^\gamma L^\gamma \\
&\leq a_\lambda \lambda^{s_1(p-1)}A^{p-1}
\end{aligned} \tag{2.6}$$

and

$$\lambda f_2(x,\bar{u}(x),\bar{v}(x)) \leq a_\lambda \lambda^{s_2(q-1)}B^{q-1}. \tag{2.7}$$

But, as $a_\lambda < 1$ for $\lambda > \lambda_0$, there exists $c_\lambda > 0$ such that

$$a_\lambda\lambda^{s_1(p-1)}A^{p-1}+c_\lambda \leq \lambda^{s_1(p-1)}A^{p-1}, \quad a_\lambda\lambda^{s_2(q-1)}B^{q-1}+c_\lambda \leq \lambda^{s_2(q-1)}B^{q-1}. \tag{2.8}$$

Also, since that g_i, $i = 1,2$, are bounded on bounded sets, there exists $\mu_0 = \mu_0(\lambda) > 0$ such that

$$|\mu||g_1(x,\bar{u}(x),\bar{v}(x))| \leq c_\lambda, \quad |\mu||g_2(x,\bar{u}(x),\bar{v}(x))| \leq c_\lambda \tag{2.9}$$

for all $|\mu| < \mu_0$. Then, by (2.6), (2.8), and (2.9) we obtain

$$\begin{aligned}
&\lambda f_1(x,\bar{u}(x),\bar{v}(x)) + \mu g_1(x,\bar{u}(x),\bar{v}(x)) \\
&\leq a_\lambda\lambda^{s_1(p-1)}A^{p-1} + |\mu g_1(x,\bar{u}(x),\bar{v}(x))| \\
&\leq a_\lambda\lambda^{s_1(p-1)}A^{p-1} + c_\lambda \\
&\leq \lambda^{s_1(p-1)}A^{p-1}\,.
\end{aligned} \tag{2.10}$$

From (2.7), (2.8), and (2.9), we obtain

$$\lambda f_2(x,\bar{u}(x),\bar{v}(x)) + \mu g_2(x,\bar{u}(x),\bar{v}(x)) \leq \lambda^{s_2(q-1)}B^{q-1}, \tag{2.11}$$

for all $|\mu| < \mu_0$. Hence, by (2.4) and (2.10), we conclude that

$$\int_\Omega |\nabla \bar{u}|^{p-2}\nabla\bar{u}\nabla w\,dx \geq \lambda\int_\Omega f_1(x,\bar{u}(x),\bar{v}(x))w\,dx + \mu\int_\Omega g_1(x,\bar{u}(x),\bar{v}(x))w\,dx. \tag{2.12}$$

Analogously, from (2.5) and (2.11), we obtain

$$\int_\Omega |\nabla \bar{v}|^{q-2}\nabla\bar{v}\nabla z\,dx \geq \lambda\int_\Omega f_2(x,\bar{u}(x),\bar{v}(x))z\,dx + \mu\int_\Omega g_2(x,\bar{u}(x),\bar{v}(x))z\,dx. \tag{2.13}$$

Thus, from (2.12) and (2.13), we see that $(\bar{u},\bar{v})$ is a upper-solution of (1.1) with $\bar{u} \in C^{1,\alpha_1}(\overline{\Omega})$ and $\bar{v} \in C^{1,\alpha_2}(\overline{\Omega})$.

2.2. **Lower-solution.** In this subsetion, we prove that (1.1) possesses a lower-solution. Let us fix ξ and η such that

$$1<\xi<\frac{p}{p-1},\quad 1<\eta<\frac{q}{q-1}\cdot \tag{2.14}$$

From (1.8) and (1.9) we have $a_1>0$, $a_2>0$, and $R>0$ such that

$$f_i(x,s,t)\geq a_1,\quad \text{for } i=1,2 \text{ an all } s>R\ t>R, \tag{2.15}$$

$$f_i(x,s,t)\geq -a_2,\quad \text{for } i=1,2 \text{ and all } s,t\in(0,+\infty), \tag{2.16}$$

uniformly in $x\in\Omega$.

Consider λ_p the eigenvalue associated to positive eigenfunction φ_p of the problem of eigenvalue of p-Laplacian operator, and λ_q the eigenvalue associated with positive eigenfunction φ_q of the problem of eigenvalue of q-Laplacian operator. We take a_3 and a_4 positive constants satisfying

$$a_3>2\frac{\lambda_p(a_2+1)\xi^{p-1}}{a_1},\quad a_4>2\frac{\lambda_q(a_2+1)\eta^{q-1}}{a_1}, \tag{2.17}$$

and define

$$(\underline{u}(x),\underline{v}(x)):=(c_\lambda\varphi_p^{\xi}(x),d_\lambda\varphi_q^{\eta}(x)),$$

where

$$c_\lambda=\Big(\frac{\lambda a_2+1}{a_3}\Big)^{\frac{1}{p-1}},\quad d_\lambda=\Big(\frac{\lambda a_2+1}{a_4}\Big)^{\frac{1}{q-1}}. \tag{2.18}$$

Thus, for $w\in W_0^{1,p}(\Omega)$ and $z\in W_0^{1,q}(\Omega)$ with $w(x)\geq 0$ and $z(x)\geq 0$ for a.e. $x\in\Omega$, we obtain

$$\begin{aligned}&\int_\Omega|\nabla\underline{u}|^{p-2}\nabla\underline{u}\nabla w\,dx\\&=c_\lambda^{p-1}\xi^{p-1}\int_\Omega\Big[\lambda_p\varphi_p^{\xi(p-1)}-(\xi-1)(p-1)\varphi_p^{(\xi-1)(p-1)-1}|\nabla\varphi_p|^p\Big]w\,dx\end{aligned} \tag{2.19}$$

and

$$\begin{aligned}&\int_\Omega|\nabla\underline{v}|^{q-2}\nabla\underline{v}\nabla z\,dx\\&=d_\lambda^{q-1}\eta^{q-1}\int_\Omega\Big[\lambda_q\varphi_q^{\eta(q-1)}-(\eta-1)(q-1)\varphi_q^{(\eta-1)(q-1)-1}|\nabla\varphi_q|^q\Big]zdx.\end{aligned} \tag{2.20}$$

We know that $\varphi_p,\varphi_q>0$ in Ω and $|\nabla\varphi_p|,|\nabla\varphi_q|\geq\sigma$ on $\partial\Omega$ for some $\sigma>0$. Also, we can suppose that $\|\varphi_p\|_\infty=\|\varphi_q\|_\infty=1$. Furthermore, by using (2.14), it is easy to prove that there exists $\zeta>0$ such that

$$\lambda_p\varphi_p^{\xi(p-1)}-(\xi-1)(p-1)\varphi_p^{(\xi-1)(p-1)-1}|\nabla\varphi_p|^p\leq -a_3, \tag{2.21}$$

$$\lambda_q\varphi_q^{\eta(q-1)}-(\eta-1)(q-1)\varphi_q^{(\eta-1)(q-1)-1}|\nabla\varphi_q|^q\leq -a_4, \tag{2.22}$$

in $\Omega_\zeta:=\{x\in\Omega:\operatorname{dist}(x,\partial\Omega)\leq\zeta\}$. But, we have by (2.14), (2.16), and (2.18) that

$$-c_\lambda^{p-1}\xi^{p-1}a_3=-(\lambda a_2+1)\xi^{p-1}\leq-(\lambda a_2+1)\leq\lambda f_1(x,\underline{u},\underline{v})-1 \tag{2.23}$$

and

$$-d_\lambda^{q-1}\eta^{q-1}a_4\leq\lambda f_2(x,\underline{u},\underline{v})-1, \tag{2.24}$$

for all $x \in \Omega$. Therefore, from (2.21), (2.22), (2.23), and (2.24), we obtain

$$c_\lambda^{p-1}\xi^{p-1}\Big[\lambda_p\varphi_p^{\xi(p-1)} - (\xi-1)(p-1)\varphi_p^{(\xi-1)(p-1)-1}|\nabla\varphi_p|^p\Big] \le \lambda f_1(x,\underline{u},\underline{v}) - 1 \tag{2.25}$$

and

$$d_\lambda^{q-1}\eta^{q-1}\Big[\lambda_q\varphi_q^{\eta(q-1)} - (\eta-1)(q-1)\varphi_q^{(\eta-1)(q-1)-1}|\nabla\varphi_q|^q\Big] \le \lambda f_2(x,\underline{u},\underline{v}) - 1, \tag{2.26}$$

in $\Omega_\zeta := \{x \in \Omega : \operatorname{dist}(x,\partial\Omega) \le \zeta\}$.

On the other hand, there exists $a_5 > 0$ such that $\varphi_p(x), \varphi_q(x) \ge a_5$ for all $x \in \Omega \setminus \Omega_\zeta$. Then, if $\lambda_0 > 0$ is provided of proof of existence of upper-solution, and by taking $\lambda_0 > 0$ greater than one, if necessary, we can suppose

$$\lambda_0 \ge \max\{1, \frac{2}{a_1}, \frac{R^{p-1}a_5^{-\xi(p-1)}a_3^{-1}}{a_2}, \frac{R^{q-1}a_5^{-\eta(q-1)}a_4^{-1}}{a_2}\} > 0.$$

Thus

$$\underline{u}(x) = c_\lambda\varphi_p^\xi(x) \ge c_\lambda a_5^\xi > R, \quad \underline{v}(x) = d_\lambda\varphi_p^\xi(x) \ge d_\lambda a_5^\eta > R,$$

for all $x \in \Omega \setminus \Omega_\zeta$ and $\lambda > \lambda_0$. Therefore, by (2.15), we have

$$\lambda f_1(x,\underline{u}(x),\underline{v}(x)) - 1 \ge \lambda a_1 - 1, \quad \lambda f_2(x,\underline{u}(x),\underline{v}(x)) - 1 \ge \lambda a_1 - 1 \tag{2.27}$$

for all $x \in \Omega \setminus \Omega_\zeta$ and $\lambda > \lambda_0$.

Claim. By (2.17) and $\lambda > \lambda_0 \ge \max\{1, \frac{2}{a_1}, \frac{R^{p-1}a_5^{-\xi(p-1)}a_3^{-1}}{a_2}, \frac{R^{q-1}a_5^{-\eta(q-1)}a_4^{-1}}{a_2}\}$, we have

$$a_3 > \frac{\lambda_p\xi^{p-1}(\lambda a_2+1)}{\lambda a_1 - 1} \quad\text{and}\quad a_4 > \frac{\lambda_q\eta^{q-1}(\lambda a_2+1)}{\lambda a_1 - 1}. \tag{2.28}$$

In fact, since that $\lambda > \frac{2}{a_1}$, we obtain

$$a_1 - \frac{1}{\lambda} > a_1 - \frac{a_1}{2} = \frac{a_1}{2},$$

so, as $\lambda > 1$ and by (2.17),

$$\begin{aligned}
\frac{\lambda_p\xi^{p-1}(\lambda a_2+1)}{\lambda a_1 - 1} &= \frac{\lambda_p\xi^{p-1}(a_2+\frac{1}{\lambda})}{a_1 - \frac{1}{\lambda}} \\
&< \frac{\lambda_p\xi^{p-1}(a_2+1)}{a_1 - \frac{1}{\lambda}} \\
&< \frac{\lambda_p\xi^{p-1}(a_2+1)}{\frac{a_1}{2}} \\
&= \frac{2\lambda_p(a_2+1)\xi^{p-1}}{a_1} < a_3,
\end{aligned}$$

and similarly

$$a_4 > \frac{\lambda_q\eta^{q-1}(\lambda a_2+1)}{\lambda a_1 - 1},$$

which prove the claim.

Then, from (2.19), (2.27), and (2.28), we achieve

$$\begin{aligned}
&c_\lambda^{p-1}\xi^{p-1}\big[\lambda_p\varphi_p^{\xi(p-1)}-(\xi-1)(p-1)\varphi_p^{(\xi-1)(p-1)-1}|\nabla\varphi_p|^p\big](x)\\
&\le c_\lambda^{p-1}\xi^{p-1}\lambda_p\varphi_p^{\xi(p-1)}(x)\\
&\le \lambda_p c_\lambda^{p-1}\xi^{p-1}\\
&\le \lambda_p\frac{\lambda a_2+1}{a_3}\xi^{p-1}\\
&\le \lambda a_1-1\\
&\le \lambda f_1(x,\underline{u}(x),\underline{v}(x))-1
\end{aligned}\tag{2.29}$$

and, by (2.20), (2.27), and (2.28),

$$\begin{aligned}
&d_\lambda^{q-1}\eta^{q-1}\big[\lambda_q\varphi_q^{\eta(q-1)}-(\eta-1)(q-1)\varphi_q^{(\eta-1)(q-1)-1}|\nabla\varphi_q|^q\big](x)\\
&\le \lambda_q\frac{\lambda a_2+1}{a_4}\eta^{q-1}\\
&\le \lambda f_2(x,\underline{u}(x),\underline{v}(x))-1,
\end{aligned}\tag{2.30}$$

for all $x\in\Omega\setminus\Omega_\varsigma$. Thus, by combining (2.25), (2.26), (2.29), and (2.30), we obtain

$$\begin{aligned}
&c_\lambda^{p-1}\xi^{p-1}\Big[\lambda_p\varphi_p^{\xi(p-1)}-(\xi-1)(p-1)\varphi_p^{(\xi-1)(p-1)-1}|\nabla\varphi_p|^p\Big](x)\\
&\le \lambda f_1(x,\underline{u}(x),\underline{v}(x))-1
\end{aligned}\tag{2.31}$$

and

$$\begin{aligned}
&d_\lambda^{q-1}\eta^{q-1}\Big[\lambda_q\varphi_q^{\eta(q-1)}-(\eta-1)(q-1)\varphi_q^{(\eta-1)(q-1)-1}|\nabla\varphi_q|^q\Big](x)\\
&\le \lambda f_2(x,\underline{u}(x),\underline{v}(x))-1,
\end{aligned}\tag{2.32}$$

for all $\lambda>\lambda_0$ and $x\in\Omega$. Moreover, if $\mu_0=\mu_0(\lambda)>0$ is provided of proof of existence of upper-solution; for each $\lambda>\lambda_0$, since that g_i, $i=1,2$, are bounded on bounded sets, replacing $\mu_0>0$ by another smaller, if necessary, we have

$$|\mu||g_1(x,\underline{u}(x),\underline{v}(x))|\le 1,\quad |\mu||g_2(x,\underline{u}(x),\underline{v}(x))|\le 1\tag{2.33}$$

for all $|\mu|<\mu_0$. Therefore, by (2.33) it follows that

$$\lambda f_1(x,\underline{u}(x),\underline{v}(x))-1\le \lambda f_1(x,\underline{u}(x),\underline{v}(x))+\mu g_1(x,\underline{u}(x),\underline{v}(x)),\tag{2.34}$$

$$\lambda f_2(x,\underline{u}(x),\underline{v}(x))-1\le \lambda f_2(x,\underline{u}(x),\underline{v}(x))+\mu g_2(x,\underline{u}(x),\underline{v}(x)),\tag{2.35}$$

for all $|\mu|<\mu_0$ and $x\in\Omega$.

Hence, substituting (2.34) and (2.35) in (2.31) and (2.32), respectively, and by using (2.19) and (2.20) , we achieve

$$\begin{aligned}
\int_\Omega|\nabla\underline{u}|^{p-2}\nabla\underline{u}\nabla w\,dx\le{}&\lambda\int_\Omega f_1(x,\underline{u}(x),\underline{v}(x))w\,dx\\
&+\mu\int_\Omega g_1(x,\underline{u}(x),\underline{v}(x))w\,dx
\end{aligned}\tag{2.36}$$

and

$$\begin{aligned}
\int_\Omega|\nabla\underline{v}|^{q-2}\nabla\underline{v}\nabla z\,dx\le{}&\lambda\int_\Omega f_2(x,\underline{u}(x),\underline{v}(x))z\,dx\\
&+\mu\int_\Omega g_2(x,\underline{u}(x),\underline{v}(x))z\,dx,
\end{aligned}\tag{2.37}$$

so, we conclude that $(\underline{u},\underline{v})$ is a lower-solution of (1.1) with $\underline{u},\underline{v}\in C^1(\Omega)$.

2.3. **Proof of Theorem 1.1.** In subsections 2.1 and 2.2 we proved that there exists $\lambda_0 > 0$ such that for each $\lambda > \lambda_0$ there exist $\mu_0 = \mu_0(\lambda) > 0$ and $(\bar{u}, \bar{v})$, $(\underline{u}, \underline{v})$ that are upper-solution and lower-solution, respectively, of system (1.1), with $\bar{u} \in C^{1,\alpha_1}(\overline{\Omega})$, $\bar{v} \in C^{1,\alpha_2}(\overline{\Omega})$, and $\underline{u}, \underline{v} \in C^1(\Omega)$, whenever $|\mu| < \mu_0$.

Let $w \in W_0^{1,p}(\Omega)$ and $z \in W_0^{1,q}(\Omega)$ satisfy $w, z \geq 0$ for a.e. in Ω. Then, from (2.17), (2.25), and (2.29), we have

$$\begin{aligned} \int_\Omega |\nabla \underline{u}|^{p-2} \nabla \underline{u} \nabla w dx &\leq \lambda_p \frac{(\lambda a_2 + 1)}{a_3} \xi^{p-1} \int_\Omega w dx \\ &\leq \lambda \frac{a_2 + \frac{1}{\lambda}}{a_2 + 1} \frac{a_1}{2} \int_\Omega w dx \\ &\leq \lambda \frac{a_1}{2} \int_\Omega w dx\,. \end{aligned} \tag{2.38}$$

By (2.17), (2.26), and (2.30), we have

$$\int_\Omega |\nabla \underline{v}|^{q-2} \nabla \underline{v} \nabla z dx \leq \lambda \frac{a_1}{2} \int_\Omega z dx. \tag{2.39}$$

However, since that $s_1(p-1) > 1$ and $s_2(q-1) > 1$, changing $\lambda_0 > 0$ by another greater than 1, if necessary, we can suppose that

$$\lambda \frac{a_1}{2} \leq \min\{\lambda^{s_1(p-1)} A^{p-1},\ \lambda^{s_2(q-1)} B^{q-1}\} \tag{2.40}$$

for all $\lambda \geq \lambda_0$. Hence, from (2.4), (2.38), and (2.40), we conclude that

$$\int_\Omega |\nabla \underline{u}|^{p-2} \nabla \underline{u} \nabla w \, dx \leq \int_\Omega |\nabla \bar{u}|^{p-2} \nabla \bar{u} \nabla w \, dx \tag{2.41}$$

and by (2.5), (2.39), and (2.40),

$$\int_\Omega |\nabla \underline{v}|^{q-2} \nabla \underline{v} \nabla z dx \leq \int_\Omega |\nabla \bar{v}|^{q-2} \nabla \bar{v} \nabla z dx, \tag{2.42}$$

so, by the weak comparison principle (see [3, Lemma 2.2]), we obtain $\underline{u} \leq \bar{u}$ and $\underline{v} \leq \bar{v}$ for all $x \in \Omega$. Thus, by using (1.2), we obtain by the standard theorem of lower and upper solution (see [10, Theorem 2.4]) a solution $(u, v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ of system (1.1) with $\underline{u} \leq u \leq \bar{u}$ and $\underline{v} \leq v \leq \bar{v}$ for almost everywhere in Ω. In particular, we see that $u, v \in L^\infty(\Omega)$ and $u(x) > 0$, $v(x) > 0$ for a.e. $x \in \Omega$. Then, by [12, Theorem 1], we obtain $u \in C^{1,\rho_1}(\Omega)$ and $v \in C^{1,\rho_2}(\Omega)$ for some $\rho_1, \rho_2 > 0$, so $u(x) > 0$, $v(x) > 0$ for all $x \in \Omega$.

3. Proof of Theorem 1.2

Supposing by contradiction that there exists a nontrivial solution (u, v) of (1.1), for some λ, μ satisfying (1.11), then by variational characterization of λ_p and λ_q, we achieve

$$\begin{aligned} \lambda_p \int_\Omega |u|^p dx &\leq \int_\Omega |\nabla u|^p dx \\ &\leq \int_\Omega [(|\lambda| k_1 + |\mu| k_5)|u|^p + (|\lambda| k_2 + |\mu| k_6)|v|^q] dx \end{aligned} \tag{3.1}$$

and similarly

$$\lambda_q \int_\Omega |v|^q dx \leq \int_\Omega [(|\lambda| k_3 + |\mu| k_7)|u|^p + (|\lambda| k_4 + |\mu| k_8)|v|^q] \, dx. \tag{3.2}$$

From (3.1) and (3.2), we have

$$
\begin{aligned}
0 < \{\lambda_p - [|\lambda|(k_1+k_3) + |\mu|(k_5+k_7)]\} \int_\Omega |u|^p dx \\
+ \{\lambda_q - [|\lambda|(k_2+k_4) + |\mu|(k_6+k_8)]\} \int_\Omega |v|^q dx \le 0,
\end{aligned}
$$

which is a contradiction.

References

[1] J. Ali, R. Shivaji; *Positive solutions for a class of p-Laplacian systems with multiple parameters*, J. Math. Anal. Appl. 335 (2007) 1013-1019.

[2] F. Brock, L. Iturriaga, J. Sánchez, P. Ubilla; *Existence of positive solutions for p-laplacian problems with weights*, Comm. Pure Appl. Anal. 5 (2006) 941-952.

[3] A. Cañada, P. Drábek, J. L. Gámez; *Existence of positive solutions for some problems with nonlinear diffusion*, Trans. Amer. Math. Soc. 349 (1997) 4231-4249.

[4] A. Castro, M. Hassanpour, R. Shivaji; *Uniqueness of non-negative solutions for a semipositone problems with concave nonlinearity*, Commun. Partial Diff. Eqns. 20 (1995) 1927-1936.

[5] C. Chen; *On positive weak solutions for a class of quasilinear elliptic systems*, Nonlinear Anal. 62 (2005) 751-756.

[6] M. Chhetri, D. D. Hai, R. Shivaji; *On positive solutions for classes of p-laplacian semipositone systems*, Discr. Cont. Dyn. Syst. 9 (2003) 1063-1071.

[7] M. Cuesta; *Eigenvalue problems for the p-Laplacian with indefinite weights*, Electronic J. Diff. Eqns. 2001 (2001) No. 33, 1-9.

[8] J. Fleckinger, R. Pardo, F. Thélin; *Four-parameter bifurcation for a p-Laplacian system*, Eletronic J. Diff. Eqns. 2001 (2001) (06) 1-15.

[9] D. D. Hai, R. Shivaji; *An existence result on positive solutions for a class of p-laplacian systems*, Nonlinear Anal. TMA 56 (2004) 1007-1010.

[10] O. H. Miyagaki, R. S. Rodrigues; *On positive solutions for a class of singular quasilinear elliptic systems*, J. Math. Anal. Appl. 334 (2007) 818-833.

[11] K. Perera, R. Shivaji; *Positive solutions of multiparameter semipositone p-Laplacian problems*, J. Math. Anal. Appl. 338 (2008) 1397-1400.

[12] P. Tolksdorf; *Regularity for a more general class of quasilinear elliptic equations*, J. Diff. Eqns. 51 (1984) 126-150.

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