Electronic Journal of Differential Equations, Vol. 2013 (2013), No. 198, pp. 1–12.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

STABILITY AND BIFURCATION ANALYSIS FOR A DISCRETE-TIME BIDIRECTIONAL RING NEURAL NETWORK MODEL WITH DELAY

YAN-KE DU, RUI XU, QI-MING LIU

Abstract. We study a class of discrete-time bidirectional ring neural network model with delay. We discuss the asymptotic stability of the origin and the existence of Neimark-Sacker bifurcations, by analyzing the corresponding characteristic equation. Employing M-matrix theory and the Lyapunov functional method, global asymptotic stability of the origin is derived. Applying the normal form theory and the center manifold theorem, the direction of the Neimark-Sacker bifurcation and the stability of bifurcating periodic solutions are obtained. Numerical simulations are given to illustrate the main results.

1. Introduction

Since Hopfiled's pioneering work [5, 12], the dynamic behavior (including stability, periodic oscillatory and chaos) of continuous-time Hopfield neural networks has received much attention due to their applications in optimization, signal processing, image processing, solving nonlinear algebraic equation, pattern recognition, associative memories and so on (see, [1, 9, 10, 17] and references therein).

It is well known that time delays in the information processing of neurons exist. The delayed axonal signal transmissions in the neural networks make the dynamic behaviors more complicated, and may destabilize stable equilibria and give rise to periodic oscillation, bifurcation and chaos (see [2, 4, 10, 14]). Therefore, the delay is inevitable and cannot be neglected. For computer simulations, experimental or computational purposes, it is common to discretize the continuous-time neural networks. In some sense, the discrete-time model inherits the dynamical characteristics of the continuous-time networks. We refer the reader to [3, 6, 11, 16] for related discussions on the need and importance of discrete-time analogues to reflect the dynamics of their continuous-time counterparts.

In the field of neural networks, rings are studied to gain insight into the mechanisms underlying the behavior of recurrent networks. In [15], Wang and Han investigated the following continuous-time bidirectional ring network model with

2000 *Mathematics Subject Classification.* 92B20, 34K18, 34K20, 37G05.
Key words and phrases. Neural network; time delay; stability; bifurcation.

Submitted January 3, 2012. Published September 5, 2013.

delay

$$\begin{aligned} \dot{x} &= -x + \alpha f(y(t-\tau)) + \beta f(z(t-\tau)), \\ \dot{y} &= -y + \alpha f(z(t-\tau)) + \beta f(x(t-\tau)), \\ \dot{z} &= -z + \alpha f(x(t-\tau)) + \beta f(y(t-\tau)), \end{aligned} \tag{1.1}$$

where τ denotes the synaptic transmission delay, α and β are connection strengths, $f : R \to R$ is the activation function. In [15], some conditions on the linear stability of the trivial solution of system (1.1) were given and Hopf bifurcation, including its direction and stability, were investigated.

Motivated by the work of Wang and Han [15] and the discussions above, in the present paper, for simplicity, assuming the neurons in the network be identical (see [18]), we are concerned with the stability and bifurcation analysis of the following discrete-time bidirectional ring neural network model with delay

$$\begin{aligned} x(n+1) &= ax(n) + \beta f(y(n-k)) + \beta f(z(n-k)), \\ y(n+1) &= ay(n) + \beta f(z(n-k)) + \beta f(x(n-k)), \\ z(n+1) &= az(n) + \beta f(x(n-k)) + \beta f(y(n-k)), \end{aligned} \tag{1.2}$$

where $a \in (0,1)$ is the internal decay of the neurons, β is the connection strength, $k \in N$ is the time delay.

This paper contributes to understanding of neural networks as follows:

(1) There is a large body of work discussing the stability and bifurcation of neural networks with delays, but most of them deal only with continuous-time neural network models, or discrete-time neural network models of two neurons with or without time delays ([6, 16]). Here we discuss the dynamic behavior of a tri-neuron discrete-time bidirectional ring neural network with delay. The characteristic equation of the neural network is a polynomial equation with high order terms. Using a new approach, sufficient and necessary conditions are derived to ensure that all the roots of the characteristic equation stay inside or on the unit circle.

(2) We remove some restrictions on the conditions required by [15], and in a sense, our results on the asymptotic stability of the origin are less restrictive than those for the corresponding continuous system in [15].

(3) Employing M-matrix theory and the Lyapunov functional method, global asymptotic stability of the origin is derived, which was not taken into account in [15]. The stability criterion is simple and can be easily checked.

The rest of this paper is organized as follows. In Section 2, we analyze the location of roots of a class of polynomial equation. In Section 3, the local stability of the origin and the existence of Neimark-Sacker bifurcations are discussed by analyzing the corresponding characteristic equation, and global asymptotic stability is derived using the method of M-matrix and Lyapunov function. In Section 4, we discuss the stability and direction of the Neimark-Sacker bifurcation by employing the normal form method and the center manifold theorem. Some numerical simulations are carried out in Section 5 to illustrate the main results. In Section 6, a brief discussion is given to conclude the work of this paper.

2. Analysis of polynomial equations

In this section, we analyze the location of the roots of the polynomial equation

$$\lambda^{k+1} - a\lambda^k - b = 0, \ a \in (0,1), \ b \in R, \tag{2.1}$$

which will be used to determine the asymptotic stability of system (1.2).

Suppose that $\lambda = \mathrm{e}^{i\theta}$ is a root of (2.1). Substituting it into (2.1) and separating the real and imaginary parts, we have

$$\begin{gathered} \cos((k+1)\theta) - a\cos(k\theta) = b, \\ \sin((k+1)\theta) - a\sin(k\theta) = 0. \end{gathered} \tag{2.2}$$

Using the identities $\cos((k+1)\theta) = \cos(k\theta)\cos\theta - \sin(k\theta)\sin\theta$ and $\sin((k+1)\theta) = \sin(k\theta)\cos\theta + \cos(k\theta)\sin\theta$, we rewrite (2.2) as

$$\sqrt{a^2+1-2a\cos\theta}\Big[\frac{\cos\theta - a}{\sqrt{a^2+1-2a\cos\theta}}\cos(k\theta) - \frac{\sin\theta}{\sqrt{a^2+1-2a\cos\theta}}\sin(k\theta)\Big] = b,$$
$$\sqrt{a^2+1-2a\cos\theta}\Big[\frac{\cos\theta - a}{\sqrt{a^2+1-2a\cos\theta}}\sin(k\theta) + \frac{\sin\theta}{\sqrt{a^2+1-2a\cos\theta}}\cos(k\theta)\Big] = 0.$$

It is easy to see that if $\theta \in (0,\pi)$, (2.2) is equivalent to the equations

$$\begin{gathered} \sqrt{a^2+1-2a\cos\theta}\cdot\cos(h(\theta)) = b, \\ \sin(h(\theta)) = 0, \end{gathered} \tag{2.3}$$

where

$$h(\theta) = \operatorname{arccot}\frac{\cos\theta - a}{\sin\theta} + k\theta.$$

Since

$$h'(\theta) = \frac{1}{1 + \big(\frac{\cos\theta - a}{\sin\theta}\big)^2}\cdot\frac{1 - a\cos\theta}{\sin^2\theta} + k > 0$$

for $\theta \in (0,\pi)$ and

$$\lim_{\theta\to 0^+} h(\theta) = 0, \quad \lim_{\theta\to\pi^-} h(\theta) = (k+1)\pi,$$

we derive that $h(\theta) : (0,\pi) \to (0,(k+1)\pi)$ is an increasing bijective function. From the second equation in (2.3), we know that $h(\theta) = j\pi$, $j = 1,2,\dots,k$. Denote $\theta_j = h^{-1}(j\pi)$, $j = 1,2,\dots,k$. Then θ_j satisfies the equation

$$j\pi = \operatorname{arccot}\frac{\cos\theta - a}{\sin\theta} + k\theta,$$

which yields $f(\theta) = 0$, where

$$f(\theta) = \sin((k+1)\theta) - a\sin(k\theta).$$

Obviously,

$$f(0) = 0,\; f'(0^+) = (1-a)k + 1 > 0, \quad f(\frac{j\pi}{k+1}) = \begin{cases} a\sin\frac{j\pi}{k+1} > 0, & \text{if } j \text{ is even} \\ -a\sin\frac{j\pi}{k+1} < 0, & \text{if } j \text{ is odd} \end{cases}$$

$j = 1,2,\dots,k$. Therefore, we can deduce that $\theta_j \in (\frac{(j-1)\pi}{k+1}, \frac{j\pi}{k+1})$, $j = 1,2,\dots,k$. From the first equation in (2.3), we get that

$$b = b_j = (-1)^j\sqrt{a^2+1-2a\cos\theta_j},\; j = 1,2,\dots,k. \tag{2.4}$$

If $\theta = 0$, then $b = b_0 = 1 - a > 0$; if $\theta = \pi$, then $b = b_{k+1} = (-1)^{k+1}(1+a)$.

Obviously, if θ is a root of (2.2), $-\theta$ is also a root of (2.2). Hence, we only need to consider the roots $\lambda = \mathrm{e}^{i\theta}$ of (2.1) in $[0,\pi]$. Further, from (2.4), we deduce that

$$\cdots < b_3 < b_1 < b_0 < b_2 < b_4 < \dots. \tag{2.5}$$

On the other hand, from (2.1), we have
$$\frac{\mathrm{d}|\lambda|^2}{\mathrm{d}b} = \frac{\mathrm{d}(\lambda\bar{\lambda})}{\mathrm{d}b} = \lambda\frac{\mathrm{d}\bar{\lambda}}{\mathrm{d}b} + \bar{\lambda}\frac{\mathrm{d}\lambda}{\mathrm{d}b} = \frac{2\,\mathrm{Re}(\lambda P(\lambda))}{|P(\lambda)|^2},$$
where $P(\lambda) = \lambda^{k-1}[(k+1)\lambda - ak]$. It follows that
$$\frac{\mathrm{d}|\lambda|^2}{\mathrm{d}b}\Big|_{b=b_j} = \frac{2b_j}{|P(\mathrm{e}^{i\theta_j})|^2}\Big(\frac{1-a\cos\theta_j}{a^2+1-2a\cos\theta_j} + k\Big).$$
Hence, we have
$$\mathrm{sign}\,\Big\{\frac{\mathrm{d}|\lambda|^2}{\mathrm{d}b}\Big|_{b=b_j}\Big\} = \mathrm{sign}\{b_j\} = (-1)^j, \quad j = 0,1,\dots,k+1.$$
If $b=0$, then system (2.1) has two roots a and 0, which are inside the unit circle. As the parameter b varies, there exist roots of (2.1) which appear on or cross the unit circle.

On the other hand, it can be easily verified that both $\pi/2$ and $2\pi/3$ are not roots of the second equation of (2.2), which implies that $\mathrm{e}^{\pm is\theta_j} \neq 1$ for $s = 1,2,3,4$ and $j = 1,2,\dots,k$.

From the discussions above, we have the following results.

Theorem 2.1. *For equation* (2.1) *with $a \in (0,1)$ and $b \in R$, we have*

- (i) *if $b \in (b_1, b_0)$, all the roots of* (2.1) *are inside the unit circle;*
- (ii) *if $b = b_0 = 1-a$, there is a simple root $\lambda = 1$ of* (2.1) *on the unit circle, and all the other roots are inside the unit circle;*
- (iii) *if $b = b_j$, $j = 1,2,\dots,k$, there is a pair of complex roots $\mathrm{e}^{\pm i\theta_j}$ on the unit circle. Further, $\mathrm{e}^{\pm is\theta_j} \neq 1$ for $s = 1,2,3,4$. Moreover, for the case of $b = b_1$, all other roots are inside the unit circle;*
- (iv) *if $b = b_{k+1} = (-1)^{k+1}(1+a)$,* (2.1) *has a simple root $\lambda = -1$ on the unit circle;*
- (v) *if $|b| > |b_{k+1}| = 1+a$, all roots of* (2.1) *are outside the unit circle;*
- (vi) $\frac{\mathrm{d}|\lambda|^2}{\mathrm{d}b}\big|_{b=b_j} \neq 0$ *for $j = 0,1,\dots,k+1$, where $b_j = (-1)^j\sqrt{a^2+1-2a\cos\theta_j}$, and θ_j is the unique solution in $(\frac{(j-1)\pi}{k+1}, \frac{j\pi}{k+1})$ of the equation $\sin((k+1)\theta) - a\sin(k\theta) = 0$ for $j = 1,2,\dots,k$.*

3. Stability analysis and existence of bifurcations

Throughout this paper, we assume that

(H1) $f(0) = 0$, $f(\cdot) \in C^3(R)$.

Denote $x_0(n) = x(n)$, $x_j(n+1) = x_{j-1}(n)$; $y_0(n) = y(n)$, $y_j(n+1) = y_{j-1}(n)$; $z_0(n) = z(n)$, $z_j(n+1) = z_{j-1}(n)$, $j = 1,2,\dots,k$. Then we can transform system (1.2) into the following system of $3k+3$ difference equations without delays
$$\begin{aligned}
x_0(n+1) &= ax_0(n) + \beta f(y_k(n)) + \beta f(z_k(n)),\\
y_0(n+1) &= ay_0(n) + \beta f(z_k(n)) + \beta f(x_k(n)),\\
z_0(n+1) &= az_0(n) + \beta f(x_k(n)) + \beta f(y_k(n)),\\
x_j(n+1) &= x_{j-1}(n),\\
y_j(n+1) &= y_{j-1}(n),\\
z_j(n+1) &= z_{j-1}(n), \quad j = 1,2,\dots,k.
\end{aligned} \tag{3.1}$$

For convenience, we denote $c = \beta f'(0)$. The Jacobian matrix of system (3.1) at the equilibrium $E = (0, \dots, 0)$ is as follows

$$A = \begin{bmatrix} B & 0 & 0 & c & 0 & c \\ I_k & 0 & 0 & 0 & 0 & 0 \\ 0 & c & B & 0 & 0 & c \\ 0 & 0 & I_k & 0 & 0 & 0 \\ 0 & c & 0 & c & B & 0 \\ 0 & 0 & 0 & 0 & I_k & 0 \end{bmatrix}, \tag{3.2}$$

where $B = (a, 0, \dots, 0)_{1\times k}$, I_k is a $k \times k$ identity matrix, 0 is a zero matrix of appropriate size.

The associated characteristic equation of system (3.1) is

$$(\lambda^{k+1} - a\lambda^k + c)^2(\lambda^{k+1} - a\lambda^k - 2c) = 0. \tag{3.3}$$

Applying Theorem 2.1 to (3.3) and noting that $d_1 \le -d_0$, we can obtain the following results.

Theorem 3.1. *Assume* (H1) *and that* $0 < a < 1$. *Then we have*

- (i) *if* $\max\{-b_0, b_1/2\} < c < b_0/2$, *the origin of system* (1.2) *is asymptotically stable;*
- (ii) *if* $c = -b_0$ *or* $b_0/2$, *a fold bifurcation occurs at the origin in system* (1.2)*;*
- (iii) *if* $c = b_1/2$, *a Neimark-Sacker bifurcation occurs at the origin in system* (1.2), *where* $b_0 = 1 - a$, $b_1 = -\sqrt{a^2 + 1 - 2a\cos\theta_1}$, *in which* θ_1 *is the unique solution in* $(0, \frac{\pi}{k+1})$ *of the equation* $\sin((k+1)\theta) - a\sin(k\theta) = 0$.

Remark 3.2. As to the case of $c = b_1/2$, if $c = -b_j$ or $b_j/2$, Neimark-Sacker bifurcations occur at the origin in system (1.2), where $b_j = (-1)^j\sqrt{a^2 + 1 - 2a\cos\theta_j}$, in which θ_j is the unique solution in $(\frac{(j-1)\pi}{k+1}, \frac{j\pi}{k+1})$ of the equation $\sin((k+1)\theta) - a\sin(k\theta) = 0$ for $j = 1, 2, \dots, k$. If $c = -b_{k+1}$ or $b_{k+1}/2$, system (1.2) has a Flip bifurcation at the origin, where $d_{k+1} = (-1)^{k+1}(1 + a)$.

In what follows, we investigate the global asymptotic stability of system (1.2).

Theorem 3.3. *Under assumption* (H1), *the origin of* (1.2) *is globally asymptotically stable if the following conditions hold.*

- (H2) *There exists a constant* $L > 0$ *such that* $|f'(\cdot)| \le L$.
- (H3) $1 - a - 2L|\beta| > 0$.

Proof. Since $1 - a - 2L|\beta| > 0$, the matrix

$$A = \begin{pmatrix} 1-a & -L|\beta| & -L|\beta| \\ -L|\beta| & 1-a & -L|\beta| \\ -L|\beta| & -L|\beta| & 1-a \end{pmatrix}$$

is an M-matrix, and there exists a vector $p = (p_i)_{1\times 3} > 0$ such that $pA > 0$ ([13]); that is,

$$p_i(1-a) - (\sum_{j=1}^{3} p_j - p_i)L|\beta| > 0, \; i = 1, 2, 3.$$

Hence, we can choose $\lambda > 1$ such that

$$p_i(1 - a\lambda) - \Big(\sum_{j=1}^{3} p_j - p_i\Big)L|\beta|\lambda^{k+1} > 0, \quad i = 1, 2, 3. \tag{3.4}$$

Let $U_1(n) = \lambda^n|x(n)|$, $U_2(n) = \lambda^n|y(n)|$, $U_3(n) = \lambda^n|z(n)|$. From (1.2), we have

$$U_i(n+1) \le a\lambda U_i(n) + L|\beta|\lambda^{k+1}\Big[\sum_{j=1}^{3} U_j(n-k) - U_i(n-k)\Big], \quad i = 1,2,3. \tag{3.5}$$

Define a Lyapunov function

$$V(n) = \sum_{i=1}^{3} p_i U_i(n) + \sum_{l=n-k}^{n-1}\sum_{i=1}^{3}\Big[\Big(\sum_{j=1}^{3} p_j - p_i\Big)L|\beta|\lambda^{k+1}U_i(l)\Big].$$

Then from (3.4) and (3.5), we deduce that

$$\Delta V(n) = V(n+1) - V(n) = -\sum_{i=1}^{3}\Big[p_i(1-a\lambda) - (\sum_{j=1}^{3} p_j - p_i)L|\beta|\lambda^{k+1}\Big]U_i(n) \le 0,$$

which implies that $V(n) \le V(0)$. Note that

$$V(n) \ge m_0\lambda^n(|x(n)| + |y(n)| + |z(n)|),$$
$$V(0) = \sum_{i=1}^{3} p_i U_i(0) + \sum_{l=-k}^{-1}\sum_{i=1}^{3}\Big[(\sum_{j=1}^{3} p_j - p_i)L|\beta|\lambda^{k+1}U_i(l)\Big] := M_0,$$

where $m_0 = \min_{i=1,2,3}\{p_i\}$, M_0 is a positive constant. Thus,

$$|x(n)| + |y(n)| + |z(n)| \le \frac{M_0}{m_0}\lambda^{-n}.$$

Noting that $\lambda > 1$, we get $\lim_{n\to+\infty} x(n) = 0$, $\lim_{n\to+\infty} y(n) = 0$, $\lim_{n\to+\infty} z(n) = 0$. Then, the origin of system (1.2) is globally attractive. On the other hand, it is easy to verify that $|c| < b_0/2$ under conditions (H2) and (H3). Considering that $b_1 \le b_0$, we have $\max\{-b_0, b_1/2\} < c < b_0/2$ if $|c| < b_0/2$. Then the origin of system (1.2) is asymptotically stable. Consequently, the origin of system (1.2) is globally asymptotically stable. □

4. Direction and stability of Neimark-Sacker bifurcation

In this section, employing the normal form theory and the center manifold theorem for discrete-time system developed by Kuznetsov [7], we study the direction of Neimark-Sacker bifurcation and the stability of periodic solutions bifurcating from the origin of system (1.2).

From the discussions in Section 3, we know that if $c = b_1/2$, a Neimark-Sacker bifurcation occurs at the origin in system (1.2). For convenience, we denote $b_1/2$ by $\tilde{b}$, and the pair of imaginary roots of the associated characteristic equation for system (1.2) by $e^{\pm i\tilde{\theta}}$.

Denote $\lambda = e^{i\tilde{\theta}}$. Let q be an eigenvector of A corresponding to eigenvalue λ, and p be an eigenvector of A^{T} corresponding to eigenvalue $\bar{\lambda}$, that is, $Aq = \lambda q$ and $A^{\mathrm{T}}p = \bar{\lambda}p$. Obviously, q and p are $3(k+1)$-dimensional complex column vectors.

By direct calculations, we obtain that

$$
\begin{aligned}
q &= \Big(\frac{2c}{\lambda-a}, \frac{2c}{\lambda(\lambda-a)}, \dots, \frac{2c}{\lambda^{k-1}(\lambda-a)}, \frac{2c}{\lambda^{k}(\lambda-a)},\\
&\quad \lambda^k, \lambda^{k-1}, \dots, \lambda, 1, \lambda^k, \lambda^{k-1}, \dots, \lambda, 1\Big)^{\mathrm{T}},\\
p &= E\Big(\frac{2c}{\bar\lambda^{2k-1}(\bar\lambda-a)^2}, \frac{2c}{\bar\lambda^{2k-1}(\bar\lambda-a)}, \frac{2c}{\bar\lambda^{2k-2}(\bar\lambda-a)}, \dots, \frac{2c}{\bar\lambda^{k+1}(\bar\lambda-a)},\\
&\quad \frac{2c}{\bar\lambda^{k}(\bar\lambda-a)}, \frac{1}{\bar\lambda^{k-1}(\bar\lambda-a)}, \frac{1}{\bar\lambda^{k-1}}, \frac{1}{\bar\lambda^{k-2}}, \dots, \frac{1}{\bar\lambda}, 1,\\
&\quad \frac{1}{\bar\lambda^{k-1}(\bar\lambda-a)}, \frac{1}{\bar\lambda^{k-1}}, \frac{1}{\bar\lambda^{k-2}}, \dots, \frac{1}{\bar\lambda}, 1\Big)^{\mathrm{T}},
\end{aligned}
\tag{4.1}
$$

in which

$$
E = \frac{1}{2k + \frac{2\bar\lambda}{\bar\lambda-a} + \frac{4kc^2}{\bar\lambda^{2k}(\bar\lambda-a)^2} + \frac{4c^2}{\bar\lambda^{2k-1}(\bar\lambda-a)^3}},
$$

where p and q satisfy $\langle p, q\rangle = 1$.

System (3.1) can be rewritten as

$$
u(n+1) = Au(n) + F(u(n)),
$$

where A is defined by (3.2), and

$$
\begin{gathered}
u(n) = (x_0(n), x_1(n), \dots, x_k(n), y_0(n), y_1(n), \dots, y_k(n), z_0(n), z_1(n), \dots, z_k(n))^{\mathrm{T}},\\
F(u(n)) = (F_1(u(n)), 0\dots, 0, F_2(u(n)), 0\dots, 0, F_3(u(n)), 0\dots, 0)^{\mathrm{T}},
\end{gathered}
$$

in which F_i $(i = 1, 2, 3)$ can be expanded in the form

$$
\begin{aligned}
F_1(\xi) &= \frac{1}{2}\beta f''(0)[\xi_{2k+2}^2 + \xi_{3k+3}^2] + \frac{1}{6}\beta f'''(0)[\xi_{2k+2}^3 + \xi_{3k+3}^3] + \text{h.o.t.},\\
F_2(\xi) &= \frac{1}{2}\beta f''(0)[\xi_{k+1}^2 + \xi_{3k+3}^2] + \frac{1}{6}\beta f'''(0)[\xi_{k+1}^3 + \xi_{3k+3}^3] + \text{h.o.t.},\\
F_3(\xi) &= \frac{1}{2}\beta f''(0)[\xi_{k+1}^2 + \xi_{2k+2}^2] + \frac{1}{6}\beta f'''(0)[\xi_{k+1}^3 + \xi_{2k+2}^3] + \text{h.o.t.}.
\end{aligned}
$$

Simple calculations yield

$$
\begin{aligned}
B_1(x,y) &= \sum_{j,k=1}^{3k+3} \frac{\partial^2 F_1(\xi)}{\partial\xi_j \partial\xi_k}\Big|_{\xi=0} x_j y_k = \beta f''(0)[x_{2k+2}y_{2k+2} + x_{3k+3}y_{3k+3}],\\
B_2(x,y) &= \sum_{j,k=1}^{3k+3} \frac{\partial^2 F_2(\xi)}{\partial\xi_j \partial\xi_k}\Big|_{\xi=0} x_j y_k = \beta f''(0)[x_{k+1}y_{k+1} + x_{3k+3}y_{3k+3}],\\
B_3(x,y) &= \sum_{j,k=1}^{3k+3} \frac{\partial^2 F_3(\xi)}{\partial\xi_j \partial\xi_k}\Big|_{\xi=0} x_j y_k = \beta f''(0)[x_{k+1}y_{k+1} + x_{2k+2}y_{2k+2}],\\
C_1(x,y,z) &= \sum_{j,k,l=1}^{3k+3} \frac{\partial^3 F_1(\xi)}{\partial\xi_j \partial\xi_k \partial\xi_l}\Big|_{\xi=0} x_j y_k z_l\\
&= \beta f'''(0)[x_{2k+2}y_{2k+2}z_{2k+2} + x_{3k+3}y_{3k+3}z_{3k+3}],
\end{aligned}
$$

$$\begin{aligned} C_2(x,y,z) &= \sum_{j,k,l=1}^{3k+3} \frac{\partial^3 F_2(\xi)}{\partial \xi_j \partial \xi_k \partial \xi_l}\Big|_{\xi=0} x_j y_k z_l \\ &= \beta f'''(0)[x_{k+1}y_{k+1}z_{k+1} + x_{3k+3}y_{3k+3}z_{3k+3}], \\ C_3(x,y,z) &= \sum_{j,k,l=1}^{3k+3} \frac{\partial^3 F_3(\xi)}{\partial \xi_j \partial \xi_k \partial \xi_l}\Big|_{\xi=0} x_j y_k z_l \\ &= \beta f'''(0)[x_{k+1}y_{k+1}z_{k+1} + x_{2k+2}y_{2k+2}z_{2k+2}]. \end{aligned} \tag{4.2}$$

If $b = \tilde{b}$, the restriction of system (3.1) to its two-dimensional center manifold at the critical parameter value can be transformed into the normal form written in complex coordinates (see [7, 8]):

$$w \to \lambda w(1 + \frac{1}{2} d|w|^2) + O(|w|^4), \quad w \in C,$$

in which

$$d = \bar{\lambda} \left\langle p, \hat{C}(q,q,\bar{q}) + 2\hat{B}\big(q, (I-A)^{-1}\hat{B}(q,\bar{q})\big) + \hat{B}\big(\bar{q}, (\lambda^2 I - A)^{-1}\hat{B}(q,q)\big) \right\rangle,$$

with p and q defined by (4.1) and $\hat{B} = (B_1, 0, \dots, 0, B_2, 0, \dots, 0, B_3, 0, \dots, 0,)^{\mathrm{T}}$, $\hat{C} = (C_1, 0, \dots, 0, C_2, 0, \dots, 0, C_3, 0, \dots, 0,)^{\mathrm{T}}$ with $B_i,\ C_i$ $(i = 1, 2, 3)$ defined by (4.2).

Theorem 4.1 ([8])**.** *The direction and stability of the Neimark-Sacker bifurcation are determined by the sign of* $\mathrm{Re}(d)$*. If* $\mathrm{Re}(d) < 0$*, then the bifurcation is supercritical, that is, the closed invariant curve bifurcating from the origin is asymptotically stable. If* $\mathrm{Re}(d) > 0$*, then the bifurcation is subcritical; that is, the closed invariant curve bifurcating from the origin is unstable.*

5. Numerical simulations

In this section, we give two examples to illustrate the results derived in Sections 3 and 4. In system (1.2), we choose the activation function as the type of inverse tangent function or hyperbolic tangent function; i.e., $f(v) = \tanh(v)$, then $f'(0) = 1$, $f''(0) = 0$, $f'''(0) = -2$, and $|f'(x)| \le 1$. In addition, in the following simulations, the numerical results of $y(n)$ and $z(n)$ are similar to those of $x(n)$, so they are omitted.

Example 5.1. For system (1.2), If $a = 0.5$, $k = 2$, then $b_0 = 0.5$, $b_1 \approx -0.7808$. Choose $\beta = -0.38$ and 0.24, respectively, we have $\max\{-b_0, b_1/2\} = -0.3904 < c < 0.25 = b_0/2$. By Theorem 3.1, the origin of system (1.2) is asymptotically stable (see Figure 1). If $\beta = -0.3904 \approx b_1/2$, a Neimark-Sacker bifurcation occurs at the origin in system (1.2). Furthermore, from the formulae in Sections 3 and 4, and by direct computations, we obtain

$$\tilde{\theta} \approx 0.8758, \ d \approx -0.6168 + 0.1248i.$$

Therefore, $\mathrm{Re}(d) \approx -0.6168 < 0$, and from Theorem 4.1, the Neimark-Sacker bifurcation is supercritical and the bifurcating periodic solution is asymptotically stable (see Figure 2).

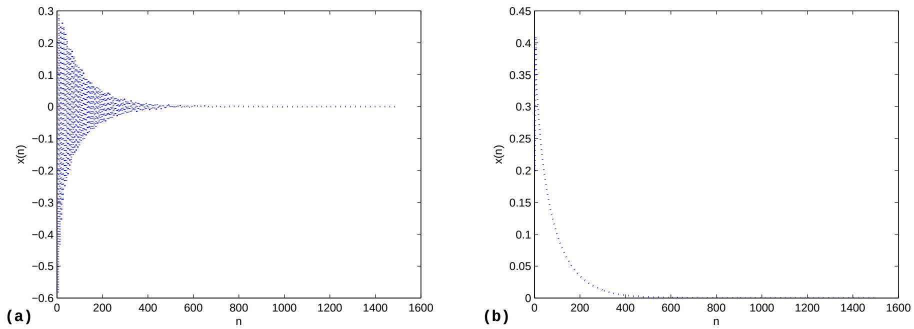

FIGURE 1. (a) Dynamic behavior of system (1.2) with $a = 0.5$, $k = 2$, $\beta = -0.38$. (b) Dynamic behavior of system (1.2) with $a = 0.5$, $k = 2$, $\beta = 0.24$. The origin of system (1.2) is asymptotically stable

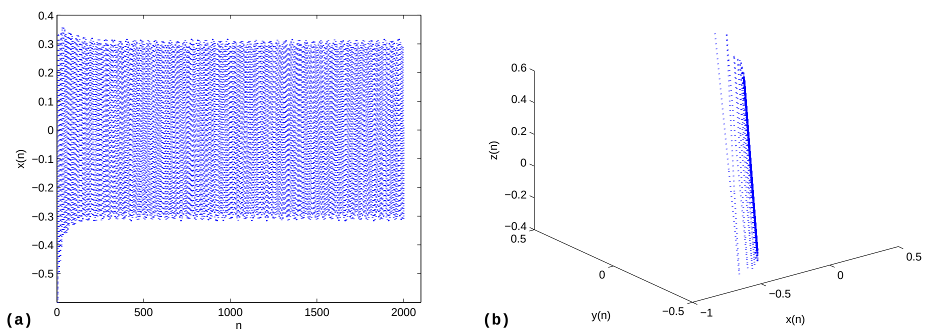

FIGURE 2. Dynamic behavior of system (1.2) with $a = 0.5$, $k = 2$, $\beta = -0.3904$. A Neimark-Sacker bifurcation occurs at the origin in system (1.2). (a) Diagram in (n, x)-plane. (b) Phase portrait

Example 5.2. Consider the following generalized system with different connection weights through the neurons

$$
\begin{aligned}
x(n+1) &= ax(n) + \alpha f(y(n-k)) + \beta f(z(n-k)),\\
y(n+1) &= ay(n) + \alpha f(z(n-k)) + \beta f(x(n-k)),\\
z(n+1) &= az(n) + \alpha f(x(n-k)) + \beta f(y(n-k)).
\end{aligned}
\tag{5.1}
$$

Let $a = 0.5$, $\alpha = -0.4$, and choose $\beta = 0.25, 0.3, 0.5, 1.2$, respectively, numerical simulations show that Neimark-Sacker bifurcations occur at the origin in system (5.1). We further find that with the increasing of β, the velocity of the trajectory going to the bifurcating periodic solution is increasing (see Figure 3). In Figure 4, we exhibit a typical bifurcation and chaos diagram when we fix other parameters and choose β as a bifurcation parameter ($-4 < \beta < 4$). It clearly shows that system (5.1) admits rich dynamics including period-doubling bifurcation and chaos.

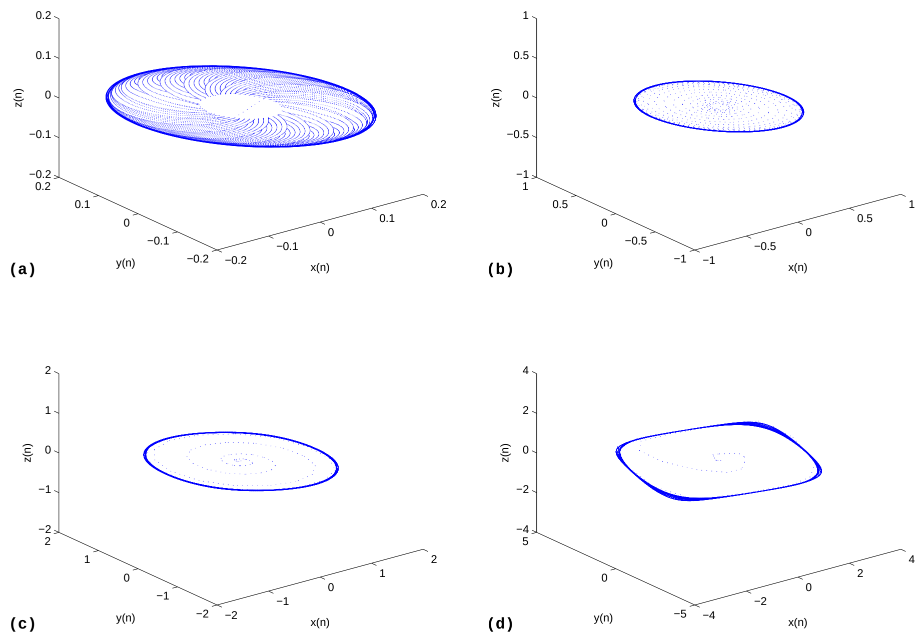

FIGURE 3. Phase plot in space (x, y, z) for system (5.1) with (a) $c = 0.25$, (b) $c = 0.3$, (c) $c = 0.5$ and (d) $c = 1.2$

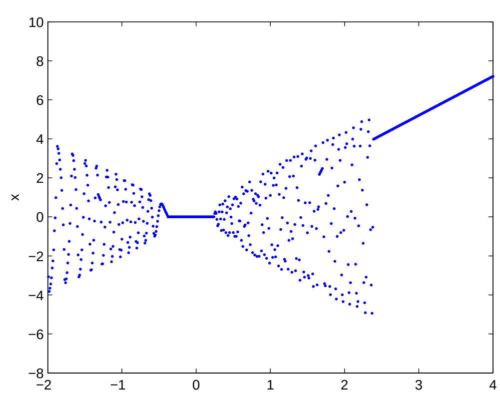

FIGURE 4. Bifurcation diagram in (c, x)-plane for system (5.1) with step size of 0.01 for c. It shows the effect of parameter c on the dynamic behavior

Conclusion. In this article, a class of discrete-time bidirectional ring neural network model with delay was investigated. Analyzing the corresponding characteristic equation, the asymptotic stability of the origin was discussed, and choosing

$c = \beta f'(0)$ as a bifurcation parameter, we showed that system (1.2) undergoes Neimark-Sacker bifurcations at the origin. Using the M-matrix method and the Lyapunov function, global asymptotic stability of the origin was derived. Applying the normal form theory and the center manifold theorem, the direction of the Neimark-Sacker bifurcation and the stability of bifurcating periodic solution were obtained.

We removed the restrictions of the conditions that $f'(0) = 1$ and $xf(x) \neq 0$ for $x \neq 0$ in [15], and derived conditions under which the origin is asymptotically stable. Noting that $b_1 > 1a$, the hypotheses of Theorem 3.1 are less restrictive than those for the corresponding continuous system ($\alpha = \beta$) in [15]. Moreover, when the connection weights through the neurons in a bidirectional ring neural network model are different, numerical simulations show that the corresponding system still undergoes Neimark-Sacker bifurcations at the origin. We leave for future work the study of (1.2) with different connection weights, activation functions and time delays.

Acknowledgements. This work was supported by the National Natural Science Foundation of China (No. 11071254), the Scientific Research Foundation for the Returned Overseas Chinese Scholars, State Education Ministry and the Natural Science Foundation of Young Scientist of Hebei Province (No. A2013506012).

References

[1] J. Cao, W. Yu, Y. Qu; A new complex network model and convergence dynamics for reputation computation in virtual organizations, *Phys. Lett. A* **356** (2006) 414-425.

[2] Q. Gan, R. Xu, W. Hu, P. Yang; Bifurcation analysis for a tri-neuron discrete-time BAM neural network with delays, *Chaos Solitons Fractals* **42** (2009) 2502-2511.

[3] S. Guo, Y. Chen; Stability and bifurcation of a discrete-time three-neuron system with delays, *Int. J. Appl. Math. Eng. Sci.* **1** (2007) 103-115.

[4] S. Guo, X. Tang, L. Huang; Bifurcation analysis in a discrete-time single-directional network with delays, *Neurocomputing* **71** (2008) 1422-1435.

[5] J. Hopfield; Neural networks and physical systems with emergent collective computational abilities, *Proc. Natl. Acad. Sci.* **79** (1982) 2554-2558.

[6] E. Kaslik, St. Balint; Bifurcation analysis for a discrete-time Hopfield neural network of two neurons with two delays and self-connections, *Chaos Solitons Fractals* **39** (2009) 83-91.

[7] Y. A. Kuznetsov; Elements of applied bifurcation theory, vol. 112 of Applied Mathematical Sciences, Springer, New York, USA, 1998.

[8] Y. A. Kuznetsov, H. G. E. Meijer; Numerical normal forms for codim 2 bifurcations of fixed points with at most two critical eigenvalues, *SIAM J. Sci. Comput.* **26** (2005) 1932-1954.

[9] S. M. Lee, O. M. Kwonb, Ju H. Park; A novel delay-dependent criterion for delayed neural networks of neutral type, *Phys. Lett. A* **374** (2010) 1843-1848.

[10] X. Li, J. Cao; Delay-dependent stability of neural networks of neutral type with time delay in the leakage term, *Nonlinearity* **23** (2010) 1709-1726.

[11] A. Stuart, A. Humphries; Dynamical systems and numerical analysis, Cambridge University Press, Cambridge, 1996.

[12] D. Tank, J. Hopfield; Simple neural optimization networks: an A/D converter, signal decision circuit and a linear programming circuit, *IEEE Trans. Circ. Syst.* **33** (1986) 533-541.

[13] R. S. Varga; Matrix iterative analysis, vol. 27 of Springer Series in Computational Mathematics, Springer, Berlin, Germany, 2000.

[14] B. Wang, J. Jian; Stability and Hopf bifurcation analysis on a four-neuron BAM neural network with distributed delays, *Commun. Nonlinear Sci. Numer. Simul.* **15** (2010) 189-204.

[15] L. Wang, X. Han; Stability and Hopf bifurcation analysis in bidirectional ring network model, *Commun. Nonlinear Sci. Numer. Simul.* **16** (2011) 3684-3695.

[16] H. Zhao, L. Wang, C. Ma; Hopf bifurcation and stability analysis on discrete-time Hopfield neural network with delay, *Nonlinear Anal. Real World Appl.* **9** (2008) 103-113.
[17] J. Zhou, S. Li, Z. Yang; Global exponential stability of Hopfield neural networks with distributed delays, *Appl. Math. Model.* **33** (2009) 1513-1520.
[18] H. Zhu, L. Huang; Stability and bifurcation in a tri-neuron network model with discrete and distributed delays, *Appl. Math. Comput.* **188** (2007) 1742-1756.

Yan-Ke Du
Institute of Applied Mathematics, Shijiazhuang Mechanical Engineering College, Shijiazhuang. 050003, China
E-mail address: yankedu2011@163.com

Rui Xu
Institute of Applied Mathematics, Shijiazhuang Mechanical Engineering College, Shijiazhuang. 050003, China
E-mail address: rxu88@163.com

Qi-Ming Liu
Institute of Applied Mathematics, Shijiazhuang Mechanical Engineering College, Shijiazhuang. 050003, China
E-mail address: lqmmath@163.com