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EXISTENCE OF GLOBAL SOLUTIONS TO FREE BOUNDARY VALUE PROBLEMS FOR BIPOLAR NAVIER-STOKES-POSSION SYSTEMS

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Abstract. In this article, we consider the free boundary value problem for one-dimensional compressible bipolar Navier-Stokes-Possion (BNSP) equations with density-dependent viscosities. For general initial data with finite energy and the density connecting with vacuum continuously, we prove the global existence of the weak solution. This extends the previous results for compressible NS [27] to NSP.

1. Introduction

Bipolar Navier-Stokes-Possion (BNSP) has been used to simulate the transport of charged particles under the influence of electrostatic force governed by the self-consistent Possion equation. In this article, we consider the free boundary value problem for one-dimensional isentropic compressible BNSP with density-dependent viscosities:

$$\begin{gathered}
\rho_\tau + (\rho u)_\xi = 0,\\
(\rho u)_\tau + (\rho u^2)_\xi + p(\rho)_\xi = \rho\Phi_\xi + (\mu(\rho)u_\xi)_\xi,\\
n_\tau + (nv)_\xi = 0,\\
(nv)_\tau + (nv^2)_\xi + p(n)_\xi = -n\Phi_\xi + (\mu(n)v_\xi)_\xi,\\
\Phi_{\xi\xi} = \rho - n,
\end{gathered} \tag{1.1}$$

where the unknown functions are the charges densities $\rho(\xi,\tau) \geq 0$, $n(\xi,\tau) \geq 0$, the velocities u, v and the electrostatic potential Φ. Here, $p(\rho) = \rho^\gamma(\gamma > 1)$ and $p(n) = n^\gamma(\gamma > 1)$ are the pressure functions, and $\mu(\rho)$, $\mu(n)$ are the viscosity coefficients.

There are many progress made recently concerned with the existence of solution to the free boundary value problem for the compressible Navier-Stokes equations with density-dependent viscosities. When the fluid density connects to vacuum with discontinuity, Liu-Xin-Yang [21] proved the existence and uniqueness of the local weak solution. Under certain assumptions imposed on the viscosities, Yang-Yao-Zhu [29] established the global existence and uniqueness of weak solution. As for

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related results, the reader can refer to [14, 26] and references therein. When the fluid density connects with vacuum continuously, under some conditions imposed on the viscosities, Yang-Zhao [30] proved the existence of the local solution. Yang-Zhu [31] established the global existence of weak solution when viscosities satisfied certain condition. For more results about fluid density connects with vacuum continuously, the reader can refer to [19, 27] and references therein.

There also have been extensive studies on the global existence and asymptotic behavior of weak solution to the unipolar Navier-Stokes-Possion system (NSP). The global existence of weak solution to NSP with general initial data was proved in [5, 33]. The quasi-neutral and some related asymptotic limits were studied in [4, 6, 12, 28]. In the case when the Possion equation describes the self-gravitonal force for stellar gases, the global existence of weak solution and asymptotic behavior were also investigated together with the stability analysis, refer to [7, 15] and the references therein. In addition, the global well-posedness of NSP was proved in the Besov space in [9]. The global existence and the optimal time convergence rates of the classical solution were obtained recently in [18].

For bipolar Navier-Stokes-Possion system (BNSP), there are also abundant results concerned with the existence and asymptotic behavior of the global weak solution. Li-Yang-Zou [17] proved optimal L^2 time convergence rate for the global classical solution for a small initial perturbation of the constant equilibrium state. The optimal time decay rate of global strong solution is established in [22, 32]. Liu-Lian-Qian [20] established global existence of solution to Bipolar Navier-Stokes-Possion system. Lin-Hao-Li [24] studied the global existence and uniqueness of the strong solution in hybrid Besov spaces with the initial data close to an equilibrium state. As a continuation of the study in this direction, in this paper, we will study the free boundary value problem for BNSP.

The rest of this article is as follows. In Section 2, we state the main results of this paper. The global existence of the weak solution is proven in Section 3.

2. Main results

For simplicity, the viscosity terms are assumed to satisfy $\mu(\rho) = \rho^\alpha$, $\mu(n) = n^\alpha, \alpha > 0$. In this situation, (1.1) becomes

$$\begin{gathered} \rho_\tau + (\rho u)_\xi = 0, \\ (\rho u)_\tau + (\rho u^2)_\xi + (\rho^\gamma)_\xi = \rho\Phi_\xi + (\rho^\alpha u_\xi)_\xi, \\ n_\tau + (nv)_\xi = 0, \\ (nv)_\tau + (nv^2)_\xi + (n^\gamma)_\xi = -n\Phi_\xi + (n^\alpha v_\xi)_\xi, \\ \Phi_{\xi\xi} = \rho - n, \end{gathered} \tag{2.1}$$

for $(\xi, \tau) \in \Omega_\tau$, with

$$\Omega_\tau = \{(\xi, \tau) : a(\tau) \le \xi \le b(\tau),\ \tau \ge 0\}. \tag{2.2}$$

The boundary condition is

$$(\rho, n)(a(\tau), \tau) = (\rho, n)(b(\tau), \tau) = 0, \quad \Phi_\xi(a(\tau), \tau) = \Phi_\xi(b(\tau), \tau) = 0, \tag{2.3}$$

where $a(\tau)$ and $b(\tau)$ are free boundary defined by

$$\frac{\mathrm{d}}{\mathrm{d}\tau} a(\tau) = u(a(\tau), \tau) = v(a(\tau), \tau), \quad \frac{\mathrm{d}}{\mathrm{d}\tau} b(\tau) = u(b(\tau), \tau) = v(b(\tau), \tau), \quad \tau > 0, \tag{2.4}$$

and

$$a(0) = a_0, \quad b(0) = b_0, \quad a_0 < b_0. \tag{2.5}$$

The initial data is

$$(\rho, u, n, v, \Phi_\xi)(\xi, 0) = (\rho_0(\xi), u_0(\xi), n_0(\xi), v_0(\xi), \Phi_{\xi 0}(\xi)), \quad \xi \in \Omega_0 = [a_0, b_0]. \tag{2.6}$$

Throughout the present paper, the initial data is assumed to satisfy:

(A1) $0 \le \rho_0(\xi) \le C\big((\xi - a_0)(b_0 - \xi)\big)^{\frac{\sigma}{1-\sigma}}$, $0 \le n_0(\xi) \le C\big((\xi - a_0)(b_0 - \xi)\big)^{\frac{\sigma}{1-\sigma}}$ with $0 < \sigma < 1$,

$$\Phi_{\xi 0} \in L^2(\Omega_0),\ (\rho_0^{\gamma - \frac{1}{2}})_\xi \in L^2(\Omega_0), \quad (n_0^{\gamma - \frac{1}{2}})_\xi \in L^2(\Omega_0)$$

where C is a positive constant;

(A2) for sufficiently large positive integer m,

$$\begin{gathered}\big((\xi - a_0)(b_0 - \xi)\big)^{\frac{2m-1}{1-\sigma}} [(\rho_0^\alpha)_\xi]^{2m} \rho_0^{1-2m} \in L^1(\Omega_0),\\ \big((\xi - a_0)(b_0 - \xi)\big)^{\frac{2m-1}{1-\sigma}} [(n_0^\alpha)_\xi]^{2m} n_0^{1-2m} \in L^1(\Omega_0);\end{gathered}$$

(A3) $u_0(\xi) \in L^\infty(\Omega_0)$, $v_0(\xi) \in L^\infty(\Omega_0)$, $\rho_0^{-1/2}(\rho_0^\alpha u_{0\xi})_\xi \in L^2(\Omega_0)$ and $n_0^{-1/2}(n_0^\alpha v_{0\xi})_\xi \in L^2(\Omega_0)$;

(A4) $0 < \alpha < 1/3$, $\gamma > 1$.

Without the loss of generality, the total initial mass is renormalized to be one throughout the present paper; i.e.,

$$\int_{\Omega_0} \rho_0 \mathrm{d}\xi = 1, \quad \int_{\Omega_0} n_0 \mathrm{d}\xi = 1.$$

We define the global weak solution to the FBVP for the compressible BNSP (2.1) as follows.

Definition 2.1. For any $T > 0$, $(\rho, n, u, v, \Phi_\xi)$ is said to be a weak solution to free boundary value problem (2.1)–(2.6) on $\Omega_\tau \times [0, T]$, provided that there holds

$$\begin{gathered}\rho \in C^0(\Omega_\tau \times [0,T]), \quad u \in C^0(\Omega_\tau \times [0,T]),\\ u_\xi \in L^1(\Omega_\tau \times [0,T]), \quad \rho^\alpha u_\xi \in L^\infty(\Omega_\tau \times [0,T]),\\ n \in C^0(\Omega_\tau \times [0,T]), \quad v \in C^0(\Omega_\tau \times [0,T]),\\ v_\xi \in L^1(\Omega_\tau \times [0,T]), \quad n^\alpha v_\xi \in L^\infty(\Omega_\tau \times [0,T]),\\ \Phi_\xi \in L^\infty([0,T]; H^1(\Omega_\tau)),\end{gathered} \tag{2.7}$$

and (2.1) are satisfied in the sense of distributions. Namely, it holds for all $\varphi \in C_0^\infty\big((a(\tau), b(\tau)) \times [0, T)\big)$ that

$$\int_{\Omega_0} \rho_0 \varphi(\xi, 0) \mathrm{d}\xi + \int_0^T \int_{\Omega_\tau} (\rho \varphi_\tau + \rho u \varphi_\xi) \mathrm{d}\xi \mathrm{d}\tau = 0, \tag{2.8}$$

$$\int_{\Omega_0} n_0 \varphi(\xi, 0) \mathrm{d}\xi + \int_0^T \int_{\Omega_\tau} (n \varphi_\tau + n v \varphi_\xi) \mathrm{d}\xi \mathrm{d}\tau = 0, \tag{2.9}$$

$$\int_0^T \int_{\Omega_\tau} \varphi_\xi \Phi_\xi \mathrm{d}\xi \mathrm{d}\tau + \int_0^T \int_{\Omega_\tau} \varphi(\rho - n) \mathrm{d}\xi \mathrm{d}\tau = 0, \tag{2.10}$$

and for all $\psi \in C_0^\infty\big((a(\tau), b(\tau)) \times [0, T)\big)$ that

$$\int_{\Omega_0} \rho_0 u_0 \psi(\xi, 0) \mathrm{d}\xi + \int_0^T \int_{\Omega_\tau} \Big(\rho u \psi_\tau + \rho \Phi_\xi \psi + (\rho u^2 + \rho^\gamma - \rho^\alpha u_\xi)\psi_\xi\Big) \mathrm{d}\xi \mathrm{d}\tau = 0, \tag{2.11}$$

$$\int_{\Omega_0} n_0 v_0 \psi(\xi, 0) \mathrm{d}\xi + \int_0^T \int_{\Omega_\tau} \Big(n v \psi_\tau - n \Phi_\xi \psi + (n v^2 + n^\gamma - n^\alpha v_\xi)\psi_\xi\Big) \mathrm{d}\xi \mathrm{d}\tau = 0. \tag{2.12}$$

Then, we have the following results for global weak solution.

Theorem 2.2. Assume that (A1)–(A4) hold. Then for any $T > 0$, there exists a global weak solution $(\rho, n, u, v, \Phi_\xi)$ to the FBVP for the compressible BNSP (2.1) with initial data (2.6) and boundary condition (2.3) in $\Omega_\tau \times (0, T)$ in the sense of Definition 2.1. In addition, there hold

$$0 \le \rho(\xi, \tau) \le C(T)\big((\xi - a(\tau))(b(\tau) - \xi)\big)^{\frac{\sigma}{1-\sigma}}, \tag{2.13}$$

$$0 \le n(\xi, \tau) \le C(T)\big((\xi - a(\tau))(b(\tau) - \xi)\big)^{\frac{\sigma}{1-\sigma}}. \tag{2.14}$$

In particular,

$$\rho(\xi, \tau) > 0, \quad n(\xi, \tau) > 0, \quad \xi \in (a(\tau), b(\tau)), \tag{2.15}$$

where C_T is a positive constant dependent of the time and initial data.

3. Existence of weak global solutions

The proof of Theorem 2.2 consists of the construction of approximate solution, the basic a priori estimates, and compactness arguments. We establish the a priori estimates for any solution $(\rho, n, u, v, \Phi_\xi)$ to FBVP (2.1)–(2.6) in this section.

3.1. **Some a priori estimates.**

Lemma 3.1. *Assume conditions in Theorem 2.2, and that* $(\rho, n, u, v, \Phi_\xi)$ *is any weak solution to the FBVP* (2.1)–(2.6) *for* $\tau \in [0, T]$. *Then*

$$\int_{\Omega_\tau} \big(\rho u^2 + n v^2 + \Phi_\xi^2 + \rho^\gamma + n^\gamma\big) \mathrm{d}\xi + \int_0^\tau \int_{\Omega_\tau} (\rho^\alpha u_\xi^2 + n^\alpha v_\xi^2) \mathrm{d}\xi \mathrm{d}s \le C E(0), \tag{3.1}$$

where $C > 0$ *is a constant independent of* τ, *and*

$$E(0) = \int_{\Omega_0} \big(\rho_0 u_0^2 + n_0 v_0^2 + \Phi_{\xi 0}^2 + \rho_0^\gamma + n_0^\gamma\big) \mathrm{d}\xi.$$

Proof. Multiplying $(2.1)_2$ by u and integrating it with respect to ξ over Ω_τ and using $(2.1)_1$, we have

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \int_{\Omega_\tau} \big(\frac{1}{2} \rho u^2 + \frac{1}{\gamma - 1} \rho^\gamma\big) \mathrm{d}\xi + \int_{\Omega_\tau} \rho^\alpha u_\xi^2 \mathrm{d}\xi = \int_{\Omega_\tau} \rho_\tau \Phi \mathrm{d}\xi, \tag{3.2}$$

using a similar method, we have

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \int_{\Omega_\tau} \big(\frac{1}{2} n v^2 + \frac{1}{\gamma - 1} n^\gamma\big) \mathrm{d}\xi + \int_{\Omega_\tau} n^\alpha v_\xi^2 \mathrm{d}\xi = -\int_{\Omega_\tau} n_\tau \Phi \mathrm{d}\xi.$$

We have also

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{\Omega_\tau}\Big(\frac{1}{2}(\rho u^2+nv^2)+\frac{1}{\gamma-1}(\rho^\gamma+n^\gamma)\Big)\mathrm{d}\xi+\int_{\Omega_\tau}\Big(\rho^\alpha u_\xi^2+n^\alpha v_\xi^2\Big)\mathrm{d}\xi\\
&=\int_{\Omega_\tau}(\rho-n)_\tau\Phi\mathrm{d}\xi=\int_{\Omega_\tau}\Phi_{\xi\xi\tau}\Phi\mathrm{d}\xi\\
&=-\int_{\Omega_\tau}\Phi_{\xi\tau}\Phi_\xi\mathrm{d}\xi=-\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{\Omega_\tau}\frac{1}{2}\Phi_\xi^2\mathrm{d}\xi,
\end{aligned}\tag{3.3}$$

integrating it over $[0,\tau]$, we obtain (3.1). $\square$

Lemma 3.2. *Under the same assumptions as in Lemma 3.1, there holds*

$$\rho(\xi,\tau)\le C(T),\quad n(\xi,\tau)\le C(T),\quad (\xi,\tau)\in\Omega_\tau\times[0,T],$$

where $C(T)>0$ is a constant dependent of time.

Proof. Define characteristic line $\frac{\mathrm{d}\xi(\tau)}{\mathrm{d}\tau}=u(\xi(\tau),\tau)$, then (2.1) becomes

$$\begin{gathered}
\dot\rho+\rho u_\xi=0,\\
\rho\dot u+(\rho^\gamma)_\xi=\rho\Phi_\xi+(\rho^\alpha u_\xi)_\xi,
\end{gathered}\tag{3.4}$$

where

$$\dot f(\xi(\tau),\tau)=\frac{\mathrm{d}f(\xi(\tau),\tau)}{\mathrm{d}\tau}=f_\tau+\frac{\mathrm{d}\xi(\tau)}{\mathrm{d}\tau}f_\xi=f_\tau+uf_\xi.$$

Then we have

$$\rho^\alpha u_\xi=\int_{a(\tau)}^{\xi(\tau)}\rho\dot u\mathrm{d}y+\rho^\gamma-\int_{a(\tau)}^{\xi(\tau)}\rho\Phi_y\mathrm{d}y,$$

with the help of (3.4), we have

$$-\frac{1}{\alpha}\frac{\mathrm{d}\rho^\alpha}{\mathrm{d}\tau}=\frac{\mathrm{d}\int_{a(\tau)}^{\xi(\tau)}\rho u\mathrm{d}y}{\mathrm{d}\tau}+\rho^\gamma-\int_{a(\tau)}^{\xi(\tau)}\rho\Phi_\xi\mathrm{d}y,\tag{3.5}$$

Integrating (3.5) over $[0,\tau]$ to obtain

$$\rho^\alpha+\alpha\int_0^\tau\rho^\gamma\mathrm{d}s=\rho_0^\alpha-\alpha\int_{a(\tau)}^{\xi(\tau)}\rho u\mathrm{d}y+\alpha\int_{a_0}^{\xi(0)}\rho_0u_0\mathrm{d}y+\alpha\int_0^\tau\int_{a(\tau)}^{\xi(\tau)}\rho\Phi_y\mathrm{d}y\mathrm{d}s,\tag{3.6}$$

which implies

$$\rho^\alpha\le\rho_0^\alpha-\alpha\int_{a(\tau)}^{\xi(\tau)}\rho u\mathrm{d}y+\alpha\int_{a_0}^{\xi(0)}\rho_0u_0\mathrm{d}y+\alpha\int_0^\tau\int_{a(\tau)}^{\xi(\tau)}\rho\Phi_y\mathrm{d}y\mathrm{d}s\le C(T).\tag{3.7}$$

Using the same idea, we obtain

$$n^\alpha\le C(T).\tag{3.8}$$

The combination of (3.7) and (3.8) gives rise to Lemma 3.2. $\square$

Lemma 3.3. *Under the same assumptions as in Lemma 3.1, there holds*

$$\int_{\Omega_\tau}\rho u^{2m}\mathrm{d}\xi+m(2m-1)\int_0^\tau\int_{\Omega_\tau}\rho^\alpha u^{2m-2}u_\xi^2\mathrm{d}\xi\le C(T),\quad m\in N^+,\tag{3.9}$$

$$\int_{\Omega_\tau}nv^{2m}\mathrm{d}\xi+m(2m-1)\int_0^\tau\int_{\Omega_\tau}n^\alpha v^{2m-2}v_\xi^2\mathrm{d}\xi\le C(T),\quad m\in N^+.\tag{3.10}$$

Proof. To show (3.9), we multiplying $(2.1)_2$ with $2mu^{2m-1}$ and integrating over the Ω_τ respect to ξ, using $(2.1)_1$ and (2.3), we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{\Omega_\tau}\rho u^{2m}\mathrm{d}\xi + 2m(2m-1)\int_{\Omega_\tau}\rho^\alpha u^{2m-2}u_\xi^2\mathrm{d}\xi \\
&= 2m(2m-1)\int_{\Omega_\tau}\rho^\gamma u^{2m-2}u_\xi\mathrm{d}\xi + 2m\int_{\Omega_\tau}\rho u^{2m-1}\Phi_\xi\mathrm{d}\xi \\
&\le m(2m-1)\int_{\Omega_\tau}\rho^\alpha u^{2m-2}u_\xi^2\mathrm{d}\xi + C(T)\int_{\Omega_\tau}\rho u^{2m}\mathrm{d}\xi + C(T),
\end{aligned} \tag{3.11}$$

with the help of Gronwall's inequality, we obtain

$$\int_{\Omega_\tau}\rho u^{2m}\mathrm{d}\xi \le C(T). \tag{3.12}$$

Similarly, we have

$$\int_{\Omega_\tau} nv^{2m}\mathrm{d}\xi \le C(T). \tag{3.13}$$

The combination of (3.12) and (3.13), yiels Lemma 3.3. □

To obtain the lower bound of density conveniently, we solve the FBVP (2.1) in Lagrangian coordinates, we just deal with $(2.1)_1$–$(2.1)_2$, with the same idea, we also can deal with $(2.1)_3$–$(2.1)_4$.

Let us introduce the Lagrangian coordinates transform

$$x = \int_{a(t)}^{\xi}\rho(z,\tau)\mathrm{d}z, \quad t = \tau.$$

Then the free boundaries $\xi = a(\tau)$ and $\xi = b(\tau)$ become $x = 0$ and $x = 1$ by the conservation of mass.

Hence, in the Lagrangian coordinates, the free boundary problem (2.1) becomes

$$\begin{gathered}
\rho_t + \rho^2 u_x = 0, \\
u_t + (\rho^\gamma)_x = \rho\Phi_x + (\rho^{1+\alpha}u_x)_x, \quad 0 < x < 1, \quad t > 0,
\end{gathered} \tag{3.14}$$

with the boundary conditions

$$\rho(0,t) = \rho(1,t) = 0, \tag{3.15}$$

and initial data

$$(\rho,u)(x,0) = (\rho_0(x), u_0(x)), \quad 0 \le x \le 1. \tag{3.16}$$

Note that in Lagrange coordinates the condition (A1)–(A4) of ρ_0, u_0 is equivalent to

- (B1) $0 \le \rho_0(x) \le C(x(1-x))^\sigma$ with $C > 0$ and $(\rho_0^\gamma(x))_x \in L^2([0,1])$;
- (B2) for sufficiently large positive integer m, we have $(x(1-x))^{k_1}\rho_0^{-1}(x) \in L^1([0,1])$, where $k_1 > \frac{1}{2m}$ and $(x(1-x))^{2m-1}((\rho_0^\alpha(x))_x)^{2m} \in L^1([0,1])$;
- (B3) $u_0(x) \in L^\infty([0,1])$, $(\rho_0^{\alpha+1}(x)u_{0x})_x \in L^2([0,1])$;
- (B4) $0 < \alpha < \frac{1}{3}$, $\quad \gamma > 1$.

First, making use of similar arguments as in [27] with modifications, we can establish the following Lemmas.

Lemma 3.4. *Assume* (B1)–(B4) *hold. Let* (ρ, u, Φ_x) *be any weak solution to the free boundary problem* (3.14)–(3.16) *for* $t \in [0, T]$*, then*

$$\int_0^1 \frac{1}{2}u^2 \mathrm{d}x + \int_0^1 \frac{1}{\gamma-1}\rho^{\gamma-1}\mathrm{d}x + \int_0^t \int_0^1 \rho^{1+\alpha}u_x^2 \mathrm{d}x\mathrm{d}s \le C(T), \tag{3.17}$$

$$\rho^{1+\alpha}u_x = \int_0^x u_t \mathrm{d}y + \rho^\gamma - \int_0^x \rho\Phi_y \mathrm{d}y, \tag{3.18}$$

$$\rho^\alpha + \alpha \int_0^t \rho^\gamma \mathrm{d}s = \rho_0^\alpha - \alpha \int_0^t \int_0^x u_s \mathrm{d}y\mathrm{d}s + \alpha \int_0^t \int_0^x \rho\Phi_y \mathrm{d}y\mathrm{d}s, \tag{3.19}$$

$$\int_0^1 u^{2m}\mathrm{d}x + m(2m-1)\int_0^t \int_0^1 u^{2(m-1)}\rho^{1+\alpha}u_x^2 \mathrm{d}x\mathrm{d}s \le C(T). \tag{3.20}$$

The proof of Lemma 3.4 is similar to that of Lemmas 3.1–3.3; we omit them here.

Lemma 3.5. *Under the same assumptions as Lemma 3.4, it holds*

$$\rho(x,t) \le C(T)(x(1-x))^{\sigma_0},$$

where $\sigma_0 = \min\{\sigma, \frac{2m-1}{2\alpha m}\}$.

Proof. From (3.19), we have

$$\begin{aligned}
\rho^\alpha(x,t) &\le \rho_0^\alpha(x) - \alpha\int_0^x u(x,t)\mathrm{d}y + \alpha\int_0^x u_0 \mathrm{d}y + \alpha\int_0^t\int_0^x \rho\Phi_y \mathrm{d}y\mathrm{d}s \\
&\le \rho_0^\alpha(x) + C\Big(\int_0^1 u^{2m}(x,t)\mathrm{d}x\Big)^{1/(2m)}(x(1-x))^{\frac{2m-1}{2m}} + C(T)x(1-x) \\
&\le C(T)(x(1-x))^{\sigma\alpha} + C(T)(x(1-x))^{\frac{2m-1}{2m}},
\end{aligned}$$

which implies

$$\rho(x,t) \le C(T)(x(1-x))^{\sigma} + C(T)(x(1-x))^{\frac{2m-1}{2\alpha m}}.$$

Then Lemma 3.5 follows. □

Lemma 3.6. *Under the same assumptions as Lemma 3.4, for any integer* $m > 0$*, it holds*

$$\int_0^1 (x(1-x))^{2m-1}((\rho^\alpha)_x)^{2m}\mathrm{d}x \le C(T). \tag{3.21}$$

Proof. From $(3.14)_1$, we have

$$(\rho^\alpha)_t = -\alpha\rho^{1+\alpha}u_x,$$

which by using $(3.14)_2$ implies

$$(\rho^\alpha)_{xt} = -\alpha\big(u_t + (\rho^\gamma)_x - \rho\Phi_x\big). \tag{3.22}$$

Integrating (3.22) in t over $[0, t]$, we have

$$(\rho^\alpha)_x = (\rho_0^\alpha)_x - \alpha(u - u_0) - \alpha\int_0^t (\rho^\gamma)_x \mathrm{d}s + \alpha\int_0^t \rho\Phi_x \mathrm{d}s. \tag{3.23}$$

Multiplying (3.23) by $(x(1-x))^{2m-1}((\rho^\alpha)_x)^{2m-1}$ and integrating it in x over [0,1], we have

$$\begin{aligned}
&\int_0^1 (x(1-x))^{2m-1}((\rho^\alpha)_x)^{2m}\mathrm{d}x \\
&= \int_0^1 (x(1-x))^{2m-1}((\rho^\alpha)_x)^{2m-1}(\rho_0^\alpha)_x\mathrm{d}x \\
&\quad - \alpha\int_0^1 (x(1-x))^{2m-1}((\rho^\alpha)_x)^{2m-1}(u-u_0)\mathrm{d}x \\
&\quad - \alpha\int_0^1 (x(1-x))^{2m-1}((\rho^\alpha)_x)^{2m-1}\int_0^t(\rho^\gamma)_x\mathrm{d}s\mathrm{d}x \\
&\quad + \alpha\int_0^1 (x(1-x))^{2m-1}((\rho^\alpha)_x)^{2m-1}\int_0^t\rho\Phi_x\mathrm{d}s\mathrm{d}x.
\end{aligned} \tag{3.24}$$

Using Young's inequality, we have

$$\begin{aligned}
&\int_0^1 (x(1-x))^{2m-1}((\rho^\alpha)_x)^{2m}\mathrm{d}x \\
&\le \frac{1}{2}\int_0^1 (x(1-x))^{2m-1}((\rho^\alpha)_x)^{2m}\mathrm{d}x + C\int_0^1 (x(1-x))^{2m-1}((\rho_0^\alpha)_x)^{2m}\mathrm{d}x \\
&\quad + C\int_0^1 (x(1-x))^{2m-1}u^{2m}\mathrm{d}x + C\int_0^1 (x(1-x))^{2m-1}\Big(\int_0^t(\rho^\gamma)_x\mathrm{d}s\Big)^{2m}\mathrm{d}x \\
&\quad + C\int_0^1 (x(1-x))^{2m-1}u_0^{2m}\mathrm{d}x + C\int_0^1 (x(1-x))^{2m-1}\Big(\int_0^t\rho\Phi_x\mathrm{d}s\Big)^{2m}\mathrm{d}x.
\end{aligned} \tag{3.25}$$

By using Lemma 3.4, we have

$$\begin{aligned}
&\int_0^1 (x(1-x))^{2m-1}((\rho^\alpha)_x)^{2m}\mathrm{d}x \\
&\le C(T)\int_0^1 (x(1-x))^{2m-1}\int_0^t[(\rho^\gamma)_x]^{2m}\mathrm{d}s\mathrm{d}x + C(T) \\
&\le C(T)\int_0^t \max_{[0,1]}(\rho^{\gamma-\alpha})^{2m}\int_0^1 (x(1-x))^{2m-1}[(\rho^\alpha)_x]^{2m}\mathrm{d}x\mathrm{d}s + C(T),
\end{aligned}$$

Gronwall inequality implies Lemma 3.6. □

Lemma 3.7. *Under the same assumptions as Lemma 3.4, for any $k_1 > \frac{1}{2m}$, it holds*

$$\int_0^1 \frac{(x(1-x))^{k_1}}{\rho(x,t)}\mathrm{d}x \le C(T). \tag{3.26}$$

Proof. From (3.14), we have

$$\Big(\frac{(x(1-x))^{k_1}}{\rho(x,t)}\Big)_t = (x(1-x))^{k_1}u_x(x,t). \tag{3.27}$$

Integrating (3.27) over $[0,1]\times[0,t]$ and using Young's inequality, we have

$$\int_0^1 \frac{(x(1-x))^{k_1}}{\rho(x,t)}\mathrm{d}x = \int_0^1 \frac{(x(1-x))^{k_1}}{\rho_0(x,t)}\mathrm{d}x + \int_0^t\int_0^1 (x(1-x))^{k_1}u_x\mathrm{d}x\mathrm{d}s$$

$$
\begin{aligned}
&\le \int_0^1 \frac{(x(1-x))^{k_1}}{\rho_0(x,t)}\mathrm{d}x + C\int_0^t\int_0^1 (x(1-x))^{k_1-1}|u|\mathrm{d}x\mathrm{d}s \\
&\le C + C\int_0^t\int_0^1 u^{2m}\mathrm{d}x\mathrm{d}s + C\int_0^t\int_0^1 (x(1-x))^{\frac{2m(k_1-1)}{2m-1}}\mathrm{d}x\mathrm{d}s.
\end{aligned}
$$

By using Lemma 3.4 and noticing when $k_1 > \frac{1}{2m}$; i.e., $\frac{2m(k_1-1)}{2m-1} > -1$, we have

$$
\int_0^1 \frac{(x(1-x))^{k_1}}{\rho(x,t)}\mathrm{d}x \le C(T), \tag{3.28}
$$

which proves Lemma 3.7. □

If we choose $k_1 = \frac{1}{2m-1} (> \frac{1}{2m})$ in Lemma 3.7, then we have the following result which is used to get the lower bound estimate of the density function $\rho(x,t)$.

Corollary 3.8. *The following estimate holds:*

$$
\int_0^1 \frac{(x(1-x))^{\frac{1}{2m-1}}}{\rho(x,t)}\mathrm{d}x \le C(T). \tag{3.29}
$$

The next Lemma gives an estimate on the lower bound for the density function $\rho(x,t)$.

Lemma 3.9. *Under the same assumptions as Lemma 3.4, for any $0 < \alpha < 1$, there exists a positive integer m such that $\alpha < \frac{2m-1}{2m}$. Let $k_2 \ge \frac{2m\alpha+1}{2m-1-2m\alpha}$, then the following estimate holds:*

$$
\rho(x,t) \ge C(T)(x(1-x))^{1+k_2}. \tag{3.30}
$$

Proof. Now by using Sobolev's embedding theorem $W^{1,1}[0,1] \hookrightarrow L^\infty[0,1]$ and Hölder's inequality, we have by Corollary 3.8 and Lemma 3.6

$$
\begin{aligned}
\frac{(x(1-x))^{1+k_2}}{\rho(x,t)} &\le \int_0^1 \frac{(x(1-x))^{1+k_2}}{\rho(x,t)}\mathrm{d}x + \int_0^1 \Big|\Big(\frac{(x(1-x))^{1+k_2}}{\rho(x,t)}\Big)_x\Big|\mathrm{d}x \\
&\le \max_{[0,1]}(x(1-x))^{1+k_2-\frac{1}{2m-1}} \int_0^1 \frac{(x(1-x))^{\frac{1}{2m-1}}}{\rho(x,t)}\mathrm{d}x \\
&\quad + \int_0^1 \frac{(x(1-x))^{1+k_2}\big|\rho_x(x,t)\big|}{\rho^2(x,t)}\mathrm{d}x \\
&\quad + (1+k_2)\max_{[0,1]}(x(1-x))^{k_2-\frac{1}{2m-1}} \int_0^1 \frac{(x(1-x))^{\frac{1}{2m-1}}}{\rho(x,t)}\mathrm{d}x \\
&\le C(T) + \frac{1}{\alpha}\int_0^1 \frac{(x(1-x))^{1+k_2}\big|(\rho^\alpha(x,t))_x\big|}{\rho^{1+\alpha}(x,t)}\mathrm{d}x \\
&\le C(T) + \frac{1}{\alpha}\Big(\int_0^1 (x(1-x))^{2m-1}[(\rho^\alpha)_x]^{2m}\mathrm{d}x\Big)^{1/(2m)} \\
&\quad \times \Big(\int_0^1 (x(1-x))^{(k_2+\frac{1}{2m})q}\rho^{-(1+\alpha)q}\mathrm{d}x\Big)^{1/q} \\
&\le C(T) + C(T)\Big(\int_0^1 \frac{(x(1-x))^{\frac{1}{2m-1}}}{\rho(x,t)}\mathrm{d}x\Big)^{1/q} \\
&\times \max_{[0,1]}\Big(\frac{(x(1-x))^{(k_2+\frac{1}{2m})q-\frac{1}{2m-1}}}{\rho^{(1+\alpha)q-1}}\Big)^{1/q}
\end{aligned}
$$

$$\leq C(T) + C(T)\max_{[0,1]}\Big(\frac{(x(1-x))^{1+k_2}}{\rho(x,t)}\Big)^{1+\alpha-q}(x(1-x))^{k_3}, \quad (3.31)$$

where $q = \frac{2m}{2m-1}$ and $k_3 = k_2 - (1+k_2)(\alpha + \frac{1}{2m})$.

When $k_2 \geq \frac{2m\alpha+1}{2m-1-2m\alpha}$ and m sufficiently large, we have

$$k_3 = k_2 - (1+k_2)(\alpha + \frac{1}{2m}) \geq 0.$$

This and (3.31) show that

$$\max_{[0,1]}\frac{(x(1-x))^{1+k_2}}{\rho(x,t)} \leq C(T) + C(T)\Big(\max_{[0,1]}\frac{(x(1-x))^{1+k_2}}{\rho(x,t)}\Big)^{\alpha+\frac{1}{2m}}. \quad (3.32)$$

For $0 < \alpha < 1$, there exists a positive integer m, such that $\alpha < \frac{2m-1}{2m}$; i.e., $0 < \alpha + \frac{1}{2m} < 1$. Therefore, (3.32) implies

$$\max_{[0,1]}\frac{(x(1-x))^{1+k_2}}{\rho(x,t)} \leq C(T). \quad (3.33)$$

This proves (3.30) and the proof of Lemma 3.9 is complete. □

Lemma 3.10. *Under the same assumptions as Lemma 3.4, for $0 < \alpha < \frac{1}{3}$, $k_2 < \frac{1}{2\alpha} - 1$, we have*

$$\int_0^1 u_t^2 \mathrm{d}x + \int_0^t\int_0^1 \rho^{1+\alpha}u_{xs}^2 \mathrm{d}x\mathrm{d}s \leq C(T). \quad (3.34)$$

Proof. Differentiating (3.14) with respect to time t and then integrating it after multiplying by $2u_t$ with respect to x and t over $[0,1]\times[0,t]$, we deduce

$$\begin{aligned}
&\int_0^1 u_t^2\mathrm{d}x + 2\int_0^t\int_0^1 \rho^{1+\alpha}u_{xs}^2\mathrm{d}x\mathrm{d}s \\
&= 2(1+\alpha)\int_0^t\int_0^1 \rho^{2+\alpha}u_x u_{xs}\mathrm{d}x\mathrm{d}s - 2\gamma\int_0^t\int_0^1 \rho^{1+\gamma}u_x u_{xs}\mathrm{d}x\mathrm{d}s \\
&\quad + 2\int_0^t\int_0^1 (\rho\Phi_x)_s u_s\mathrm{d}x\mathrm{d}s + \int_0^1 u_{0t}^2\mathrm{d}x.
\end{aligned} \quad (3.35)$$

From assumptions (B1) and (B2), we have

$$\int_0^1 u_{0t}^2\mathrm{d}x \leq C. \quad (3.36)$$

From Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&2(1+\alpha)\int_0^t\int_0^1 \rho^{2+\alpha}u_x u_{xs}\mathrm{d}x\mathrm{d}s \\
&\leq \frac{1}{2}\int_0^t\int_0^1 \rho^{1+\alpha}u_{xs}^2\mathrm{d}x\mathrm{d}s + 2(1+\alpha)^2\int_0^t\int_0^1 \rho^{3+\alpha}u_x^4\mathrm{d}x\mathrm{d}s,
\end{aligned} \quad (3.37)$$

and

$$\begin{aligned}
&-2\gamma\int_0^t\int_0^1 \rho^{1+\gamma}u_x u_{xs}\mathrm{d}x\mathrm{d}s \\
&\leq \frac{1}{2}\int_0^t\int_0^1 \rho^{1+\alpha}u_{xs}^2\mathrm{d}x\mathrm{d}s + 2\gamma^2\int_0^t\int_0^1 \rho^{2\gamma+1-\alpha}u_x^2\mathrm{d}x\mathrm{d}s,
\end{aligned} \quad (3.38)$$

and

$$2\int_0^t\int_0^1(\rho\Phi_x)_s u_s \mathrm{d}x\mathrm{d}s \le C(T)+\int_0^t\int_0^1 u_s^2\mathrm{d}x\mathrm{d}s. \tag{3.39}$$

Therefore,

$$\begin{aligned}
&\int_0^1 u_t^2\mathrm{d}x+\int_0^t\int_0^1\rho^{1+\alpha}u_{xs}^2\mathrm{d}x\mathrm{d}s\\
&\le C(T)+2(1+\alpha)^2\int_0^t\int_0^1\rho^{3+\alpha}u_x^4\mathrm{d}x\mathrm{d}s\\
&\quad+2\gamma^2\int_0^t\int_0^1\rho^{2\gamma+1-\alpha}u_x^2\mathrm{d}x\mathrm{d}s+\int_0^t\int_0^1 u_s^2\mathrm{d}x\mathrm{d}s\\
&=C(T)+2(1+\alpha)^2J_1+2\gamma^2J_2+\int_0^t\int_0^1 u_s^2\mathrm{d}x\mathrm{d}s.
\end{aligned}\tag{3.40}$$

Now we estimate J_1 and J_2 as follows: By Hölder's inequality, we have

$$J_1=\int_0^t\int_0^1\rho^{3+\alpha}u_x^4\mathrm{d}x\mathrm{d}s\le\int_0^t\max_{[0,1]}(\rho^2u_x^2)V(s)\mathrm{d}s, \tag{3.41}$$

where

$$V(s)=\int_0^1\rho^{1+\alpha}u_x^2\mathrm{d}x.$$

On the other hand, from (3.18), Lemma 3.5 and Lemma 3.9, we have

$$\begin{aligned}
\rho^2u_x^2&=\rho^{-2\alpha}(\rho^{1+\alpha}u_x)^2\\
&=\rho^{-2\alpha}\Big(\int_0^x u_t\mathrm{d}y+\rho^\gamma-\int_0^x\rho\Phi_y\mathrm{d}y\Big)^2\\
&\le C\rho^{-2\alpha}\Big(x(1-x)\int_0^1 u_t^2\mathrm{d}x+\rho^{2\gamma}+x(1-x)\int_0^x(\rho\Phi_y)^2\mathrm{d}y\Big)\\
&\le C(T)(x(1-x))^{1-2\alpha(1+k_2)}\int_0^1 u_t^2\mathrm{d}x\\
&\quad+C\rho^{2\gamma-2\alpha}+C(T)(x(1-x))^{1-2\alpha(1+k_2)}.
\end{aligned}\tag{3.42}$$

When $0<\alpha<\frac{1}{3}$ and $k_2\le\frac{1}{2\alpha}-1$, for sufficiently large m, we have

$$1-2\alpha(1+k_2)\ge 0,$$

which implies

$$\max_{[0,1]}\rho^2u_x^2\le C(T)\int_0^1 u_t^2\mathrm{d}x+C(T).$$

Therefore,

$$J_1\le C(T)\int_0^t V(s)\int_0^1 u_s^2\mathrm{d}x\mathrm{d}s+C(T)\int_0^t V(s)\mathrm{d}s. \tag{3.43}$$

Similarly, we have

$$J_2=\int_0^t\int_0^1\rho^{2\gamma+1-\alpha}u_x^2\mathrm{d}x\mathrm{d}s\le C(T)\int_0^t V(s)\int_0^1 u_s^2\mathrm{d}x\mathrm{d}s+C(T)\int_0^t V(s)\mathrm{d}s. \tag{3.44}$$

From (3.40), (3.43) and (3.44) and Lemma 3.4, we have

$$\int_0^1 u_t^2\mathrm{d}x+\int_0^t\int_0^1\rho^{1+\alpha}u_{xs}^2\mathrm{d}x\mathrm{d}s\le C(T)\Big(1+\int_0^t\big(1+V(s)\big)\int_0^1 u_s^2\mathrm{d}x\mathrm{d}s\Big). \tag{3.45}$$

Gronwall's inequality and Lemma 3.4 give

$$\int_0^1 u_t^2 \mathrm{d}x \le C(T)\exp\Big(C(T)\int_0^t (V(s)+1)\mathrm{d}s\Big) \le C(T). \tag{3.46}$$

Combining (3.45) with (3.46), we can get (3.34) immediately. This completes the proof of Lemma 3.10. □

Lemma 3.11. *Under the same assumptions as Lemma 3.4, we have*

$$\int_0^1 |\rho_x(x,t)|\mathrm{d}x \le C(T), \tag{3.47}$$

$$\|\rho^{1+\alpha}u_x(x,t)\|_{L^\infty([0,1]\times[0,T])} \le C(T), \tag{3.48}$$

$$\int_0^1 |(\rho^{1+\alpha}u_x)_x(x,t)|\mathrm{d}x \le C(T). \tag{3.49}$$

Proof. Since

$$\begin{gathered}\rho^{1+\alpha}u_x = \int_0^x u_t \mathrm{d}y + \rho^\gamma - \int_0^x \rho\Phi_y \mathrm{d}y,\\ (\rho^{1+\alpha}u_x)_x = u_t + (\rho^\gamma)_x - \rho\Phi_x,\end{gathered} \tag{3.50}$$

Inequalities (3.48) and (3.49) follows from Lemma 3.5 and Lemma 3.10.

On the other hand, by using Young's inequality, from Lemma 3.5 and Lemma 3.6, we have

$$\begin{aligned}&\int_0^1 |\rho_x|\mathrm{d}x\\ &= \frac{1}{\alpha}\int_0^1 |(x(1-x))^{\frac{2m-1}{2m}}(\rho^\alpha)_x|(x(1-x))^{-\frac{2m-1}{2m}}\rho^{1-\alpha}\mathrm{d}x\\ &\le \frac{1}{2m}\int_0^1 (x(1-x))^{2m-1}[(\rho^\alpha)_x]^{2m}\mathrm{d}x + \frac{2m-1}{2m}\int_0^1 (x(1-x))^{-1}\rho^{\frac{2m(1-\alpha)}{2m-1}}\mathrm{d}x\\ &\le C(T) + C\int_0^1 (x(1-x))^{-1+\frac{2m(1-\alpha)}{2m-1}\sigma}\mathrm{d}x \le C(T),\end{aligned} \tag{3.51}$$

which implies (3.47), and completes the proof. □

Lemma 3.12. *Under the same assumptions as Lemma 3.4, for* $0 < \alpha < \frac{1}{3}$ *and* $k_2 \le \frac{1}{2\alpha} - 1 - \frac{1}{(2m-1)\alpha}$*, we have*

$$\int_0^1 |u_x(x,t)|\mathrm{d}x \le C(T), \tag{3.52}$$

$$\|u(x,t)\|_{L^\infty([0,1]\times[0,T])} \le C(T). \tag{3.53}$$

Proof. From (3.18), we have

$$u_x(x,t) = \rho^{-1-\alpha}\int_0^x u_t(y,t)\mathrm{d}y + \rho^{\gamma-\alpha-1} - \rho^{-1-\alpha}\int_0^x \rho\Phi_y \mathrm{d}y. \tag{3.54}$$

By Lemma 3.10 and using Hölder's inequality, we have

$$
\begin{aligned}
&\int_0^1 |u_x(x,t)|\mathrm{d}x \\
&\le \int_0^1 \rho^{\gamma-\alpha-1}(x,t)\mathrm{d}x + \int_0^1 \rho^{-1-\alpha}(x,t)\int_0^x u_t \mathrm{d}y\mathrm{d}x \\
&\quad - \int_0^1 \rho^{-1-\alpha}\int_0^x \rho\Phi_y \mathrm{d}y\mathrm{d}x \\
&\le \int_0^1 \rho^{\gamma-\alpha-1}(x,t)\mathrm{d}x + \int_0^1 \rho^{-\alpha-1}(x,t)(x(1-x))^{1/2}\mathrm{d}x(\int_0^1 u_t^2\mathrm{d}x)^{1/2} \\
&\quad + \int_0^1 \rho^{-\alpha-1}(x,t)(x(1-x))^{1/2}\mathrm{d}x(\int_0^1 (\rho\Phi_x)^2\mathrm{d}x)^{1/2} \\
&\le \int_0^1 \rho^{\gamma-\alpha-1}(x,t)\mathrm{d}x + C(T)\int_0^1 \rho^{-\alpha-1}(x,t)(x(1-x))^{1/2}\mathrm{d}x.
\end{aligned}
\tag{3.55}
$$

The next we will prove (3.52).

Case 1: If $\gamma-\alpha-1<0$, then by Lemma 3.9 we have

$$\int_0^1 \rho^{\gamma-\alpha-1}(x,t)\mathrm{d}x \le C(T)\int_0^1 (x(1-x))^{(\gamma-\alpha-1)(1+k_2)}\mathrm{d}x.$$

Since

$$k_2 \le \frac{1}{2\alpha} - 1 - \frac{1}{(2m-1)\alpha},$$

for $\gamma>1$ we have

$$(\gamma-\alpha-1)(1+k_2) \ge \frac{\gamma-\alpha-1}{2\alpha} + \frac{\gamma-\alpha-1}{(2m-1)\alpha} > -1.$$

Therefore,

$$\int_0^1 \rho^{\gamma-\alpha-1}(x,t)\mathrm{d}x \le C(T). \tag{3.56}$$

Case 2: If $\gamma-\alpha-1\ge 0$, then (3.52) follows from Lemma 3.5.

On the other hand, by Corollary 3.8 and Lemma 3.9, we have

$$
\begin{aligned}
&\int_0^1 \rho^{-\alpha-1}(x,t)(x(1-x))^{1/2}\mathrm{d}x \\
&\le \max_{[0,1]}\{(x(1-x))^{\frac12-\frac{1}{2m-1}}\rho^{-\alpha}(x,t)\}\int_0^1 (x(1-x))^{\frac{1}{2m-1}}\rho^{-1}\mathrm{d}x \\
&\le C(T)\max_{[0,1]}\{(x(1-x))^{\frac12-\frac{1}{2m-1}}\rho^{-\alpha}(x,t)\} \\
&\le C(T)\max_{[0,1]}\{(x(1-x))^{\frac12-\frac{1}{2m-1}-\alpha(1+k_2)}\}.
\end{aligned}
\tag{3.57}
$$

When $k_2 \le \frac{1}{2\alpha}-1-\frac{1}{(2m-1)\alpha}$, we have

$$\frac12 - \frac{1}{2m-1} - \alpha(1+k_2) \ge 0.$$

Inequalities (3.55)–(3.57) show that

$$\int_0^1 |u_x(x,t)|\mathrm{d}x \le C(T). \tag{3.58}$$

On the other hand, by Sobolev's embedding theorem $W^{1,1}([0,1]) \hookrightarrow L^\infty([0,1])$ and Young's inequality, we have from (3.58) and Lemma 3.4, $|u(x,t)| \le C(T)$. This completes the proof of Lemma 3.12. $\square$

By coordinates transform, from Lemma 3.11–Lemma 3.12, we obtain

$$u_x \in L^1([a(t),b(t)] \times [0,T]) \text{ and } \rho^\alpha u_x \in L^\infty([a(t),b(t)] \times [0,T]),$$

and then from Lemma 3.10–Lemma 3.12 and Aubin's Lemma, we have

$$\rho, u \in C^0([a(t),b(t)] \times [0,T]).$$

By similar arguments, we have

$$v_x \in L^1([a(t),b(t)] \times [0,T]), \quad n^\alpha v_x \in L^\infty([a(t),b(t)] \times [0,T]),$$
$$n, v \in C^0([a(t),b(t)] \times [0,T]).$$

From $(2.1)_5$ and above regularities of ρ and n, we have

$$\Phi_x \in L^\infty([0,T], H^1[a(t),b(t)]).$$

3.2. **Proof of Theorem 2.2.** With the estimates obtained in Sections 3.1, we can apply the method in [8] and references therein, to prove the existence of weak solutions to the FBVP (2.1), we omit its proof here.

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