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ANISOTROPIC PROBLEMS WITH VARIABLE EXPONENTS AND CONSTANT DIRICHLET CONDITIONS

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Abstract. We study a general class of anisotropic problems involving $\vec{p}(\cdot)$-Laplace type operators. We search for weak solutions that are constant on the boundary by introducing a new subspace of the anisotropic Sobolev space with variable exponent and by proving that it is a reflexive Banach space. Our argumentation for the existence of weak solutions is mainly based on a variant of the mountain pass theorem of Ambrosetti and Rabinowitz.

1. Introduction

In this article, we consider $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ a rectangular-like domain; that is, a union of finitely many rectangular domains (or cubes) with edges parallel to the coordinate axes. We will analyze the existence of solutions of the nonhomogeneous anisotropic problem

$$\begin{gathered} -\sum_{i=1}^{N} \partial_{x_i} a_i(x, \partial_{x_i} u) + b(x)|u|^{p_M(x)-2}u = \lambda f(x,u), \quad \text{for } x \in \Omega \\ u(x) = \text{constant, for } x \in \partial\Omega \end{gathered} \tag{1.1}$$

where $\lambda \geq 0$ and the functions involved in this problem will be described in Section 3. We mention that the assumptions that will be imposed on functions a_i allow us to take

$$a_i(x,s) = |s|^{p_i(x)-2}s \quad \text{for all } i \in \{1, \dots, N\},$$

so that the operator

$$\sum_{i=1}^{N} \partial_{x_i} a_i\left(x, \partial_{x_i} u\right) \tag{1.2}$$

becomes in particular the $\vec{p}(\cdot)$ - Laplace operator

$$\Delta_{\vec{p}(x)}(u) = \sum_{i=1}^{N} \partial_{x_i}\Big(|\partial_{x_i} u|^{p_i(x)-2}\partial_{x_i} u\Big).$$

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This is why the operators (1.2) are often known as generalized $\vec{p}(\cdot)$ - Laplace type operators. At the same time, when choosing

$$a_i(x,s) = (1+|s|^2)^{(p_i(x)-2)/2} s \quad \text{for all } i \in \{1,\dots,N\},$$

we are led to the anisotropic mean curvature operator with variable exponent

$$\sum_{i=1}^{N} \partial_{x_i} \big[\big(1+|\partial_{x_i} u|^2\big)^{(p_i(x)-2)/2} \partial_{x_i} u \big].$$

Note that the space in which we work is a subspace of the anisotropic Sobolev space, $W^{1,\vec{p}(\cdot)}(\Omega)$, where $\vec{p}(\cdot) = \big(p_1(\cdot),\dots,p_N(\cdot)\big)$ is a vector with variable components.

The problem considered here extends [5, Theorem 4], where the discussion was conducted in the framework of the isotropic Sobolev space with variable exponent and actually goes back to [22, Theorem 3.1], where the authors worked in the classical Sobolev space. The interest in transposing the problems into new problems with variable exponents is linked to a large scale of applications that are involving some nonhomogeneous materials. It was established that for an appropriate treatment of these materials we can not rely on the classical Sobolev space and that we have to allow the exponent to vary instead. We can refer here to the electrorheological fluids or to the thermorheological fluids that have multiple applications to hydraulic valves and clutches, brakes, shock absorbers, robotics, space technology, tactile displays etc (see for example [1, 16, 19, 20, 21]). Moreover, the variable exponent spaces are involved in studies that provide other types of applications, like the ones in elastic materials [23], image restoration [6], contact mechanics [4] etc. Lately, a new development of the theory appeared due to the preoccupation for the nonhomogeneous materials that behave differently on different space directions. As a result, the anisotropic spaces with variable exponent were introduced, see [7, 10, 17].

It is not a surprise that, when passing from a variable exponent to an anisotropic variable exponent, new difficulties occur. To overpass these difficulties, we combine the classical techniques with the recent techniques that appeared when treating anisotropic problems with variable exponents. Two such problems that are related to our study were presented in [2, 3]. Nonetheless, the problem handled here is more complicated. That is because, on the one hand, we work on the anisotropic with variable exponent of the functions that are constant on the boundary (further denoted by V), instead of the anisotropic space with variable exponent of the functions that are vanishing on the boundary (later we will prove that V is a reflexive Banach space). On the other hand, we use more general hypotheses than in [2, 3] on the functions involved in (1.1). As an example, in [2, 3] it is used the critical exponent $P_{-,\infty}$, which is a constant and it is optimal when dealing with constant exponents. Here we replace it by a variable critical exponent, which is more appropriate. Other improvements are made to the assumptions on functions f and a_i and, of course, some of them generate more difficulties. However, the discussion of our results is better to be made after we present the functional framework of the variable exponent spaces and after we remind some of their properties in the next section.

2. Preliminary results

In what follows, we will recall the definition and the main properties of the spaces with variable exponents together with some results that are needed for the proof of our main results.

For $r \in C_+(\overline{\Omega})$, we introduce the Lebesgue space with variable exponent defined by

$$L^{r(\cdot)}(\Omega) = \{u : u \text{ is a measurable real-valued function, } \int_\Omega |u(x)|^{r(x)}\,dx < \infty\},$$

where

$$C_+(\overline{\Omega}) = \{r \in C(\overline{\Omega};\mathbb{R}) : \inf_{x\in\Omega} r(x) > 1\}.$$

This space, endowed with the Luxemburg norm,

$$\|u\|_{L^{r(\cdot)}(\Omega)} = \inf\{\mu > 0 : \int_\Omega |\frac{u(x)}{\mu}|^{r(x)}\,dx \le 1\},$$

is a separable and reflexive Banach space [13, Theorem 2.5, Corollary 2.7]. We also have an embedding result.

Theorem 2.1 ([13, Theorem 2.8])**.** *Assume that Ω is bounded and r_1, $r_2 \in C_+(\overline{\Omega})$ such that $r_1 \le r_2$ in Ω. Then the embedding $L^{r_2(\cdot)}(\Omega) \hookrightarrow L^{r_1(\cdot)}(\Omega)$ is continuous.*

Furthermore, the Hölder-type inequality

$$\Big|\int_\Omega u(x)v(x)\,dx\Big| \le 2\|u\|_{L^{r(\cdot)}(\Omega)}\|v\|_{L^{r'(\cdot)}(\Omega)} \tag{2.1}$$

holds for all $u \in L^{r(\cdot)}(\Omega)$ and $v \in L^{r'(\cdot)}(\Omega)$ (see [13, Theorem 2.1]), where we denoted by $L^{r'(\cdot)}(\Omega)$ the conjugate space of $L^{r(\cdot)}(\Omega)$, obtained by conjugating the exponent pointwise; that is, $1/r(x)+1/r'(x) = 1$ (see [13, Corollary 2.7]). Moreover, we denote

$$r^+ = \sup_{x\in\Omega} r(x), \quad r^- = \inf_{x\in\Omega} r(x)$$

and for $u \in L^{r(\cdot)}(\Omega)$, we have the following properties (see for example [9, Theorem 1.3, Theorem 1.4]):

$$\|u\|_{L^{r(\cdot)}(\Omega)} < 1\ (= 1; > 1) \iff \int_\Omega |u(x)|^{r(x)}\,dx < 1\ (= 1; > 1); \tag{2.2}$$

$$\|u\|_{L^{r(\cdot)}(\Omega)} > 1 \Rightarrow \|u\|^{r^-}_{L^{r(\cdot)}(\Omega)} \le \int_\Omega |u(x)|^{r(x)}\,dx \le \|u\|^{r^+}_{L^{r(\cdot)}(\Omega)}; \tag{2.3}$$

$$\|u\|_{L^{r(\cdot)}(\Omega)} < 1 \Rightarrow \|u\|^{r^+}_{L^{r(\cdot)}(\Omega)} \le \int_\Omega |u(x)|^{r(x)}\,dx \le \|u\|^{r^-}_{L^{r(\cdot)}(\Omega)}; \tag{2.4}$$

$$\|u\|_{L^{r(\cdot)}(\Omega)} \to 0\ (\to \infty) \iff \int_\Omega |u(x)|^{r(x)}\,dx \to 0\ (\to \infty). \tag{2.5}$$

To recall the definition of the isotropic Sobolev space with variable exponent, $W^{1,r(\cdot)}(\Omega)$, we set

$$W^{1,r(\cdot)}(\Omega) = \{u \in L^{r(\cdot)}(\Omega) : \partial_{x_i} u \in L^{r(\cdot)}(\Omega) \text{ for all } i \in \{1,\dots,N\}\},$$

endowed with the norm

$$\|u\|_{W^{1,r(\cdot)}(\Omega)} = \|u\|_{L^{r(\cdot)}(\Omega)} + \sum_{i=1}^N \|\partial_{x_i} u\|_{L^{r(\cdot)}(\Omega)}.$$

The space $\big(W^{1,r(\cdot)}(\Omega),\, \|\cdot\|_{W^{1,r(\cdot)}(\Omega)}\big)$ is a separable and reflexive Banach space (see [13, Theorem 1.3]).

To pass to the anisotropic spaces with variable exponent, everywhere below we consider $\vec{p}:\overline{\Omega}\to\mathbb{R}^N$ to be the vectorial function

$$\vec{p}(x)=(p_1(x),\dots,p_N(x))$$

with $p_i\in C_+(\overline{\Omega})$ for all $i\in\{1,\dots,N\}$ and we put

$$p_M(x)=\max\{p_1(x),\dots,p_N(x)\},\quad p_m(x)=\min\{p_1(x),\dots,p_N(x)\}.$$

The anisotropic space with variable exponent is

$$W^{1,\vec{p}(\cdot)}(\Omega)=\{u\in L^{p_M(\cdot)}(\Omega):\partial_{x_i}u\in L^{p_i(\cdot)}(\Omega)\text{ for all } i\in\{1,\dots,N\}\}$$

and it is endowed with the norm

$$\|u\|_{W^{1,\vec{p}(\cdot)}(\Omega)}=\|u\|_{L^{p_M(\cdot)}(\Omega)}+\sum_{i=1}^N\|\partial_{x_i}u\|_{L^{p_i(\cdot)}(\Omega)}.$$

The space $\big(W^{1,\vec{p}(\cdot)}(\Omega),\|\cdot\|_{W^{1,\vec{p}(\cdot)}(\Omega)}\big)$ is a reflexive Banach space (see [7, Theorems 2.1 and 2.2]). Furthermore, an embedding theorem takes place for all the exponents that are strictly less than a variable critical exponent, which is introduced with the help of the notations

$$\overline{p}(x)=\frac{N}{\sum_{i=1}^N 1/p_i(x)},\quad r^\star(x)=\begin{cases}Nr(x)/[N-r(x)] & \text{if } r(x)<N,\\ \infty & \text{if } r(x)\ge N.\end{cases}$$

Theorem 2.2 ([7, Theorem 2.5])**.** *Let $\Omega\subset\mathbb{R}^N$ be a rectangular-like domain and $p_i\in C_+(\overline{\Omega})$ for all $i\in\{1,\dots,N\}$. If $q\in C(\overline{\Omega};\mathbb{R})$, $1\le q(x)<\max\{\overline{p}^*(x),p_M(x)\}$ for all $x\in\overline{\Omega}$, then we have the compact embedding $W^{1,\vec{p}(\cdot)}(\Omega)\hookrightarrow L^{q(\cdot)}(\Omega)$.*

An important subspace of $W^{1,\vec{p}(\cdot)}(\Omega)$ is $W_0^{1,\vec{p}(\cdot)}(\Omega)$, that is, the subspace of the functions that are vanishing on the boundary. According to [17], the space $\big(W_0^{1,\vec{p}(\cdot)}(\Omega),\|u\|_{W_0^{1,\vec{p}(\cdot)}(\Omega)}\big)$ is a reflexive Banach space, where

$$\|u\|_{W_0^{1,\vec{p}(\cdot)}(\Omega)}=\sum_{i=1}^N\|\partial_{x_i}u\|_{L^{p_i(\cdot)}(\Omega)}.$$

We introduce a new subspace of $W^{1,\vec{p}(\cdot)}(\Omega)$, that is,

$$V=\{u\in W^{1,\vec{p}(\cdot)}(\Omega):u\big|_{\partial\Omega}\equiv\text{constant}\}.\tag{2.6}$$

As announced at the beginning of this section, we are going to find a weak solution to our problem in the space V. The main tool in finding such a solution is represented by the following Ambrosetti-Rabinowitz mountain pass theorem (see for example [11, 14, 18]).

Theorem 2.3. *Let $(X,\|\cdot\|_X)$ be a Banach space. Assume that $\Phi\in C^1(X;\mathbb{R})$ satisfies the Palais-Smale condition; that is, any sequence $(u_n)_n\subset X$ such that $\big(\Phi(u_n)\big)_n$ is bounded and $\Phi'(u_n)\to 0$ in $X^\star$ as $n\to\infty$, contains a subsequence converging to a critical point of Φ. Also, assume that Φ has a mountain pass geometry; that is,*

(i) *there exist two constants $\tau>0$ and $\rho\in\mathbb{R}$ such that $\Phi(u)\ge\rho$ if $\|u\|_X=\tau$;*
(ii) *$\Phi(0)<\rho$ and there exists $e\in X$ such that $\|e\|_X>\tau$ and $\Phi(e)<\rho$.*

Then Φ has a critical point $u_0 \in X \setminus \{0, e\}$ with critical value

$$\Phi(u_0) = \inf_{\gamma \in \mathcal{P}} \sup_{u \in \gamma} \Phi(u) \geq \rho > 0,$$

where $\mathcal{P}$ denotes the class of the paths $\gamma \in C([0,1]; X)$ joining 0 to e.

3. Main results

For presenting our main result, we have to describe the functions involved in our problem. Let us denote by $A_i : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$, $i \in \{1, \dots, N\}$, and by $F : \Omega \times \mathbb{R} \to \mathbb{R}$ the antiderivatives of the Carathéodory functions $a_i : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$, respectively $f : \Omega \times \mathbb{R} \to \mathbb{R}$; that is,

$$A_i(x,s) = \int_0^s a_i(x,t)\,dt, \quad F(x,s) = \int_0^s f(x,t)\,dt.$$

For every $i \in \{1, \dots, N\}$, we work under the following hypotheses.

- (B1) $b \in L^\infty(\Omega)$ and there exists $b_0 > 0$ such that $b(x) \geq b_0$ for all $x \in \Omega$.
- (A1) There exists a positive constant $\bar{c}_i$ such that a_i fulfills
$$|a_i(x,s)| \leq \bar{c}_i \left(d_i(x) + |s|^{p_i(x)-1} \right),$$
for all $x \in \Omega$ and all $s \in \mathbb{R}$, where $d_i \in L^{p_i'(\cdot)}(\Omega)$ (with $1/p_i(x) + 1/p_i'(x) = 1$) is a nonnegative function.
- (A2) There exists $k_i > 0$ such that
$$k_i |s|^{p_i(x)} \leq a_i(x,s)s \leq p_i(x)\ A_i(x,s),$$
for all $x \in \Omega$ and all $s \in \mathbb{R}$.
- (A3) The monotonicity condition
$$[a_i(x,s) - a_i(x,t)](s-t) > 0$$
takes place for all $x \in \Omega$ and all $s, t \in \mathbb{R}$ with $s \neq t$.
- (A4) $a_i(x,0) = 0$ for all $x \in \partial\Omega$.
- (F1) There exist $k > 0$ and $q \in C_+(\overline{\Omega})$ with $p_M^+ < q^- < q^+ < \overline{p}^*(x)$ for all $x \in \overline{\Omega}$, such that f verifies
$$|f(x,s)| \leq k\big(1 + |s|^{q(x)-1}\big)$$
for all $x \in \Omega$ and all $s \in \mathbb{R}$.
- (F2) There exist $\gamma > p_M^+$ and $s_0 > 0$ such that the Ambrosetti-Rabinowitz condition
$$0 < \gamma F(x,s) \leq s f(x,s)$$
holds for all $x \in \Omega$ and for all $s \in \mathbb{R}$ with $|s| > s_0$.
- (F3) $\lim_{|s| \to 0} \frac{f(x,s)}{|s|^{p_M^+ - 1}} = 0$ for all $x \in \Omega$.

Taking into consideration condition (A4) we can introduce the notion of weak solution to our problem.

Definition 3.1. We define the weak solution for problem (1.1) as a function $u \in V$ satisfying:

$$\int_\Omega \sum_{i=1}^N a_i(x, \partial_{x_i} u) \partial_{x_i} v\,dx + \int_\Omega b(x)|u|^{p_M(x)-2} uv\,dx - \lambda \int_\Omega f(x,u)v\,dx = 0,$$

for all $v \in V$.

The energy functional corresponding to (1.1) is defined as $I : V \to \mathbb{R}$,

$$I(u) = \int_\Omega \sum_{i=1}^N A_i(x, \partial_{x_i} u)\,dx + \int_\Omega \frac{b(x)}{p_M(x)}|u|^{p_M(x)}\,dx - \lambda \int_\Omega F(x,u)\,dx. \tag{3.1}$$

By a standard calculus one can see that functional I is well defined and of class C^1 (see for example [22, Lemma 3.4]), its Gâteaux derivative being described by

$$\langle I'(u), v\rangle = \int_\Omega \sum_{i=1}^N a_i(x, \partial_{x_i} u)\partial_{x_i} v\,dx + \int_\Omega b(x)|u|^{p_M(x)-2}uv\,dx - \lambda \int_\Omega f(x,u)v\,dx,$$

for all u, $v \in V$.

Theorem 3.2. *Let $p_i \in C_+(\overline{\Omega})$ for all $i \in \{1,\dots,N\}$ with $p_M^+ < \overline{p}^*(x)$ for all $x \in \Omega$. Assume that $b : \Omega \to \mathbb{R}$ satisfies* (B1) *and that $f : \Omega\times\mathbb{R} \to \mathbb{R}$ and $a_i : \overline{\Omega}\times\mathbb{R} \to \mathbb{R}$, $i \in \{1,\dots,N\}$, are Carathéodory functions satisfying* (F1)-(F3)*, respectively* (A1)–(A4)*. Then, problem* (1.1) *has at least one nontrivial weak solution in V for every $\lambda > 0$.*

Given the assumptions of Theorem 3.2 we can show that functional I satisfies the Palais-Smale condition and it has a mountain pass geometry, which we will accomplish by proving three lemmas. But first we need two theorems.

Theorem 3.3. *$(V, \|\cdot\|_{W^{1,\vec{p}(\cdot)}(\Omega)})$ is a reflexive Banach space.*

Proof. Our goal is to prove that V is a closed subspace of the reflexive Banach space $W^{1,\vec{p}(\cdot)}(\Omega)$ with respect to $\|\cdot\|_{W^{1,\vec{p}(\cdot)}(\Omega)}$. The idea of the proof is taken from [22, Lemma 2.1] and it is adapted to the case of anisotropic spaces with variable exponents (see also [5, Theorem 3]).

We consider a sequence $(v_n)_n \subset V$ which converges to a function $v \in W^{1,\vec{p}(\cdot)}(\Omega)$ and we will prove that $v \in V$. We note that V can be represented in a different way than it is in (2.6), that is,

$$V = \{u + c : u \in W_0^{1,\vec{p}(\cdot)}(\Omega),\ c \in \mathbb{R}\}.$$

As a consequence, there exist $(u_n)_n \in W_0^{1,\vec{p}(\cdot)}(\Omega)$ and $(c_n)_n \subset \mathbb{R}$ such that, for all $n \in \mathbb{N}$, $v_n = u_n + c_n$. We have

$$\begin{aligned}
&\|u_n - u_m\|_{W_0^{1,\vec{p}(\cdot)}(\Omega)} \\
&\le \sum_{i=1}^N \|\partial_{x_i}(u_n - u_m - c_n + c_m)\|_{L^{p_i(\cdot)}(\Omega)} + \|u_n - u_m - c_n + c_m\|_{L^{p_M(\cdot)}(\Omega)} \\
&= \|v_n - v_m\|_{W^{1,\vec{p}(\cdot)}(\Omega)}.
\end{aligned}$$

Keeping in mind that $(v_n)_n$ is a Cauchy sequence in $\big(W^{1,\vec{p}(\cdot)}(\Omega), \|\cdot\|_{W^{1,\vec{p}(\cdot)}(\Omega)}\big)$, the previous relation implies that $(u_n)_n$ is a Cauchy sequence in the Banach space $\Big(W_0^{1,\vec{p}(\cdot)}(\Omega), \|\cdot\|_{W_0^{1,\vec{p}(\cdot)}(\Omega)}\Big)$. Hence

$$(u_n)_n \text{ converges to a function } \tilde{u} \in W_0^{1,\vec{p}(\cdot)}(\Omega). \tag{3.2}$$

At the same time, by the Poincaré inequality, there exists a positive constant m_1 such that

$$\|c_n - c_m\|_{L^{p_m^-}(\Omega)} \le \|u_n - c_n - u_m + c_m\|_{L^{p_m^-}(\Omega)} + m_1 \sum_{i=1}^N \|\partial_{x_i}(u_m - u_n)\|_{L^{p_m^-}(\Omega)}.$$

Then, by Theorem 2.1,

$$\|c_n - c_m\|_{L^1(\Omega)} \le m_2\|v_n - v_m\|_{L^{p_M(\cdot)}(\Omega)} + m_3 \sum_{i=1}^{N} \|\partial_{x_i}(v_n - v_m)\|_{L^{p_i(\cdot)}(\Omega)},$$

where m_2, m_3 are positive constants. The sequence $(v_n)_n$ being Cauchy in the space $\big(W^{1,\vec{p}(\cdot)}(\Omega), \|\cdot\|_{W^{1,\vec{p}(\cdot)}(\Omega)}\big)$ and Ω being bounded, it follows from the above that $(c_n)_n$ is a Cauchy sequence in $(\mathbb{R}, |\cdot|)$, thus

$$(c_n)_n \text{ converges to a number } \tilde{c} \in \mathbb{R}. \tag{3.3}$$

Using (3.2) and (3.3), the uniqueness of the limit yields that $v = \tilde{u} + \tilde{c}$. Therefore, $v \in V$ and the proof is complete. ☐

We introduce the second useful theorem.

Theorem 3.4. *Let $\Omega \subset \mathbb{R}^N$, ($N \ge 2$) be a rectangular-like domain. Assume that $a_i : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$, $i \in \{1, \dots, N\}$, are Carathéodory functions satisfying* (A3). *If $u_n \rightharpoonup n$ (weakly) in $W^{1,\vec{p}(\cdot)}(\Omega)$ and*

$$\limsup_{n\to\infty} \int_\Omega \sum_{i=1}^{N} a_i(x, \partial_{x_i} u)(\partial_{x_i} u_n - \partial_{x_i} u)dx \le 0,$$

then $u_n \to u$ (strongly) in $W^{1,\vec{p}(\cdot)}(\Omega)$.

Proof. The same property was proved in the framework of the space $W_0^{1,\vec{p}(\cdot)}(\Omega)$ by applying Vitali Theorem to obtain

$$\limsup_{n\to\infty} \int_\Omega \sum_{i=1}^{N} |\partial_{x_i} u_n - \partial_{x_i} u|^{p_i(x)} dx = 0, \tag{3.4}$$

see [2, Lemma 2, relation (11)]. In our case, in order to complete the proof, we use Theorem 2.2 to establish that $W^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow L^{p_M(\cdot)}(\Omega)$ compactly. Since $u_n \rightharpoonup u$ in $W^{1,\vec{p}(\cdot)}(\Omega)$, we deduce that

$$u_n \to u \quad \text{in } L^{p_M(\cdot)}(\Omega). \tag{3.5}$$

Then, by (2.5), (3.4) and (3.5) we conclude that $u_n \to u$ in $W^{1,\vec{p}(\cdot)}(\Omega)$. ☐

Remark 3.5. In [2, Lemma 2], the author considers Ω to be a bounded domain with smooth boundary, but this does not change the proof of relation (3.4) in the situation when Ω is a rectangular-like domain. However, in the case when Ω is a bounded domain with smooth boundary, [2, Lemma 2] could not be extended to $W^{1,\vec{p}(\cdot)}(\Omega)$ due to the lack of a compactness embedding between $W^{1,\vec{p}(\cdot)}(\Omega)$ and $L^{p_M(\cdot)}(\Omega)$.

Now we can proceed with our first lemma. Everywhere below we work under the hypotheses of Theorem 3.2.

Lemma 3.6. *The energy functional I introduced by* (3.1) *satisfies the Palais-Smale condition.*

Proof. Let $\beta \in \mathbb{R}$ and $(u_n)_n \subset V$ be such that

$$|I(u_n)| < \beta, \quad I'(u_n) \to 0 \quad \text{in } V^\star \text{ as } n \to \infty. \tag{3.6}$$

Our goal is to show that $(u_n)_n$ is strongly convergent in V. The first step is to show that $(u_n)_n$ is bounded. To this end, we assume by contradiction that, passing eventually to a subsequence still denoted by $(u_n)_n$, we have

$$\|u_n\|_{W^{1,\vec{p}(\cdot)}(\Omega)} \to \infty \quad \text{as } n \to \infty.$$

Using relation (3.6) and assumptions (B1), (A2), for n large enough we infer

$$\begin{aligned}
1+\beta+\|u_n\|_{W^{1,\vec{p}(\cdot)}(\Omega)} &\geq I(u_n) - \frac{1}{\gamma}\langle I'(u_n), u_n\rangle \\
&\geq \sum_{i=1}^N \int_\Omega \big(\frac{1}{p_i(x)} - \frac{1}{\gamma}\big) a_i(x, \partial_{x_i} u_n)\partial_{x_i} u_n \, dx \\
&\quad + b_0\big(\frac{1}{p_M^+} - \frac{1}{\gamma}\big) \int_\Omega |u_n|^{p_M(x)}\, dx \\
&\quad - \lambda \int_{\{x\in\Omega: |u_n(x)|>s_0\}} \big[F(x,u_n) - \frac{1}{\gamma} f(x,u_n)u_n\big]\, dx \\
&\quad - \lambda|\Omega| \sup\{|F(x,t) - \frac{1}{\gamma} f(x,t)t| : x\in\Omega, |t|\leq s_0\},
\end{aligned}$$

where γ and s_0 are the constants from (F2). Using (A2) and (F2) we deduce that, for n large enough,

$$\begin{aligned}
1+\beta+\|u_n\|_{W^{1,\vec{p}(\cdot)}(\Omega)} &\geq \big(\frac{1}{p_M^+} - \frac{1}{\gamma}\big) \min\{k_i : i\in\{1,\dots,N\}\} \sum_{i=1}^N \int_\Omega |\partial_{x_i} u_n|^{p_i(x)}\, dx \\
&\quad + b_0\big(\frac{1}{p_M^+} - \frac{1}{\gamma}\big) \int_\Omega |u_n|^{p_M(x)}\, dx - C_1,
\end{aligned} \tag{3.7}$$

where $C_1 = \lambda|\Omega|\sup\{|F(x,t) - \frac{1}{\gamma} f(x,t)t| : x\in\Omega, |t|\leq s_0\} > 0$. We denote

$$\begin{aligned}
\mathcal{I}_1 &= \{i\in\{1,\dots,N\} : \|\partial_{x_i} u_n\|_{L^{p_i(\cdot)}(\Omega)} \leq 1\}, \\
\mathcal{I}_2 &= \{i\in\{1,\dots,N\} : \|\partial_{x_i} u_n\|_{L^{p_i(\cdot)}(\Omega)} > 1\}.
\end{aligned}$$

Then, by (2.2), (2.3) and (2.4),

$$\begin{aligned}
\sum_{i=1}^N \int_\Omega |\partial_{x_i} u_n|^{p_i(x)} dx &= \sum_{i\in\mathcal{I}_1} \int_\Omega |\partial_{x_i} u_n|^{p_i(x)} dx + \sum_{i\in\mathcal{I}_2} \int_\Omega |\partial_{x_i} u_n|^{p_i(x)}\, dx \\
&\geq \sum_{i\in\mathcal{I}_1} \|\partial_{x_i} u_n\|_{L^{p_i(\cdot)}}^{p_M^+} + \sum_{i\in\mathcal{I}_2} \|\partial_{x_i} u_n\|_{L^{p_i(\cdot)}}^{p_m^-} \\
&\geq \sum_{i=1}^N \|\partial_{x_i} u_n\|_{L^{p_i(\cdot)}}^{p_m^-} - \sum_{i\in\mathcal{I}_1} \|\partial_{x_i} u_n\|_{L^{p_i(\cdot)}}^{p_m^-}.
\end{aligned}$$

Thus,

$$\sum_{i=1}^N \int_\Omega |\partial_{x_i} u_n|^{p_i(x)} dx \geq \sum_{i=1}^N \|\partial_{x_i} u_n\|_{L^{p_i(\cdot)}}^{p_m^-} - N. \tag{3.8}$$

On the other hand, we analyze the two cases corresponding to the values of $\|u_n\|_{L^{p_M(\cdot)}(\Omega)}$. By (2.3),

$$\int_\Omega |u_n|^{p_M(x)}\, dx \geq \|u_n\|_{L^{p_M(\cdot)}(\Omega)}^{p_m^-}, \quad \text{when } \|u_n\|_{L^{p_M(\cdot)}(\Omega)} > 1. \tag{3.9}$$

In addition,

$$\int_\Omega |u_n|^{p_M(x)}\,dx \geq \|u_n\|^{p_m^-}_{L^{p_M(\cdot)}(\Omega)} - 1, \quad \text{when } \|u_n\|_{L^{p_M(\cdot)}(\Omega)} \leq 1. \tag{3.10}$$

No matter if $\|u_n\|_{L^{p_M(\cdot)}(\Omega)}$ is subunitary or superunitary, by (3.7), (3.8), (3.9) and (3.10) we deduce that there exists a positive constant C_2 such that

$$\begin{aligned}
&1+\beta+\|u_n\|_{W^{1,\vec{p}(\cdot)}(\Omega)}\\
&\geq \big(\frac{1}{p_M^+}-\frac{1}{\gamma}\big)\min\{b_0,k_i : i\in\{1,\dots,N\}\}\Big(\sum_{i=1}^N \|\partial_{x_i}u_n\|^{p_m^-}_{L^{p_i(\cdot)}(\Omega)}\,dx + \|u_n\|^{p_m^-}_{L^{p_M(\cdot)}(\Omega)}\Big)\\
&\quad - C_2.
\end{aligned}$$

Due to the fact that

$$\begin{aligned}
&\Big(\sum_{i=1}^N \|\partial_{x_i}u_n\|_{L^{p_i(\cdot)}(\Omega)} + \|u_n\|_{L^{p_M(\cdot)}(\Omega)}\Big)^{p_m^-}\\
&\leq (N+1)^{p_m^-}\Big(\max\{\|u_n\|_{L^{p_M(\cdot)}(\Omega)}, \|\partial_{x_i}u_n\|_{L^{p_i(\cdot)}(\Omega)} : i\in\{1,\dots,N\}\}\Big)^{p_m^-},
\end{aligned}$$

there exist two positive constants C_3 and C_4 such that

$$1+\beta+\|u_n\|_{W^{1,\vec{p}(\cdot)}(\Omega)} \geq C_3\|u_n\|^{p_m^-}_{W^{1,\vec{p}(\cdot)}(\Omega)} - C_4.$$

Then, by dividing the previous inequality by $\|u_n\|_{W^{1,\vec{p}(\cdot)}(\Omega)}$ we obtain a contradiction when n goes to ∞. Consequently, $(u_n)_n$ is bounded in $W^{1,\vec{p}(\cdot)}(\Omega)$. Also, $W^{1,\vec{p}(\cdot)}(\Omega)$ is a reflexive space, so this implies that there exists a subsequence, still denoted by $(u_n)_n$ and $u\in W^{1,\vec{p}(\cdot)}(\Omega)$ such that

$$u_n \rightharpoonup u \quad \text{weakly in } W^{1,\vec{p}(\cdot)}(\Omega). \tag{3.11}$$

By Theorem 2.2, we know that $W^{1,\vec{p}(\cdot)}(\Omega)$ is compactly embedded in $L^1(\Omega)$, $L^{q(\cdot)}(\Omega)$ and $L^{p_M(\cdot)}(\Omega)$, where q is given in (F1). Therefore, since $u_n \rightharpoonup u$ in the Banach space $W^{1,\vec{p}(\cdot)}(\Omega)$, we infer that

$$u_n \to u \quad \text{in } L^1(\Omega),\ L^{q(\cdot)}(\Omega), \text{ respectively } L^{p_M(\cdot)}(\Omega). \tag{3.12}$$

Using (3.6) and (3.11) and the fact that

$$|\langle I'(u_n), u_n-u\rangle| \leq \|I'(u_n)\|_{V^*}\,\|u_n-u\|_{W^{1,\vec{p}(\cdot)}(\Omega)},$$

we obtain

$$\lim_{n\to\infty}|\langle I'(u_n), u_n-u\rangle| = 0.$$

The previous relation can be rewritten as

$$\begin{aligned}
&\lim_{n\to\infty}\int_\Omega\Big[\sum_{i=1}^N a_i(x,\partial_{x_i}u_n)\big(\partial_{x_i}u_n-\partial_{x_i}u\big) + b(x)|u_n|^{p_M(x)-2}u_n(u_n-u)\\
&-\lambda f(x,u_n)(u_n-u)\Big]\,dx = 0.
\end{aligned} \tag{3.13}$$

Applying (B1) and (2.1), we find that

$$\begin{aligned}
&|\int_\Omega b(x)|u_n|^{p_M(x)-2}u_n(u_n-u)\,dx|\\
&\leq 2\|b\|_{L^\infty(\Omega)}\||u_n|^{p_M(x)-1}\|_{L^{p'_M(\cdot)}(\Omega)}\|u_n-u\|_{L^{p_M(\cdot)}(\Omega)}.
\end{aligned} \tag{3.14}$$

Suppose by contradiction that $\||u_n|^{p_M(x)-1}\|_{L^{p'_M(\cdot)}(\Omega)} \to \infty$. So, by relation (2.5),

$$\int_\Omega \left(|u_n|^{p_M(x)-1}\right)^{p'_M(x)} dx \to \infty \Leftrightarrow \int_\Omega \left(|u_n|^{p_M(x)}\right) dx \to \infty \Leftrightarrow \|u_n\|_{L^{p_M(\cdot)}} \to \infty.$$

But $\|u_n\|_{L^{p_M(\cdot)}(\Omega)} \to \|u\|_{L^{p_M(\cdot)}(\Omega)}$, thus we have obtained a contradiction. Consequently, by (3.14) and (3.12),

$$\lim_{n\to\infty} \int_\Omega b(x)|u_n|^{p_M(x)-2}u_n(u_n-u)\,dx = 0. \tag{3.15}$$

At the same time, by (F1) and (2.1), we arrive at

$$\begin{aligned} &|\int_\Omega f(x,u_n)(u_n-u)\,dx| \\ &\le \int_\Omega |f(x,u_n)||u_n-u|\,dx \\ &\le k\int_\Omega |u_n-u|\,dx + k\int_\Omega |u_n|^{q(x)-1}|u_n-u|\,dx \\ &\le k\|u_n-u\|_{L^1(\Omega)} + 2k\||u_n|^{q(x)-1}\|_{L^{q'(\cdot)}(\Omega)}\|u_n-u\|_{L^{q(\cdot)}(\Omega)}. \end{aligned}$$

By (3.12) and (2.5), we conclude as above that

$$\lim_{n\to\infty} \int_\Omega f(x,u_n)(u_n-u)\,dx = 0. \tag{3.16}$$

Combining (3.13), (3.15) and (3.16), we obtain

$$\lim_{n\to\infty} \int_\Omega \sum_{i=1}^N a_i(x,\partial_{x_i}u_n)\,(\partial_{x_i}u_n - \partial_{x_i}u)\,dx = 0. \tag{3.17}$$

Relations (3.11) and (3.17) and Theorem 3.4 give us

$$u_n \to u \quad \text{strongly in } W^{1,\vec{p}(\cdot)}(\Omega).$$

Since V is a closed subspace of $W^{1,\vec{p}(\cdot)}(\Omega)$ and $(u_n)_n \subset V$ we obtain that $u \in V$, therefore the proof of Lemma 1 is complete. □

After the Palais-Smale condition, we are concerned with the mountain pass geometry of functional I. The other two lemmas take care of this matter.

Lemma 3.7. *There exist τ, $\rho > 0$ such that $I(u) \ge \rho$ for all $u \in W^{1,\vec{p}(\cdot)}(\Omega)$ with $\|u\|_{W^{1,\vec{p}(\cdot)}(\Omega)} = \tau$.*

Proof. By (A2) and (B1), we infer that

$$\begin{aligned} I(u) \ge{}& \frac{\min\{k_i : i \in \{1,\dots,N\}\}}{p_M^+} \int_\Omega \sum_{i=1}^N |\partial_{x_i}u|^{p_i(x)}\,dx + \frac{b_0}{p_M^+}\int_\Omega |u|^{p_M(x)}\,dx \\ &- \lambda\int_\Omega F(x,u)\,dx. \end{aligned}$$

Choosing $\tau < 1$ we have $\|u\|_{L^{p_M(\cdot)}(\Omega)} < 1$ and $\|\partial_{x_i} u\|_{L^{p_i(\cdot)}(\Omega)} < 1$. Using relation (2.4) in the above inequality,

$$\begin{aligned} I(u) &\geq \frac{\min\{k_i : i \in \{1,\dots,N\}\}}{p_M^+} \sum_{i=1}^N \|\partial_{x_i} u\|_{L^{p_i(\cdot)}(\Omega)}^{p_M^+} + \frac{b_0}{p_M^+} \|u\|_{L^{p_M(\cdot)}(\Omega)}^{p_M^+} \\ &\quad - \lambda \int_\Omega F(x,u)\,dx \\ &\geq \frac{\min\{b_0, k_i : i \in \{1,\dots,N\}\}}{(N+1)^{p_m^+} p_M^+} \|u\|_{W^{1,\vec{p}(\cdot)}(\Omega)}^{p_M^+(x)} - \lambda \int_\Omega F(x,u)\,dx, \end{aligned} \tag{3.18}$$

for all $u \in W^{1,\vec{p}(\cdot)}(\Omega)$ with $\|u\|_{W^{1,\vec{p}(\cdot)}(\Omega)} = \tau < 1$. Let us now deal with the last term of this inequality by keeping in mind that the continuous embedding from Theorem 2.2 generates the existence of two constants α_1, $\alpha_2 > 0$ such that

$$\|u\|_{L^{p_M^+}(\Omega)} \leq \alpha_1 \|u\|_{W^{1,\vec{p}(\cdot)}(\Omega)}, \quad \|u\|_{L^{q^+}(\Omega)}, \|u\|_{L^{q^-}(\Omega)} \leq \alpha_2 \|u\|_{W^{1,\vec{p}(\cdot)}(\Omega)} \tag{3.19}$$

for all $u \in V$. From (F1), we know that

$$F(x,s) \leq k\big(|s| + \frac{|s|^{q(x)}}{q(x)}\big)$$

for all $x \in \Omega$ and all $s \in \mathbb{R}$. Hence

$$F(x,s) \leq 2k|s|^{q(x)} \quad \text{for all } x \in \Omega \text{ and all } s \in \mathbb{R} \text{ with } |s| > 1.$$

Let us take $\varepsilon = \frac{\min\{b_0, k_i : i \in \{1,\dots,N\}\}}{2(N+1)^{p_m^+} \alpha_1 \lambda}$. By (F3), there exists $\delta > 0$ such that

$$|f(x,s)| \leq \varepsilon |s|^{p_M^+ - 1} \quad \text{for all } x \in \Omega \text{ and all } s \in \mathbb{R} \text{ with } |s| < \delta.$$

By the previous two inequalities we deduce that

$$\int_\Omega F(x,u)\,dx \leq \frac{\varepsilon}{p_M^+} \int_\Omega |u|^{p_M^+}\,dx + \alpha_3 \int_\Omega |u|^{q(x)}\,dx \quad \text{for all } u \in V,$$

where α_3 is a positive constant. Using relations (2.2), (2.3) and (2.4),

$$\int_\Omega F(x,u)\,dx \leq \frac{\varepsilon}{p_M^+} \|u\|_{L^{p_M^+}}^{p_M^+} + \alpha_3 \Big[\|u\|_{L^{q(\cdot)}(\Omega)}^{q^+} + \|u\|_{L^{q(\cdot)}(\Omega)}^{q^-}\Big] \quad \text{for all } u \in V.$$

From this and (3.19), there exists $\alpha_4 > 0$ such that

$$\int_\Omega F(x,u)\,dx \leq \frac{\varepsilon \alpha_1}{p_M^+} \|u\|_{W^{1,\vec{p}(\cdot)}(\Omega)}^{p_M^+} + \alpha_4 \|u\|_{W^{1,\vec{p}(\cdot)}(\Omega)}^{q^-}, \tag{3.20}$$

for all $u \in V$ with $\|u\|_{W^{1,\vec{p}(\cdot)}(\Omega)} = \tau < 1$. Putting together (3.18) and (3.20) we come to

$$I(u) \geq \frac{\min\{b_0, k_i : i \in \{1,\dots,N\}\}}{2(N+1)^{p_m^+} p_M^+} \|u\|_{W^{1,\vec{p}(\cdot)}(\Omega)}^{p_M^+} - \alpha_4 \lambda \|u\|_{W^{1,\vec{p}(\cdot)}(\Omega)}^{q^-}$$

for all $u \in V$ with $\|u\|_{W^{1,\vec{p}(\cdot)}(\Omega)} = \tau < 1$. We have assumed that $1 < p_M^+ < q^-$, thus it is clear that for τ sufficiently small we can choose $\rho > 0$ such that $I(u) \geq \rho$ for all $u \in V$ with $\|u\|_{W^{1,\vec{p}(\cdot)}(\Omega)} = \tau$. $\square$

Finally, we prove the last lemma.

Lemma 3.8. *There exists $e \in V$ with $\|e\|_{W^{1,\vec{p}(\cdot)}(\Omega)} > \tau$ (τ given in Lemma 2) such that $I(e) < 0$.*

Proof. By (F2), there exists $\tilde{\alpha} = \tilde{\alpha}(x) > 0$ such that

$$F(x,s) \geq \tilde{\alpha}(x)|s|^{\gamma} \quad \text{for all } s \in \mathbb{R} \text{ with } |s| > s_0 \text{ and all } x \in \Omega. \tag{3.21}$$

Then, due to (A1), (3.21) and the Hölder-type inequality (2.1), for any $t > 1$ we have

$$\begin{aligned}
I(tu) &\leq t \sum_{i=1}^{N} \int_{\Omega} \bar{c}_i |d_i(x)||\partial_{x_i} u| \, dx + t^{p_M^+} \sum_{i=1}^{N} \int_{\Omega} \bar{c}_i \frac{|\partial_{x_i} u|^{p_i(x)}}{p_i(x)} \, dx \\
&\quad + t^{p_M^+} \int_{\Omega} \frac{b(x)}{p_M(x)} |u|^{p_M(x)} \, dx - \lambda t^{\gamma} \int_{\{x \in \Omega : |u(x)| > s_0\}} \tilde{\alpha}(x)|u|^{\gamma} \, dx \\
&\quad - \lambda |\Omega| \inf\{F(x,s) : x \in \Omega, \, |s| \leq s_0\} \\
&\leq 2t \max\{\bar{c}_i : i \in \{1, \dots, N\}\} \sum_{i=1}^{N} \|d_i\|_{L^{p_i'(\cdot)}(\Omega)} \|\partial_{x_i} u\|_{L^{p_i(\cdot)}(\Omega)} \\
&\quad + t^{p_M^+} \frac{\max\{\bar{c}_i : i \in \{1, \dots, N\}\}}{p_m^-} \sum_{i=1}^{N} \int_{\Omega} |\partial_{x_i} u|^{p_i(x)} \, dx \\
&\quad + t^{p_M^+} \frac{\|b\|_{L^{\infty}(\Omega)}}{p_M^-} \int_{\Omega} |u|^{p_M(x)} \, dx - \lambda t^{\gamma} \int_{\{x \in \Omega : |u(x)| > s_0\}} \tilde{\alpha}(x)|u|^{\gamma} \, dx \\
&\quad - \lambda |\Omega| \inf\{F(x,s) : x \in \Omega, \, |s| \leq s_0\}.
\end{aligned}$$

Since $\gamma > p_M^+ > 1$, for t sufficiently large, we can find $e \in V$ such that $\|e\|_{W^{1,\vec{p}(\cdot)}(\Omega)} > \tau$ and $I(e) < 0$. $\square$

Taking into account Theorem 2.3, one can easily see that Lemmas 3.6-3.8 are sufficient to conclude that Theorem 3.2 holds, therefore our work is complete.

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References

[1] S. N. Antontsev, J. F. Rodrigues; On stationary thermorheological viscous flows, *Ann. Univ. Ferrara Sez. VII Sci. Mat.*, **52** (2006), 19–36.

[2] M.-M. Boureanu; Infinitely many solutions for a class of degenerate anisotropic elliptic problems with variable exponent, *Taiwanese Journal of Mathematics*, **5** (2011), 2291–2310.

[3] M.-M Boureanu; Critical point methods in degenerate anisotropic problems with variable exponent *Studia Univ. "Babeş Bolyai", Mathematica*, **LV** (2010) 27–39.

[4] M.-M. Boureanu, A. Matei and M. Sofonea, Nonlinear problems with $p(\cdot)$-growth conditions and applications to antiplane contact models, *Advanced Nonlinear Studies*, accepted.

[5] M.-M Boureanu, D.-N.Udrea; Existence and multiplicity result for elliptic problems with $p(\cdot)$-Growth conditions, *Nonlinear Anal.: Real World Applications*, **14** (2013) 1829–1844.

[6] Y. Chen, S. Levine, R. Rao; Variable Exponent, Linear Growth Functionals in Image Restoration, *SIAM Journal of Applied Mathematics*, **66** (2006), 1383–1406.

[7] X. Fan; Anisotropic variable exponent Sobolev spaces and $\vec{p}(\cdot)$–Laplacian equations, *Complex Variables and Elliptic Equations*, **55** (2010), 1–20.

[8] X. Fan; Solutions for $p(x)$–Laplacian Dirichlet problems with singular coefficients, *J. Math. Anal. Appl.*, **312** (2005), 464–477.

[9] X. Fan, D. Zhao; On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$, *J. Math. Anal. Appl.*, **263** (2001), 424–446.

[10] I. Fragalà, F. Gazzola, B. Kawohl; Existence and nonexistence results for anisotropic quasilinear equations, *Ann. Inst. H. Poincaré, Analyse Non Linéaire*, **21** (2004), 715–734.

[11] Y. Jabri; *The Mountain Pass Theorem. Variants, Generalizations and some Applications*, Cambridge University Press, 2003.

[12] B. Kone, S. Ouaro, S. Traore; Weak solutions for anisotropic nonlinear elliptic equations with variable exponents, *Electronic Journal of Differential Equations*, **2009** (2009), No. 144, 1–11.
[13] O. Kováčik, J. Rákosník; On spaces $L^{p(x)}$ and $W^{k,p(x)}$, *Czechoslovak Math. J.*, **41** (1991), 592–618.
[14] A. Kristály, V. Rădulescu, C. Varga; Variational Principles in Mathematical Physics, Geometry, and Economics: Qualitative Analysis of Nonlinear Equations and Unilateral Problems, Encyclopedia of Mathematics and its Applications, **136**, Cambridge University Press, Cambridge, 2010.
[15] A. J. Kurdila, M. Zabarankin; *Convex Functional Analysis*, Birkhäuser Verlag, 2005.
[16] Y. Liu, R. Davidson, P. Taylor; Investigation of the touch sensitivity of ER fluid based tactile display, *Proceedings of SPIE, Smart Structures and Materials: Smart Structures and Integrated Systems*, **5764** (2005), 92–99.
[17] M. Mihăilescu, P. Pucci, V. Rădulescu; Eigenvalue problems for anisotropic quasilinear elliptic equations with variable exponent, *J. Math. Anal. Appl.*, **340** (2008), 687–698.
[18] P. Pucci, V. Rădulescu; The impact of the mountain pass theory in nonlinear analysis: a mathematical survey, *Boll. Unione Mat. Ital.* Series IX, **3** (2010), 543–584.
[19] M. Růžička; *Electrorheological Fluids: Modeling and Mathematical Theory*, Springer–Verlag, Berlin, 2002.
[20] A. J. Simmonds; Electro-rheological valves in a hydraulic circuit, IEE Proceedings-D **138** (1991) 400–404.
[21] R. Stanway, J. L. Sproston, A. K. El-Wahed; Applications of electrorheological fluids in vibration control: a survey, Smart Mater. Struct. **5** (1996) 464–482.
[22] L. Zhao, P. Zhao, X. Xie; Existence and multiplicity of solutions for divergence type elliptic equations, Electronic Journal of Differential Equations **2011** (2011) no. 43, 1–9.
[23] V. V. Zhikov; Averaging of functionals in the calculus of variations and elasticity, Math. USSR Izv. **29** (1987) 33–66.

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In what follows we correct an error that occurs in our paper. Thus, to our problem we add the condition

$$\int_{\partial\Omega} \sum_{i=1}^{N} \partial_{x_i} a_i(x, \partial_{x_i} u)\nu_i \, dS = 0,$$

where ν_i, $i \in \{1, \dots, N\}$, represent the components of the unit outer normal vector. This means that the problem under consideration becomes

$$-\sum_{i=1}^{N} \partial_{x_i} a_i(x, \partial_{x_i} u) + b(x)|u|^{p_M(x)-2}u = \lambda f(x, u), \quad \text{for } x \in \Omega$$
$$u(x) \equiv \text{constant, for } x \in \partial\Omega$$
$$\int_{\partial\Omega} \sum_{i=1}^{N} \partial_{x_i} a_i(x, \partial_{x_i} u)\nu_i \, dS = 0.$$

This is a “no-flux” type of problem. In addition, we remove hypothesis (A4) because it is not needed. We mention that the rest of the paper will not suffer alterations and we point out that various problems with ”no-flux” boundary conditions received a lot of interest lately, see for example [1, 2, 3, 4, 5, 6] below.

References

[1] X. Fan, S.-G. Deng; *Remarks on Ricceri's variational principle and applications to the $p(x)$-Laplacian equations*, Nonlinear Analysis TMA 67 (2007) 3064–3075.
[2] V. K. Le; *On a sub-supersolution method for variational inequalities with Leray-Lions operators in variable exponent spaces*, Nonlinear Anal. TMA 71 (2009) 3305–3321.

[3] V. K. Le, K. Schmitt; *Sub-supersolution theorems for quasilinear elliptic problems: a variational approach* Electronic Journal of Differential Equations 2004 (2004) no. 118, 1–7.
[4] L. H. Nguyen, K. Schmitt; *Nonlinear elliptic Dirichlet and no-flux boundary value problems*, Ann. Univ. Buchar., Math. Ser. 3(61), No. 2 (2012) 201–217.
[5] Q. Zhang, Y. Guo, G. Chen; *Existence and multiple solutions for a variable exponent system*, Nonlinear Analysis TMA 73 (2010) 3788–3804.
[6] L. Zhao, P. Zhao, X. Xie; *Existence and multiplicity of solutions for divergence type elliptic equations*, Electronic Journal of Differential Equations 2011 (2011) no. 43, 1–9.

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