ENERGY DECAY FOR DEGENERATE KIRCHHOFF EQUATIONS WITH WEAKLY NONLINEAR DISSIPATION

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Abstract. In this article we consider a degenerate Kirchhoff equation wave equation with a weak frictional damping,
\[
(|u_t|^{l-2} u_t)_t - \left( \int_\Omega |\nabla u|^2 \, dx \right)^{\gamma} \Delta u + \alpha(t) g(u_t) = 0.
\]
We prove general stability estimates using some properties of convex functions, without imposing any growth condition at the frictional damping term.

1. Introduction

In this article, we consider the initial-boundary value problem for the nonlinear Kirchhoff equation
\[
(|u_t|^{l-2} u_t)_t - \left( \int_\Omega |\nabla u|^2 \, dx \right)^{\gamma} \Delta u + \alpha(t) g(u_t) = 0, \quad \text{in } \Omega \times (0, \infty)
\]
\[
u = 0, \quad \text{on } \partial\Omega \times (0, \infty) \tag{1.2}
\]
\[
u(x, 0) = u^0(x), \quad u_t(x, 0) = u^1(x), \quad x \in \Omega, \tag{1.3}
\]
where \(l \geq 2, \gamma \geq 0\) are given constants, \(\Omega\) is a bounded domain in \(\mathbb{R}^n\) with a smooth boundary \(\partial\Omega\), and \(g\) and \(\alpha\) are one-variable functions satisfying some conditions to be specified later. This problem has been studied by many authors and several existence, nonexistence, and decay results have appeared. For instance, when \(l = 2\) and \(\gamma = 0\), the problem was treated by Mustafa and Massaoudi [11]. By using some properties of convex functions, they established a general decay result without imposing any growth condition on \(g\) at the origin. Abdelli and Benaissa [1] treated system (1.1)-(1.3) for \(g\) having a polynomial growth near the origin and established energy decay results depending on \(\alpha\) and \(g\) under appropriate relations between \(l\) and \(\gamma\). In a related work, Amroun and Benaissa [3] constructed an exact solution of (1.1)-(1.3) in the presence of a nonlinear source term and for \(\alpha \equiv 0\). They also proved a finite-time blow-up result for some specific initial data. Benaissa and Guesmia [2] proved the existence of global solution, as well as, a general stability result for the following equation
\[
(|u'|^{l-2} u')' - \Delta u + \alpha(t) g(u') = 0, \quad \text{in } \Omega \times \mathbb{R}_+.
\]
where $\Delta \phi = \sum_{i=1}^{n} \partial_{x_i}(\partial_i(\partial_{x_i}|^2)\partial_{x_i})$.

In this article, we use some technique from [11] to establish an explicit and general decay result, depending on $g$ and $\alpha$. The proof is based on the multiplier method and makes use of some properties of convex functions, the general Young inequality and Jensen’s inequality. These convexity arguments were introduced and developed by Lasiecka and co-workers ([7, 8, 9]) and used, with appropriate modifications, by Liu and Zuazua [10], Alabau-Boussouira [2] and others.

The paper is organized as follows: in section 2, we give our hypotheses and establish a useful lemma. In section 3, we state and prove our main result.

2. Preliminaries

To state and prove our result, we need the following hypotheses:

(H1) $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a nonincreasing differentiable function.

(H2) $g : \mathbb{R} \rightarrow \mathbb{R}$ is a nondecreasing $C^0$ function such that there exist $\varepsilon, c_1, c_2 > 0$, and a convex and increasing function $G : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ of class $C^1(\mathbb{R}_+) \cap C^2([0, +\infty[)$ satisfying $G(0) = G'(0) = 0$ or $G$ is linear on $[0, \varepsilon]$ such that

\[ c_1|s|^{l-1} \leq |g(s)| \leq c_2|s|^p, \text{ if } |s| \geq \varepsilon; \]
\[ |s|^l + |g(s)|^{\frac{l}{l-1}} \leq G^{-1}(sg(s)), \text{ if } |s| \leq \varepsilon; \]

with $p$ satisfying

\[ l - 1 \leq p \leq \frac{n+2}{n-2}, \text{ if } n > 2; \]
\[ l - 1 \leq p < \infty, \text{ if } n \leq 2. \]

Now we define the energy associated to the solution of the system (1.1)-(1.3) by

\[ E(t) = \frac{l-1}{l} \left\| u_t \right\|^l_l + \frac{1}{1+\gamma} \left\| \nabla_x u \right\|_2^{2(\gamma+1)} \]  
(2.1)

Lemma 2.1. Let $u$ be the solution of (1.1)-(1.3). Then

\[ E'(t) = -\alpha(t) \int_{\Omega} u g(u_t) \, dx \leq 0. \]  
(2.2)

Proof. Multiplying (1.1) by $u_t$ and integrating over $\Omega$, using the boundary conditions, the assertion of the lemma follows. \qed

3. Main Result

To prove our main result, first prove the following lemma.

Lemma 3.1. Assume that (H1), (H2) hold and that $l \geq 2(\gamma + 1)$. Then the functional

\[ F(t) = ME(t) + \int_{\Omega} u|u_t|^{l-2} u_t \, dx, \]

defined along the solution of (1.1)-(1.3), satisfies the following estimate, for some positive constants $M, c, m$:

\[ F'(t) \leq -mE(t) + c \int_{\Omega} (|u_t|^l + |ug(u_t)|^{\frac{l}{l-1}}) \, dx \]

and $F(t) \sim E(t)$. 

Proof. Using system (1.1)-(1.3), (2.1) and (2.2), we obtain
\[
F'(t) = ME'(t) + \int_\Omega |u_t|^l dx + \int_\Omega u(|u_t|^l t) dx \\
\leq \int_\Omega |u_t|^l dx + \left( \int_\Omega |\nabla u|^2 dx \right)^\gamma \int_\Omega u \Delta u dx - \alpha(t) \int_\Omega ug(u_t) dx \\
\leq \int_\Omega |u_t|^l dx - \int_\Omega |\nabla u|^{2(\gamma+1)} dx - \alpha(t) \int_\Omega ug(u_t) dx \\
\leq -m E(t) + c \int_\Omega (|u_t|^l + |u g(u_t)|) dx
\]

To prove that \( F(t) \sim E(t) \), we show that for some positive constants \( \lambda_1 \) and \( \lambda_2 \),
\[
\lambda_1 E(t) \leq E(t) \leq \lambda_2 E(t). \tag{3.1}
\]
We use (2.1), Poincaré’s and Young’s inequalities with exponents \( \frac{l}{l-1} \) and \( \frac{l}{2} \), and recall that \( 2 \leq l \leq p + 1 \leq 2 \frac{\gamma+1}{\gamma+2} \), to obtain
\[
\int_\Omega u |u_t|^{-2} u_t dx \leq C \int_\Omega |u_t|^l dx + \varepsilon \int_\Omega |u_t|^l dx \\
\leq C \| \nabla u \|_2^\gamma + \varepsilon \| u_t \|_2^l \\
\leq C \varepsilon \frac{E^{1-\frac{1}{2(l-1)}}(t)}{E(t)} + c \varepsilon E(t) \\
\leq C \varepsilon \frac{E^{1-\frac{1}{2(l-1)}}(t)}{E(t)} + c \varepsilon E(t).
\]

By noting that \( l \geq 2(\gamma+1) \) and using (2.2), we have
\[
\int_\Omega u |u_t|^{-2} u_t dx \leq C \varepsilon \frac{E^{1-\frac{1}{2(\gamma+1)}}(t)}{E(t)} + c \varepsilon E(t),
\]
and
\[
\int_\Omega u |u_t|^{-2} u_t dx \geq -C \int_\Omega |u_t|^l dx - \varepsilon \int_\Omega |u_t|^l dx \\
\geq -C \varepsilon \frac{E^{1-\frac{1}{2(l-1)}}(t)}{E(t)} + c \varepsilon E(t) \\
\geq -C \varepsilon \frac{E^{1-\frac{1}{2(l-1)}}(t)}{E(t)} + c \varepsilon E(t).
\]

Then, for \( m \) large enough, we obtain (3.1). This completes the proof. \( \square \)

Taking \( 0 < \varepsilon_1 < \varepsilon \) such that
\[
sg(s) \leq \min \{ \varepsilon, G(\varepsilon) \}, \quad \text{if } |s| \leq \varepsilon_1, \tag{3.2}
\]
and
\[
\begin{cases}
|s|^{l-1} \leq |g(s)| \leq c_g |s|^p, & \quad \text{if } |s| \leq \varepsilon_1 \\
|s|^l + |g(s)|^{\frac{1}{l-1}} \leq G^{-1}(sg(s)), & \quad \text{if } |s| \leq \varepsilon_1.
\end{cases} \tag{3.3}
\]

Considering the following partition of \( \Omega \),
\[
\Omega_1 = \{ x \in \Omega : |u_t| \leq \varepsilon_1 \}, \quad \Omega_2 = \{ x \in \Omega : |u_t| > \varepsilon_1 \}
\]
and using the embedding \( H^0_0(\Omega) \hookrightarrow L^{p+1}(\Omega) \) and Hölder’s inequality, we obtain
\[
\int_{\Omega_2} |ug(u_t)| dx \leq \left( \int_{\Omega_2} |u|^{p+1} dx \right)^{\frac{1}{p+1}} \left( \int_{\Omega_2} |g(u_t)|^{\frac{p}{p+1}} dx \right)^{\frac{p}{p+1}}
\]
By Lemma 3.1, (3.4) and (3.5), for $\delta$ positive constants

then, we use Young’s inequality and the fact that $p$

Using Poincaré’s inequality and (3.3) yields

Here, $G$ satisfies

Similarly, using (2.1) and Young’s inequality, we have, for any $\alpha > 0,$

By Lemma 3.1, (3.4) and (3.5), for $\delta$ small enough, the function $L = F + C_\delta E$

\[ L'(t) \leq \left( - m + \delta E^{\frac{p-1}{p+1}}(0) + \delta E^{\frac{1}{p+1}}(0) \right) E(t) \]

and

\[ L(t) \sim E(t). \]  

**Theorem 3.2.** Assume that (H1), (H2) hold and $l \geq 2(\gamma + 1).$ Then there exist positive constants $k_1, k_2, k_3$ and $\varepsilon_0$ such that the solution of $1.1$  \[ J.3 \] satisfies

\[ E(t) \leq k_3 G^{-1}_1 \left( k_1 \int_0^t \alpha(s) \, ds + k_2 \right) \quad \forall t \geq 0, \]  

where

\[ G_1(t) = \int_t^1 \frac{1}{G_2(s)} \, ds, \quad G_2(t) = tG'(\varepsilon_0 t). \]  

Here, $G_1$ is strictly decreasing and convex on $(0, 1]$ with $\lim_{t \to 0} G_1(t) = +\infty.$
Proof. Multiplying (3.6) by $\alpha(t)$, we have
\[
\alpha(t)L'(t) \leq -d\alpha(t)E(t) + \alpha(t) \int_{\Omega_1} \left[ |u_t|^\alpha + |g(u_t)|^\frac{\alpha}{\alpha-1} \right] dx
\] (3.10)

Case 1. $G$ is linear on $[0, \varepsilon]$, then we deduce that
\[
\alpha(t)L'(t) \leq -d\alpha(t)E(t) + \alpha(t) \int_{\Omega_1} u_t g(u_t) dx = -d\alpha(t)E(t) - cE'(t)
\]
Consequently, we arrive at
\[
(\alpha L + cE)'(t) \leq -d\alpha(t)E(t).
\]
Recalling that
\[
\alpha L + cE \sim E,
\]
we obtain
\[
E(t) \leq c' e^{-c'' \int_0^t \alpha(s) ds}
\]
Thus, we have
\[
E(t) \leq c' e^{-c'' \int_0^t \alpha(s) ds} = c' G^{-1}(c'' \int_0^t \alpha(s) ds)
\]
by a simple computation.

Case 2. $G$ is nonlinear on $[0, \varepsilon]$. In this case, we define
\[
I(t) = \frac{1}{|\Omega_1|} \int_{\Omega_1} u_t g(u_t) dx.
\]
and exploit Jensen’s inequality and the concavity of $G^{-1}$ to obtain
\[
G^{-1}(I(t)) \geq c \int_{\Omega_1} G^{-1}(u_t g(u_t)) dx.
\]
By using this inequality and (3.3), we obtain
\[
\alpha(t) \int_{\Omega_1} |u_t|^\alpha + |g(u_t)|^\frac{\alpha}{\alpha-1} dx \leq \alpha(t) \int_{\Omega_1} G^{-1}(u_t g(u_t)) dx \leq c\alpha(t) G^{-1}(I(t)).
\]
(3.12)
Let us set $H_0 = \alpha L + E$ and exploit (2.2), (3.10), (3.12), and $\alpha$ being nonincreasing, to obtain
\[
H_0'(t) \leq -d\alpha(t)E(t) + \alpha(t)G^{-1}(I(t)) + E'(t)
\]
\[
\leq -d\alpha(t)E(t) + \alpha(t)G^{-1}(I(t)),
\]
(3.13)
and recall (3.7), to deduce that $H_0 \sim E$.

For $\varepsilon_0 < \varepsilon$ and $c_0 > 0$, we define $H_1$ by
\[
H_1(t) = G'(\varepsilon_0 \frac{E(t)}{E(0)}) H_0(t) + c_0 E(t).
\]
Then, we see easily that, for $a_1, a_2 > 0$,
\[
a_1 H_1(t) \leq E(t) \leq a_2 H_1(t),
\]
(3.14)
By recalling that $E' \leq 0$, $G' > 0$, $G'' > 0$ on $(0, \varepsilon]$ and making use of (2.1) and (3.13), we obtain
\[
H_1'(t) = \varepsilon_0 E'(t) E(0) G''(\varepsilon_0 \frac{E(t)}{E(0)}) H_0(t) + G'(\varepsilon_0 \frac{E(t)}{E(0)}) H_0'(t) + c_0 E'(t)
\]
\[
\leq -d\alpha(t)E(t) G'(\varepsilon_0 \frac{E(t)}{E(0)}) + \alpha(t) G'(\varepsilon_0 \frac{E(t)}{E(0)}) G^{-1}(I(t)) + c_0 E'(t).
\]
(3.15)
Let $G^*$ be the convex conjugate of $G$ in the sense of Young (see Arnold [4] p. 61-64), then
\[ G^*(s) = s(G')^{-1}(s) - G[(G')^{-1}(s)], \quad \text{if } s \in (0, G'(\varepsilon)], \quad (3.16) \]
and $G^*$ satisfies the generalized Young’s inequality
\[ AB \leq G^*(A) + G(B), \quad \text{if } A \in (0, G'(\varepsilon)], \ B \in (0, \varepsilon]. \quad (3.17) \]
with $A = G'(\varepsilon_0 E(t)/E(0))$ and $B = G^{-1}(I(t))$, using (2.2), (3.2) and (3.15)–(3.16), we obtain
\[
H'(t) \leq -d\alpha(t)E(t)G'(\varepsilon_0 \frac{E(t)}{E(0)}) + c\alpha(t)G^*(G'(\varepsilon_0 \frac{E(t)}{E(0)})) + c\alpha(t)I(t) + c_0 E'(t)
\]
\[
\leq -d\alpha(t)E(t)G'(\varepsilon_0 \frac{E(t)}{E(0)}) + \varepsilon_0 \alpha(t)E(t) - cE'(t) + c_0 E'(t).
\]
Choosing $c_0 > c$ and $\varepsilon_0$ small enough, we obtain
\[ H'(t) \leq -k\alpha(t)E(t)G'(\varepsilon_0 \frac{E(t)}{E(0)}) = -k\alpha(t)G_2\left(\frac{E(t)}{E(0)}\right) \quad (3.18) \]
where $G_2(t) = tG'(\varepsilon_0 t)$. Since
\[ G_2'(t) = G'(\varepsilon_0 t) + \varepsilon_0 tG''(\varepsilon_0 t). \]
and $G$ is convex on $(0, \varepsilon]$, we find that $G_2'(t) > 0$ and $G_2(t) > 0$ on $(0, 1]$. By setting $H(t) = \frac{a_1 H_1(t)}{E(0)}$ ($a_1$ is given in (3.14)), we easily see that, by (3.14), we have
\[ H(t) \sim E(t) \quad (3.19) \]
Using (3.18), we arrive at
\[ H'(t) \leq -k_1\alpha(t)G_2(H(t)) \]
By recalling (3.9), we deduce $G_2(t) = -1/G_1'(t)$, hence
\[ H'(t) \leq k_1\alpha(t) \frac{1}{G_1''(H(t))}, \]
which gives
\[ [G_1(H(t))]' = H'(t)G_1'(H(t)) \leq k_1\alpha(t) \]
A simple integration leads to
\[ G_1(H(s)) \leq k_1 \int_0^t \alpha(s) \, ds + G_1(H(0)) \]
Consequently,
\[ H(t) \leq G_1^{-1}(k_1 \int_0^t \alpha(s) \, ds + k_2) \quad (3.20) \]
Using (3.19) and (3.20) we obtain (3.8). The proof is complete. \qed

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