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GROUND STATE SOLUTIONS FOR ASYMPTOTICALLY PERIODIC SCHRÖDINGER EQUATIONS WITH CRITICAL GROWTH

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Abstract. Using the Nehari manifold and the concentration compactness principle, we study the existence of ground state solutions for asymptotically periodic Schrödinger equations with critical growth.

1. Introduction and statement of main results

In recent years, there have been many works on the existence of non-trivial solutions for the nonlinear Schrödinger equation

$$-\Delta u+V(x)u=g(x,u), \tag{1.1}$$

due to its physical and mathematical interests; see for example the references in this article. Especially, the study of ground state solutions has made great progress and attracted many authors' attention. Ground state solution is such a non-trivial solution with least energy, which has great physical interests. The results mainly depend on the spectrum of the operator $A:=-\Delta+V$ in $L^2(\mathbb{R}^N)$ (denoted by $\sigma(A)$), the periodicity of V and g, and the growth condition of g since they effect the restore of compactness.

According to the location of 0 in $\sigma(A)$, we have three cases: $\inf\sigma(A)>0$; 0 lies in a gap of $\sigma(A)$; and 0 is a boundary point of a gap of $\sigma(A)$. For details, see [7, 11]. In this paper, we are concerned with the first case.

Many authors focus on the case that V and g are periodic in the variable x. When g satisfies subcritical growth condition, there are many results, see [6, 13, 15, 19, 23]. Moreover, the authors made many efforts to weaken the classical Ambrosetti-Rabinowitz condition for g and the differentiability of g. Recently, Szulkin and Weth [19] showed that problem (1.1) possesses a ground state solution under Nehari-type conditions for merely continuous g. Later, for g with critical growth, in [25] we considered the existence of ground state solutions of (1.1).

Other authors turn to study asymptotically periodic Schrödinger equations of the form (1.1). Equation (1.1) is called asymptotically periodic if it can approach to a periodic equation in some sense as $|x|\to\infty$; in the sequel we explain the meaning

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of asymptotic periodicity. Silva and Vieira [18], Lins and Silva [14] considered equation (1.1) with subcritical growth and critical growth g respectively. However, they obtained only the existence of non-trivial solutions for (1.1).

Motivated by these works, we focused our interest in ground state solutions for asymptotically periodic equation (1.1) with critical growth. Note that for asymptotically periodic equation (1.1) with subcritical growth, we can have similar result. But it is much simpler than the critical growth case and then we ignore this case. We consider the equation

$$\begin{gathered} -\Delta u + V(x)u = K(x)|u|^{2^*-2}u + f(x,u), \quad x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{gathered} \tag{1.2}$$

where V, K and f are asymptotically periodic in the variable x.

In what follows, the notation inf is understood as the essential infimum. First we make some assumptions on the functions V and K:

- (H1) $V \in L^\infty(\mathbb{R}^N)$, $\inf_{\mathbb{R}^N} V > 0$,
- (H2) $K \in L^\infty(\mathbb{R}^N)$, $\inf_{\mathbb{R}^N} K > 0$, and there exists a point $x_0 \in \mathbb{R}^N$ such that $K(x) = |K|_\infty + O(|x-x_0|^{N-2})$, as $x \to x_0$.

For the nonlinearity f, setting $F(x,u) = \int_0^u f(x,s)ds$, we assume that

- (H3) $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$, $|f(x,u)| \le a(1+|u|^{q-1})$, for some $a > 0$ and $2 < q < 2^*$, where $2^* = 2N/(N-2)$, $N \ge 3$,
- (H4) $f(x,u) = o(u)$ uniformly in x as $u \to 0$,
- (H5) $u \mapsto \frac{K(x)|u|^{2^*-2}u + f(x,u)}{|u|}$ is increasing on $(-\infty,0)$ and $(0,\infty)$,
- (H6) (i) $\frac{F(x,u)}{|u|^{2^*-2}} \to \infty$ uniformly in x as $|u| \to \infty$, if $N = 3$,
  (ii) $\frac{F(x,u)}{u^2 \log|u|} \to \infty$ uniformly in x as $|u| \to \infty$, if $N = 4$,
  (iii) $\frac{F(x,u)}{u^2} \to \infty$ uniformly in x as $|u| \to \infty$, if $N > 4$.

Let $\mathcal{F}$ be the class of functions $\tilde{h} \in L^\infty(\mathbb{R}^N)$ such that, for every $\epsilon > 0$ the set $\{x \in \mathbb{R}^N : |\tilde{h}(x)| \ge \epsilon\}$ has finite Lebesgue measure. The asymptotic periodicity of V, K and f as $|x| \to \infty$ is given by the condition

- (H7) there exist functions $V_p, K_p \in L^\infty(\mathbb{R}^N)$ and $f_p \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ such that
  (i) V_p, K_p and f_p is 1-periodic in $x_i, 1 \le i \le N$,
  (ii) $V - V_p, K - K_p \in \mathcal{F}$, $|f(x,u) - f_p(x,u)| \le |h(x)|(|u| + |u|^{q-1})$, $x \in \mathbb{R}^N$, $h \in \mathcal{F}$,
  (iii) $V \le V_p$, $K \ge K_p$ and $F(x,u) \ge F_p(x,u) := \int_0^u f_p(x,s)ds$.
  (iv) $u \mapsto \frac{K_p(x)|u|^{2^*-2}u + f_p(x,u)}{|u|}$ is nondecreasing on $(-\infty,0)$ and $(0,\infty)$.

Theorem 1.1. *If* (H1)-(H7) *are satisfied, then the problem* (1.2) *has a ground state solution.*

Remark 1.2. The assumption (H5) is inspired by [19]. As in [19], under the condition (H5), the method of Nehari manifold is valid. Then we use the method of Nehari manifold to find ground state solutions for (1.2). In addition, the condition (H6) is inspired by [19] and [14], and it will be used to restrict the functional level to a suitable interval and then overcome the difficulties brought by the critical term $K(x)|u|^{2^*-2}u$. The condition (H6) is stronger than the condition (g_5) in [14], since here we need to obtain the boundedness of PS sequence without the condition (g_3) in [14].

The proof of Theorem 1.1 is mainly based on the method of Nehari manifold and the concentration-compactness principle. We first follow the same outline as in [19] to reduce our problem of looking for a ground state solution into that of finding a minimizer on the Nehari manifold. We then apply concentration-compactness principle to solve this minimization problem. As far as we know, for similar problems, in most of the previous papers the weak limit of minimizing sequence is nontrivial and the weak limit is the desired ground state solution (see, for instance, [13] and [19]). However, inspired by [14], here we allow the case that the weak limit is trivial.

The article is organized as follows. In Section 2 we give some preliminaries. In Section 3 we give our variational framework. In Section 4 we estimate the least energy on Nehari manifold. In Section 5 we prove Theorem 1.1.

2. Notation and preliminaries

In this article we use the following notation. Denote $\mathbb{R}^+ = [0, +\infty)$. For $1 \leq p \leq \infty$, the norm in $L^p(\mathbb{R}^N)$ is denoted by $|\cdot|_p$. For any $r > 0$ and $x \in \mathbb{R}^N$, $B_r(x)$ denotes the ball centered at x with the radius r. $\int_{\mathbb{R}^N} f(x)dx$ will be represented by $\int f(x)dx$. Let E be a Hilbert space, the Fréchet derivative of a functional Φ at u, $\Phi'(u)$, is an element of the dual space E^* and we shall denote $\Phi'(u)$ evaluated at $v \in E$ by $\langle \Phi'(u), v\rangle$.

By (H1), we define the inner product and norm of the Sobolev space $X := H^1(\mathbb{R}^N)$ by

$$(u, v) = \int \big(\nabla u \nabla v + V(x)uv\big)dx, \quad \|u\|^2 = \int \big(|\nabla u|^2 + V(x)u^2\big)dx.$$

Define $S_1 = \{u \in X : \|u\| = 1\}$. Moreover,

$$\|u\|_p^2 = \int \big(|\nabla u|^2 + V_p(x)u^2\big)dx,$$

is an equivalent norm in X by (H7)-(iii).

Let $g(x, u) = K(x)|u|^{2^*-2}u + f(x, u)$. The functional corresponding to our problem is

$$I(u) = \frac{1}{2}\|u\|^2 - \int G(x, u)dx, \quad u \in X,$$

where

$$G(x, u) := \int_0^u g(x, s)ds = \frac{1}{2^*}K(x)|u|^{2^*} + F(x, u).$$

By (H1)–(H4), I is differentiable and its critical points are solutions of (1.2).

A solution $\tilde{u} \in X$ of (1.2) is called a ground state solution if

$$I(\tilde{u}) = \min\{I(u) : u \in X \setminus \{0\},\ I'(u) = 0\}.$$

In the process of finding ground state solutions for (1.2), the corresponding periodic equation of (1.2) is very important. The corresponding periodic equation is

$$\begin{gathered} -\Delta u + V_p(x)u = K_p(x)|u|^{2^*-2}u + f_p(x, u), \quad x \in \mathbb{R}^N, \\ u \in X. \end{gathered} \tag{2.1}$$

Let $g_p(x, u) = K_p(x)|u|^{2^*-2}u + f_p(x, u)$. The functional corresponding to (2.1) is

$$I_p(u) = \frac{1}{2}\|u\|_p^2 - \int G_p(x, u)dx,$$

where $G_p(x,u) := \int_0^u g_p(x,s)ds$. Below we give some instruction for our conditions.

Lemma 2.1. *If* (H3), (H4) *are satisfied, then for all* $\epsilon > 0$ *there exist* $a_\epsilon, b_\epsilon > 0$ *such that*

$$|f(x,u)| \le \epsilon|u| + a_\epsilon |u|^{q-1}, \quad \forall u \in \mathbb{R}, \tag{2.2}$$

$$|g(x,u)| \le \epsilon|u| + b_\epsilon |u|^{2^*-1}, \quad \forall u \in \mathbb{R}. \tag{2.3}$$

If (H4) *and* (H5) *are satisfied, then*

$$0 < G(x,u) < \frac{1}{2} g(x,u)u, \quad \forall u \neq 0. \tag{2.4}$$

Moreover, if (H7)-(iv) *is satisfied, then*

$$G_p(x,u) \le \frac{1}{2} g_p(x,u)u, \quad \forall u \in \mathbb{R}. \tag{2.5}$$

Proof. The inequalities (2.2) and (2.3) follow easily from (H3) and (H4). Now we prove (2.4) and (2.5). By (H4) and (H5), we have $g(x,u) > 0$ for all $u > 0$, and $g(x,u) < 0$ for all $u < 0$. Thus $G(x,u) = \int_0^u g(x,s)\,ds > 0$ for all $u \neq 0$. Again for all $u > 0$ we have

$$G(x,u) = \int_0^u \frac{g(x,s)}{s} s\,ds < \int_0^u \frac{g(x,u)}{u} s\,ds = \frac{1}{2} g(x,u)u. \tag{2.6}$$

For $u < 0$, the above inequality still holds. Thus (2.4) follows. In a similar way we deduce that (2.5) holds. $\square$

For the derivative of the functional I we have the following lemma.

Lemma 2.2. *Let* $V, K \in L^\infty(\mathbb{R}^N)$ *and* $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ *satisfy*

$$|f(x,u)| \le C(|u| + |u|^{q-1}), \quad \forall u \in \mathbb{R}. \tag{2.7}$$

There holds the following results:

- (i) I' *maps bounded sets in* X *into bounded sets in* X^**;*
- (ii) I' *is weakly sequentially continuous; i.e., if* $u_n \rightharpoonup u$ *in* X*, then* $I'(u_n) \rightharpoonup I'(u)$ *in* X^**.*

Proof. (i) Let $\{u_n\}$ be a bounded sequence in X. By (2.7) we have

$$|g(x,u)| \le C(|u| + |u|^{2^*-1}) \le C(1 + |u|^{2^*-1}), \quad \forall u \in \mathbb{R}, \tag{2.8}$$

here and below we use the same C to indicate may various positive constants. For any $v \in X$, it follows that

$$\begin{aligned}
\Big|\int g(x,u_n)v dx\Big| &\le \int C\big(|u_n||v| + |u_n|^{2^*-1}|v|\big)dx \\
&\le C\big(|u_n|_2 |v|_2 + |u_n|_{2^*}^{2^*-1}|v|_{2^*}\big) \\
&\le C\big(\|u_n\|\|v\| + \|u_n\|^{2^*-1}\|v\|\big) \le C\|v\|,
\end{aligned}$$

since $\{u_n\}$ is bounded. Note that

$$\langle I'(u_n), v\rangle = (u_n, v) - \int g(x,u_n)v dx.$$

Then we obtain $|\langle I'(u_n), v\rangle| \le C\|v\|$ and so $\{I'(u_n)\}$ is bounded in X^*.

(ii) Assume that $u_n \rightharpoonup u$ in X. For any $v \in C_0^\infty(\mathbb{R}^N)$ with the support Ω, we may assume that $u_n \to u$ in $L^{2^*-1}(\Omega)$. By (2.8) we have $g(x,u_n) \to g(x,u)$ in $L^1(\Omega)$. Then

$$\int g(x,u_n)v dx \to \int g(x,u)v dx.$$

Hence, we obtain

$$\langle I'(u_n), v\rangle \to \langle I'(u), v\rangle, \quad \forall v \in C_0^\infty(\mathbb{R}^N). \tag{2.9}$$

By (i), we have $\{I'(u_n)\}$ is bounded in X^*. Combining the fact that $C_0^\infty(\mathbb{R}^N)$ is dense in X, we conclude that (2.9) holds for any $v \in X$ and so $I'(u_n) \rightharpoonup I'(u)$ in X^*. □

If f satisfies (H3) and (H4), then (2.7) holds. In addition, if f_p satisfies (H7)-(ii), then

$$|f_p(x,u)| \le |f(x,u) - f_p(x,u)| + |f(x,u)| \le C(|u| + |u|^{q-1}), \ \forall u \in \mathbb{R}.$$

Hence, we have the following result.

Remark 2.3. If (H3) and (H4) are satisfied, then I' is weakly sequentially continuous. If (H3), (H4) and (H7)-(ii) are satisfied, then I'_p is weakly sequentially continuous.

3. Variational setting

In this section, we describe the variational framework for our problem. To find ground state solutions, we use the method of Nehari manifold. As in [19], we reduce our variational problem to the minimization problem on a Nehari manifold. Then we take advantage of concentration compactness lemma to deal with the minimizing problem.

The Nehari manifold M corresponding to I is defined by

$$M = \{u \in X \setminus \{0\} : \langle I'(u), u\rangle = 0\}.$$

Moreover, the least energy on M is given by $c := \inf_M I$. As in the proof of [26, Lemma 3.3], we have the following lemma.

Lemma 3.1. *Under assumptions* (H1)–(H6)*, the following results hold:*

(i) *For all $u \in X \setminus \{0\}$, there exists a unique $t_u > 0$ such that $t_u u \in M$ and $I(t_u u) = \max_{t\ge 0} I(tu)$.*

(ii) *For each compact subset $W \subset S_1$, there exists a constant C_W such that $t_u \le C_W$ for all $u \in W$.*

From Lemma 3.1 (i), for any $u \in X\setminus\{0\}$ we define the mapping $\hat{m} : X\setminus\{0\} \to M$ by $\hat{m}(u) = t_u u$. In addition, for all $v \in \mathbb{R}^+ u$ we have $\hat{m}(v) = \hat{m}(u)$. Define $m := \hat{m}|_{S_1}$. Then m is a bijection from S_1 to M. Then

$$c = \inf_M I = \inf_{u\in S_1} I(\hat{m}(u)) = \inf_{u\in X\setminus\{0\}} I(\hat{m}(u)).$$

By Lemma 3.1 (i), we have $I(\hat{m}(u)) = \max_{t\ge 0} I(tu)$. Therefore,

$$c = \inf_{u\in X\setminus\{0\}} \max_{t\ge 0} I(tu). \tag{3.1}$$

Since the nonlinearity f is merely continuous, the Nehari manifold M may not be differentiable and it may hve no differential structure. So the restriction of I on M may have no derivative. As before, we find that there is a one-to-one correspondence between S_1 and M. Noting that S_1 is differentiable, we replace M with S_1. Thus, we introduce the functional $\Psi : S_1 \to \mathbb{R}$ by $\Psi(u) = I(m(u))$. The lemma below shows that the PS sequences and critical points of Ψ on S_1 and those of I on M are corresponded by the mapping m.

Proposition 3.2. *If* (H1)–(H6) *are satisfied, then the following results hold:*

(i) *If $\{w_n\}$ is a Palais-Smale (PS) sequence for Ψ, then $\{m(w_n)\}$ is a PS sequence for I.*
(ii) *$\inf_{S_1} \Psi = \inf_M I$. Moreover, if w is a critical point of Ψ, then $m(w)$ is a nontrivial critical point of I.*

Results (i) and (ii) in the above propositions follow from [19, Corollaries 3.3(b) and 3.3(c)]; so we refer to [19] and omit its proof.

Lemma 3.3. *A minimizer of $I|_M$ is a ground state solution of* (1.2).

Proof. Let $u \in M$ be such that $I(u) = \inf_M I = c$. We claim that $I'(u) = 0$. Indeed, by $I(u) = c$, we obtain $\Psi(w) = c$, where $w = m^{-1}(u) \in S_1$. By Proposition 3.2 (ii), we have $\Psi(w) = \inf_{S_1} \Psi$. So $\Psi'(w) = 0$. Using Proposition 3.2 (ii) again, we deduce that $I'(u) = 0$. For any $v \in X \setminus \{0\}$ satisfying $I'(v) = 0$, we obtain $v \in M$. So $I(v) \geq c = I(u)$. Thus, u is a ground state solution of (1.2). □

From Lemma 3.3, we know that the problem of seeking for a ground state solution for (1.2) can be transformed into that of finding a minimizer of $I|_M$. In the process of finding the minimizer, comparing with previous related work (see [13, 19]), we mainly need to overcome the difficulties which brought by the critical growth term $K(x)|u|^{2^*-2}u$ and the non-periodicity of (1.2).

First we deal with the difficulty brought by the critical growth term. Recall that the best constant for the Sobolev embedding $\mathcal{D}^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$ is given by

$$S = \inf_{u \in \mathcal{D}^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{|\nabla u|_2^2}{|u|_{2^*}^2}.$$

We shall prove that when the above infimum c lies in a certain interval, the minimizing sequence of c is bounded.

Lemma 3.4. *Let* (H1)-(H6) *hold. If $c < \frac{1}{N}|K|_\infty^{-\frac{N-2}{2}} S^{N/2}$, then the minimizing sequence of c is bounded.*

Proof. Let $\{u_n\}$ be a minimizing sequence of I on M. Namely,

$$I(u_n) \to c, \quad \langle I'(u_n), u_n \rangle = 0.$$

We argue by contradiction. Suppose that $\{u_n\}$ is unbounded. Without loss of generality, we may assume that $\|u_n\| \to \infty$. Set $v_n := u_n/\|u_n\|$. Then we may suppose $v_n \rightharpoonup v$ in X, $v_n \to v$ in $L^2_{\rm loc}(\mathbb{R}^N)$, and $v_n \to v$ a.e. in $\mathbb{R}^N$. Moreover, from Lemma 3.1 (i) we have

$$I(u_n) \geq I(tv_n), \quad \text{for all } t \geq 0. \tag{3.2}$$

If v_n is vanishing; i.e.,

$$\lim_{n\to\infty} \sup_{y \in \mathbb{R}^N} \int_{B_1(y)} v_n^2(x)\,dx = 0,$$

then Lions Compactness Lemma implies that $v_n \to 0$ in $L^q(\mathbb{R}^N)$. For any $\epsilon > 0$, $t > 0$ given in (3.2), by (2.2) we obtain

$$\Big|\int F(x, tv_n)dx\Big| \le \epsilon t^2 |v_n|_2^2 + a_\epsilon t^q |v_n|_q^q < C\epsilon + a_\epsilon t^q |v_n|_q^q, \tag{3.3}$$

since $\{v_n\}$ is bounded in $L^2(\mathbb{R}^N)$. Noting that $v_n \to 0$ in $L^q(\mathbb{R}^N)$, then for above ϵ, there exists $J \in \mathbb{N}$ such that $a_\epsilon t^q |v_n|_q^q < C\epsilon$, for $n > J$. So by (3.3) for $n > J$ we obtain that $|\int F(x, tv_n)dx| < C\epsilon$. Then

$$\int F(x, tv_n)dx \to 0.$$

Combining with (3.2), for large n we have

$$\begin{aligned}
c + o_n(1) = I(u_n) &\ge \sup_{t\ge 0} I(tv_n) \\
&= \sup_{t\ge 0}\Big[\frac{t^2}{2} - \frac{t^{2^*}}{2^*}\int K(x)|v_n|^{2^*}dx\Big] + o_n(1) \\
&\ge \sup_{t\ge 0}\Big[\frac{t^2}{2} - \frac{t^{2^*}}{2^*}|K|_\infty \int |v_n|^{2^*}dx\Big] + o_n(1) \\
&\ge \sup_{t\ge 0}\Big[\frac{t^2}{2} - \frac{t^{2^*}}{2^*}|K|_\infty S^{-\frac{2^*}{2}}\Big] + o_n(1) \\
&= \frac{1}{N}|K|_\infty^{-\frac{N-2}{2}} S^{N/2} + o_n(1).
\end{aligned}$$

Then $c \ge \frac{1}{N}|K|_\infty^{-\frac{N-2}{2}} S^{N/2}$ contradicting the fact that $c < \frac{1}{N}|K|_\infty^{-\frac{N-2}{2}} S^{N/2}$. Hence $\{v_n\}$ is non-vanishing. Then there exists $x_n \in \mathbb{R}^N$ and $\delta_0 > 0$ such that

$$\int_{B_1(x_n)} v_n^2(x)dx > \delta_0. \tag{3.4}$$

Set $\tilde{v}_n(\cdot) = v_n(\cdot + x_n)$. Passing to a subsequence, we may assume that $\tilde{v}_n \rightharpoonup \tilde{v}$ in X, $\tilde{v}_n \to \tilde{v}$ in $L^2_{\rm loc}(\mathbb{R}^N)$, and $\tilde{v}_n \to \tilde{v}$ a.e. in $\mathbb{R}^N$. By (3.4) we obtain

$$\int_{B_1(0)} \tilde{v}_n^2(x)dx > \delta_0.$$

So $\tilde{v} \ne 0$. Then there exists a positive measure set Ω_1 such that $\tilde{v}(x) \ne 0$, $\forall x \in \Omega_1$. Set $\tilde{u}_n = \|u_n\|\tilde{v}_n$. Noting that $\|u_n\| \to \infty$ and $\tilde{v}_n \to \tilde{v}$ a.e. in $\mathbb{R}^N$, then $\tilde{u}_n(x) \to \infty$, $x \in \Omega_1$. We need to discuss for the dimension N by (H6). First we assume that $N > 4$. By (H6) we have

$$\int_{\Omega_1} \liminf \frac{F(x + x_n, \tilde{u}_n)}{\tilde{u}_n^2}\tilde{v}_n^2 dx = \infty.$$

Then

$$\int_{\Omega_1} \liminf\Big(\frac{1}{2^*}\frac{K(x + x_n)|\tilde{u}_n|^{2^*}}{\|u_n\|^2} + \frac{F(x + x_n, \tilde{u}_n)}{\tilde{u}_n^2}\tilde{v}_n^2\Big)dx = \infty.$$

Therefore, (2.4) and Fatou's Lemma yield that

$$\liminf \int_{\Omega_1}\Big(\frac{1}{2^*}\frac{K(x + x_n)|\tilde{u}_n|^{2^*}}{\|u_n\|^2} + \frac{F(x + x_n, \tilde{u}_n)}{\|u_n\|^2}\Big)dx = \infty.$$

Again using (2.4) we have

$$\liminf \int \Big(\frac{1}{2^*}\frac{K(x+x_n)|\tilde{u}_n|^{2^*}}{\|u_n\|^2} + \frac{F(x+x_n,\tilde{u}_n)}{\|u_n\|^2}\Big)dx = \infty.$$

Then

$$\begin{aligned}
\frac{c}{\|u_n\|^2} + o_n(1) &= \frac{I(u_n)}{\|u_n\|^2} \\
&= \frac{1}{2} - \frac{1}{2^*}\int \frac{K(x)|u_n|^{2^*}}{\|u_n\|^2}dx - \int \frac{F(x,u_n)}{\|u_n\|^2}dx \\
&= \frac{1}{2} - \frac{1}{2^*}\int \frac{K(x+x_n)|\tilde{u}_n|^{2^*}}{\|u_n\|^2}dx - \int \frac{F(x+x_n,\tilde{u}_n)}{\|u_n\|^2}dx \to -\infty.
\end{aligned}$$

This is a contradiction. Similarly, when $N = 3$ or $N = 4$, there is also a contradiction by (H6). The proof is complete. □

Next we treat the difficulty caused by the non-periodicity of (1.2). Since V, K and f in (1.2) are non-periodic, we cannot use the invariance of the functional under translation to look for a minimizer. However, the approached equation of (1.2) as $|x| \to \infty$ is periodic, we shall take advantage of the periodicity of the equation (2.1) and the relationship of the functionals and derivatives between (1.2) and (2.1) to find the minimizer. By (H7)-(iii), one easily has the following lemma.

Lemma 3.5. *Let* (H7)-(iii) *hold. Then $I(u) \le I_p(u)$, for all $u \in X$.*

As in the proof of [14, Lemma 5.1], and [26, Lemma 4.1], we have the following two lemmas, respectively.

Lemma 3.6. *Let* (H7)-(ii) *hold. Assume that $\{u_n\} \subset X$ is bounded and $\varphi_n(x) = \varphi(x - x_n)$, where $\varphi \in X$ and $x_n \in \mathbb{R}^N$. If $|x_n| \to \infty$, then*

$$\int (V(x) - V_p(x))u_n\varphi_n dx \to 0,$$
$$\int (K(x) - K_p(x))|u_n|^{2^*-2}u_n\varphi_n dx \to 0.$$

Lemma 3.7. *Let* (H7)-(ii) *hold. Assume that $\{u_n\} \subset X$ satisfies $u_n \rightharpoonup 0$ and $\varphi_n \in X$ is bounded. Then*

$$\int [f(x,u_n) - f_p(x,u_n)]\varphi_n dx \to 0.$$

Remark 3.8. Let (H7)-(ii) hold. Assume that $\{u_n\} \subset X$ satisfies $u_n \rightharpoonup 0$. Note that

$$\int [F(x,u_n) - F_p(x,u_n)]dx = \int_{\mathbb{R}^N}\int_0^1 \big[f(x,tu_n)u_n - f_p(x,tu_n)u_n\big]dtdx.$$

Then similar to the proof of Lemma 3.7, we have

$$\int [F(x,u_n) - F_p(x,u_n)]dx \to 0.$$

4. Estimates

This section, we estimate the least energy c, provided that

$$c \in (0, \frac{1}{N}|K|_\infty^{-\frac{N-2}{2}} S^{N/2}),$$

where S is the best constant for the Sobolev embedding $\mathcal{D}^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$ given in Section 3.

Lemma 4.1. *Let* (H3) *and* (H4) *hold. Then* $c > 0$*.*

Proof. By (H3) and (H4), Inequality (2.3) holds. Then for all $\epsilon > 0$ there exists $d_\epsilon > 0$ such that

$$|G(x,u)| \le \epsilon u^2 + d_\epsilon |u|^{2^*}, \quad \forall u \in \mathbb{R}.$$

Consequently

$$\begin{aligned} I(u) &= \frac{1}{2}\|u\|^2 - \int G(x,u)dx \\ &\ge (\frac{1}{2} - \epsilon)\|u\|^2 - d_\epsilon |u|_{2^*}^{2^*} \\ &\ge (\frac{1}{2} - \epsilon)\|u\|^2 - d_\epsilon C\|u\|^{2^*}. \end{aligned}$$

Let $\epsilon < 1/4$, then there exist small $r > 0$ and $\varrho > 0$ such that $I(u) \ge \varrho$, for $\|u\| = r$. For any $w \in X \setminus \{0\}$, there exists t_0 such that $\|t_0 w\| = r$. Then $I(t_0 w) \ge \varrho$. So

$$c = \inf_{w \in X \setminus \{0\}} \max_{t \ge 0} I(tw) \ge \varrho > 0$$

by (3.1). $\square$

To show that $c < \frac{1}{N}|K|_\infty^{-\frac{N-2}{2}} S^{N/2}$, by the definition of c, we shall choose a function $u \in M$ and show that $I(u) < \frac{1}{N}|K|_\infty^{-\frac{N-2}{2}} S^{N/2}$. The construct of u is based on a test function in $X \setminus \{0\}$. The test function is standard, see [23].

Without loss of generality, in the condition (H2), we assume that $x_0 = 0$. For $\epsilon > 0$, the function $w_\epsilon : \mathbb{R}^N \to \mathbb{R}$ defined by

$$w_\epsilon(x) = C(N) \frac{\epsilon^{\frac{N-2}{4}}}{(\epsilon + |x|^2)^{\frac{N-2}{2}}},$$

where $C(N) = [N(N-2)]^{\frac{N-2}{4}}$, is a family of functions on which S is attained. Let $\phi \in C_0^\infty(\mathbb{R}^N, [0,1])$, $\phi \equiv 1$ in $B_{\rho/2}(0)$, $\phi \equiv 0$ in $\mathbb{R}^N \setminus B_\rho(0)$. Then define the test function by

$$v_\epsilon = \frac{u_\epsilon}{(\int K(x) u_\epsilon^{2^*} dx)^{\frac{1}{2^*}}},$$

where $u_\epsilon = \phi w_\epsilon$.

The following lemma gives some properties for v_ϵ and u_ϵ, proved in [14].

Lemma 4.2. *Suppose that* (H2) *is satisfied. Then the following results hold:*

$$\int |\nabla v_\epsilon|^2 dx \le |K|_\infty^{\frac{2-N}{N}} S + O(\epsilon^{\frac{N-2}{2}}), \quad \text{as } \epsilon \to 0^+,$$

$$|v_\epsilon|_2^2 = \begin{cases} O(\epsilon^{\frac{N-2}{2}}), & \text{if } N = 3, \text{ as } \epsilon \to 0^+; \\ O(\epsilon|\log \epsilon|), & \text{if } N = 4, \text{ as } \epsilon \to 0^+; \\ O(\epsilon), & \text{if } N > 4, \text{ as } \epsilon \to 0^+. \end{cases} \tag{4.1}$$

Moreover, there exist positive constants k_1, k_2 and ϵ_0 such that

$$k_1 < \int K u_\epsilon^{2^*} dx < k_2, \quad \text{for all } 0 < \epsilon < \epsilon_0. \tag{4.2}$$

Now we are ready to prove that $c < \frac{1}{N}|K|_\infty^{-\frac{N-2}{2}} S^{N/2}$.

Lemma 4.3. *Suppose* (H1)–(H6) *are satisfied. Then*

$$c < \frac{1}{N}|K|_\infty^{-\frac{N-2}{2}} S^{N/2}.$$

Proof. By the definition of c, we just need to verify that there exists $v \in M$ such that

$$I(v) < \frac{1}{N}|K|_\infty^{-\frac{N-2}{2}} S^{N/2}. \tag{4.3}$$

We first claim that for $\epsilon > 0$ small enough, there exists constants $t_\epsilon > 0$, A_1 and A_2 independent of ϵ such that

$$I(t_\epsilon v_\epsilon) = \max_{t \geq 0} I(t v_\epsilon),$$

and

$$0 < A_1 < t_\epsilon < A_2 < \infty. \tag{4.4}$$

In fact, by Lemma 3.1 (i), there exists $t_\epsilon > 0$ such that

$$t_\epsilon v_\epsilon \in M, \quad I(t_\epsilon v_\epsilon) = \max_{t \geq 0} I(t v_\epsilon). \tag{4.5}$$

Then $I(t_\epsilon v_\epsilon) \geq c > 0$. So $t_\epsilon^2 \|v_\epsilon\|^2 \geq 2c$ by (2.4). Moreover, since $\|v_\epsilon\|$ is bounded for ϵ small enough by (4.1), then we conclude that there exists $A_1 > 0$ such that $t_\epsilon \geq A_1$, for every $\epsilon > 0$ sufficiently small. On the other hand, since $t_\epsilon v_\epsilon \in M$, we obtain $\langle I'(t_\epsilon v_\epsilon), t_\epsilon v_\epsilon \rangle = 0$. Noting that $\int K(x) v_\epsilon^{2^*}\, dx = 1$, we have

$$t_\epsilon^2 \|v_\epsilon\|^2 = t_\epsilon^{2^*} + \int f(x, t_\epsilon v_\epsilon) t_\epsilon v_\epsilon dx.$$

By (2.2) we find that

$$t_\epsilon^{2^*} \leq t_\epsilon^2 \|v_\epsilon\|^2 + \delta t_\epsilon^2 |v_\epsilon|_2^2 + C_\delta t_\epsilon^q |v_\epsilon|_q^q \leq (1+\delta) t_\epsilon^2 \|v_\epsilon\|^2 + C C_\delta t_\epsilon^q \|v_\epsilon\|^q.$$

Noting that $q < 2^*$, then there exists $A_2 > 0$ such that $t_\epsilon \leq A_2$ since $\|v_\epsilon\|$ is bounded for small ϵ.

Now we estimate $I(t_\epsilon v_\epsilon)$. Note that

$$\begin{aligned} I(t_\epsilon v_\epsilon) &\leq \frac{t_\epsilon^2}{2}(B_\epsilon + |V|_\infty |v_\epsilon|_2^2) - \frac{t_\epsilon^{2^*}}{2^*} - \int F(x, t_\epsilon v_\epsilon) dx \\ &= \Big(\frac{t_\epsilon^2}{2} B_\epsilon - \frac{t_\epsilon^{2^*}}{2^*}\Big) + \Big(\frac{t_\epsilon^2}{2}|V|_\infty |v_\epsilon|_2^2 - \int F(x, t_\epsilon v_\epsilon) dx\Big) \\ &:= I_1 + I_2, \end{aligned} \tag{4.6}$$

where $B_\epsilon := \int |\nabla v_\epsilon|^2 dx$. For I_1, considering the function $\theta : [0, \infty) \to \mathbb{R}$, $\theta(t) = \frac{1}{2} B_\epsilon t^2 - \frac{1}{2^*} t^{2^*}$, we have that $t_0 = B_\epsilon^{\frac{1}{2^*-2}}$ is a maximum point of θ and $\theta(t_0) = \frac{1}{N} B_\epsilon^{N/2}$. Then $I_1 \leq \frac{1}{N} B_\epsilon^{N/2}$. Combining with (4.1) we have

$$I_1 \leq \frac{1}{N}(|K|_\infty^{\frac{2-N}{N}} S + O(\epsilon^{\frac{N-2}{2}}))^{N/2} \leq \frac{1}{N}(|K|_\infty^{\frac{2-N}{2}} S^{N/2} + O(\epsilon^{\frac{N-2}{2}})), \tag{4.7}$$

where we applying the inequality

$$(a_1+a_2)^\zeta \le a_1^\zeta + \zeta(a_1+a_2)^{\zeta-1}a_2, \quad a_1,a_2\ge 0, \zeta\ge 1.$$

For I_2, given $A_0>0$, we invoke (H6) to obtain $R=R(A_0)>0$ such that, for $x\in\mathbb{R}^N$, $s\ge R$,

$$F(x,s)\ge \begin{cases} A_0 s^{2^*-2}, & \text{if N=3;}\\ A_0 s^2\log s, & \text{if N=4;}\\ A_0 s^2, & \text{if N>4.}\end{cases} \tag{4.8}$$

By (4.1) and (4.8), we estimate I_2 in the three cases about the dimension. First we assume that $N=3$.

For $|x|<\epsilon^{1/2}<\rho/2$, noting that $\phi\equiv 1$ in $B_{\rho/2}(0)$, by the definition of v_ϵ and (4.2), we find a constant $\alpha>0$ such that

$$t_\epsilon v_\epsilon(x) \ge \frac{A_1}{(k_2)^{\frac{1}{2^*}}}u_\epsilon(x) \ge \frac{A_1}{(k_2)^{\frac{1}{2^*}}}w_\epsilon(x) = \frac{A_1 C(N)}{(k_2)^{\frac{1}{2^*}}}\frac{\epsilon^{\frac{N-2}{4}}}{(\epsilon+|x|^2)^{\frac{N-2}{2}}} \ge \alpha\epsilon^{-\frac{N-2}{4}}, \tag{4.9}$$

here A_1 is given by (4.4). Then we may choose $\epsilon_1>0$ such that

$$t_\epsilon v_\epsilon(x)\ge \alpha\epsilon^{-\frac{N-2}{4}}\ge R,$$

for $|x|<\epsilon^{1/2}, 0<\epsilon<\epsilon_1$. From (4.8) it follows that

$$F(x,t_\epsilon v_\epsilon(x)) \ge A_0 t_\epsilon^{2^*-2} v_\epsilon^{2^*-2},$$

for $|x|<\epsilon^{1/2}$, $0<\epsilon<\epsilon_1$. Then for any $0<\epsilon<\epsilon_1$, by (4.9) we infer that

$$\begin{aligned} \int_{B_{\epsilon^{1/2}}(0)} F(x,t_\epsilon v_\epsilon)dx &\ge A_0\int_{B_{\epsilon^{1/2}}(0)} t_\epsilon^{2^*-2}v_\epsilon^{2^*-2}dx\\ &\ge A_0\alpha^{2^*-2}\int_{B_{\epsilon^{1/2}}(0)} \epsilon^{-\frac{N-2}{4}(2^*-2)}dx\\ &\ge A_0\alpha^{2^*-2}\epsilon^{-1}\omega_N\int_0^{\epsilon^{1/2}} r^{N-1}dr = A_0\alpha^{2^*-2}\frac{\omega_N}{N}\epsilon^{\frac{N-2}{2}}. \end{aligned}\tag{4.10}$$

For $|x|>\epsilon^{1/2}$, by (H4) and (H6), there exists $\eta>0$ such that

$$F(x,s)+\eta s^2\ge 0, \quad s\in\mathbb{R}.$$

Then by (4.1) we obtain

$$\int_{\mathbb{R}^N\setminus B_{\epsilon^{1/2}}(0)} F(x,t_\epsilon v_\epsilon)dx \ge -\eta t_\epsilon^2\int_{\mathbb{R}^N\setminus B_{\epsilon^{1/2}}(0)} v_\epsilon^2dx \ge -\eta A_2^2|v_\epsilon|_2^2 \ge -\tilde\eta\epsilon^{\frac{N-2}{2}}, \tag{4.11}$$

where A_2 is given by (4.4).

Combining (4.1), (4.10) and (4.11), we have

$$I_2\le C\epsilon^{\frac{N-2}{2}} - (A_0\alpha^{2^*-2}\frac{\omega_N}{N}-\tilde\eta)\epsilon^{\frac{N-2}{2}}.$$

Inserting the above inequality and (4.7) into (4.6), we find there exists a constant $C_1>0$ such that

$$I(t_\epsilon v_\epsilon)\le \frac{1}{N}|K|_\infty^{\frac{2-N}{2}}S^{N/2} + (C_1 - A_0\alpha^{2^*-2}\frac{\omega_N}{N}+\tilde\eta)\epsilon^{\frac{N-2}{2}}.$$

Since $A_0 > 0$ is arbitrary, we choose large enough A_0 such that $C_1 - A_0\alpha^{2^*-2}\frac{\omega_N}{N} + \tilde{\eta} < 0$. Then for small $\epsilon > 0$ we have

$$I(t_\epsilon v_\epsilon) < \frac{1}{N}|K|_\infty^{\frac{2-N}{2}} S^{N/2}.$$

Noting that $t_\epsilon v_\epsilon \in M$ by (4.5), then (4.3) establishes. Similarly, we can yield (4.3) for the other two cases with $N > 4$ and $N = 4$. This ends the proof. □

5. Proof of main theorem

Proof of Theorem 1.1. By Lemma 3.3, we only need to show that the infimum c is attained. Assume that $\{w_n\} \subset S_1$ is a minimizing sequence satisfying $\Psi(w_n) \to \inf_{S_1}\Psi$. By the Ekeland variational principle, we suppose $\Psi'(w_n) \to 0$. Then, from Proposition 3.2 (i) it follows that $I'(u_n) \to 0$, where $u_n = m(w_n) \in M$. Moreover, by Proposition 3.2 (ii), we have $I(u_n) = \Psi(w_n) \to c$. Applying Lemmas 4.3 and 3.4, we obtain that $\{u_n\}$ is bounded in X. Up to a subsequence, we assume that $u_n \rightharpoonup \tilde{u}$ in X, $u_n \to \tilde{u}$ in $L^2_{\rm loc}(\mathbb{R}^N)$ and $u_n \to \tilde{u}$ a.e. on $\mathbb{R}^N$. Using Remark 2.3, we have $I'(\tilde{u}) = 0$. Below we shall prove that if $\tilde{u} \neq 0$, it is just a minimizer. Otherwise, if $\tilde{u} = 0$, by concentration compactness principle and the periodicity of (2.1), we can still find a minimizer. Namely, we distinguish two cases that $\tilde{u} \neq 0$ and $\tilde{u} = 0$.

Case 1: $\tilde{u} \neq 0$. So $\tilde{u} \in M$ and then $I(\tilde{u}) \geq c$. Let

$$\tilde{G}(x,u) = \frac{1}{2}g(x,u)u - G(x,u). \tag{5.1}$$

Note that $\tilde{G}(x,u_n) \geq 0$. By Fatou's lemma, $\int \tilde{G}(x,\tilde{u})\,dx \leq \liminf \int \tilde{G}(x,u_n)\,dx$. Note that

$$c + o_n(1) = I(u_n) - \frac{1}{2}\langle I'(u_n), u_n\rangle = \int \tilde{G}(x,u_n)\,dx \tag{5.2}$$

and

$$I(\tilde{u}) = I(\tilde{u}) - \frac{1}{2}\langle I'(\tilde{u}), \tilde{u}\rangle = \int \tilde{G}(x,\tilde{u})\,dx.$$

It follows that $I(\tilde{u}) \leq c$. Therefore, $I(\tilde{u}) = c$.

Case 2: $\tilde{u} = 0$. This case is more complicated. We discuss that $\{u_n\}$ is vanishing or non-vanishing. It is easy to see that the case of vanishing does not happen since the energy $c \in (0, \frac{1}{N}|K|_\infty^{-\frac{N-2}{2}} S^{N/2})$ by Lemmas 4.1 and 4.3. In the case of non-vanishing, we can follow the similar idea in [14] to construct a minimizer.

Suppose $\{u_n\}$ is vanishing. Namely

$$\lim_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{B_1(y)} u_n^2(x)\,dx = 0.$$

Then Lions Compactness Lemma implies that $u_n \to 0$ in $L^q(\mathbb{R}^N)$. As in the proof of Lemma 3.4, we easily have $\int F(x,u_n)dx \to 0$ and $\int f(x,u_n)u_n dx \to 0$. Note that

$$I(u_n) \to c, \quad \langle I'(u_n), u_n\rangle \to 0.$$

Therefore,

$$c = \frac{1}{2}\|u_n\|^2 - \frac{1}{2^*}\int K(x)|u_n|^{2^*}dx + o_n(1), \tag{5.3}$$

$$\|u_n\|^2 = \int K(x)|u_n|^{2^*}dx + o_n(1). \tag{5.4}$$

By (5.4) we obtain

$$\|u_n\|^2 \le |K|_\infty |u_n|_{2^*}^{2^*} + o_n(1) \le |K|_\infty S^{-\frac{2^*}{2}} \|u_n\|^{2^*} + o_n(1). \tag{5.5}$$

If $\|u_n\| \to 0$, then it follows from (5.3) and (5.4) that $c = 0$. However, from Lemma 4.1 we obtain $c > 0$. This is a contradiction. Then $\|u_n\| \not\to 0$. So $\|u_n\| \ge |K|_\infty^{-\frac{N-2}{4}} S^{\frac{N}{4}} + o_n(1)$ by (5.5). Then from (5.3) and (5.4) we easily conclude that $c \ge \frac{1}{N}|K|_\infty^{-\frac{N-2}{2}} S^{N/2}$ contradicting the fact that $c < \frac{1}{N}|K|_\infty^{-\frac{N-2}{2}} S^{N/2}$ by Lemma 4.3.

Hence $\{u_n\}$ is non-vanishing. Then there exists $x_n \in \mathbb{R}^N$ and $\delta_0 > 0$ such that

$$\int_{B_1(x_n)} u_n^2(x)dx > \delta_0. \tag{5.6}$$

Without loss of generality, we assume that $x_n \in \mathbb{Z}^N$. Since $u_n \to \tilde{u}$ in $L^2_{\rm loc}(\mathbb{R}^N)$ and $\tilde{u} = 0$, we may suppose that $|x_n| \to \infty$ up to a subsequence. Denote $\bar{u}_n$ by $\bar{u}_n(\cdot) = u_n(\cdot + x_n)$. Similarly, passing to a subsequence, we assume that $\bar{u}_n \rightharpoonup \bar{u}$ in X, $\bar{u}_n \to \bar{u}$ in $L^2_{\rm loc}(\mathbb{R}^N)$, and $\bar{u}_n \to \bar{u}$ a.e. on $\mathbb{R}^N$. By (5.6) we have

$$\int_{B_1(0)} \bar{u}_n^2(x)dx > \delta_0.$$

So $\bar{u} \ne 0$.

We first claim that

$$I_p'(\bar{u}) = 0. \tag{5.7}$$

Indeed, for all $\psi \in X$, set $\psi_n(\cdot) := \psi(\cdot + x_n)$. From Lemma 3.6, replacing φ_n by ψ_n it follows that

$$\int (V(x) - V_p(x))u_n\psi_n dx \to 0,$$

$$\int (K(x) - K_p(x))|u_n|^{2^*-2}u_n\psi_n dx \to 0.$$

Moreover, replacing φ_n by ψ_n again, Lemma 3.7 implies

$$\int [f(x, u_n) - f_p(x, u_n)]\psi_n dx \to 0.$$

Consequently,

$$\langle I'(u_n), \psi_n\rangle - \langle I_p'(u_n), \psi_n\rangle \to 0.$$

Noting that $I'(u_n) \to 0$ and $\|\psi_n\| = \|\psi\|$, we have $\langle I'(u_n), \psi_n\rangle \to 0$. So

$$\langle I_p'(u_n), \psi_n\rangle \to 0.$$

Moreover, by the periodicity of V_p, K_p and f_p in the variable x and $x_n \in \mathbb{Z}^N$, we obtain

$$\langle I_p'(\bar{u}_n), \psi\rangle = \langle I_p'(u_n), \psi_n\rangle.$$

Then $\langle I_p'(\bar{u}_n), \psi\rangle \to 0$. By the arbitrary of ψ, $I_p'(\bar{u}_n) \rightharpoonup 0$ in X^*. Since I_p' is weakly sequentially continuous (Remark 2.3), (5.7) holds.

Now we prove that

$$I_p(\bar{u}) \le c. \tag{5.8}$$

Replacing φ_n by u_n, Lemma 3.7 yields

$$\int [f(x, u_n)u_n - f_p(x, u_n)u_n]dx \to 0. \tag{5.9}$$

It follows from Remark 3.8 that

$$\int [F(x,u_n) - F_p(x,u_n)]dx \to 0. \tag{5.10}$$

Set $\tilde{G}_p(x,u) = \frac{1}{2}g_p(x,u)u - G_p(x,u)$. By the condition $K \geq K_p$, (5.9) and (5.10), we obtain

$$\begin{aligned}
\int \tilde{G}_p(x,u_n) &= \frac{1}{N}\int K_p(x)|u_n|^{2^*}dx + \int (\frac{1}{2}f_p(x,u_n)u_n - F_p(x,u_n))\,dx \\
&\leq \frac{1}{N}\int K(x)|u_n|^{2^*}dx + \int (\frac{1}{2}f_p(x,u_n)u_n - F_p(x,u_n))\,dx \\
&= \frac{1}{N}\int K(x)|u_n|^{2^*}dx + \int (\frac{1}{2}f(x,u_n)u_n - F(x,u_n))\,dx + o_n(1) \\
&= \int \tilde{G}(x,u_n)dx + o_n(1),
\end{aligned}$$

where $\tilde{G}$ is given in (5.1). Noting that $\tilde{G}_p$ is 1-periodic in x, we have

$$\int \tilde{G}_p(x,\bar{u}_n)dx = \int \tilde{G}_p(x,u_n)dx.$$

Therefore,

$$\int \tilde{G}_p(x,\bar{u}_n)dx \leq \int \tilde{G}(x,u_n)dx + o_n(1).$$

From (2.5) and Fatou's Lemma it follows that

$$\int \tilde{G}_p(x,\bar{u})dx + o_n(1) \leq \int \tilde{G}_p(x,\bar{u}_n)dx.$$

So

$$\int \tilde{G}_p(x,\bar{u})dx \leq \int \tilde{G}(x,u_n)dx + o_n(1).$$

Combining with (5.2) and (5.7) we obtain

$$\begin{aligned}
c + o_n(1) = \int \tilde{G}(x,u_n)dx &\geq \int \tilde{G}_p(x,\bar{u})dx + o_n(1) \\
&= I_p(\bar{u}) - \frac{1}{2}\langle I_p'(\bar{u}),\bar{u}\rangle + o_n(1) = I_p(\bar{u}) + o_n(1).
\end{aligned}$$

So we have $I_p(\bar{u}) \leq c$.

We shall verify that $\max_{t\geq 0} I_p(t\bar{u}) = I_p(\bar{u})$. Indeed, let $\chi(t) = I_p(t\bar{u})$, $t > 0$. Then

$$\chi'(t) = t\big(\|\bar{u}\|_p^2 - \int \frac{g_p(x,t\bar{u})\bar{u}}{t}dx\big) := t\tilde{A}(t).$$

Since $I_p'(\bar{u}) = 0$ by (5.7), $\tilde{A}(1) = 0$. By (H7)-(iv), $\tilde{A}$ is non-increasing in $(0,\infty)$, then $\tilde{A}(t) \geq 0$ when $0 < t < 1$ and $\tilde{A}(t) \leq 0$ when $t > 1$. Hence $\chi'(t) \geq 0$ when $0 < t < 1$ and $\chi'(t) \leq 0$ when $t > 1$. Therefore, $\max_{t\geq 0} I_p(t\bar{u}) = I_p(\bar{u})$.

Using Lemma 3.1 (i), there exists $t_{\bar{u}} > 0$ such that $t_{\bar{u}}\bar{u} \in M$. Then by Lemma 3.5 we infer

$$I(t_{\bar{u}}\bar{u}) \leq I_p(t_{\bar{u}}\bar{u}) \leq \max_{t\geq 0} I_p(t\bar{u}) = I_p(\bar{u}).$$

With the use of (5.8), we have $I(t_{\bar{u}}\bar{u}) \leq c$. Noting that $t_{\bar{u}}\bar{u} \in M$, we obtain $I(t_{\bar{u}}\bar{u}) \geq c$. Then $I(t_{\bar{u}}\bar{u}) = c$.

In a word, we deduce that c is attained, and the corresponding minimizer is a ground state solution of (1.2). This completes the proof. $\square$

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