COEFFICIENTS OF SINGULARITIES FOR A SIMPLY SUPPORTED PLATE PROBLEMS IN PLANE SECTORS

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Abstract. This article represents the solution to a plate problem in a plane sector that is simple supported, as a series. By using appropriate Green’s functions, we establish a biorthogonality relation between the terms of the series, which allows us to calculate the coefficients.

1. Introduction

Let $S$ be the truncated plane sector of angle $\omega \leq 2\pi$, and radius $\rho$ ($\rho$ is positive and fixed) defined by:

$$S = \{(r \cos \theta, r \sin \theta) \in \mathbb{R}^2, 0 < r < \rho, \ 0 < \theta < \omega\}$$

(1.1)

and $\Sigma$ the circular boundary part

$$\Sigma = \{(\rho \cos \theta, \rho \sin \theta) \in \mathbb{R}^2, 0 < \theta < \omega\}.$$  

(1.2)

We are interested in the study of a function $u$, belonging to the Sobolev space $H^2(S)$, and being the solution of

$$\Delta^2 u = 0, \quad \text{in } S$$

$$u = Mu = 0, \quad \text{for } \theta = 0, \omega,$$

(1.3)

where the operator $M$ represents the bending moment and is defined as

$$Mu = \nu \Delta u + (1 - \nu)(\partial_{11} u n_1^2 + 2\partial_{12} u n_1 n_2 + \partial_{22} u n_2^2).$$

(1.4)

Here $\nu$ is a real number ($0 < \nu < 1/2$) called Poisson coefficient and $n = (n_1, n_2)$ is the unit outward normal vector to $\Gamma_0$ and $\Gamma_1$ (See Figure 1).

The boundary conditions $u = 0$ and $Mu = 0$, for $\theta = 0, \theta = \omega$ mean that the plate is simply supported.

This type of boundary conditions arises in problems of linear or non linear vibrations of thin imperfect plates. See for example [2, pages 5,6] and the references therein.
We show that the solutions \( u \) of this problems can be written as a series of the form
\[
u(r, \theta) = \sum_{\alpha \in E} c_{\alpha} r^\alpha \phi_{\alpha} (\theta).
\] (1.5)

Here \( E \) stands for the set of solutions of the equation in a complex variable \( \alpha \):
\[
\sin^2 (\alpha - 1) \omega = \sin^2 \omega, \quad \text{Re} \alpha > 1 \quad (1.6)
\]

For further studies of the set \( E \), see for example Blum and Rannacher [1], and Grisvard [4].

We will compute the coefficients \( c_{\alpha} \) in (1.5). This sort of calculations have already been done by Tcha-Kondor [5] for the Dirichlet’s boundary conditions, and by Chikouche-Aibeche [3] for the Neumann’s boundary conditions. These authors have established, thanks to the Green’s formula, a relation of biorthogonality between any two functions \( \phi_{\alpha} \), which allows them to calculate the coefficients \( c_{\alpha} \). We follow the same approach. Thus we need to write the appropriate Green formula for the domain \( S \). Using this formula, we establish a relation of biorthogonality between the functions \( \phi_{\alpha} \).

In the case of a crack domain \( (\omega = 2\pi) \) this relation reduces to the simple one obtained by Tcha-Kondor. This enables us, in this particular situation, to find an explicit formula for the coefficients \( c_{\beta} \).

2. Separation of variables

Replacing \( u \) by \( r^\alpha \phi_{\alpha}(\theta) \) in problem (1.3) leads us to the boundary value problem
\[
\phi_{\alpha}^{(3)}(\theta) + (2\alpha^2 - 4\alpha + 4)\phi_{\alpha}^{(2)}(\theta) + \alpha^2 (\alpha - 2)^2 \phi_{\alpha}(\theta) = 0, \quad (2.1)
\]
\[
\phi_{\alpha}^{(2)}(\theta) + [\nu \alpha^2 + (1 - \nu)\alpha] \phi_{\alpha}(\theta) = 0, \quad \theta = 0, \quad \theta = \omega \quad (2.2)
\]
\[
\phi_{\alpha}(\theta) = 0, \quad \theta = 0, \quad \theta = \omega \quad (2.3)
\]
A relation similar to the orthogonality obtained for the biharmonic operator is given, in the following theorem.

**Theorem 2.1.** Let \( \phi_\alpha \) and \( \phi_\beta \) be solutions of (2.1) with \( \alpha \) and \( \beta \) solutions of (1.6). Then, for \( \alpha \neq \beta \), one has

\[
[\phi_\alpha, \phi_\beta] = \int_0^\omega \left[ (\alpha^2 - 2\alpha)\phi_\alpha - \frac{\nu(\alpha + \beta) + (3 - \nu) - 2\alpha}{\alpha - \beta} \phi_\alpha'' \phi_\beta'' + \frac{\nu(\alpha + \beta) + (3 - \nu) - 2\beta}{\alpha - \beta} \phi_\alpha' \phi_\beta' \right] d\theta = 0. \tag{2.4}
\]

**Proof.** We use the Green formula

\[
\int_S (v\Delta^2 u - u\Delta^2 v) dx = \int_{\Gamma} \left[ (uNv + \frac{\partial u}{\partial n} Mv) - (vNu + \frac{\partial v}{\partial n} Mu) \right] d\sigma, \tag{2.5}
\]

where

\[
Nu = -\frac{\partial \Delta u}{\partial n} + (1 - \nu)(\partial_1^2 u n_1 n_2 - \partial_2^2 u (n_1^2 - n_2^2) + \partial_2^2 u n_1 n_2),
\]

and \( \Gamma \) is the boundary of \( S \). For two functions \( u,v \) which are solutions of (1.3), using the Green’s formula we obtain

\[
\int_\Sigma \left[ (uNv + \frac{\partial u}{\partial n} Mv) - (vNu + \frac{\partial v}{\partial n} Mu) \right] d\sigma = 0 \tag{2.6}
\]

On \( \Sigma \), for the function \( u_\alpha = r^\alpha \phi_\alpha \), we have

\[
\frac{\partial u_\alpha}{\partial n} = \frac{\partial u_\alpha}{\partial r} = \alpha r^{\alpha-1} \phi_\alpha,
\]

\[
M_{u_\alpha} = r^{\alpha-2} \{ [\alpha^2 - (1 - \nu)\alpha] \phi_\alpha + \nu \phi_\alpha'' \}, \tag{2.7}
\]

\[
Nu_\alpha = r^{\alpha-3} \{ -\alpha^2 (\alpha - 2) \phi_\alpha + [(\nu - 2)\alpha + (3 - \nu)] \phi_\alpha'' \}.
\]

The results follow from the application of formula (2.6) to the biharmonic functions \( u_\alpha = r^\alpha \phi_\alpha \) and \( u_\beta = r^\beta \phi_\beta \), and by using relations (2.7). \( \square \)

**Remark 2.2.** The relation (2.4) between the functions \( \phi_\alpha \) and \( \phi_\beta \) is similar to the relation of biorthogonality obtained when the functions \( \phi_\alpha \) and \( \phi_\beta \) satisfying (2.1) with the Dirichlet boundary conditions \( \phi_\alpha = \phi'_\alpha = \phi_\beta = \phi'_\beta = 0 \) for \( \theta = 0 \) and \( \theta = \omega \). In this latter case, the relation is given by

\[
\int_0^\omega \phi_\alpha \phi_\beta'' d\theta = \int_0^\omega \phi_\alpha'' \phi_\beta d\theta \tag{2.8}
\]

which is obtained by a double integration by parts:

\[
\int_0^\omega \phi_\alpha \phi_\beta'' d\theta = \int_0^\omega \phi_\alpha'' \phi_\beta d\theta + [\phi_\alpha, \phi_\beta']_0^\omega - [\phi_\alpha', \phi_\beta]_0^\omega, \tag{2.9}
\]

and using the Dirichlet’s boundary conditions.

The following corollary is an immediate consequence of remark 2.2.

**Corollary 2.3.** Let \( \phi_\alpha \) and \( \phi_\beta \) be solutions of (2.1) with \( \alpha \) and \( \beta \) solutions of (2.6). Suppose in addition that

\[
[\phi_\alpha, \phi_\beta]_0^\omega - [\phi'_\alpha, \phi_\beta]_0^\omega = 0, \tag{2.10}
\]
and \( \alpha \neq \beta \), then
\[
[\phi_\alpha, \phi_\beta] = \int_0^{2\pi} \left\{ (\alpha^2 - 2\alpha)\phi_\alpha + \phi_\alpha''' \overline{\phi_\beta} + ((\beta^2 - 2\beta)\phi_\beta + \overline{\phi_\beta}''\phi_\alpha) \right\} d\theta = 0 \quad (2.11)
\]

**Remark 2.4.** For \( u_\alpha = r^\alpha \phi_\alpha \) we have
\[
\Delta u_\alpha - \frac{2}{r} \frac{\partial u_\alpha}{\partial r} = r^{\alpha-2}[(\alpha^2 - 2\alpha)\phi_\alpha + \phi_\alpha'']. \quad (2.12)
\]

Let \( P \) be the operator \( P = \Delta - \frac{2}{r} \frac{\partial}{\partial r} \). From the corollary 2.3 and Remark 2.4, we deduce the following result.

**Corollary 2.5.** Under the hypotheses of corollary 2.3, if \( \alpha \neq \beta \), we have
\[
\int_\Sigma \left( P u_\alpha \cdot \overline{u_\beta} + u_\alpha \cdot P \overline{u_\beta} \right) d\sigma = 0. \quad (2.13)
\]

### 3. Formula for the coefficients in the crack case

The crack case (\( \omega = 2\pi \)) is an important one, among singular domains, in the applications. Moreover in this case the solutions of (2.6) are explicitly known and we have
\[
E = \{ \frac{k}{2}, k \in \mathbb{N}, k > 2 \}
\]
and these roots are of multiplicity 2.

In this framework we assume that the solution \( u \) admits the representation
\[
u = \sum_{\alpha \in E} (c_\alpha u_\alpha + d_\alpha v_\alpha), \quad E = \{ \frac{k}{2}, k \in \mathbb{N}, k > 2 \},
\]

\[
\begin{align*}
    u_\alpha &= r^\alpha \phi_\alpha(\theta), \\
    v_\alpha &= r^\alpha \psi_\alpha(\theta)
\end{align*}
\]
the solutions \( \phi_\alpha \) and \( \psi_\alpha \), in terms of \( \theta \), are the odd functions:
\[
\begin{align*}
    \phi_\alpha(\theta) &= \sin(\alpha - 2)\theta, \\
    \psi_\alpha(\theta) &= \sin \alpha \theta.
\end{align*}
\]
and since \( \alpha = k/2 \), we obtain
\[
\begin{align*}
    \phi_\alpha(0) &= \phi_\alpha(\omega) = \psi_\alpha(0) = \psi_\alpha(\omega) = 0
\end{align*}
\]
and thus
\[
\begin{align*}
    [\phi_\alpha, \phi_\beta]^\omega_0 &= [\phi_\alpha', \phi_\beta]^\omega_0 = 0, \\
    [\psi_\alpha, \psi_\beta]^\omega_0 &= [\psi_\alpha', \psi_\beta]^\omega_0 = 0, \\
    [\phi_\alpha, \psi_\beta]^\omega_0 &= [\phi_\alpha', \psi_\beta]^\omega_0 = 0.
\end{align*}
\]

From here comes the idea of decomposing the solution \( u \) of (1.3) into two parts as follows:
\[
\begin{align*}
    w_1 &= \sum_{\alpha \in E_1} (c_\alpha u_\alpha + d_\alpha v_\alpha), \quad i = 1, 2, \\
    E_1 &= \{ 2m, m > 1 \}, \quad E_2 = \{ 2m + 1, 2m > 1 \}
\end{align*}
\]
Calculation of $c_\beta$ and $d_\beta$. From the expressions of $\phi_\alpha$, $\psi_\alpha$ one easily sees that:

if $\alpha \in E_1$, then $\phi'_\alpha(0) = \phi'_\alpha(\omega)$ and $\psi'_\alpha(0) = \psi'_\alpha(\omega)$,
if $\alpha \in E_2$, then $\phi'_\alpha(0) = -\phi'_\alpha(\omega)$ and $\psi'_\alpha(0) = -\psi'_\alpha(\omega)$.

Equations (3.4) and (3.6) allow us to apply corollary 2.5 to functions $u_\alpha$ and $u_\beta$ (resp. $u_\alpha, v_\alpha$ and $u_\beta, v_\beta$) and get the relations:

\[
\int_{\sigma}(Pw_1 \cdot u_\beta + w_1 \cdot Pu_\beta)\,d\sigma = 2c_\beta \int_{\sigma}(u_\beta \cdot Pu_\beta)\,d\sigma + d_\beta \int_{\sigma}(Pv_\beta \cdot u_\beta + v_\beta \cdot Pu_\beta)\,d\sigma, \\
\int_{\sigma}(Pw_1 \cdot v_\beta + w_1 \cdot Pv_\beta)\,d\sigma = c_\beta \int_{\sigma}(Pu_\beta \cdot v_\beta + u_\beta \cdot Pv_\beta)\,d\sigma + 2d_\beta \int_{\sigma}(Pv_\beta \cdot v_\beta)\,d\sigma.
\]

By direct calculations we obtain

\[
\int_{\sigma}(Pu_\beta \cdot v_\beta + u_\beta \cdot Pv_\beta)\,d\sigma = 0, \\
\int_{\sigma}(u_\beta \cdot Pu_\beta)\,d\sigma = (\beta - 2)\omega \rho^{2\beta - 1} \tag{3.8}
\]

and from this we get our main the result.

**Theorem 3.1.** Let $u$ be a the solution of (1.3) written in the form

\[
u = w_1 + w_2 \tag{3.9}
\]

where

\[
w_i = \sum_{\alpha \in E_i} (c_\alpha u_\alpha + d_\alpha v_\alpha), \quad i = 1, 2 \tag{3.10}
\]

Suppose that the series that gives $w_i$ is uniformly convergent in $S$. Then for any $\alpha \in E_i, \ i = 1, 2$ we have

\[
c_\alpha = \frac{\rho^{1-2\alpha}}{2(\alpha - 2)\omega} \int_{\sigma}(Pw_1 \cdot u_\alpha + w_1 \cdot Pu_\alpha)\,d\sigma \tag{3.11}
\]

\[
d_\alpha = \frac{\rho^{1-2\alpha}}{2\alpha \omega} \int_{\Sigma}(Pu_1 \cdot v_\alpha + w_1 \cdot Pv_\alpha)\,d\sigma
\]

**Remark 3.2.** Let $\zeta \in H^{3/2}(\Sigma) \cap H^1_0(\Sigma)$ be the trace of $u$ on $\Sigma$ and $\chi \in H^{-1/2}(\Sigma)$ the trace of $Pu$ on $\Sigma$.

If $u$ is regular in order that $\zeta \in H^4([0, 2\pi])$ and $\chi \in H^2([0, 2\pi])$, then we have a uniform convergence of the series in $S_{\rho_0}$ for all $\rho_0 < \rho$, see [3].

### 3.1. Independence of the coefficients.

**Proposition 3.3.** The coefficients $c_\beta$ (resp $d_\beta$) are independent of $\rho$.

**Proof.** Let us prove that the derivative of $c_\beta$ with respect to $\rho$ is zero. Observing the expression of $c_\beta$ in Theorem 3.1, we just have to prove that the derivative, with respect to $\rho$, of

\[
\gamma_\beta = \rho^{1-2\beta} \int_{\sigma}(Pw_1 \cdot u_\beta + w_1 \cdot Pu_\beta)\,d\sigma. \tag{3.12}
\]
vanishes. By derivation with respect to \( r \) we have
\[
\gamma_\beta' = \int_0^\omega \left\{ \frac{\partial}{\partial r} (\Delta w_i) r^{2-\beta} \phi_\beta + [(2-\beta) \Delta w_i - 2 \frac{\partial^2 w_i}{\partial r^2} + (\beta^2 - 2) \frac{1}{r} \frac{\partial w_i}{\partial r}] r^{1-\beta} \phi_\beta \\
+ \frac{\partial u_i}{\partial r} r^{-\beta} \phi_\beta'' - \beta w_i r^{-1-\beta} [(\beta^2 - 2\beta) \phi_\beta + \phi_\beta''] \right\} d\theta.
\]
(3.13)

On \( \Sigma \), we have
\[
\frac{\partial}{\partial r} (\Delta w_i) = -N w_i + (1 - v) \left[ \frac{1}{r^3} \frac{\partial^2 w_i}{\partial \theta^2} - \frac{1}{r^2} \frac{\partial^2 w_i}{\partial r \partial \theta^2} \right],
\]
and
\[
(2 - \beta) \Delta w_i - 2 \frac{\partial^2 w_i}{\partial r^2} = -\beta M w_i + [2 - (1 - v)\beta] \left[ \frac{1}{r} \frac{\partial w_i}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w_i}{\partial \theta^2} \right].
\]
(3.14)

Using these formulas in the expression of \( \gamma_\beta' \) we obtain
\[
\gamma_\beta' = -\int_0^\omega (\beta M w_i r^{1-\beta} \phi_\beta + N w_i r^{2-\beta} \phi_\beta) d\theta \\
+ \int_0^\omega \left\{ [(\beta^2 - (1 - v)\beta) \phi_\beta + \phi_\beta''] \frac{\partial u_i}{\partial r} - (1 - v) \frac{\partial^3 u_i}{\partial r \partial \theta^2} \phi_\beta \right\} r^{-\beta} d\theta \\
+ \int_0^\omega \left\{ [2 - (1 - v)(\beta - 1)] \frac{\partial^2 w_i}{\partial \theta^2} \phi_\beta - \beta w_i [\beta^2 - 2\beta] + \phi_\beta'' \right\} r^{-1-\beta} d\theta.
\]
(3.15)

By a double integration by parts, we verify that
\[
\int_0^\omega \frac{\partial^2 w_i}{\partial \theta^2} \phi_\beta d\theta = \int_0^\omega w_i \phi_\beta'' d\theta
\]
(3.16)
\[
\int_0^\omega \frac{\partial^3 w_i}{\partial r \partial \theta^2} \phi_\beta d\theta = \int_0^\omega \frac{\partial w_i}{\partial r} \phi_\beta'' d\theta
\]
(3.17)

Using (3.15)–(3.17) in the expression of \( \gamma_\beta' \) and putting the \( \rho^{1-2\beta} \), we obtain
\[
\gamma_\beta' = -\rho^{1-2\beta} \omega (\beta M w_i r^{\beta-1} \phi_\beta + N w_i r^{\beta} \phi_\beta) \rho d\theta \\
+ \rho^{1-2\beta} \int_0^\omega [(\beta^2 - (1 - v)\beta) \phi_\beta + v \phi_\beta''] r^{\beta-2} \frac{\partial w_i}{\partial r} \rho d\theta \\
+ \rho^{1-2\beta} \int_0^\omega \{-[\beta^2 (\beta - 2)] \phi_\beta + [-2 - (2 - \beta) + (3 - v)] \phi_\beta'' \} r^{\beta-3} w_i \rho d\theta.
\]
(3.18)

Taking into account of (2.7), whose expressions appear explicitly in \( \gamma_\beta' \), we obtain
\[
\gamma_\beta' = \rho^{1-2\beta} \int_\Sigma \{ (u_\beta N w_i + \frac{\partial u_\beta}{\partial n} M w_i) - (u_i N u_\beta + \frac{\partial u_i}{\partial n} M u_\beta) \} \, d\sigma = 0.
\]
(3.19)

We follow the same analysis to prove the independence of \( d_\beta \) with respect to \( \rho \). □
3.2. Convergence of the series. We first write $c_\alpha$ and $d_\alpha$ in the form
\begin{equation}
    c_\alpha = I_i \rho^{-\alpha}, \quad d_\alpha = J_i \rho^{-\alpha}
\end{equation}
with
\begin{align}
    I_i &= \frac{\rho}{2\omega(\alpha - 2)} \int_\sigma (Pw_i \phi_\alpha + w_i \rho^{-2}[(\alpha^2 - 2\alpha)\phi_\alpha + \phi''_\alpha]) d\sigma, \\
    J_i &= \frac{-\rho}{2\omega \alpha} \int_\sigma (Pw_i \psi_\alpha + w_i \rho^{-2}[(\alpha^2 - 2\alpha)\psi_\alpha + \psi''_\alpha]) d\sigma.
\end{align}
The solution $u$ of (1.3) is then written as
\begin{equation}
    u = w_1 + w_2
\end{equation}
and we have the following result.

**Theorem 3.4.** The series (3.23) converges uniformly in $S_{\rho_0}$ for all $\rho_0 < \rho$.

**Proof.** Set
\begin{equation}
    H_{i,\alpha} = \int_0^\omega (Pw_i \phi_\alpha + u_i \rho^{-2}[(\alpha^2 - 2\alpha)\phi_\alpha + \phi''_\alpha]) d\theta
\end{equation}
We show that $H_{i,\alpha}$ is $1/\alpha$ times by bounded term, for $\alpha$ large enough. According to (3.17), we have
\begin{equation}
    \int_0^\omega u_i \phi''_\alpha d\theta = \int_0^\omega u_i \phi_\alpha d\theta
\end{equation}
Replacing $\phi_\alpha$ by its expression and integrating by parts we obtain
\begin{equation}
    \int_0^\omega u_i' \phi''_\alpha d\theta = \frac{1}{\alpha \omega(\alpha - 2)} \int_0^\omega u_i'' \cos(\alpha - 2) d\theta
\end{equation}
On the other hand, by a triple integration by parts, we have
\begin{equation}
    (\alpha^2 - 2\alpha) \int_0^\omega u_i \phi_\alpha d\theta = \frac{1}{\alpha \omega(\alpha - 2)} \int_0^\omega u_i'' \cos(\alpha - 2) d\theta
\end{equation}
Also, integrating by parts, we obtain
\begin{equation}
    \int_0^\omega (∆ u_i - \frac{2}{r} \frac{\partial u_i}{\partial r}) \phi_\alpha d\theta = \frac{1}{\alpha} \int_0^\omega \nabla^2 u_i \cos(\alpha - 2) d\theta
\end{equation}
Then, we deduce the existence of a constant $C_0$ such that:
\begin{equation}
    |H_{i,\alpha}| \leq \frac{C_0}{\alpha}
\end{equation}
Using this last inequality and the fact that $\phi_\alpha$ is bounded as well as the term $1/(2\omega(\alpha - 2))$ for large $\alpha$ we deduce the existence of a constant $C$ such that
\begin{equation}
    \left| \sum_{\alpha \in E_i} c_\alpha r^\alpha \phi_\alpha \right| \leq \sum_{\alpha \in E_i} \frac{C}{\alpha} (\frac{r}{\rho})^\alpha
\end{equation}
which converges uniformly in $S_{\rho_0}$ for $\rho_0 < \rho$. Convergence of $\sum_{\alpha \in E_i} d_\alpha r^\alpha \psi_\alpha$ is proved by the same way. □
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