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GENERALIZED PICONE'S IDENTITY AND ITS APPLICATIONS

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Abstract. In this article we give a generalized version of Picone's identity in a nonlinear setting for the p-Laplace operator. As applications we give a Sturmian Comparison principle and a Liouville type theorem. We also study a related singular elliptic system.

1. Introduction

The classical Picone's identity states that, for differentiable functions $v > 0$ and $u \geq 0$, we have

$$|\nabla u|^2 + \frac{u^2}{v^2}|\nabla v|^2 - 2\frac{u}{v}\nabla u \nabla v = |\nabla u|^2 - \nabla(\frac{u^2}{v})\nabla v \geq 0 \tag{1.1}$$

Later Allegreto-Huang [1] presented a Picone's identity for the p-Laplacian, which is an extension of (1.1). As an immediate consequence, they obtained a wide array of applications including the simplicity of the eigenvalues, Sturmian comparison principles, oscillation theorems and Hardy inequalities to name a few. This work motivated a lot of generalization of the Picone's identity in different cases see [3, 6, 7] and the reference therein. In a recent paper Tyagi [7] proved a generalized version of Picone's identity in the nonlinear framework, asking the question about the Picone's identity which can deal with problems of the type:

$$\begin{gathered} -\Delta u = a(x)f(u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered}$$

where Ω is a open, bounded subset of $\mathbb{R}^n$.

They proved that for differentiable functions $v > 0$ and $u \geq 0$ we have

$$|\nabla u|^2 + \frac{|\nabla u|^2}{f'(v)} + (\frac{u\sqrt{f'(v)}\nabla v}{f(v)} - \frac{\nabla u}{\sqrt{f'(v)}})^2 = |\nabla u|^2 - \nabla(\frac{u^2}{f(v)}) \cdot \nabla v \geq 0 \tag{1.2}$$

where $f(y) \neq 0$ and $f'(y) \geq 1$ for all $y \neq 0$; $f(0) = 0$.

Moreover $|\nabla u|^2 - \nabla(u^2/f(v)) \cdot \nabla v = 0$ holds if and only if $u = cv$ for an arbitrary constant c. In this article, we generalize the main result of Tyagi [7] for the p-laplacian operator; i.e, we will give a nonlinear analogue of the Picone's identity for the p-Laplacian operator.

In this work, we assume the following hypothesis:

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- Ω denotes any domain in $\mathbb{R}^n$.
- $1 < p < \infty$.
- $f : (0,\infty) \to (0,\infty)$ be a C^1 function.

2. Main Results

We first start with the Picone's identity for p-Laplacian.

Theorem 2.1. *Let $v > 0$ and $u \geq 0$ be two non-constant differentiable functions in Ω. Also assume that $f'(y) \geq (p-1)[f(y)^{\frac{p-2}{p-1}}]$ for all y. Define*

$$L(u,v) = |\nabla u|^p - \frac{pu^{p-1}\nabla u|\nabla v|^{p-2}\nabla v}{f(v)} + \frac{u^p f'(v)|\nabla v|^p}{[f(v)]^2}.$$

$$R(u,v) = |\nabla u|^p - \nabla(\frac{u^p}{f(v)})|\nabla v|^{p-2}\nabla v.$$

Then $L(u,v) = R(u,v) \geq 0$. Moreover $L(u,v) = 0$ a.e. in Ω if and only if $\nabla(\frac{u}{v}) = 0$ a.e. in Ω.

Remark 2.2. When $p = 2$ and $f(y) = y$ we get the Classical Picone's Identity (1.1) for Laplacian and when $p = 2$ we get back its nonlinear version (1.2).

Proof of Theorem 2.1. Expanding $R(u,v)$ by direct calculation we get $L(u,v)$. To show $L(u,v) \geq 0$ we proceed as follows,

$$\begin{aligned}
L(u,v) &= |\nabla u|^p - \frac{pu^{p-1}\nabla u|\nabla v|^{p-2}\nabla v}{f(v)} + \frac{u^p f'(v)|\nabla v|^p}{[f(v)]^2} \\
&= |\nabla u|^p + \frac{u^p f'(v)|\nabla v|^p}{[f(v)]^2} - \frac{pu^{p-1}|\nabla u||\nabla v|^{p-1}}{f(v)} \\
&\quad + \frac{pu^{p-1}|\nabla v|^{p-2}}{f(v)}\{|\nabla u||\nabla v| - \nabla u\nabla v\} \\
&= p\Big(\frac{|\nabla u|^p}{p} + \frac{(u|\nabla v|)^{(p-1)q}}{q[f(v)]^q}\Big) - \frac{p}{q}\frac{(u|\nabla v|)^{(p-1)q}}{[f(v)]^q} - \frac{pu^{p-1}|\nabla u||\nabla v|^{p-1}}{f(v)} \\
&\quad + \frac{u^p f'(v)|\nabla v|^p}{[f(v)]^2} + \frac{pu^{p-1}|\nabla v|^{p-2}}{f(v)}\{|\nabla u||\nabla v| - \nabla u.\nabla v\}
\end{aligned}$$

Recall from Young's inequality, for non-negative a and b, we have

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \tag{2.1}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Equality holds if $a^p = b^q$.
So using Young's Inequality we have,

$$p\Big(\frac{|\nabla u|^p}{p} + \frac{(u|\nabla v|)^{(p-1)q}}{q[f(v)]^q}\Big) \geq \frac{pu^{p-1}|\nabla u||\nabla v|^{p-1}}{f(v)} \tag{2.2}$$

Which is possible since both u and f are non negative. Equality holds when

$$|\nabla u| = \frac{u}{[f(v)]^{\frac{q}{p}}}|\nabla v| \tag{2.3}$$

Again using the fact that, $f'(y) \geq (p-1)[f(y)^{\frac{p-2}{p-1}}]$ we have

$$\frac{u^p f'(v)|\nabla v|^p}{[f(v)]^2} \geq \frac{p}{q}\frac{(u|\nabla v|)^{(p-1)q}}{[f(v)]^q} \tag{2.4}$$

Equality holds when

$$f'(y) = (p-1)[f(y)^{\frac{p-2}{p-1}}]\,. \tag{2.5}$$

Combining (2.2) and (2.4) we obtain $L(u,v) \geq 0$. Equality holds when (2.3) and (2.5) together with $|\nabla u||\nabla v| = \nabla u.\nabla v$ holds simultaneously.

Solving for (2.5) one obtains $f(v) = v^{p-1}$. So when, $L(u,v)(x_0) = 0$ and $u(x_0) \neq 0$, then (2.2) together with $f(v) = v^{p-1}$ and $|\nabla u||\nabla v| = \nabla u.\nabla v$ yields,

$$\nabla\big(\frac{u}{v}\big)(x_0) = 0.$$

If $u(x_0) = 0$, then $\nabla u = 0$ a.e. on $\{u(x) = 0\}$ and $\nabla\big(\frac{u}{v}\big)(x_0) = 0$. □

3. Applications

We begin this section with the application of the above Picone's identity in the nonlinear framework. As is well understood today that Picone's identity plays a significant role in the proof of Sturmian comparison theorems, Hardy-Sobolev inequalities, eigenvalue problems, determining Morse index etc. In this section, following the spirit of [1], we will give some applications of the nonlinear Picone's identity.

Hardy type result. We start this part with a theorem which can be applied to prove Hardy type inequality following the same method as in [1].

Theorem 3.1. *Assume that there is a $v \in C^1$ satisfying*

$$-\Delta_p v \geq \lambda g f(v) \quad v > 0 \quad \text{in } \Omega.$$

for some $\lambda > 0$ and nonnegative continuous function g. Then for any $u \in C_c^\infty(\Omega)$; $u \geq 0$ it holds that

$$\int_\Omega |\nabla u|^p \geq \lambda \int_\Omega g|u|^p \tag{3.1}$$

where, f satisfies $f'(y) \geq (p-1)[f(y)^{\frac{p-2}{p-1}}]$.

Proof. Let $\Omega_0 \subset \Omega$, Ω_0 be compact. Take $\phi \in C_0^\infty(\Omega)$, $\phi > 0$. By Theorem 2.1, we have

$$\begin{aligned}
0 &\leq \int_{\Omega_0} L(\phi, v) \leq \int_\Omega L(\phi, v) \\
&= \int_\Omega R(\phi, v) = \int_\Omega |\nabla \phi|^p - \nabla\big(\frac{\phi^p}{f(v)}\big)|\nabla v|^{p-2}\nabla v \\
&= \int_\Omega |\nabla \phi|^p + \nabla\big(\frac{\phi^p}{f(v)}\big)\Delta_p v \\
&\leq \int_\Omega |\nabla \phi|^p - \lambda \int_\Omega g\phi^p.
\end{aligned}$$

Letting $\phi \to u$, we have (3.1). □

Sturmium Comparison Principle. Comparison principles always played an important role in the qualitative study of partial differential equation. We present here a nonlinear version of the Sturmium comparison principle.

Theorem 3.2. *Let f_1 and f_2 are the two weight functions such that $f_1 < f_2$ and f satisfies $f'(y) \geq (p-1)[f(y)^{\frac{p-2}{p-1}}]$. If there is a positive solution u satisfying*

$$-\Delta_p u = f_1(x)|u|^{p-2}u \text{ for } x \in \Omega, \quad u = 0 \quad \text{on } \partial\Omega.$$

Then any nontrivial solution v of

$$\begin{gathered} -\Delta_p v = f_2(x)f(v) \quad \text{for } x \in \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{3.2}$$

must change sign.

Proof. Let us assume that there exists a solution $v > 0$ of (3.2) in Ω. Then by Picone's identity we have

$$\begin{aligned} 0 \leq \int_\Omega L(u,v) &= \int_\Omega R(u,v) \\ &= \int_\Omega |\nabla u|^p - \nabla\Big(\frac{u^p}{f(v)}\Big)|\nabla v|^{p-2}\nabla v \\ &= \int_\Omega f_1(x)u^p - f_2(x)u^p \\ &= \int_\Omega (f_1 - f_2)u^p < 0, \end{aligned}$$

which is a contradiction. Hence, v changes sign in Ω. □

Liouville type result. In this section we present a Liouville type result for p-Laplacian. Existence of solution for some equation having non-variational structure is generally obtained using the bifurcation method and by obtaining a priori estimates. With this in mind we give a proof of Liouville type result motivated by [5].

Theorem 3.3. *Let $c_0 > 0$, $p > 1$ and f satisfy $f'(y) \geq (p-1)[f(y)^{\frac{p-2}{p-1}}]$. Then the inequality*

$$-\Delta_p v \geq c_0 f(v) \tag{3.3}$$

has no positive solution in $W^{1,p}_{\rm loc}(\mathbb{R}^n)$.

Proof. We start by assuming that v is a positive solution of (3.3). Choose $R > 0$ and let ϕ_1 be the first eigenfunction corresponding to the first eigenvalue $\lambda_1(B_R(y))$ such that $\lambda_1(B_R(y)) < c_0$.

Taking $\frac{\phi_1^p}{f(v)}$ as a test function, which is valid since by Vazquez maximum principle [8], $\frac{\phi_1^p}{f(v)} \in W^{1,p}(B_R(y))$. Hence,

$$c_0 \int_{B_R(y)} \phi_1^p - \int_{B_R(y)} |\nabla \phi_1|^p \leq -\int_{B_R(y)} R(\phi_1, v) \leq 0\,.$$

Tt follows that

$$c_0 \leq \frac{\int_{B_R(y)} |\nabla \phi_1|^p}{\int_{B_R(y)} \phi_1^p} = \lambda_1(B_R(y)) < c_0,$$

which is a contradiction. □

Quasilinear system with singular nonlinearity. In this part we will start with a singular system of elliptic equations often occurring in chemical heterogeneous catalyst dynamics. We will show that Picone's Identity yields a linear relationship between u and v. For more information on the singular elliptic equations we refer to [2, 4] and the reference therein.

Consider the singular system of elliptic equations

$$\begin{gathered}
-\Delta_p u = f(v) \quad \text{in } \Omega \\
-\Delta_p v = \frac{[f(v)]^2}{u^{p-1}} \quad \text{in } \Omega \\
u > 0, \quad v > 0 \quad \text{in } \Omega \\
u = 0, \quad v = 0 \quad \text{on } \partial\Omega.
\end{gathered} \tag{3.4}$$

where f satisfies $f'(y) \geq (p-1)[f(y)^{\frac{p-2}{p-1}}]$. We have the following result.

Theorem 3.4. *Let (u,v) be a weak solution of (3.4) and f satisfy $f'(y) \geq (p-1)[f(y)^{\frac{p-2}{p-1}}]$. Then $u = c_1 v$ where c_1 is a constant.*

Proof. Let (u,v) be the weak solution of (3.4). Now for any ϕ_1 and ϕ_2 in $W_0^{1,p}(\Omega)$, we have

$$\int_\Omega |\nabla u|^{p-2}|\nabla u|\nabla\phi_1 dx = \int_\Omega f(v)\phi_1 dx, \tag{3.5}$$

$$\int_\Omega |\nabla u|^{p-2}|\nabla u|\nabla\phi_2 dx = \int_\Omega \frac{[f(v)]^2}{u^{p-1}}\phi_2 dx. \tag{3.6}$$

Choosing $\phi_1 = u$ and $\phi_2 = u^p/f(v)$ in (3.5) and (3.6) we obtain

$$\int_\Omega |\nabla u|^p dx = \int_\Omega u f(v) dx = \int_\Omega \nabla(\frac{u^p}{f(v)})|\nabla v|^{p-2}\nabla v dx.$$

Hence we have

$$\int_\Omega R(u,v) dx = \int_\Omega \big(|\nabla u|^p - \nabla(\frac{u^p}{f(v)})|\nabla v|^{p-2}\nabla v\big) dx = 0.$$

By the positivity of $R(u,v)$ we have that $R(u,v) = 0$ and hence

$$\nabla(\tfrac{u}{v}) = 0$$

which gives $u = c_1 v$ where c_1 is a constant. $\square$

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